Intermediate ALGEBRA

FOURTH EDITION

Y

TUSSY • GUSTAFSON • KOENIG

Get the most out of each worked example by using all of its features.

EXAMPLE 1 Here, we state the given problem.

Strategy Then, we explain what will be done to solve the problem.

WHY Next, we explain why it will be done this way.

Solution The steps that follow show how the problem is solved by using the given strategy.

1ST STEP

The given problem = **The result of 1ST STEP** *This author note explains the 1ST Step*

2ND STEP

= **The result of 2ND STEP** *This author note explains the 2ND Step*

3RD STEP

= **The result of 3RD STEP *(the answer)*** *This author note explains the 3RD Step*

Self Check 1 A Similar Problem

Now Try **Problem 45**

After reading the example, try the Self Check problem to test your understanding. The answer is given at the end of the section, right before the Study Set.

After you work the Self Check, you are ready to try a similar problem in the Guided Practice section of the Study Set.

EDITION

4

INTERMEDIATE ALGEBRA

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To the mathematics instructors of Edgewood High School, the University of Redlands, and California State University at Los Angeles, whose love of their discipline and dedication to their students continue to be an inspiration to me

ALAN S. TUSSY

To my wife, Carol, whose support has been invaluable

R. DAVID GUSTAFSON

To my husband, Brian, with love; and my daughters, Ashley, Brianna, and Carly, whom I love very much and who bring me great joy

DIANE R. KOENIG

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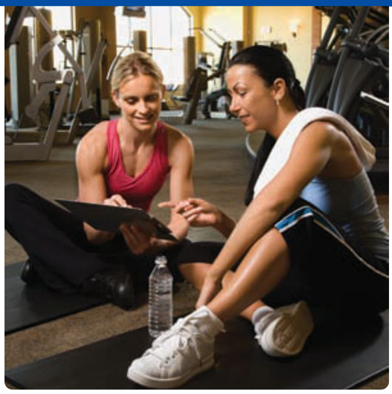

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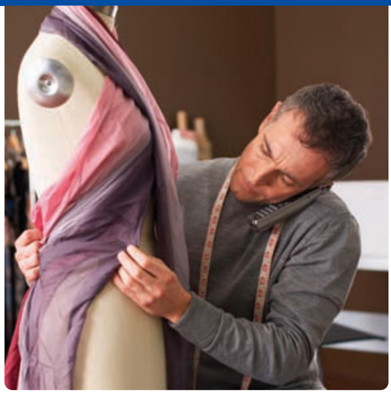

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PREFACE

Intermediate Algebra, Fourth Edition, is more than a simple upgrade of the third edition. Substantial changes have been made to the worked example structure, the *Study Sets,* and the pedagogy. Throughout the revision process, our objective has been to ease teaching challenges and meet students' educational needs.

Algebra, for many of today's developmental math students, is like a foreign language. They have difficulty translating the words, their meanings, and how they apply to problem solving. With these needs in mind (and as educational research suggests), our fundamental goal is to have students read, write, think, and speak using the *language of algebra.* Instructional approaches that include vocabulary, practice, and well-defined pedagogy, along with an emphasis on reasoning, modeling, communication, and technology skills have been blended to address this need.

The most common question that students ask as they watch their instructors solve problems and as they read the textbook is . . . *Why?* The new fourth edition addresses this question in a unique way. Experience teaches us that it's not enough to know *how* a problem is solved. Students gain a deeper understanding of algebraic concepts if they know *why* a particular approach is taken. This instructional truth was the motivation for adding a **Strategy** and **Why** explanation to the solution of each worked example. The fourth edition now provides, on a consistent basis, a concise answer to that all-important question: *Why?*

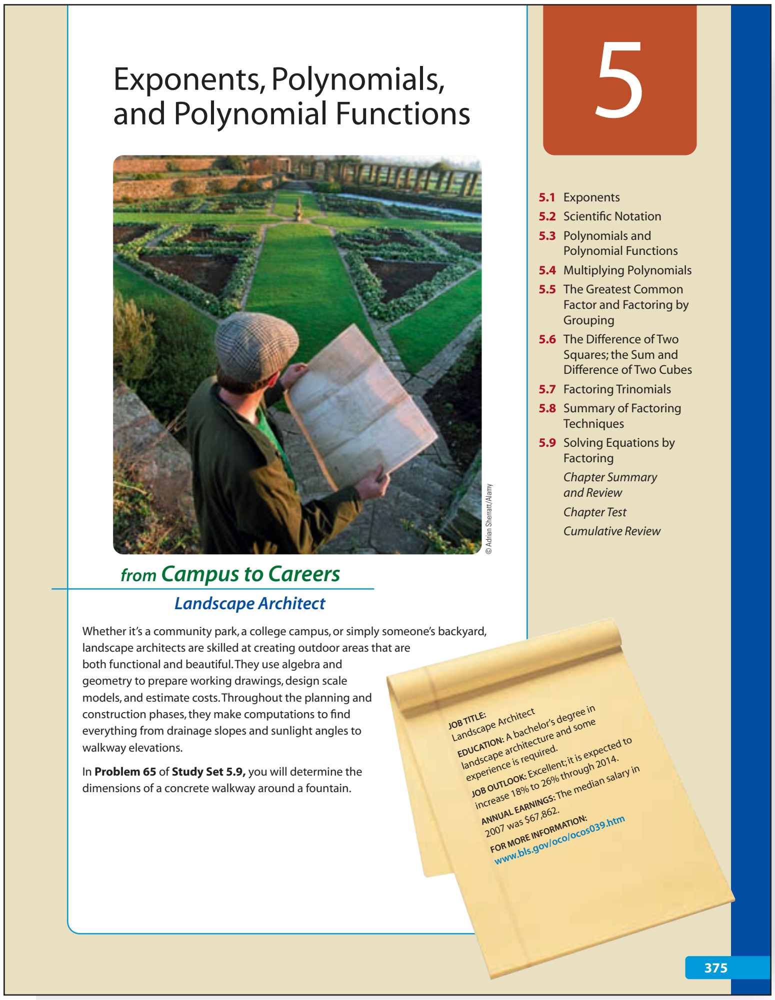
These are just two of several reasons we trust that this revision will make this course a better experience for both instructors and students.

NEW TO THIS EDITION

- **New Chapter Openers**
- **New Worked Example Structure**
- **New Study Skills Workshop Module**
- **New Language of Algebra, Success Tip, and Caution Boxes**
- **New Chapter Objectives**
- **New Guided Practice and Try It Yourself Sections in the Study Sets**
- **New Chapter Summary and Review**
- **New Study Skills Checklists**

Chapter Openers That Answer the Question: When Will I Use This?

Instructors are asked this question time and again by students. In response, we have written chapter openers called *From Campus to Careers.* This feature highlights vocations that require various algebraic skills. Designed to inspire career exploration, each includes job outlook, educational requirements, and annual earnings information. Careers presented in the openers are tied to an exercise found later in the *Study Sets.*



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Exponents, Polynomials, and Polynomial Functions

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from **Campus to Careers**

Landscape Architect

Whether it's a community park, a college campus, or simply someone's backyard, landscape architects are skilled at creating outdoor areas that are both functional and beautiful. They use algebra and geometry to prepare working drawings, design scale models, and estimate costs. Throughout the planning and construction phases, they make computations to find everything from drainage slopes and sunlight angles to walkway elevations.

In **Problem 65** of **Study Set 5.9,** you will determine the dimensions of a concrete walkway around a fountain.

JOB TITLE: Landscape Architect
EDUCATION: A bachelor's degree in landscape architecture and some experience is required.
JOB OUTLOOK: Excellent; it is expected to increase 18% to 26% through 2014.
ANNUAL EARNINGS: The median salary in 2007 was $67,862.
FOR MORE INFORMATION: www.bls.gov/oco/ocos039.htm

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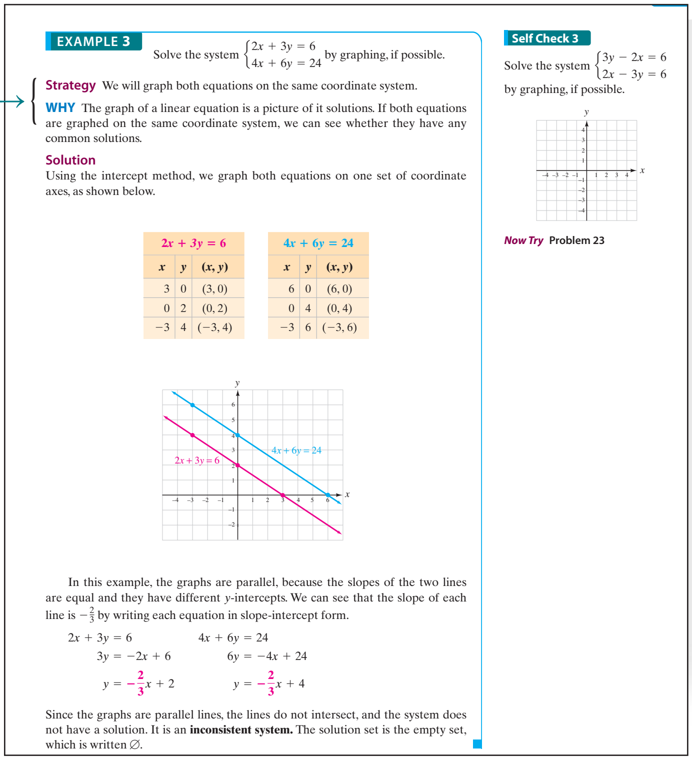
Examples That Tell Students Not Just How, But WHY

Why? That question is often asked by students as they watch their instructor solve problems in class and as they are working on problems at home. It's not enough to know *how* a problem is solved. Students gain a deeper understanding of the algebraic concepts if they know *why* a particular approach was taken. This instructional truth was the motivation for adding a *Strategy* and *Why* explanation to each worked example.


EXAMPLE 3 Solve the system $\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 24 \end{cases}$ by graphing, if possible.

Strategy We will graph both equations on the same coordinate system.

WHY The graph of a linear equation is a picture of it solutions. If both equations are graphed on the same coordinate system, we can see whether they have any common solutions.

Solution

Using the intercept method, we graph both equations on one set of coordinate axes, as shown below.

$2x + 3y = 6$		
x	y	(x, y)
3	0	(3, 0)
0	2	(0, 2)
−3	4	(−3, 4)

$4x + 6y = 24$		
x	y	(x, y)
6	0	(6, 0)
0	4	(0, 4)
−3	6	(−3, 6)

In this example, the graphs are parallel, because the slopes of the two lines are equal and they have different y-intercepts. We can see that the slope of each line is $-\frac{2}{3}$ by writing each equation in slope-intercept form.

$$2x + 3y = 6 \qquad 4x + 6y = 24$$
$$3y = -2x + 6 \qquad 6y = -4x + 24$$
$$y = -\frac{2}{3}x + 2 \qquad y = -\frac{2}{3}x + 4$$

Since the graphs are parallel lines, the lines do not intersect, and the system does not have a solution. It is an **inconsistent system.** The solution set is the empty set, which is written ∅.

Self Check 3

Solve the system $\begin{cases} 3y - 2x = 6 \\ 2x - 3y = 6 \end{cases}$ by graphing, if possible.

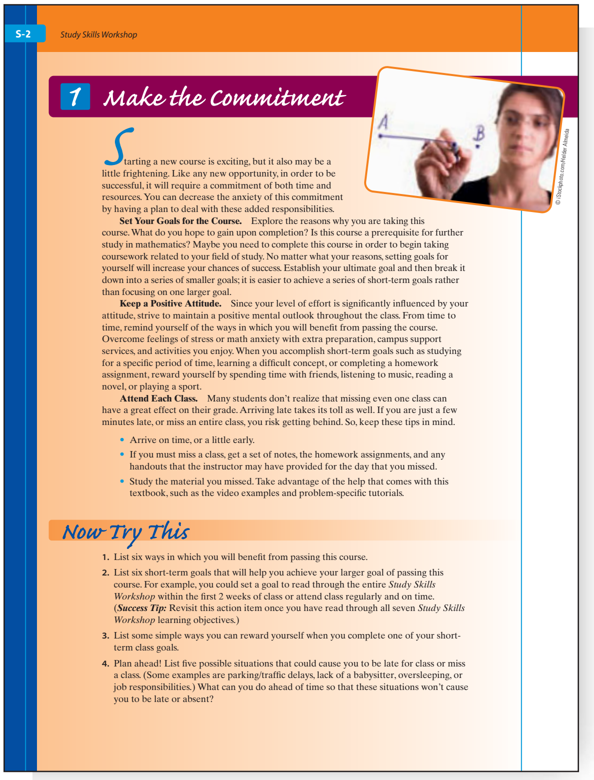
Now Try Problem 23

Examples That Offer Immediate Feedback

Each worked example includes a *Self Check.* These can be completed by students on their own or as classroom lecture examples, which is how Alan Tussy uses them. Alan asks selected students to read aloud the *Self Check* problems as he writes what the student says on the board. The other students, with their books open to that page, can quickly copy the *Self Check* problem to their notes. This speeds up the note-taking process and encourages student participation in his lectures. It also teaches students how to read mathematical symbols. Each *Self Check* answer is printed adjacent to the corresponding problem in the *Annotated Instructor's Edition* for easy reference. *Self Check* solutions can be found at the end of each section in the student edition before each *Study Set.*

Examples That Ask Students to Work Independently

Each worked example ends with a *Now Try* problem. These are the final step in the learning process. Each one is linked to a similar problem found within the *Guided Practice* section of the *Study Sets.*


S-2 Study Skills Workshop

1 Make the Commitment

Starting a new course is exciting, but it also may be a little frightening. Like any new opportunity, in order to be successful, it will require a commitment of both time and resources. You can decrease the anxiety of this commitment by having a plan to deal with these added responsibilities.

Set Your Goals for the Course. Explore the reasons why you are taking this course. What do you hope to gain upon completion? Is this course a prerequisite for further study in mathematics? Maybe you need to complete this course in order to begin taking coursework related to your field of study. No matter what your reasons, setting goals for yourself will increase your chances of success. Establish your ultimate goal and then break it down into a series of smaller goals; it is easier to achieve a series of short-term goals rather than focusing on one larger goal.

Keep a Positive Attitude. Since your level of effort is significantly influenced by your attitude, strive to maintain a positive mental outlook throughout the class. From time to time, remind yourself of the ways in which you will benefit from passing the course. Overcome feelings of stress or math anxiety with extra preparation, campus support services, and activities you enjoy. When you accomplish short-term goals such as studying for a specific period of time, learning a difficult concept, or completing a homework assignment, reward yourself by spending time with friends, listening to music, reading a novel, or playing a sport.

Attend Each Class. Many students don't realize that missing even one class can have a great effect on their grade. Arriving late takes its toll as well. If you are just a few minutes late, or miss an entire class, you risk getting behind. So, keep these tips in mind.

- Arrive on time, or a little early.
- If you must miss a class, get a set of notes, the homework assignments, and any handouts that the instructor may have provided for the day that you missed.
- Study the material you missed. Take advantage of the help that comes with this textbook, such as the video examples and problem-specific tutorials.

Now Try This

1. List six ways in which you will benefit from passing this course.
2. List six short-term goals that will help you achieve your larger goal of passing this course. For example, you could set a goal to read through the entire *Study Skills Workshop* within the first 2 weeks of class or attend class regularly and on time. (***Success Tip:*** Revisit this action item once you have read through all seven *Study Skills Workshop* learning objectives.)
3. List some simple ways you can reward yourself when you complete one of your short-term class goals.
4. Plan ahead! List five possible situations that could cause you to be late for class or miss a class. (Some examples are parking/traffic delays, lack of a babysitter, oversleeping, or job responsibilities.) What can you do ahead of time so that these situations won't cause you to be late or absent?

Emphasis on Study Skills

Intermediate Algebra begins with a *Study Skills Workshop* module. Instead of simple, unrelated suggestions printed in the margins, this module contains one-page discussions of study skills topics followed by a *Now Try This* section offering students actionable skills, assignments, and projects that will impact their study habits throughout the course.

> **The Language of Algebra** *Commutative* is a form of the word *commute*, meaning to go back and forth. *Commuter* trains take people to and from work.

Integrated Focus on the Language of Algebra

Language of Algebra boxes draw connections between mathematical terms and everyday references to reinforce the language of algebra approach that runs throughout the text.

Guidance When Students Need It Most

Appearing at key teaching moments, *Success Tips* and *Caution* boxes improve students' problem-solving abilities, warn students of potential pitfalls, and increase clarity.

> **Success Tip** When asked to factor a polynomial, we must be sure to factor it completely. After factoring a polynomial, always check to see whether any of the factors in the result can be factored further.

> **Caution!** If two variables are used to represent two unknown quantities, we must form a system of two equations to find the unknowns.

Useful Objectives Help Keep Students Focused

Each section begins with a set of numbered *Objectives* that focus students' attention on the skills that they will learn. As each objective is discussed in the section, the number and heading reappear to the reader to remind them of the objective at hand.

Objectives

1. Read and interpret inequality symbols.
2. Graph intervals and use interval and set-builder notation.
3. Solve linear inequalities using properties of inequality.
4. Use linear inequalities to solve problems.

SECTION 4.1
Solving Linear Inequalities in One Variable

Traffic signs often appear in front of schools. From the figure a motorist knows that

- A speed *greater than* 25 miles per hour breaks the law and could possibly result in a ticket for speeding.
- A speed *less than or equal to* 25 miles per hour is within the posted speed limit.

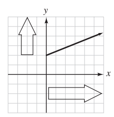
Statements such as these can be expressed using *inequality symbols.*

SECTION 2.5 **STUDY SET**

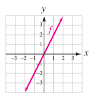
VOCABULARY

Fill in the blanks.

1. Sets of ordered pairs are called ________.
2. In a relation, the set of all first components is called the ________ of the relation.
3. In a relation, the set of all second components is called the ______ of the relation.
4. A ________ is a set of ordered pairs (a relation) in which to each first component there corresponds exactly one second component.
5. In a function, the set of first components (the input values) is called the ________ of the function. The set of second components (the output values) is called the ______.
6. If y is a function of x, x is called the ___________ variable and y is called the __________ variable.
7. $y = f(x)$ is read as y is a ________ of x or y _______ f of x.
8. A ______ function is a function that can be written in the f
9. The
beca



16. Use the graph of function f on the right to find each of the following.
 a. $f(-2)$
 b. $f(0)$
 c. $f(1)$



NOTATION

Complete each solution.

17. If $f(x) = x^2 - 3x$, find $f(-5)$.

$f(x) = x^2 - 3x$
$) = (-5)^2 - 3(\ \ )$
$= \ \ + 15$

TRY IT YOURSELF

Complete each table.

81. $f(t) = |t - 2|$

t	$f(t)$
-1.7	
0.9	
5.4	

82. $f(r) = -2r^2 + 1$

Input	Output
-1.7	
0.9	
5.4	

83. $g(x) = x^3$

Input	Output
$-\frac{3}{4}$	
$\frac{1}{6}$	
$\frac{5}{2}$	

84. $g(x) = 2\left(-x - \frac{1}{4}\right)$

x	$g(x)$
$-\frac{3}{4}$	
$\frac{1}{8}$	
$\frac{5}{2}$	

Find the domain of each function.

85. $g(x) = \dfrac{x}{6 - x}$
86. $H(x) = \dfrac{4}{3 - 2x}$
87. $s(x) = |x - 7|$
88. $t(x) = \left|\dfrac{2x}{3} + 1\right|$

Thoroughly Revised Study Sets

The *Study Sets* have been thoroughly revised to ensure that every example type covered in the section is represented in the *Guided Practice* problems. Particular attention was paid to developing a gradual level of progression within problem types.

Guided Practice Problems

All of the problems in the *Guided Practice* portion of the *Study Sets* are linked to an associated worked example or objective from that section. This feature promotes student success by referring them to the proper worked example(s) or objective(s) if they encounter difficulties solving homework problems.

Try It Yourself

To promote problem recognition, many *Study Sets* now include a collection of *Try It Yourself* problems that *do not* link to worked examples. These problem types are thoroughly mixed, giving students an opportunity to practice decision making and strategy selection as they would when taking a test or quiz.

Comprehensive End-of-Chapter Summary with Integrated Chapter Review

The end-of-chapter material has been redesigned to function as a complete study guide for students. New chapter summaries that include definitions, concepts, and examples, by section, have been written. Review problems for each section immediately follow the summary for that section.

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CHAPTER 5 SUMMARY AND REVIEW

SECTION 5.1 Exponents

DEFINITIONS AND CONCEPTS	EXAMPLES
An **exponent** indicates repeated multiplication. It tells how many times the **base** is to be used as a factor. If n is a natural number, $x^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors of } x}$ where x is the **base** and n the **exponent.**	Identify the base and the exponent in each expression. $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ — 2 is the base, 6 is the exponent; $p^5 = p \cdot p \cdot p \cdot p \cdot p$ — p is the base, 5 is the exponent; $(-ab)^3 = (-ab)(-ab)(-ab)$ — $-ab$ is the base, 3 is the exponent; $7c^4 = 7 \cdot c \cdot c \cdot c \cdot c$ — c is the base, 4 is the exponent
Rules for Exponents: If m and n represent integers and there are no divisions by 0, then	Simplify each expression:
Product rule: $x^m x^n = x^{m+n}$; *Quotient rule:* $\frac{x^m}{x^n} = x^{m-n}$	$x^2 \cdot x^3 = x^{2+3} = x^5$; $\frac{m^9}{m^3} = m^{9-3} = m^6$
Power rule: $(x^m)^n = x^{mn}$; *Power of a product:* $(xy)^m = x^m y^m$	$(r^4)^5 = r^{4 \cdot 5} = r^{20}$; $(2y)^5 = 2^5 y^5 = 32y^5$
Power of a quotient: $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$; *Zero exponent:* $x^0 = 1$	$\left(\frac{a}{3}\right)^4 = \frac{a^4}{3^4} = \frac{a^4}{81}$; $5^0 = 1$
Negative exponent: $x^{-n} = \frac{1}{x^n}$; *Exponent of 1:* $x^1 = x$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$; $7^1 = 7$
Negative exponents appearing in fractions: $\frac{1}{x^{-n}} = x^n$; $\frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$; $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$	$\frac{1}{10^{-2}} = 10^2 = 100$; $\frac{4^{-2}}{m^{-3}} = \frac{m^3}{4^2} = \frac{m^3}{16}$; $\left(\frac{7}{a}\right)^{-2} = \left(\frac{a}{7}\right)^2 = \frac{a^2}{7^2} = \frac{a^2}{49}$

REVIEW EXERCISES

Evaluate each expression.

1. 3^5 **2.** -2^5

3. $(-4)^3$ **4.** $\left(\frac{2}{3}\right)^2$

Simplify each expression. Write answers using positive exponents.

5. $x^4 \cdot x^2$ **6.** $m^{-3}n^{-4}m^6n^{-1}$

7. $\frac{(4m^6)^3}{m^2}$ **8.** $(-t^2)^2(t^3)^3$

Study Skills That Point Out Common Student Mistakes

In Chapter 1, we have included four *Study Skills Checklists* designed to actively show students how to effectively use the key features in this text. Subsequent chapters include one checklist just before the *Chapter Summary and Review* that provides another layer of preparation to promote student success. These *Study Skills Checklists* warn students of common errors, giving them time to consider these pitfalls before taking their exam.

STUDY SKILLS CHECKLIST

Preparing for the Chapter 3 Test

In Chapter 3 you learned five methods to solve a system of linear equations. You also learned how to solve problems using systems of equations. As you prepare for the exam over this material, make sure you also review the following checklist.

☐ To check a proposed solution of a system of equations, be sure the coordinates of the ordered pair satisfies *both* equations.

Is $(3, -2)$ a solution of the system $\begin{cases} 3x + 4y = 1 \\ x + 2y = -1 \end{cases}$?

$$\begin{aligned} 3(3) + 4(-2) &= 1 & 3 + 2(-2) &= -1 \\ 9 - 8 &= 1 & 3 - 4 &= -1 \\ 1 &= 1 \quad \text{True} & -1 &= -1 \quad \text{True} \end{aligned}$$

Yes, $(3, -2)$ is a solution of the system.

☐ When using the substitution or the addition (elimination) method, remember to find the value of *both* the variables.

For the system of linear equations $\begin{cases} x = 2y - 3 \\ x + 4y = 3 \end{cases}$, the y-coordinate of the solution is $y = 1$.

To find the x-value, substitute 1 for y in either equation:

$$\begin{aligned} x &= 2y - 3 \\ x &= 2(1) - 3 \\ x &= 2 - 3 \\ x &= -1 \end{aligned}$$

The solution is $(-1, 1)$.

☐ The equations of a system must be written in standard form before the corresponding augmented matrix can be written.

$\begin{cases} x = y + 4 \\ y = -2x + 5 \end{cases}$ Subtract y / Add $2x$

$\begin{cases} x - y = 4 \\ 2x + y = 5 \end{cases}$ $\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 2 & 1 & 5 \end{array}\right]$

☐ To evaluate a 2×2 determinant, multiply the numbers on the main diagonal *minus* multiply the numbers on the other diagonal and simplify.

$$\begin{aligned} \begin{vmatrix} 3 & -2 \\ 5 & -4 \end{vmatrix} &= 3(-4) - (-2)(5) \\ &= -12 - (-10) \\ &= -12 + 10 \\ &= -2 \end{aligned}$$

TRUSTED FEATURES

- **Study Sets** found in each section offer a multifaceted approach to practicing and reinforcing the concepts taught in each section. They are designed for students to methodically build their knowledge of the section concepts, from basic recall to increasingly complex problem solving, through reading, writing, and thinking mathematically.

  Vocabulary—Each *Study Set* begins with the important *Vocabulary* discussed in that section. The fill-in-the-blank vocabulary problems emphasize the main concepts taught in the chapter and provide the foundation for learning and communicating the language of algebra.

  Concepts—In *Concepts,* students are asked about the specific subskills and procedures necessary to successfully complete the *Guided Practice* and *Try It Yourself* problems that follow.

  Notation—In *Notation,* the students review the new symbols introduced in a section. Often, they are asked to fill in steps of a sample solution. This strengthens their ability to read and write mathematics and prepares them for the *Guided Practice* problems by modeling solution formats.

  Guided Practice—The problems in *Guided Practice* are linked to an associated worked example or objective from that section. This feature promotes student success by referring them to the proper examples if they encounter difficulties solving homework problems.

  Try It Yourself—To promote problem recognition, the *Try It Yourself* problems are thoroughly mixed and are *not* linked to worked examples, giving students an opportunity to practice decision-making and strategy selection as they would when taking a test or quiz.

  Applications—The *Applications* provide students the opportunity to apply their newly acquired algebraic skills to relevant and interesting real-life situations.

  Writing—The *Writing* problems help students build mathematical communication skills.

  Review—The *Review* problems consist of randomly selected problems from previous chapters. These problems are designed to keep students' successfully mastered skills up-to-date before they move on to the next section.

- **Detailed Author Notes** that guide students along in a step-by-step process appear in the solutions to every worked example.
- **The Five-Step Problem-Solving Strategy** guides students through applied worked examples using the *analyze, form, solve, state,* and *check* process. This approach clarifies the thought process and algebra skills necessary to solve a wide variety of problems. As a result, students' confidence is increased and their problem-solving abilities are strengthened.
- **Think It Through** features make the connection between mathematics and student life. These relevant topics often require algebra skills from the chapter to be applied to a real-life situation. Topics include tuition costs, student enrollment, job opportunities, credit cards, and many more.
- **Chapter Tests,** at the end of every chapter, can be used as preparation for the class exam.

- **Cumulative Reviews** follow the end-of-chapter material and keep students' skills current before moving on to the next chapter. For the Fourth Edition, each problem is now linked to the associated section from which the problem came for ease of reference. The final *Cumulative Review* is often used by instructors as a Final Exam Review.
- **Using Your Calculator** features (formerly called *Calculator Snapshots*) are designed for instructors who wish to use calculators as part of the instruction in this course. These features introduce keystrokes and show how scientific and graphing calculators can be used to solve problems. In the *Study Sets,* icons are used to denote problems that may be solved using a calculator.

CHANGES TO THE TABLE OF CONTENTS

Based on feedback from colleagues and users of the third edition, the following changes have been made to the table of contents in an effort to further streamline the text and make it even easier to use.

- The Chapter 1 topics have been reorganized and expanded:

  1.1 *The Language of Algebra*

  1.2 *The Real Numbers*

  1.3 *Operations with Real Numbers*

  1.4 *Simplifying Algebraic Expressions Using Properties of Real Numbers*

  1.5 *Solving Linear Equations Using Properties of Equality*

  1.6 *Solving Formulas; Geometry* (formulas have been moved to their own section and geometry has been added)

  1.7 *Using Equations to Solve Problems*

  1.8 *More about Problem Solving*

- The Chapter 3 topics have been reorganized with two new sections added to emphasize problem-solving:

  3.1 *Solving Systems of Equations by Graphing*

  3.2 *Solving Systems of Equations Algebraically*

  3.3 *Problems Solving Using Systems of Two Equations* (new)

  3.4 *Solving Systems of Equations in Three Variables*

  3.5 *Problem Solving Using Systems of Three Equations* (new)

  3.6 *Solving Systems of Equations Using Matrices*

  3.7 *Solving Systems of Equations Using Determinants*

- The Chapter 6 topics have been reorganized with two new sections added, one of them emphasizing problem solving:

  6.1 *Rational Functions and Simplifying Rational Expressions*

  6.2 *Multiplying and Dividing Rational Expressions*

  6.3 *Adding and Subtracting Rational Expressions*

  6.4 *Simplifying Complex Fractions*

  6.5 *Dividing Polynomials*

  6.6 *Synthetic Division* (new)

6.7 *Solving Rational Equations*

6.8 *Problem Solving Using Rational Equations* (new)

6.9 *Proportion and Variation*

- The section *Complex Numbers* has been moved from Chapter 8 to Section 7.7.
- The Chapter 10 topics have been reorganized:

10.1 *The Circle and the Parabola*

10.2 *The Ellipse*

10.3 *The Hyperbola*

10.4 *Solving Nonlinear Systems of Equations*

GENERAL REVISIONS AND OVERALL DESIGN

- We have edited the prose so that it is even more clear and concise.
- Strategic use of color has been implemented within the new design to help the visual learner.
- Added color in the solutions highlights key steps and improves readability.
- We have updated much of the data and graphs and have added scaling to all axes in all graphs.
- We have added more real-world applications.
- We have included more problem-specific photographs and improved the clarity of the illustrations.

INSTRUCTOR RESOURCES

Print Ancillaries

Instructor's Resource Binder (0-538-73675-5)
Maria H. Andersen, *Muskegon Community College*
NEW! Each section of the main text is discussed in uniquely designed *Teaching Guides* containing instruction tips, examples, activities, worksheets, overheads, assessments, and solutions to all worksheets and activities.

Complete Solutions Manual (0-538-73401-9)
Gowribalan Ananda Vamadeva, *University of Cincinnati*
The *Complete Solutions Manual* provides worked-out solutions to all of the problems in the text.

Annotated Instructor's Edition (1-4390-4837-3)
The *Annotated Instructor's Edition* provides the complete student text with answers next to each respective exercise. New to this edition: Teaching Examples have been added for each worked example.

Electronic Ancillaries

Enhanced WebAssign
Instant feedback and ease of use are just two reasons why WebAssign is the most widely used homework system in higher education. WebAssign's homework delivery system allows you to assign, collect, grade, and record homework assignments via the web. Personal Study Plans provide diagnostic quizzing for each chapter that identifies concepts that students still need to master, and directs them to the appropriate review material. And now, this proven system has been enhanced to include links to textbook sections, video examples, and problem-specific tutorials. For further utility, students will also have the option to purchase an online multimedia eBook of the text. Enhanced WebAssign is more than a homework system—it is a complete learning system for math students. Contact your local representative for ordering details.

Solution Builder
Easily build solution sets for homework or exams using *Solution Builder's* online solutions manual. Visit www.cengage.com/solutionbuilder

PowerLecture with ExamView® (0-538-73344-6)
This CD-ROM provides the instructor with dynamic media tools for teaching. Create, deliver, and customize tests (both print and online) in minutes with *ExamView® Computerized Testing Featuring Algorithmic Equations.* Easily build solution sets for homework or exams using *Solution Builder's* online solutions manual. Microsoft® PowerPoint® lecture slides, figures from the book, and Test Bank, in electronic format, are also included on this CD-ROM.

Text Specific Videos (0-538-73400-0)
Rena Petrello, *Moorpark College*
These 10- to 20-minute problem-solving lessons cover nearly every learning objective from each chapter in the Tussy/Gustafson/Koenig text. Recipient of the "Mark Dever Award for Excellence in Teaching," Rena Petrello presents each lesson using her experience teaching online mathematics courses. It was through this online teaching experience that Rena discovered the lack of suitable content for online instructors, which caused her to develop her own video lessons—and ultimately create this video project. These videos have won four awards: two Telly Awards, one Communicator Award, and one Aurora Award (an international honor). Students will love the additional guidance and support when they have missed a class or when they are preparing for an upcoming quiz or exam. The videos are available for purchase as a set of DVDs or online via CengageBrain.com.

STUDENT RESOURCES

Print Ancillaries

Student Workbook (0-538-73403-5)
Maria H. Andersen, *Muskegon Community College*
NEW! Get a head-start. The *Student Workbook* contains all of the assessments, activities, and worksheets from the *Instructor's Resource Binder* for classroom discussions, in-class activities, and group work.

Student Solutions Manual (0-538-73402-7)
Gowribalan Ananda Vamadeva, *University of Cincinnati*
The *Student Solutions Manual* provides worked-out solutions to the odd-numbered problems in the text.

Electronic Ancillaries

Enhanced WebAssign
Get instant feedback on your homework assignments with Enhanced WebAssign (assigned by your instructor). Personal Study Plans provide diagnostic quizzing for each chapter that identifies concepts that you still need to master, and directs you to the appropriate review material. This online homework system is easy to use and includes helpful links to textbook sections, video examples, and problem-specific tutorials. For further ease of use, purchase an online multimedia eBook via WebAssign.

Website *www.cengage.com/math/tussy*
Visit us on the web for access to a wealth of learning resources, including tutorials, final exams, chapter outlines, chapter reviews, web links, videos, flashcards, study skills handouts, and more!

ACKNOWLEDGMENTS

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R. David Gustafson is Professor Emeritus of Mathematics at Rock Valley College in Illinois and coauthor of several best-selling math texts, including Gustafson/Frisk's *Beginning Algebra, Intermediate Algebra, Beginning and Intermediate Algebra: A Combined Approach, College Algebra,* and the Tussy/Gustafson developmental mathematics series. His numerous professional honors include Rock Valley Teacher of the Year and Rockford's Outstanding Educator of the Year. He earned a Master of Arts from Rockford College in Illinois, as well as a Master of Science from Northern Illinois University.

Diane R. Koenig

Diane Koenig received a Bachelor of Science degree in Secondary Math Education from Illinois State University in 1980. She began her career at Rock Valley College in 1981, when she became the Math Supervisor for the newly formed Personalized Learning Center. Earning her Master's Degree in Applied Mathematics from Northern Illinois University, Ms. Koenig in 1984 had the distinction of becoming the first full-time woman mathematics faculty member at Rock Valley College. In addition to being nominated for AMATYC's Excellence in Teaching Award, Diane Koenig was chosen as the Rock Valley College Faculty of the Year by her peers in 2005, and, in 2006, she was awarded the NISOD Teaching Excellence Award as well as the Illinois Mathematics Association of Community Colleges Award for Teaching Excellence. In addition to her teaching, Ms. Koenig has been an active member of the Illinois Mathematics Association of Community Colleges (IMACC). As a member, she has served on the board of directors, on a state-level task force rewriting the course outlines for the developmental mathematics courses, and as the association's newsletter editor.

APPLICATIONS INDEX

Examples that are applications are shown with boldface page numbers.
Exercises that are applications are shown with lightface page numbers.

Marketing

Mathematics

Measurement

Medicine and Health

Miscellaneous

Politics, Government, and the Military

Science and Engineering

Sports

Travel

Weather

Study Skills Workshop

OBJECTIVES

1 Make the Commitment

2 Prepare to Learn

3 Manage Your Time

4 Listen and Take Notes

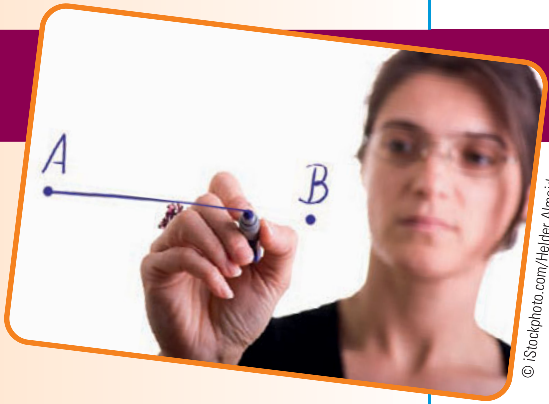
5 Build a Support System

6 Do Your Homework

7 Prepare for the Test

SUCCESS IN YOUR COLLEGE COURSES requires more than just mastery of the content. The development of strong study skills and disciplined work habits plays a crucial role as well. Good note-taking, listening, test-taking, team-building, and time management skills are habits that can serve you well, not only in this course, but throughout your life and into your future career. Students often find that the approach to learning that they used for their high school classes no longer works when they reach college. In this Study Skills Workshop, *we will discuss ways of improving and fine-tuning your study skills, providing you with the best chance for a successful college experience.*

1 Make the Commitment



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Starting a new course is exciting, but it also may be a little frightening. Like any new opportunity, in order to be successful, it will require a commitment of both time and resources. You can decrease the anxiety of this commitment by having a plan to deal with these added responsibilities.

Set Your Goals for the Course. Explore the reasons why you are taking this course. What do you hope to gain upon completion? Is this course a prerequisite for further study in mathematics? Maybe you need to complete this course in order to begin taking coursework related to your field of study. No matter what your reasons, setting goals for yourself will increase your chances of success. Establish your ultimate goal and then break it down into a series of smaller goals; it is easier to achieve a series of short-term goals rather than focusing on one larger goal.

Keep a Positive Attitude. Since your level of effort is significantly influenced by your attitude, strive to maintain a positive mental outlook throughout the class. From time to time, remind yourself of the ways in which you will benefit from passing the course. Overcome feelings of stress or math anxiety with extra preparation, campus support services, and activities you enjoy. When you accomplish short-term goals such as studying for a specific period of time, learning a difficult concept, or completing a homework assignment, reward yourself by spending time with friends, listening to music, reading a novel, or playing a sport.

Attend Each Class. Many students don't realize that missing even one class can have a great effect on their grade. Arriving late takes its toll as well. If you are just a few minutes late, or miss an entire class, you risk getting behind. So, keep these tips in mind.

- Arrive on time, or a little early.
- If you must miss a class, get a set of notes, the homework assignments, and any handouts that the instructor may have provided for the day that you missed.
- Study the material you missed. Take advantage of the help that comes with this textbook, such as the video examples and problem-specific tutorials.

Now Try This

1. List six ways in which you will benefit from passing this course.
2. List six short-term goals that will help you achieve your larger goal of passing this course. For example, you could set a goal to read through the entire *Study Skills Workshop* within the first 2 weeks of class or attend class regularly and on time. (***Success Tip:*** Revisit this action item once you have read through all seven *Study Skills Workshop* learning objectives.)
3. List some simple ways you can reward yourself when you complete one of your short-term class goals.
4. Plan ahead! List five possible situations that could cause you to be late for class or miss a class. (Some examples are parking/traffic delays, lack of a babysitter, oversleeping, or job responsibilities.) What can you do ahead of time so that these situations won't cause you to be late or absent?

2 Prepare to Learn

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Many students believe that there are two types of people—those who are good at math and those who are not—and that this cannot be changed. This is not true! You can increase your chances for success in mathematics by taking time to prepare and taking inventory of your skills and resources.

Discover Your Learning Style. Are you a visual, verbal, or auditory learner? The answer to this question will help you determine how to study, how to complete your homework, and even where to sit in class. For example, visual-verbal learners learn best by reading and writing; a good study strategy for them is to rewrite notes and examples. However, auditory learners learn best by listening, so listening to the video examples of important concepts may be their best study strategy.

Get to Know Your Textbook and Its Resources. You have made a significant investment in your education by purchasing this book and the resources that accompany it. It has been designed with you in mind. Use as many of the features and resources as possible in ways that best fit your learning style.

Know What Is Expected. Your course syllabus maps out your instructor's expectations for the course. Read the syllabus completely and make sure you understand all that is required. If something is not clear, contact your instructor for clarification.

Organize Your Notebook. You will definitely appreciate a well-organized notebook when it comes time to study for the final exam. So let's start now! Refer to your syllabus and create a separate section in the notebook for each chapter (or unit of study) that your class will cover this term. Now, set a standard order within each section. One recommended order is to begin with your class notes, followed by your completed homework assignments, then any study sheets or handouts, and, finally, all graded quizzes and tests.

Now Try This

1. To determine what type of learner you are, take the *Learning Style Survey* at http://www.metamath.com/multiple/multiple_choice_questions.html. You may also wish to take the *Index of Learning Styles Questionnaire* at http://www.engr.ncsu.edu/learningstyles/ilsweb.html, which will help you determine your learning type and offer study suggestions by type. List what you learned from taking these surveys. How will you use this information to help you succeed in class?
2. Complete the *Study Skills Checklists* found at the end of sections 1–4 of Chapter 1 in order to become familiar with the many features that can enhance your learning experience using this book.
3. Read through the list of Student Resources found in the Preface of this book. Which ones will you use in this class?
4. Read through your syllabus and write down any questions that you would like to ask your instructor.
5. Organize your notebook using the guidelines given above. Place your syllabus at the very front of your notebook so that you can see the dates over which the material will be covered and for easy reference throughout the course.

3 Manage Your Time

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Now that you understand the importance of attending class, how will you make time to study what you have learned while attending? Much like learning to play the piano, math skills are best learned by practicing a little every day.

Make the Time. In general, 2 hours of independent study time is recommended for every hour in the classroom. If you are in class 3 hours per week, plan on 6 hours per week for reviewing your notes and completing your homework. It is best to schedule this time over the length of a week rather than to try to cram everything into one or two marathon study days.

Prioritize and Make a Calendar. Because daily practice is so important in learning math, it is a good idea to set up a calendar that lists all of your time commitments, as well as the time you will need to set aside for studying and doing your homework. Consider how you spend your time each week and prioritize your tasks by importance. During the school term, you may need to reduce or even eliminate certain nonessential tasks in order to meet your goals for the term.

Maximize Your Study Efforts. Using the information you learned from determining your learning style, set up your blocks of study time so that you get the most out of these sessions. Do you study best in groups or do you need to study alone to get anything done? Do you learn best when you schedule your study time in 30-minute time blocks or do you need at least an hour before the information kicks in? Consider your learning style to set up a schedule that truly suits your needs.

Avoid Distractions. Between texting and social networking, we have so many opportunities for distraction and procrastination. On top of these, there are the distractions of TV, video games, and friends stopping by to hang out. Once you have set your schedule, honor your study times by turning off any electronic devices and letting your voicemail take messages for you. After this time, you can reward yourself by returning phone calls and messages or spending time with friends after the pressure of studying has been lifted.

Now Try This

1. Keep track of how you spend your time for a week. Rate each activity on a scale from 1 (not important) to 5 (very important). Are there any activities that you need to reduce or eliminate in order to have enough time to study this term?
2. List three ways that you learn best according to your learning style. How can you use this information when setting up your study schedule?
3. Download the *Weekly Planner Form* from www.cengage.com/math/tussy and complete your schedule. If you prefer, you may set up a schedule in Google Calendar (calendar.google.com), www.rememberthemilk.com, your cell, or your email system. Many of these have the ability to set up useful reminders and to-do lists in addition to a weekly schedule.
4. List three ways in which you are most often distracted. What can you do to avoid these distractions during your scheduled study times?

4 Listen and Take Notes


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Make good use of your class time by listening and taking notes. Because your instructor will be giving explanations and examples that may not be found in your textbook, as well as other information about your course (test dates, homework assignments, and so on), it is important that you keep a written record of what was said in class.

Listen Actively. Listening in class is different from listening in social situations because it requires that you be an *active* listener. Since it is impossible to write down everything that is said in class, you need to exercise your active listening skills to learn to write down what is *important.* You can spot important material by listening for cues from your instructor. For instance, pauses in lectures or statements from your instructor such as "This is really important" or "This is a question that shows up frequently on tests" are indications that you should be paying special attention. Listen with a pencil (or highlighter) in hand, ready to record or highlight (in your textbook) any examples, definitions, or concepts that your instructor discusses.

Take Notes You Can Use. Don't worry about making your notes really neat. After class you can rework them into a format that is more useful to you. However, you should organize your notes as much as possible as you write them. Copy the examples your instructor uses in class. Circle or star any key concepts or definitions that your instructor mentions while explaining the example. Later, your homework problems will look a lot like the examples given in class, so be sure to copy each of the steps in detail.

Listen with an Open Mind. Even if there are concepts presented that you feel you already know, keep tuned in to the presentation of the material and look for a deeper understanding of the material. If the material being presented is something that has been difficult for you in the past, listen with an open mind; your new instructor may have a fresh presentation that works for you.

Avoid Classroom Distractions. Some of the same things that can distract you from your study time can distract you, and others, during class. Because of this, be sure to turn off your cell phone during class. If you take notes on a laptop, log out of your email and social networking sites during class. In addition to these distractions, avoid getting into side conversations with other students. Even if you feel you were only distracted for a few moments, you may have missed important verbal or body language cues about an upcoming exam or hints that will aid in your understanding of a concept.

Now Try This

1. Before your next class, refer to your syllabus and read the section(s) that will be covered. Make a list of the terms that you predict your instructor will think are most important.
2. During your next class, bring your textbook and keep it open to the sections being covered. If your instructor mentions a definition, concept, or example that is found in your text, highlight it.
3. Find at least one classmate with whom you can review notes. Make an appointment to compare your class notes as soon as possible after the class. Did you find differences in your notes?
4. Go to www.cengage.com/math/tussy and read the *Reworking Your Notes* handout. Complete the action items given in this document.

5 Build a Support System

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Have you ever had the experience where you understand everything that your instructor is saying in class, only to go home and try a homework problem and be completely stumped? This is a common complaint among math students. The key to being a successful math student is to take care of these problems before you go on to tackle new material. That is why you should know what resources are available outside of class.

Make Good Use of Your Instructor's Office Hours. The purpose of your instructor's office hours is to be available to help students with questions. Usually these hours are listed in your syllabus and no appointment is needed. When you visit your instructor, have a list of questions and try to pinpoint exactly where in the process you are getting stuck. This will help your instructor answer your questions efficiently.

Use Your Campus Tutoring Services. Many colleges offer tutorial services for free. Sometimes tutorial assistance is available in a lab setting where you are able to drop in at your convenience. In some cases, you need to make an appointment to see a tutor in advance. Make sure to seek help as soon as you recognize the need, and come to see your tutor with a list of identified problems.

Form a Study Group. Study groups are groups of classmates who meet outside of class to discuss homework problems or study for tests. Get the most out of your study group by following these guidelines:

- Keep the group small—a maximum of four committed students. Set a regularly scheduled meeting day, time, and place.
- Find a place to meet where you can talk and spread out your work.
- Members should attempt all homework problems before meeting.
- All members should contribute to the discussion.
- When you meet, practice verbalizing and explaining problems and concepts to each other. The best way to really learn a topic is by teaching it to someone else.

Now Try This

1. Refer to your syllabus. Highlight your instructor's office hours and location. Next, pay a visit to your instructor during office hours this week and introduce yourself. (***Success Tip:*** Program your instructor's office phone number and email address into your cell phone or email contact list.)
2. Locate your campus tutoring center or math lab. Write down the office hours, phone number, and location on your syllabus. Drop by or give them a call and find out how to go about making an appointment with a tutor.
3. Find two to three classmates who are available to meet at a time that fits your schedule. Plan to meet 2 days before your next homework assignment is due and follow the guidelines given above. After your group has met, evaluate how well it worked. Is there anything that the group can do to make it better next time you meet?
4. Download the *Support System Worksheet* at www.cengage.com/math/tussy. Complete the information and keep it at the front of your notebook following your syllabus.

6 Do Your Homework


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Attending class and taking notes are important, but the only way that you are really going to learn mathematics is by completing your homework. Sitting in class and listening to lectures will help you to place concepts in short-term memory, but in order to do well on tests and in future math classes, you want to put these concepts in long-term memory. When completed regularly, homework assignments will help with this.

Give Yourself Enough Time. In Objective 3, you made a study schedule, setting aside 2 hours for study and homework for every hour that you spend in class. If you are not keeping this schedule, make changes to ensure that you can spend enough time outside of class to learn new material.

Review Your Notes and the Worked Examples from Your Text. In Objective 4, you learned how to take useful notes. Before you begin your homework, review or rework your notes. Then, read the sections in your textbook that relate to your homework problems, paying special attention to the worked examples. With a pencil in hand, work the *Self Check* and *Now Try* problems that are listed next to the examples in your text. Using the worked example as a guide, solve these problems and try to understand each step. As you read through your notes and your text, keep a list of anything that you don't understand.

Now Try Your Homework Problems. Once you have reviewed your notes and the textbook worked examples, you should be able to successfully manage the bulk of your homework assignment easily. When working on your homework, keep your textbook and notes close by for reference. If you have trouble with a homework question, look through your textbook and notes to see if you can identify an example that is similar to the homework question. See if you can apply the same steps to your homework problem. If there are places where you get stuck, add these to your list of questions.

Get Answers to Your Questions. At least one day before your assignment is due, seek help with the questions you have been listing. You can contact a classmate for assistance, make an appointment with a tutor, or visit your instructor during office hours.

Now Try This

1. Review your study schedule. Are you following it? If not, what changes can you make to adhere to the rule of 2 hours of homework and study for every hour of class?
2. Find five homework problems that are similar to the worked examples in your textbook. Were there any homework problems in your assignment that didn't have a worked example that was similar? (***Success Tip:*** Look for the *Now Try* and *Guided Practice* features for help linking problems to worked examples.)
3. As suggested in this Objective, make a list of questions while completing your homework. Visit your tutor or your instructor with your list of questions and ask one of them to work through these problems with you.
4. Go to www.cengage.com/math/tussy and read the *Study and Memory Techniques* handout. List the techniques that will be most helpful to you in your math course.

7 Prepare for the Test

Taking a test does not need to be an unpleasant experience. Use your time management, organization, and these test-taking strategies to make this a learning experience and improve your score.

Make Time to Prepare. Schedule at least four daily 1-hour sessions to prepare specifically for your test.

Four days before the test: Create your own study sheet using your reworked notes. Imagine you could bring one $8\frac{1}{2} \times 11$ sheet of paper to your test. What would you write on that sheet? Include all the key definitions, rules, steps, and formulas that were discussed in class or covered in your reading. Whenever you have the opportunity, pull out your study sheet and review your test material.

Three days before the test: Create a sample test using the in-class examples from your notes and reading material. As you review and work these examples, make sure you understand how each example relates to the rules or definitions on your study sheet. While working through these examples, you may find that you forgot a concept that should be on your study sheet. Update your study sheet and continue to review it.

Two days before the test: Use the *Chapter Test* from your textbook or create one by matching problems from your text to the example types from your sample test. Now, with your book closed, take a timed trial test. When you are done, check your answers. Make a list of the topics that were difficult for you and review or add these to your study sheet.

One day before the test: Review your study sheet once more, paying special attention to the material that was difficult for you when you took your practice test the day before. Be sure you have all the materials that you will need for your test laid out ahead of time (two sharpened pencils, a good eraser, possibly a calculator or protractor, and so on). The most important thing you can do today is get a good night's rest.

Test day: Review your study sheet, if you have time. Focus on how well you have prepared and take a moment to relax. When taking your test, complete the problems that you are sure of first. Skip the problems that you don't understand right away, and return to them later. Bring a watch or make sure there will be some kind of time-keeping device in your test room so that you can keep track of your time. Try not to spend too much time on any one problem.

Now Try This

1. Create a study schedule using the guidelines given above.
2. Read the *Preparing for a Test* handout at www.cengage.com/math/tussy.
3. Read the *Taking the Test* handout at www.cengage.com/math/tussy.
4. After your test has been returned and scored, read the *Analyzing Your Test Results* handout at www.cengage.com/math/tussy.
5. Take time to reflect on your homework and study habits after you have received your test score. What actions are working well for you? What do you need to improve?
6. To prepare for your final exam, read the *Preparing for Your Final Exam* handout at www.cengage.com/math/tussy. Complete the action items given in this document.

1

A Review of Basic Algebra

© Baerbel Schmidt/Getty Images

from Campus to Careers

Registered Dietitian

One of the most important things that you can do to protect yourself from cancer, diabetes, heart disease, and stroke is to eat right. No one knows this better than dietitians. They work in hospitals, health care centers, schools, and correctional facilities, where they plan dietary programs and supervise the preparation of healthy meals. The job of dietician requires mathematical skills such as calculating calorie intake, analyzing the nutritional content of food, and budgeting for the purchase of groceries and supplies.

Problem 57 of **Study Set 1.8** shows how a dietician can use algebra to determine how much super-lean hamburger (12% fat) should be added to regular hamburger (30% fat) to obtain a mixture that has a 16% fat content.

JOB TITLE:
Registered Dietitian
EDUCATION: A bachelor's degree in foods and nutrition (or a related field) and supervised practice
JOB OUTLOOK: Growth faster than the average, 18%–26% increase
ANNUAL EARNINGS: The median base salary in 2007 was $47,157.
FOR MORE INFORMATION:
www.bls.gov/oco/ocos077.htm

Objectives

1. Write verbal and mathematical models.
2. Use equations to construct tables of data.
3. Read graphical models.

SECTION 1.1

The Language of Algebra

Algebra is the result of contributions from many cultures over thousands of years. The word *algebra* comes from the title of the book *Al-jabr wa'l muquabalah,* written by the Arabian mathematician al-Khwarizmi around A.D. 800. Using the vocabulary and notation of algebra we can mathematically **model** many situations in the real world. In this section, we will review some of the basic components of the language of algebra.

1 Write verbal and mathematical models.

In the following rental agreement, we see that two operations need to be performed to calculate the cost of renting the banquet hall.

- First, we must *multiply* the \$100-per-hour rental cost by the number of hours that the hall is to be rented.
- To that result, we must *add* the cleanup fee of \$200.

Rental Agreement
ROYAL VISTA BANQUET HALL
Wedding Receptions•Dances•Reunions•Fashion Shows

Rented To________________________Date_______________
Lessee's Address______________________________________

Rental Charges
• \$100 per hour
• Nonrefundable \$200 cleanup fee

Terms and conditions
Lessor leases the undersigned lessee the above described property upon the terms and conditions set forth on this page and on the back of this page. Lessee promises to pay rental cost stated herein.

We can describe the process to calculate the rental cost in words using the following **verbal model:**

The cost of renting the hall	is	100	times	the number of hours it is rented	plus	200.

The table below lists some key words and phrases that are often used in mathematics to indicate the operations of addition, subtraction, multiplication, and division.

Addition +	Subtraction −	Multiplication ·	Division ÷
added to	subtracted from	multiplied by	divided by
sum	difference	product	quotient
plus	less than	times	ratio
more than	decreased by	percent (or fraction) of	half
increased by	reduced by	twice	into
greater than	minus	triple	per

We can use vocabulary from the table to write a verbal model that describes how to calculate the cost of renting the banquet hall. One such model is:

The cost (in dollars) of renting the hall is the *product* of 100 and the number of hours it is rented, *increased* by 200.

We can also describe the procedure for calculating the rental cost of the banquet hall using *variables* and mathematical symbols. A **variable** is a letter that is used to stand for a number. If we let the letter h represent the number of hours that the hall is rented, the cost to rent the hall can be represented by the notation $100h + 200$. We call $100h + 200$ an **algebraic expression**, or more simply, an **expression.**

The Language of Algebra Since the number of hours that the hall is rented can vary, or change, it is represented using a *variable.*

Algebraic Expressions

An **algebraic expression** is a combination of variables and/or numbers with the operations of addition, subtraction, multiplication, division, raising to a power, and finding a root.

Here are some more examples of expressions.

$5a - 12$	This expression involves the operations of multiplication and subtraction.
$\dfrac{50 - y}{3y^3}$	This expression involves the operations of subtraction, division, multiplication, and raising to a power.
$\sqrt{a^2 + b^2}$	This expression involves the operations of addition, raising to a power, and finding a root.

In the banquet hall example, if we let the letter c represent the cost to rent the hall, we can translate the verbal model to a **mathematical model.**

The cost of renting the hall	is	100	times	the number of hours it is rented	plus	200.
c	$=$	100	$\cdot$	h	$+$	200

The statement $c = 100h + 200$ is called an *equation.* An **equation** is a mathematical sentence that contains an $=$ symbol. The $=$ symbol indicates that the expressions on either side of it have the same value.

The Language of Algebra The equal symbol $=$ can be represented by verbs such as:

is *are* *gives* *yields* *was*

The symbol $\neq$ is read as *is not equal to.*

EXAMPLE 1 Translate each verbal model to a mathematical model.

a. The distance in miles traveled by a vehicle is the product of its average rate of speed in mph and the time in hours it travels at that rate.

b. The sale price of an item is the difference between the regular price and the discount.

Self Check 1

Express the following relationship as an equation: The simple interest earned by a deposit is the product of the principal, the annual rate of interest, and the time. $I = Prt$

Now Try **Problem 13**

Teaching Example 1 Translate each verbal model to a mathematical model.
a. The cost of gasoline purchased is the product of the price per gallon and the number of gallons.
b. The amount saved by purchasing an item on sale is the difference of the original price and the sale price.
Answers:
a. $c = pn$ **b.** $a = p - s$

Strategy We will look for key words and phrases that indicate arithmetic operations, and we will represent any unknown quantities using variables.

WHY To translate a verbal (word) model into a mathematical model means to represent it using mathematical symbols.

Solution

a. The word *product* indicates multiplication. If we let d represent the distance traveled in miles, r the vehicle's average rate of speed in mph, and t the length of time traveled in hours, we can write the verbal model in mathematical form as

$$d = rt$$

b. The word *difference* indicates subtraction. If we let s represent the sale price of the item, p the regular price, and d the discount, we have

$$s = p - d$$

Success Tip The answers to the Self Check problems are given at the end of each section, before each Study Set.

Many applied problems require insight and analysis to determine which mathematical operations to use when writing a verbal or mathematical model.

Self Check 2

WINNING THE LOTTERY After winning a lottery, three friends split the prize equally. Each person had to pay $2,000 in taxes on his or her share. Write a verbal model and a mathematical model that relate the amount of each person's share, after taxes, to the amount of the lottery prize.

Now Try **Problem 21**

Self Check 2 Answers
Each person's share, after taxes, is the quotient of the lottery prize and 3, decreased by 2,000; $S = \frac{p}{3} - 2{,}000$.

Teaching Example 2 MUSICALS It costs $35 per person to attend a musical. An additional handling fee of $40 is charged for group sales. Write a mathmetical model that describes the relationship between the total cost and the number of people in the group attending the show.
Answer:
$c = 35n + 40$

EXAMPLE 2 ***Catering*** It costs $16 per person to have a dinner catered. A $100 discount is given for groups of more than 200 people. Write a verbal and a mathematical model that describe the relationship between the catering cost and the number of people being served for groups larger than 200.

© Roger Bamber/Alamy

Strategy We will carefully read the problem to identify any phrases that indicate an arithmetic operation and then represent any unknown quantities using variables.

WHY To write a verbal or mathematical model, we must determine what arithmetic operations are involved.

Solution

The phrase *$16 per person* indicates multiplication by 16 and the phrase *a $100 discount* indicates subtraction. Thus, to find the catering cost c (in dollars) for groups larger than 200, we need to *multiply* the number n of people served by $16 and then *subtract* the $100 discount.

A verbal model is:

The catering cost (in dollars) is the *product* of 16 and the number of people served, *decreased by 100.*

Caution! The comma in the verbal model is absolutely essential to convey the correct meaning. Without it, it is unclear what is to be decreased by 100.

In symbols, the mathematical model for groups larger than 200 is:

$$c = 16n - 100$$

2 Use equations to construct tables of data.

In the banquet hall example, the equation $c = 100h + 200$ can be used to determine the cost of renting the banquet hall for *any* number of hours.

EXAMPLE 3 *Event Planning* Find the cost of renting the banquet hall for 3 hours and for 4 hours. Write the results in a table.

Strategy We will substitute 3 (and then 4) for h in the equation $c = 100h + 200$ and evaluate the right side.

WHY The cost c to rent the hall for h hours is given by the equation $c = 100h + 200$.

Solution

First, we construct the table shown below with the appropriate column headings: h for the number of hours the hall is rented and c for the cost (in dollars) to rent the hall. Then we enter the number of hours of each rental time in the left column.

Next, we use the equation $c = 100h + 200$ to find the total rental cost for 3 hours and for 4 hours.

$$\begin{aligned} c &= 100\boldsymbol{h} + 200 \\ c &= 100(\mathbf{3}) + 200 && \text{Substitute 3 for } h. \\ &= 300 + 200 && \text{Multiply.} \\ &= 500 \end{aligned}$$

$$\begin{aligned} c &= 100\boldsymbol{h} + 200 \\ c &= 100(\mathbf{4}) + 200 && \text{Substitute 4 for } h. \\ &= 400 + 200 && \text{Multiply.} \\ &= 600 \end{aligned}$$

Finally, we enter these results in the right column of the table: \$500 for a 3-hour rental and \$600 for a 4-hour rental.

h	c
3	500
4	600

Self Check 3

EVENT PLANNING Find the cost of renting the hall for 6 hours and for 7 hours. Write the results in a table.

h	c
6	800
7	900

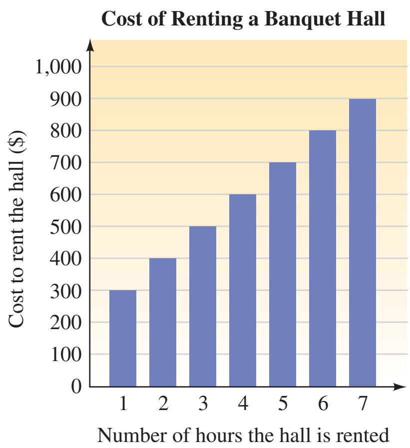
Now Try **Problem 25**

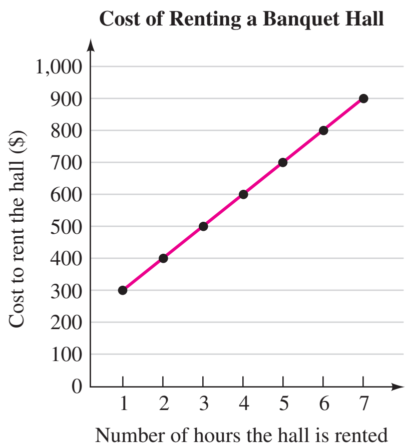
Teaching Example 3 EVENT PLANNING Using the equation in Example 3, find the cost of renting the banquet hall for 2 hours and 5 hours. Write the results in a table.
Answer:

h	c
2	400
5	700

3 Read graphical models.

The cost of renting the banquet hall for various lengths of time can also be presented graphically. The following **bar graph** has a **horizontal axis** labeled "Number of hours the hall is rented." The **vertical axis,** labeled "Cost to rent the hall (\$)," is scaled in units of 100 dollars. The bars above each of the times (1, 2, 3, 4, 5, 6, and 7 hours) extend to a height that gives the corresponding cost to rent the hall. For example, if the hall is rented for 5 hours, the bar indicates that the cost is \$700.





The **line graph** above also shows the rental costs. This type of graph consists of a series of dots drawn at the correct height, connected with line segments. We can use the line graph to find the cost of renting the banquet hall for lengths of time not shown in the bar graph.

Self Check 4

EVENT PLANNING Use the graph to find the cost of renting the banquet hall for $6\frac{1}{2}$ hours. $850

Now Try **Problem 30**

Teaching Example 4 EVENT PLANNING Use the line graph on the previous page to determine the cost of renting the hall for $3\frac{1}{2}$ hours.
Answer:
$550

EXAMPLE 4 *Event Planning*

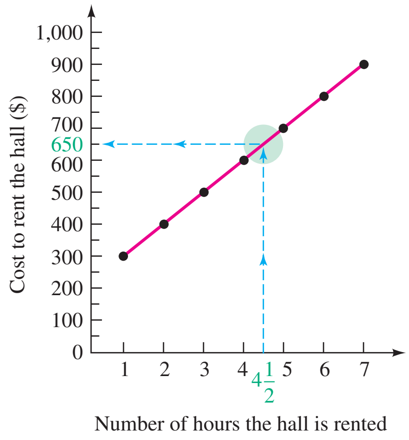
Use the line graph shown on the previous page to determine the cost of renting the hall for $4\frac{1}{2}$ hours.

Strategy Since we know the number of hours the hall is to be rented, we begin on the horizontal axis of the graph and scan up and over to read the answer on the vertical axis.

WHY We scan up and over because the scale on the vertical axis gives the cost of renting the hall.

Solution

In the figure to the right, we locate $4\frac{1}{2}$ on the horizontal axis and draw a vertical line upward to intersect the graph. From the point of intersection with the graph, we draw a horizontal line to the left that intersects the vertical axis. On the vertical axis, we can read that the rental cost is \$650 for $4\frac{1}{2}$ hours.



ANSWERS TO SELF CHECKS

1. $I = Prt$ **2.** Each person's share, after taxes, is the quotient of the lottery prize and 3, decreased by 2,000; $S = \frac{p}{3} - 2{,}000$. **3.**

h	*c*
6	800
7	900

4. $850

STUDY SKILLS CHECKLIST

Get to Know Your Textbook

Congratulations. You now own a state-of-the-art textbook that has been written especially for you. The following checklist will help you become familiar with the organization of this book. Place a check mark ☑ in each box after you answer the question.

- ☐ Turn to the **Table of Contents** on page v. How many chapters does the book have?
- ☐ Each chapter of the book is divided into **sections.** How many sections are there in Chapter 1, which begins on page 1?
- ☐ Learning **Objectives** are listed at the start of each section. How many objectives are there for Section 1.2, which begins on page 10?
- ☐ Each section ends with a **Study Set.** How many problems are there in Study Set 1.2, which begins on page 20?
- ☐ Each chapter has a **Chapter Summary and Review.** Which column of the Chapter Summary found on page 98 contains examples?
- ☐ How many review problems are there for Section 1.1 in the **Chapter Summary and Review,** which begins on page 99?
- ☐ Each chapter has a **Chapter Test.** How many problems are there in the Chapter 1 Test, which begins on page 112?
- ☐ Each chapter ends with a **Cumulative Review.** What chapters are covered by the Cumulative Review which begins on page 303?

Answers: 10, 8, 7, 94, the right, 4, 36, 1–3

SECTION 1.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. A variable is a letter that is used to stand for a number.
2. Variables and/or numbers can be combined with mathematical operations to create algebraic expressions.
3. An equation is a mathematical sentence that contains an = symbol.
4. Words such as *is, was, gives,* and *yields* translate to an = symbol.
5. Phrases such as *increased by* and *more than* are used to indicate the operation of addition.
6. Phrases such as *decreased by* and *less than* are used to indicate the operation of subtraction.

CONCEPTS

7. Classify each of the following as an expression or an equation.
   a. $6x - 5$ expression
   b. $P = a + b + c$ equation
   c. $\frac{s + 9t}{8}$ expression
   d. $\sqrt{2w^2}$ expression
8. What arithmetic operations does the expression $\frac{40 - 8n}{5}$ contain? What variable does it contain?
   subtraction, multiplication, division; n

Use the data in each table to find an equation that mathematically describes the relationship between the two quantities. Then state the relationship in words. (Answers may vary.)

9.

Tower height (ft)	Height of base (ft)
15.5	5.5
22	12
25.25	15.25
45.125	35.125

$b = t - 10$; the height of the base is 10 ft less than the height of the tower

10.

Seasonal employees	Employees
25	75
50	100
60	110
80	130

$e = 50 + s$; the number of employees is the sum of the number of seasonal employees and 50

NOTATION

11. Translate each verbal model into a mathematical model.

    a. 7 times the age of a dog in years gives the dog's equivalent human age.

    $7d = h$

    b. The take-home pay will be $2,500 minus any deductions.

    $t = 2{,}500 - d$

12. Give four verbs that can be represented by an equal symbol =. is, are, gives, yields

GUIDED PRACTICE

Translate each verbal model into a mathematical model.
See Example 1.

13. The cost each semester is the sum of $13 times the number of units taken and a student services fee of $24. $c = 13u + 24$
14. The yearly salary is $25,000 plus $75 times the number of years of experience. $s = 25{,}000 + 75y$
15. The quotient of the number of clients and seventy-five gives the number of social workers needed. $w = \frac{c}{75}$
16. The difference between 500 and the number of people in a theater gives the number of unsold tickets. $t = 500 - p$
17. Each test score was increased by 15 points to give a new adjusted test score. $A = t + 15$
18. The weight of a super-size order of french fries is twice that of a regular-size order. $s = 2r$
19. The product of the number of boxes of crayons in a case and 12 gives the number of crayons in a case. $c = 12b$
20. The perimeter of an equilateral triangle can be found by tripling the length of one of its sides. $P = 3s$

Write a verbal and mathematical model for each situation.
See Example 2.

21. TAXES A married couple has decided to split the money equally when they receive their federal income tax refund. Furthermore, the husband is going to donate $75 of his share to charity. Describe the relationship between the amount of money that the husband will keep and the amount of the couple's refund.
    The amount (in dollars) that the husband will keep is the quotient of the amount of the couple's refund and 2, decreased by 75. $h = \frac{t}{2} - 75$ (answers may vary)

Selected exercises available online at
www.webassign.net/brookscole

22. COPIERS A business is going to rent a copy machine. Under the rental agreement, the company is charged \$105 per month and 3¢ for every copy that is made. Describe the relationship between the monthly copier expense and the number of copies made.
The monthly copier expense (in dollars) is the product of 0.03 and the number of copies made, increased by 105. $C = 0.03c + 105$ (answers may vary)

23. BOTTLED WATER A driver left a production plant with 300 five-gallon bottles of drinking water on his truck. His delivery route consisted of office buildings, each of which was to receive 6 bottles of water. Describe the relationship between the number of bottles of water left on his truck and the number of stops that he has made.
The number of bottles of water left on his truck is the difference of 300 and the product of 6 and the number of stops that he has made. $b = 300 - 6s$ (answers may vary)

24. COLLECTIBLES A woman inherited 9 antique dolls. She decided to add to her collection by purchasing two more dolls each month. Describe the relationship between the number of antique dolls in her collection and the number of months since she began to purchase them.
The number of dolls in the collection is the product of two and the number of months since she began to purchase them, increased by 9. $d = 2m + 9$ (answers may vary)

Use the given equation to complete each table. See Example 3.

25. $c = \dfrac{p}{12}$

Number of packages p	Cartons c
24	2
72	6
180	15

26. $y = 100c$

Number of centuries c	Years y
1	100
6	600
21	2,100

27. $n = 22.44 - K$

K	n
0	22.44
1.01	21.43
22.44	0

28. $y = 2x + 15$

x	y
0	15
15	45
30	75

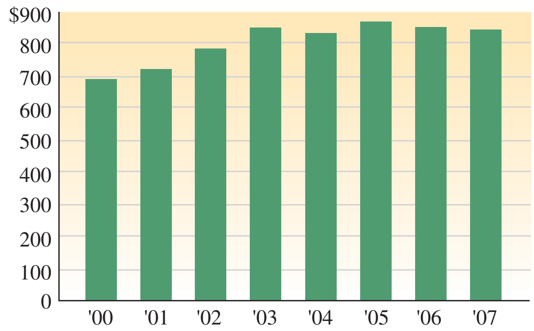
Refer to the given graph in the next column. See Example 4.

29. a. What type of graph is shown in the next column? a bar graph

b. What units are used to scale the horizontal axis? The vertical axis? 1 year, \$100

c. In what year was the average expenditure on auto insurance the least? Estimate the amount. In what year was it the greatest? Estimate the amount. 2000, \$700; 2005, \$860

U.S. Average Consumer Expenditures on Auto Insurance

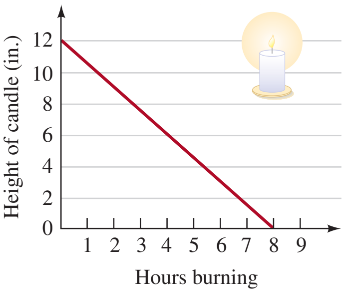


Source: Insurance Information Institute

30. a. What type of graph is shown? a line graph

b. What units are used to scale the horizontal axis? The vertical axis? 1 hr, 2 in.

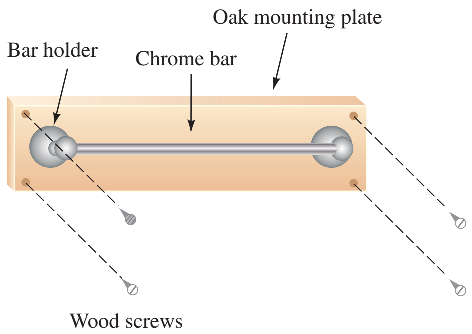
c. Estimate the height of the candle after it has burned for $3\frac{1}{2}$ hours. For 8 hours. 7 in., 0 in.



APPLICATIONS

31. PRODUCTION PLANNING Suppose r towel racks are to be manufactured. Complete the four equations that could be used to order the necessary number of oak mounting plates p, bar holders b, chrome bars c, and wood screws s.

$p = \boxed{r}$ $\quad b = \boxed{2}r$ $\quad c = \boxed{r}$ $\quad s = \boxed{4}r$



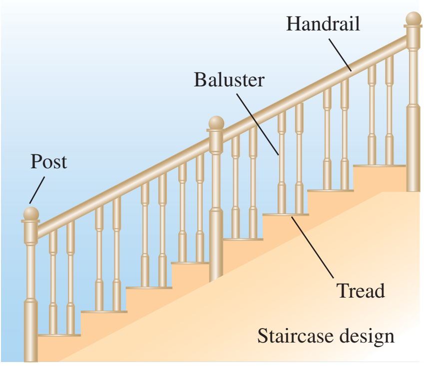
32. ORDERING STAIRCASE PARTS A builder is going to construct h new homes, each of which will have a staircase as shown. Complete the four equations that could be used to order the necessary number of balusters b, handrails r, posts p, and treads t for the entire project.

$b = 16h$

$r = 2h$

$p = 3h$

$t = 8h$



Staircase design

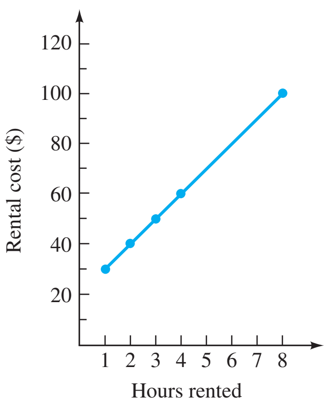
33. CARPET CLEANING See the following ad.

a. Write a verbal model that states the relationship between the cost C of renting the carpet-cleaning system and the number of hours h it is rented. The rental cost is the product of 10 and the number of hours it is rented, increased by 20.

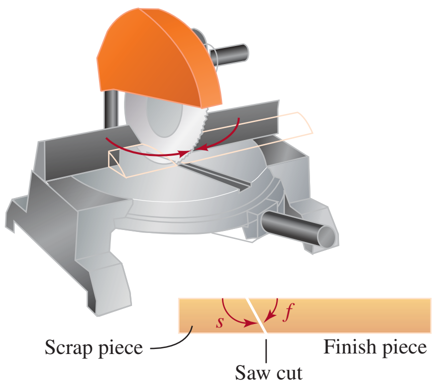
b. Translate the verbal model written in part a to a mathematical model. $C = 10h + 20$

c. Use your result from part b to complete the table, and then draw a line graph.

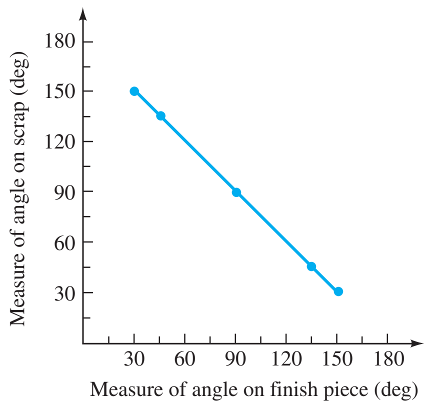
h	C
1	30
2	40
3	50
4	60
8	100



34. CARPENTRY A miter saw can pivot 180° to make angled cuts on molding. The equation that relates the angle measure s on the scrap piece of molding and the angle measure f on the finish piece of molding is $s = 180 - f$. Complete the following table and then draw a line graph.



f	s
30	150
45	135
90	90
135	45
150	30



WRITING

35. Explain the difference between an expression and an equation. Give examples.

36. Use each word below in a sentence that indicates a mathematical operation. If you are unsure of the meaning of a word, look it up in a dictionary.

quadrupled	deleted	bisected
confiscated	annexed	docked

Objectives

1. Define the set of natural numbers, whole number, and integers.
2. Define the set of rational numbers.
3. Define the set of irrational numbers.
4. Classify real numbers.
5. Graph real numbers.
6. Order the real numbers.
7. Find the opposite and the absolute value of a real number.

SECTION 1.2 The Real Numbers

In this course, we will work with *real numbers.* The set of real numbers is a collection of several other important sets of numbers.

1 Define the set of natural numbers, whole numbers, and integers.

Natural numbers are the numbers that we use for counting. To write this set, we list its **elements** (or **members**) within **braces** { }.

Natural Numbers

The set of **natural numbers,** denoted by the symbol $\mathbb{N}$, is $\{1, 2, 3, 4, 5, \ldots\}$.

Read as "the set containing one, two, three, four, five, and so on."

The Language of Algebra The symbol ... used in these definitions is called an ellipsis and it indicates that the established pattern continues forever.

The symbol $\in$ is used to indicate that an element belongs to a set. For example, we can write

$3 \in \mathbb{N}$ Read as "3 is an element of the set of natural numbers."

We can write $4.5 \notin \mathbb{N}$ to indicate that 4.5 *is not an element* of the set of natural numbers.

The natural numbers, together with 0, form the set of **whole numbers.**

Whole Numbers

The set of **whole numbers,** denoted by the symbol $\mathbb{W}$, is $\{0, 1, 2, 3, 4, 5, \ldots\}$.

When all the members of one set are also members of a second set, we say the first set is a **subset** of the second set. Since every natural number is also a whole number, the set of natural numbers is a subset of the set of whole numbers. We can use the symbol $\subseteq$ to indicate this.

$\mathbb{N} \subseteq \mathbb{W}$ Read as "The set of natural numbers is a subset of the set of whole numbers."

Since the set of whole numbers contains an element that the natural numbers do not, namely 0, we can write

$\mathbb{W} \nsubseteq \mathbb{N}$ Read as "The set of whole numbers is not a subset of the set of natural numbers."

Two other important subsets of the whole numbers are the *prime numbers* and the *composite numbers.*

Prime Numbers and Composite Numbers

A **prime number** is a natural number greater than 1 that has only itself and 1 as factors. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

A **composite number** is a natural number, greater than 1, that is not prime. The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

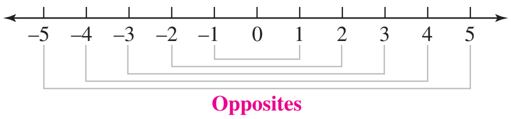
Recall from arithmetic that every composite number can be written as the product of prime numbers. For example,

$$6 = 2 \cdot 3, \quad 25 = 5 \cdot 5, \quad \text{and} \quad 168 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 7$$

Whole numbers are not adequate for describing many real-life situations. For example, if you write a check for more than what is in your account, the account balance will be less than zero.

We can use the **number line** to visualize numbers less than zero. On the number line, numbers greater than 0 are to the right of 0, and they are called **positive numbers.** Numbers less than 0 are to the left of 0, and they are called **negative numbers.**

For each natural number on the number line, there is a corresponding number, called its *opposite,* to the left of 0. In the diagram, we see that 3 and −3 (negative three) are opposites, as are −5 (negative five) and 5. Note that 0 is its own opposite.



Opposites

Two numbers that are the same distance from 0 on the number line, but on opposite sides of it, are called **opposites.**

The whole numbers, together with their opposites, form the set of *integers.*

Integers

The set of **integers,** denoted by the symbol $\mathbb{Z}$, is
$\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

Integers that are divisible by 2 are called *even integers,* and integers that are not divisible by 2 are called *odd integers.*

Even integers: $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

Odd integers: $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$

Since every whole number is also an integer, the set of whole numbers is a subset of the set of integers.

Self Check 1

Determine whether each statement is true or false.

a. $-1.7 \notin \mathbb{Z}$ true

b. $\mathbb{Z} \subseteq \mathbb{W}$ false

Now Try **Problems 27 and 31**

Teaching Example 1 Determine whether each statement is true or false.

a. $0 \in \mathbb{N}$

b. $\mathbb{N} \subseteq \mathbb{W}$

Answers:

a. false **b.** true

EXAMPLE 1 Determine whether each statement is true or false.

a. $-6 \in \mathbb{W}$ **b.** $\mathbb{Z} \not\subseteq \mathbb{N}$

Strategy We will compare each statement with the definitions of the sets of numbers to determine if the statement is true or false.

WHY The definitions of the sets of numbers tell us the symbol used to represent them and the numbers that are in the sets.

Solution

a. Since -6 is not an element of the set of whole numbers, the statement $-6 \in \mathbb{W}$ is false.

b. Since the set of integers contains at least one element that does not belong to the set of natural numbers (-1 for example), the set of integers is not a subset of the set of natural numbers. Thus, the statement $\mathbb{Z} \not\subseteq \mathbb{N}$ is true.

2 Define the set of rational numbers.

In this course, we will work with positive and negative fractions. For example, the slope of a line might be $\frac{7}{12}$ or a tank might drain at a rate of $-\frac{40}{3}$ gallons per minute. We will also work with mixed numbers. For instance, we might speak of $5\frac{7}{8}$ cups of flour or of a river that is $3\frac{1}{2}$ feet below flood stage $\left(-3\frac{1}{2}\text{ ft}\right)$. These fractions and mixed numbers are examples of *rational numbers*.

Rational Numbers

A **rational number** is any number that can be written in the form $\frac{a}{b}$, where a and b represent integers and $b \neq 0$.

Other examples of rational numbers are

$$\frac{3}{4}, \quad \frac{25}{25}, \quad \text{and} \quad \frac{19}{6}$$

To show that negative fractions are rational numbers, we can use the following fact.

Negative Fractions

Let a and b represent numbers, where $b \neq 0$,

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

To illustrate this rule, consider $-\frac{40}{3}$. It is a rational number because it can be written as $\frac{-40}{3}$, or as $\frac{40}{-3}$.

Positive and negative mixed numbers such as $5\frac{7}{8}$ and $-3\frac{1}{2}$ are rational numbers because they can be expressed as fractions.

$$5\frac{7}{8} = \frac{47}{8} \quad \text{and} \quad -3\frac{1}{2} = -\frac{7}{2} = \frac{-7}{2}$$

Any natural number, whole number, or integer can be expressed as a fraction with a denominator of 1. For example, $5 = \frac{5}{1}$, $0 = \frac{0}{1}$, and $-3 = \frac{-3}{1}$. Therefore, every natural number, whole number, and integer is also a rational number.

Throughout the book we will also work with decimals. Some examples of uses of decimals are:

- The interest rate of a loan was 11% = 0.11.
- In baseball, the distance from home plate to second base is 127.279 feet.
- The third-quarter loss for a business can be represented as −2.7 million dollars.

Terminating decimals such as 0.11, 127.279, and −2.7 are rational numbers because they can be written as fractions with integer numerators and nonzero integer denominators.

$$0.11 = \frac{11}{100} \qquad 127.279 = 127\frac{279}{1,000} = \frac{127,279}{1,000} \qquad -2.7 = -2\frac{7}{10} = \frac{-27}{10}$$

Examples of **repeating decimals** are 0.333... and 4.252525.... Any repeating decimal can be expressed as a fraction with an integer numerator and a nonzero integer denominator.

For example, $0.333\ldots = \frac{1}{3}$ and $4.252525\ldots = 4\frac{25}{99} = \frac{421}{99}$. Since every repeating decimal can be written as a fraction, repeating decimals are also rational numbers.

Rational Numbers

The set of **rational numbers,** denoted by the symbol $\mathbb{Q}$, is the set of all terminating and all repeating decimals.

We can use division to find the *decimal equivalent* of a fraction.

EXAMPLE 2

Find the decimal equivalent of each fraction to determine whether the decimal terminates or repeats. **a.** $\frac{4}{5}$ **b.** $\frac{17}{6}$

Strategy For each fraction, we will divide the numerator by the denominator.

WHY A fraction bar indicates division.

Solution

a. To change $\frac{4}{5}$ to a decimal, we divide 4 by 5.

$$\begin{array}{r} .8 \\ 5\overline{)4.0} \\ \underline{4\,0} \\ 0 \end{array}$$

Write a decimal point and a 0 to the right of 4.

In decimal form, $\frac{4}{5}$ is 0.8. This is a terminating decimal.

b. To change $\frac{17}{6}$ to a decimal, we divide 17 by 6 and obtain 2.8333.... This is a repeating decimal, because the digit 3 repeats forever. It can be written as $2.8\overline{3}$, where the **overbar** indicates that the 3 repeats.

Self Check 2

Find the decimal equivalent of each fraction to determine whether the decimal terminates or repeats.

a. $\frac{25}{990}$ $0.0\overline{25}$ repeating

b. $\frac{47}{50}$ 0.94 terminating

Now Try **Problems 35 and 37**

Teaching Example 2 Find the decimal equivalent of each fraction to determine whether the decimal terminates or repeats.

a. $\frac{5}{12}$ **b.** $\frac{3}{8}$

Answers:

a. $0.41\overline{6}$ repeating

b. 0.375 terminating

The Language of Algebra A fraction and its *decimal equivalent* are different notation that represent the same value.

The set of rational numbers is too extensive to list its members in the same way that we listed the members of the natural numbers, whole numbers, and integers. Instead, we will use the following **set-builder** notation to describe it.

Rational Numbers

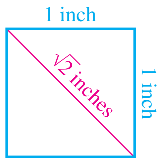
The set of rational numbers is

$$\left\{ \frac{a}{b} \,\middle|\, a \text{ and } b \text{ are integers, with } b \neq 0. \right\}.$$

Read as "the set of all numbers of the form $\frac{a}{b}$, such that a and b are integers, with $b \neq 0$."

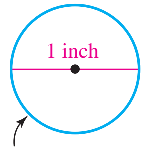
3 Define the set of irrational numbers.

Numbers that cannot be expressed as fractions with an integer numerator and a nonzero integer denominator are called **irrational numbers.** One example is $\sqrt{2}$. It can be shown that a square, with sides of length 1 inch, has a diagonal that is $\sqrt{2}$ inches long.



The length of a diagonal of the square is $\sqrt{2}$ inches.

The number represented by the Greek letter π (pi) is another example of an irrational number. It can be shown that a circle, with a 1-inch diameter, has a circumference of π inches.



The distance around the circle is π inches.

Expressed in decimal form,

$$\sqrt{2} = 1.414213562\ldots \quad \text{and} \quad \pi = 3.141592654\ldots$$

These decimals neither terminate nor repeat.

Irrational Numbers

An **irrational number** is a nonterminating, nonrepeating decimal. An irrational number cannot be expressed as a fraction with an integer numerator and a nonzero integer denominator. The set of irrational numbers is denoted by the symbol $\mathbb{H}$.

Some other examples of irrational numbers are

$$\sqrt{97} = 9.848857802\ldots$$

$$-\sqrt{7} = -2.64575131\ldots \qquad \text{This is a negative irrational number.}$$

$$2\pi = 6.283185307\ldots \qquad 2\pi \text{ means } 2 \cdot \pi.$$

Caution! Don't classify a number such as 4.12122122212222 . . . as a repeating decimal. Although it exhibits a pattern, no block of digits repeats forever. It is a nonterminating, nonrepeating decimal—an irrational number.

Not all square roots are irrational numbers. When we simplify square roots such as $\sqrt{9}$, $\sqrt{36}$, and $\sqrt{400}$, it is apparent that they are rational numbers: $\sqrt{9} = 3$, $\sqrt{36} = 6$, and $\sqrt{400} = 20$.

Using Your CALCULATOR **Approximating Irrational Numbers**

We can approximate the value of irrational numbers with a scientific calculator. To find the value of π, we press the $\boxed{\pi}$ key.

$\boxed{\pi}$ (You may have to use a $\boxed{\text{2nd}}$ or $\boxed{\text{Shift}}$ key first.) $\boxed{3.141592654}$

We see that $\pi \approx 3.141592654$. (Read $\approx$ as "is approximately equal to.") To the nearest thousandth, $\pi \approx 3.142$.

To approximate $\sqrt{2}$, we enter 2 and press the square root key $\boxed{\sqrt{\ }}$.

$2\ \boxed{\sqrt{\ }}$ $\boxed{1.414213562}$

We see that $\sqrt{2} \approx 1.414213562$. To the nearest hundredth, $\sqrt{2} \approx 1.41$.

To find π and $\sqrt{2}$ with a graphing calculator, we proceed as follows.

$\boxed{\text{2nd}}\ \pi\ \boxed{\text{ENTER}}$

π
3.141592654

$\boxed{\text{2nd}}\ \boxed{\sqrt{\ }}\ 2\ \boxed{)}\ \boxed{\text{ENTER}}$

$\sqrt{\ }(2)$
1.414213562

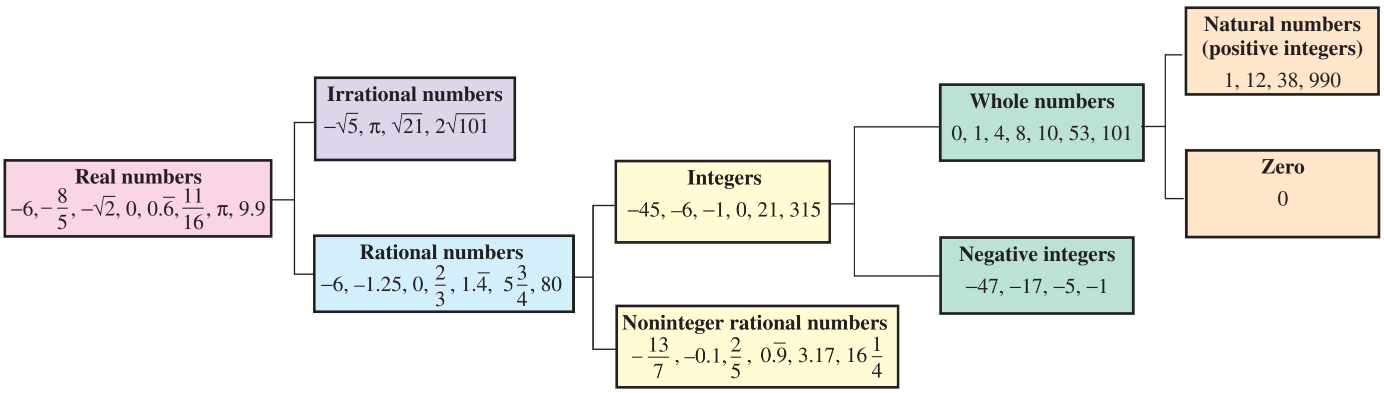
4 Classify real numbers.

The set of rational numbers together with the set of irrational numbers form the set of **real numbers.** This means that every real number can be written as either a terminating decimal, a repeating decimal, or a nonterminating, nonrepeating decimal. Thus, the set of real numbers is the set of all decimals.

The Real Numbers

A **real number** is any number that is either a rational or an irrational number. The set of real numbers is denoted by the symbol $\mathbb{R}$.

The figure below shows how the sets of numbers introduced in this section are related; it also gives some specific examples of each type of number. Note that a number can belong to more than one set. For example, -6 is an integer, a rational number, and a real number.



Self Check 3

Use the instructions for Example 3 with the following set: $\left\{-\pi,\ -5,\ 3.4,\ \sqrt{19},\ 1,\ \frac{16}{5},\ 9.\overline{7}\right\}$

Now Try **Problems 43, 45, and 47**

Self Check 3 Answers

natural numbers: 1; whole numbers: 1; integers: -5, 1; rational numbers: -5, 3.4, 1, $\frac{16}{5}$, $9.\overline{7}$; irrational numbers: $-\pi$, $\sqrt{19}$; real numbers; all

Teaching Example 3 Which numbers in the set are natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers? $\left\{-2.314,\ -1,\ 0,\ \sqrt{3},\ 4,\ 5.12122\ldots\right\}$

Answers:

natural numbers: 4; whole numbers: 0, 4; integers: -1, 0, 4; rational numbers: -2.314, -1, 0, 4; irrational numbers: $\sqrt{3}$, 5.12122 . . .; real numbers: all

EXAMPLE 3 ***Classifying Real Numbers*** Which numbers in the following set are natural numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers?

$$\left\{\frac{5}{8},\quad -0.03,\quad 45,\quad -9,\quad \sqrt{7},\quad 5\frac{2}{3},\quad 0,\quad -9.010010001\ldots,\quad 0.\overline{25}\right\}$$

Strategy We begin by scanning the given set, looking for any natural numbers. Then we scan it five more times, looking for whole numbers, for integers, for rational numbers, for irrational numbers, and, finally, for real numbers.

WHY We need to scan the given set of numbers six times, because numbers in the set can belong to more than one classification.

Solution

Natural numbers: 45 — *45 is a member of {1, 2, 3, 4, 5, . . .}.*

Whole numbers: 45, 0 — *45 and 0 are members of {0, 1, 2, 3, 4, 5, . . .}.*

Integers: 45, -9, 0 — *45, -9, and 0 are members of {. . ., -3, -2, -1, 0, 1, 2, 3, . . .}.*

Rational numbers: $\frac{5}{8}$, 45, -9, $5\frac{2}{3}$, and 0 are rational numbers because each of them can be expressed as a fraction: $45 = \frac{45}{1}$, $-9 = \frac{-9}{1}$, $5\frac{2}{3} = \frac{17}{3}$, and $0 = \frac{0}{1}$. The terminating decimal $-0.03 = \frac{-3}{100}$ and the repeating decimal $0.\overline{25} = \frac{25}{99}$ are also rational numbers.

Irrational numbers: The nonterminating, nonrepeating decimals $\sqrt{7} = 2.645751311\ldots$ and $-9.010010001\ldots$ are irrational numbers.

Real numbers: $\frac{5}{8}$, -0.03, 45, -9, $\sqrt{7}$, $5\frac{2}{3}$, 0, $-9.010010001\ldots$, $0.\overline{25}$

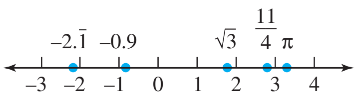
Every natural number, whole number, integer, rational number, and irrational number is a real number.

5 Graph real numbers.

We can illustrate real numbers using a number line. To each real number, there corresponds a point on the line. Furthermore, to each point on the line, there corresponds a real number, called its **coordinate.**

Self Check 4

Graph the numbers in the set $\left\{\pi,\ -2.\overline{1},\ \sqrt{3},\ \frac{11}{4},\ \text{and}\ -0.9\right\}$ on a number line.



Now Try **Problem 55**

EXAMPLE 4 Graph each number in the set $\left\{-\frac{8}{3},\ \ -1.1,\ \ 0.\overline{56},\ \ \frac{\pi}{2},\ \ -\sqrt{15},\ \ 2\sqrt{2}\right\}$ on a number line.

Strategy We locate the position of each number on the number line, draw a bold dot, and label it.

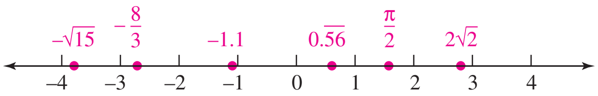
WHY To *graph a number* means to make a drawing that represents the number.

Solution

To help locate the graph of each number, we make some observations.

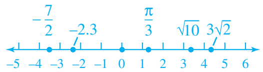
- Expressed as a mixed number, $-\frac{8}{3} = -2\frac{2}{3}$.
- Since -1.1 is less than -1, its graph is to the left of -1.
- $0.\overline{56} \approx 0.6$
- From a calculator, $\frac{\pi}{2} \approx 1.6$.

- From a calculator, $-\sqrt{15} \approx -3.9$.
- $2\sqrt{2}$ means $2 \cdot \sqrt{2}$. From a calculator, $2\sqrt{2} \approx 2.8$.



Teaching Example 4 Graph each number in the set $\left\{-\frac{7}{2}, -2.3, 0, \frac{\pi}{3}, \sqrt{10}, 3\sqrt{2}\right\}$ on a number line.

Answer:



> ***The Language of Algebra*** An example of a number that is not on the real number line is $\sqrt{-4}$. It is called an imaginary number. We will discuss such numbers in Chapter 8.

6 Order the real numbers.

As we move right on the number line, the values of the real numbers increase. As we move left, the values decrease. To compare real numbers, we often use one of the **inequality symbols** shown in the following table.

Symbol	Read as	Examples
$\neq$	"is not equal to"	$6 \neq 9$ and $0.33 \neq \frac{3}{5}$
$<$	"is less than"	$\frac{22}{3} < \frac{23}{3}$ and $-7 < -6$
$>$	"is greater than"	$19 > 5$ and $\frac{1}{2} > 0.3$
$\leq$	"is less than or equal to"	$3.5 \leq 3.5$ and $1\frac{4}{5} \leq 1.8$
$\geq$	"is greater than or equal to"	$29 \geq 29$ and $-15.2 \geq -16.7$

> ***The Language of Algebra*** If a real number x is positive, then $x > 0$. If a real number x is nonnegative, then $x \geq 0$. If a real number x is a negative number, then $x < 0$.

It is always possible to write an equivalent inequality with the inequality symbol pointing in the opposite direction. For example:

If $-3 < 4$, it is also true that $4 > -3$.

If $5.3 \geq 2.9$, it is also true that $2.9 \leq 5.3$.

EXAMPLE 5

Use one of the symbols $>$ or $<$ to make each statement true.

a. $-24 \; \square \; -25$ **b.** $\frac{3}{4} \; \square \; 0.76$

Strategy To pick the correct inequality symbol to place between a given pair of numbers, we need to determine the position of each on a number line.

WHY For any two numbers on a number line, the number to the *left* is the smaller number and the number to the *right* is the larger number.

Solution

a. Since -24 is to the right of -25 on the number line, $-24 > -25$.

b. If we express the fraction $\frac{3}{4}$ as a decimal, we can easily compare it to 0.76.

$$\text{Since } \frac{3}{4} = 0.75, \quad \frac{3}{4} < 0.76.$$

Self Check 5

Use one of the symbols $\geq$ or $\leq$ to make each statement true.

a. $\frac{2}{3} \; \boxed{\leq} \; \frac{4}{3}$

b. $8\frac{1}{2} \; \boxed{\geq} \; 8.4$

Now Try **Problems 67 and 71**

Teaching Example 5 Use one of the symbols $>$ or $<$ to make each statement true.

a. $-11 \; \square \; -10$

b. $\frac{5}{11} \; \square \; 0.44$

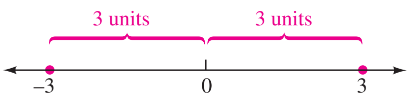
Answers:

a. $<$ **b.** $>$

7 Find the opposite and the absolute value of a real number.

Two numbers that are the same distance from 0 on the number line, but on opposite sides of it, are called **opposites** or **additive inverses.** To write the opposite of a positive number, we simply insert a negative sign $-$ in front of it. For example, the opposite of 10 is -10.

Parentheses are used to express the opposite of a negative number. For example, the opposite of -3 is written as $-(-3)$. Since -3 and 3 are the same distance from zero, the opposite of -3 is 3. In symbols, this can be written as $-(-3) = 3$.



In general, we have the following.

Opposites

The **opposite** of a number a is the number $-a$. If a is a real number, then $-(-a) = a$.

A number line can be used to measure the distance from one number to another. To express the distance that a number is from 0 on a number line, we can use absolute values.

Absolute Value

The **absolute value** of a number is its distance from 0 on the number line.

To indicate the absolute value of a number, we write the number between two vertical bars. From the previous figure, we see that $|-3| = 3$. This is read as "the absolute value of negative 3 is 3" and it tells us that the distance from 0 to -3 is 3 units. It also follows from the figure that $|3| = 3$.

The absolute value of a number can be defined more formally as follows.

Absolute Value

For any real number a, $\begin{cases} \text{If } a \geq 0, \text{ then } |a| = a \\ \text{If } a < 0, \text{ then } |a| = -a \end{cases}$

The first part of this definition states that if a is a nonnegative number (that is, if $a \geq 0$), the absolute value of a is a. The second part of the definition states that if a is a negative number (that is, if $a < 0$), the absolute value of a is the opposite of a. For example, if $a = -8$, then

$$|a| = |-8| = \underbrace{-(-8)}_{\text{The opposite of } a} = 8$$

Caution! The second part of this definition is often misunderstood. Study it carefully. It indicates that the absolute value of a negative number is the opposite (or additive inverse) of the number.

EXAMPLE 6

Find the value of each expression.

a. $|34|$ **b.** $\left|-\frac{4}{5}\right|$ **c.** $|0|$ **d.** $-|-1.8|$

Strategy We need to determine the distance that the number within the vertical absolute value bars is from 0.

WHY The absolute value of a number is the distance between 0 and the number on a number line.

Solution

a. $|34| = 34$ Because 34 is a distance of 34 from 0 on a number line.

b. $\left|-\frac{4}{5}\right| = \frac{4}{5}$ Because $-\frac{4}{5}$ is a distance of $\frac{4}{5}$ from 0 on a number line.

c. $|0| = 0$ Because 0 is a distance of 0 from 0 on a number line.

d. The negative sign outside the absolute value bars means to find the opposite of $|-1.8|$.

$$-|\mathbf{-1.8}| = -(\mathbf{1.8}) \quad \text{Find } |-1.8| \text{ first to get 1.8.}$$
$$= -1.8$$

Self Check 6

Find the value of each expression.

a. $|-9.6|$ 9.6

b. $-|-12|$ -12

c. $\left|\frac{3}{2}\right|$ $\frac{3}{2}$

Now Try **Problems 75 and 79**

Teaching Example 6 Find the value of each expression.

a. $|-13|$ **b.** $|16|$

c. $-|-3|$

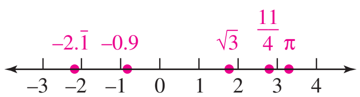
Answers:

a. 13 **b.** 16 **c.** -3

ANSWERS TO SELF CHECKS

1. a. true **b.** false **2. a.** $0.0\overline{25}$, repeating decimal **b.** 0.94, terminating decimal **3.** natural numbers: 1; whole numbers: 1; integers: -5, 1; rational numbers: -5, 3.4, 1, $\frac{16}{5}$, $9.\overline{7}$; irrational numbers: $-\pi$, $\sqrt{19}$; real numbers; all

4.



5. a. $\le$ **b.** $\ge$ **6. a.** 9.6 **b.** -12 **c.** $\frac{3}{2}$

STUDY SKILLS CHECKLIST

Learning from the Worked Examples

The following checklist will help you become familiar with the example structure in this book. Place a check mark in each box after you answer the question.

☐ Each section of the book contains worked **Examples** that are numbered. How many worked examples are there in Section 1.2, which begins on page 10?

☐ Each worked example contains a **Strategy.** Fill in the blanks to complete the following strategy for Example 2 on page 13: For each fraction, we will divide the _numerator_ _by_ _the_ _denominator_.

☐ Each Strategy statement is followed by an explanation of **Why** that approach is used. Fill in the blanks to complete the following Why for Example 2 on page 13: A fraction bar _indicates_ _division_.

☐ Each worked example has a Solution. How many lettered parts are there to the Solution in Example 2 on page 13?

☐ Each example uses red **Author Notes** to explain the steps of the solution. Fill in the blanks to complete the author note in the solution of Example 2 on page 13: Write a decimal point and a 0 _to_ _the_ _right_ _of_ _4_.

☐ After reading a worked example, you should work the **Self Check** problem. How many Self Check problems are there for Example 2 on page 13?

☐ At the end of each section, you will find the **Answers to Self Checks.** What is the answer to Self Check problem 4 on page 6?

☐ After completing a Self-Check problem, you can **Now Try** similar problems in the Study Sets. For Example 2 on page 13, which two Study Set problems are suggested?

Answers: 6, numerator by the denominator, indicates division, 2, to the right of 4, 2, $850, 35 and 37

SECTION 1.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. The set of whole numbers is $\{0, 1, 2, 3, 4, 5, \ldots\}$, the set of natural numbers is $\{1, 2, 3, 4, 5, \ldots\}$, and the set of integers is $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
2. When all the members of one set are members of a second set, we say the first set is a subset of the second set.
3. A prime number is a natural number greater than 1 that has only itself and 1 as factors. A composite number is a natural number greater than 1 that is not prime.
4. A rational number is any number that can be written as a fraction with an integer numerator and a nonzero integer denominator.
5. Irrational numbers are nonterminating, nonrepeating decimals.
6. The set of rational numbers together with the set of irrational numbers form the set of real numbers.
7. $>, \geq, <$, and $\leq$ are called inequality symbols.
8. The absolute value of any real number is the distance between the number and zero on a number line.

CONCEPTS

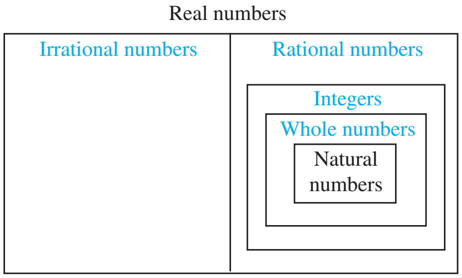
9. Name two numbers that are 6 units away from -2 on the number line. $-8, 4$
10. Show that each of the following numbers is a rational number by expressing it as a fraction with an integer numerator and a nonzero integer denominator.

$$7, \quad -7\frac{3}{5}, \quad 0.007, \quad 700.1$$

$7 = \frac{7}{1}$, $-7\frac{3}{5} = \frac{-38}{5}$, $0.007 = \frac{7}{1{,}000}$, $700.1 = \frac{7{,}001}{10}$

Determine whether each number is a repeating or a nonrepeating decimal, and whether it is a rational or an irrational number.

11. $0.090090009\ldots$ nonrepeating, irrational
12. $0.0\overline{9}$ repeating, rational
13. $5.41414141\ldots$ repeating, rational
14. $1.414213562\ldots$ nonrepeating, irrational
15. The following diagram can be used to show how the natural numbers, whole numbers, integers, rational numbers, and irrational numbers make up the set of real numbers. If the natural numbers can be represented as shown in the next column, label each of the other sets.



16. Determine whether each statement is true or false.
    a. All prime numbers are odd numbers. false
    b. $6 \geq 6$ true
    c. 0 is neither even nor odd. false
    d. Every real number is a rational number. false
17. Write each statement with the inequality symbol pointing in the opposite direction.
    a. $19 > 12$ $12 < 19$
    b. $-6 \leq -5$ $-5 \geq -6$
18. Fill in the blanks:

$$\text{For any real number } a, \begin{cases} \text{If } a \geq 0, \text{ then } |a| = a \\ \text{If } a < 0, \text{ then } |a| = -a \end{cases}$$

NOTATION

Fill in the blanks.

19. The symbol $<$ means "is less than" and the symbol $\geq$ means "is greater than or equal to."
20. $|-2|$ is read as "the absolute value of -2."
21. The symbols { } are called braces.
22. The symbol $\in$ is read as "is an element of" and the symbol $\subseteq$ is read as "is a subset of."
23. Describe the set of rational numbers using set-builder notation. $\left\{\frac{a}{b} \middle| a \text{ and } b \text{ are integers, with } b \neq 0.\right\}$
24. What set of numbers does each symbol represent?
    a. $\mathbb{N}$ Natural numbers
    b. $\mathbb{W}$ Whole numbers
    c. $\mathbb{Z}$ Integers
25. What set of numbers does each symbol represent?
    a. $\mathbb{Q}$ rational numbers
    b. $\mathbb{H}$ irrational numbers
    c. $\mathbb{R}$ real numbers
26. List two other ways that the fraction $-\frac{2}{3}$ can be written. $\frac{-2}{3}, \frac{2}{-3}$

GUIDED PRACTICE

Determine whether each statement is true or false.

See Example 1.

27. $12 \in \mathbb{N}$ true
28. $9 \in \mathbb{N}$ true
29. $-5 \notin \mathbb{Z}$ false
30. $-55 \notin \mathbb{H}$ true

Selected exercises available online at **www.webassign.net/brookscole**

31. $\mathbb{R} \subseteq \mathbb{W}$ false

32. $\mathbb{W} \subseteq \mathbb{R}$ true

33. $\mathbb{H} \not\subseteq \mathbb{Q}$ true

34. $\mathbb{Q} \not\subseteq \mathbb{Z}$ true

Find the decimal equivalent of each fraction and determine whether the decimal is terminating or repeating. See Example 2.

35. $\frac{3}{5}$ 0.6, terminating

36. $\frac{21}{50}$ 0.42, terminating

37. $-\frac{11}{15}$ $-0.7\overline{3}$, repeating

38. $-\frac{7}{30}$ $-0.2\overline{3}$, repeating

39. $\frac{27}{22}$ $1.2\overline{27}$, repeating

40. $\frac{25}{990}$ $0.0\overline{25}$, repeating

41. $\frac{2}{125}$ 0.016, terminating

42. $\frac{19}{16}$ 1.1875, terminating

List the elements of $\{-3, -\frac{8}{5}, 0, \frac{2}{3}, 1, \sqrt{3}, 2, \pi, 4.75, 9, 16.\overline{6}\}$ *that belong to the following sets.* See Example 3.

43. Natural numbers 1, 2, 9

44. Whole numbers 0, 1, 2, 9

45. Integers $-3, 0, 1, 2, 9$

46. Rational numbers $-3, -\frac{8}{5}, 0, \frac{2}{3}, 1, 2, 4.75, 9, 16.\overline{6}$

47. Irrational numbers $\sqrt{3}, \pi$

48. Real numbers $-3, -\frac{8}{5}, 0, \frac{2}{3}, 1, \sqrt{3}, 2, \pi, 4.75, 9, 16.\overline{6}$

49. Even natural numbers 2

50. Odd integers $-3, 1, 9$

51. Prime numbers 2

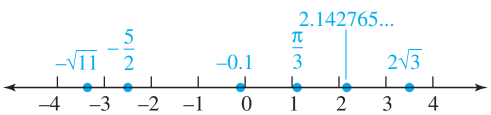
52. Composite numbers 9

53. Odd composite numbers 9

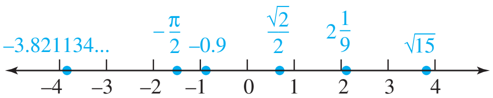
54. Odd prime numbers none

Graph each set on a number line. See Example 4.

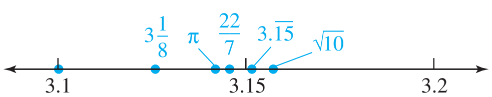
55. $\left\{-\frac{5}{2}, \ -0.1, \ 2.142765\ldots, \ \frac{\pi}{3}, \ -\sqrt{11}, \ 2\sqrt{3}\right\}$



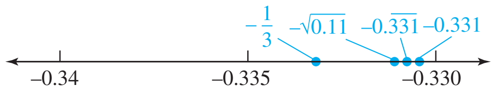
56. $\left\{2\frac{1}{9}, \ -3.821134\ldots, \ -\frac{\pi}{2}, \ \sqrt{15}, \ -0.9, \ \frac{\sqrt{2}}{2}\right\}$



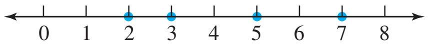
57. $\left\{3.1\overline{5}, \ \frac{22}{7}, \ 3\frac{1}{8}, \ \pi, \ \sqrt{10}, \ 3.1\right\}$



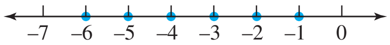
58. $\left\{-0.\overline{331}, \ -0.331, \ -\frac{1}{3}, \ -\sqrt{0.11}\right\}$



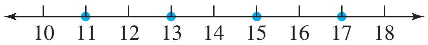
59. The set of prime numbers less than 8



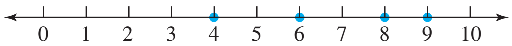
60. The set of integers between -7 and 0



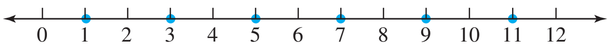
61. The set of odd integers between 10 and 18



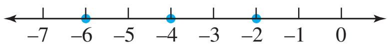
62. The set of composite numbers less than 10



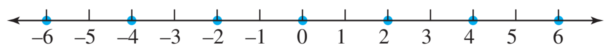
63. The set of positive odd integers less than 12



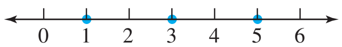
64. The set of negative even integers greater than -7



65. The set of even integers from -6 to 6



66. The set of odd natural numbers less than or equal to 5



Insert either a $<$ *or a* $>$ *symbol to make a true statement.* See Example 5.

67. $-9 \ < \ -8$

68. $-11 \ > \ -12$

69. $-(-5) \ > \ -10$

70. $|-3| \ < \ -(-6)$

71. $6.\overline{1} \ > \ -(-6)$

72. $-6.07 \ < \ -\frac{17}{6}$

73. $-7.999 \ < \ -7.1$

74. $4\frac{1}{2} \ > \ \frac{7}{2}$

Find the value of each expression. See Example 6.

75. $|20|$ 20

76. $|-20|$ 20

77. $|-5.9|$ 5.9

78. $-|1.\overline{27}|$ $-1.\overline{27}$

79. $-|-6|$ -6

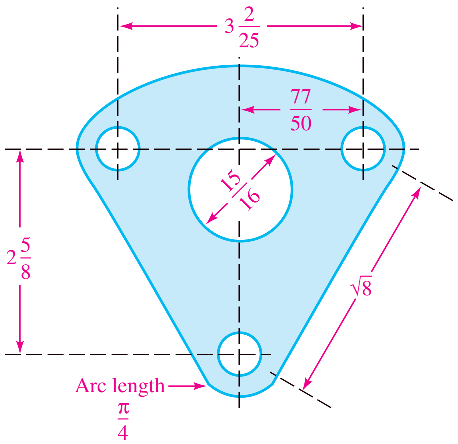
80. $-|-8|$ -8

81. $-\left|\frac{9}{4}\right|$ $-\frac{9}{4}$

82. $-\left|-\frac{5}{16}\right|$ $-\frac{5}{16}$

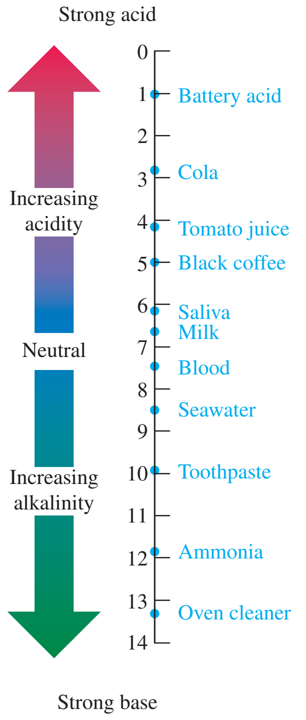
APPLICATIONS

83. DRAFTING Express each dimension in the drawing of a bracket as a four-place decimal. Approximate when necessary.
$3\frac{2}{25} = 3.0800$, $\frac{77}{50} = 1.5400$, $\frac{15}{16} = 0.9375$, $2\frac{5}{8} = 2.6250$,
$\frac{\pi}{4} \approx 0.7854$, $\sqrt{8} \approx 2.8284$

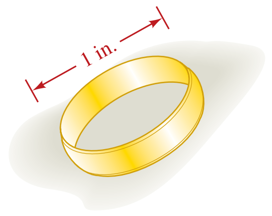


84. pH SCALE The pH scale is used to measure the strength of acids and bases (alkalines) in chemistry. It can be thought of as a number line. On the scale, graph and label each pH measurement given in the table.

Solution	pH
Seawater	8.5
Cola	2.9
Battery acid	1.0
Milk	6.6
Blood	7.4
Ammonia	11.9
Saliva	6.1
Oven cleaner	13.2
Black coffee	5.0
Toothpaste	9.9
Tomato juice	4.1



85. RINGS The formula $C = \pi D$ gives the circumference C of a circle, where D is the length of its diameter. Find the circumference of the gold wedding band. Give an *exact* answer and then an *approximate* answer, rounded to the nearest hundredth of an inch. π in. (3.141592654 … in.); 3.14 in.



86. ACCOUNTING Business losses are usually written within parentheses on financial statements. Examine the statement below for Delta Air Lines. Rank the years, in order from the greatest loss to the smallest loss. 2006, 2004, 2005, 2002, 2003

Delta Air Lines, Annual Income Statement

Net Loss in Millions of Dollars				
2006	**2005**	**2004**	**2003**	**2002**
(\$6,203)	(\$3,818)	(\$5,198)	(\$773)	(\$1,272)

WRITING

87. Explain why the whole numbers are a subset of the integers.

88. What is a real number? Give examples.

89. Explain why there are no even prime numbers greater than 2.

90. Explain why every integer is a rational number, but not every rational number is an integer.

REVIEW

91. Is $\frac{3x - 4}{2}$ an equation or an expression? expression

92. Translate into mathematical symbols: The weight of an object in ounces is 16 times its weight in pounds. $w = 16p$ (answers may vary)

Complete each table.

93. $T = x - 1.5$

x	T
3.7	2.2
10	8.5
30.6	29.1

94. $j = 3m$

m	j
0	0
15	45
300	900

SECTION 1.3
Operations with Real Numbers

Objectives

1. Add and subtract real numbers.
2. Multiply and divide real numbers.
3. Find powers and square roots of real numbers.
4. Use the order of operations rule.
5. Evaluate algebraic expressions.

Six operations can be performed with real numbers: addition, subtraction, multiplication, division, raising to a power, and finding a root. In this section, we will review the rules for performing these operations and discuss how to evaluate numerical expressions involving several operations.

1 Add and subtract real numbers.

When two numbers are added, the result is their **sum.** The rules for adding real numbers are as follows:

Adding Two Real Numbers

To **add two positive numbers,** add them in the usual way. The final answer is positive.

To **add two negative numbers,** add their absolute values and make the final answer negative.

To **add a positive number and a negative number,** subtract the smaller absolute value from the larger.

1. If the positive number has the larger absolute value, the final answer is positive.
2. If the negative number has the larger absolute value, make the final answer negative.

EXAMPLE 1 Add: **a.** $-5 + (-3)$ **b.** $8.9 + (-5.1)$ **c.** $-\frac{13}{15} + \frac{3}{5}$ **d.** $6 + (-10) + (-1)$

Strategy We will use the rules for adding positive and negative real numbers.

WHY Each sum involves signed numbers.

Solution

a. $-5 + (-3) = -8$ — Both numbers are negative. Add their absolute values, 5 and 3, to get 8, and make the final answer negative.

b. $8.9 + (-5.1) = 3.8$ — One number is positive and the other is negative. Subtract their absolute values, 5.1 from 8.9, to get 3.8. Because 8.9 has the larger absolute value, the final answer is positive.

c. $-\frac{13}{15} + \frac{3}{5} = -\frac{13}{15} + \frac{9}{15}$ — Express $\frac{3}{5}$ in terms of the lowest common denominator, 15: $\frac{3}{5} = \frac{3 \cdot 3}{5 \cdot 3} = \frac{9}{15}$.

$= -\frac{4}{15}$ — Subtract the absolute values, $\frac{9}{15}$ from $\frac{13}{15}$, to get $\frac{4}{15}$, and make the final answer negative because $-\frac{13}{15}$ has the larger absolute value.

d. To add three or more real numbers, add from left to right.

$$6 + (-10) + (-1) = -4 + (-1)$$
$$= -5$$

Self Check 1

Add:

a. $-34 + 25$ -9

b. $-70.4 + (-21.2)$ -91.6

c. $\frac{7}{4} + \left(-\frac{3}{2}\right)$ $\frac{1}{4}$

d. $-16 + 17 + (-5)$ -4

Now Try **Problems 15 and 21**

Teaching Example 1 Add:
a. $-9 + 3$ **b.** $-\frac{2}{3} + \left(-\frac{1}{4}\right)$
c. $5.6 + (-3.4)$ **d.** $-4 + (-6)$
Answers:
a. -6 **b.** $-\frac{11}{12}$
c. 2.2 **d.** -10

When two numbers are subtracted, the result is their **difference.** To find a difference, we can change the subtraction into an equivalent addition. For example, the subtraction $7 - 4$ is equivalent to the addition $7 + (-4)$, because they have the same answer:

$$7 - 4 = 3 \qquad \text{and} \qquad 7 + (-4) = 3$$

This suggests that to subtract two numbers, we can change the sign of the number being subtracted and add.

Subtracting Two Real Numbers

To **subtract two real numbers,** add the first number to the opposite (additive inverse) of the number to be subtracted.

Let a and b represent real numbers,

$$a - b = a + (-b)$$

Self Check 2

Subtract:

a. $-15 - 4$ -19

b. $-12.1 - (-7.6)$ -4.5

c. $\frac{5}{9} - \frac{7}{9}$ $-\frac{2}{9}$

d. Subtract 1 from -5 -6

e. $5 - 4 - (-15)$ 16

Now Try **Problems 23 and 29**

Teaching Example 2 Subtract:
a. $3 - 9$ **b.** $-12 - 4$
c. $-\frac{1}{7} - \left(-\frac{3}{7}\right)$ **d.** $2.6 - 5.9$
e. Subtract 4 from -5
Answers:
a. -6 **b.** -16 **c.** $\frac{2}{7}$
d. -3.3 **e.** -9

EXAMPLE 2

Subtract: **a.** $2 - 8$ **b.** $-1.3 - 5.5$ **c.** $-\frac{14}{3} - \left(-\frac{7}{3}\right)$
d. Subtract 9 from -6 **e.** $-11 - (-1) - 5$

Strategy To find each difference, we will apply the rule for subtraction: *Add the first number to the opposite of the number to be subtracted.*

WHY It is easy to make an error when subtracting signed numbers. We will probably be more accurate if we write each subtraction as addition of the opposite.

Solution

a. $2 - \overset{\text{Add}}{\mathbf{8}} = 2 + (\underbrace{\mathbf{-8}}_{\text{the opposite}})$ Here, 8 is being subtracted, so we change the sign of 8 and add. Do not change the sign of 2.

$= -6$

b. $-1.3 - \mathbf{5.5} = -1.3 + (\mathbf{-5.5})$ Change the sign of 5.5 and add. Do not change the sign of -1.3.

$= -6.8$

c. $-\frac{14}{3} - \left(-\frac{7}{3}\right) = -\frac{14}{3} + \frac{7}{3}$ Change the sign of $-\frac{7}{3}$ and add.

$= -\frac{7}{3}$

d. The number to be subtracted is 9. When we translate, we must reverse the order in which 9 and -6 appear in the sentence.

Subtract 9 from -6.

$-6 - 9 = -6 + (-9)$ Add the opposite of 9.

$= -15$

e. To subtract three or more real numbers, subtract from left to right.

$\mathbf{-11 - (-1)} - 5 = \mathbf{-10} - 5$

$= -15$

2 Multiply and divide real numbers.

When two numbers are multiplied, we call the numbers **factors** and the result is their **product.** The rules for multiplying real numbers are as follows:

Multiplying Two Real Numbers

1. **With unlike signs:** To multiply a positive number and a negative number, multiply their absolute values and make the final answer negative.
2. **With like signs:** To multiply two real numbers with the same sign, multiply their absolute values. The final answer is positive.

EXAMPLE 3

Multiply:

a. $4(-7)$ **b.** $-5.2(-3)$ **c.** $-\frac{7}{9}\left(\frac{3}{16}\right)$ **d.** $8(-2)(-3)$

Strategy We will use the rules for multiplying positive and negative real numbers.

WHY Each product involves signed numbers.

Solution

a. $4(-7) = -28$ Multiply the absolute values, 4 and 7, to get 28. Since the signs are unlike, make the final answer negative.

b. $-5.2(-3) = 15.6$ Multiply the absolute values, 5.2 and 3, to get 15.6. Since the signs are like, the final answer is positive.

c. $-\frac{7}{9}\left(\frac{3}{16}\right) = -\frac{7 \cdot 3}{9 \cdot 16}$ Multiply the numerators and multiply the denominators. Since the signs of the factors are unlike, the final answer is negative.

$= -\frac{7 \cdot \cancel{3}^{1}}{\cancel{3}_{1} \cdot 3 \cdot 16}$ Factor 9 as $3 \cdot 3$ and simplify the fraction: $\frac{3}{3} = 1$.

$= -\frac{7}{48}$ Multiply in the numerator and denominator.

d. To multiply three or more real numbers, multiply from left to right.

$$\mathbf{8(-2)}(-3) = \mathbf{-16}(-3)$$
$$= 48$$

Self Check 3

Multiply:

a. $(-6)(5)$ -30

b. $(-4.1)(-8)$ 32.8

c. $\left(\frac{4}{3}\right)\left(-\frac{1}{8}\right)$ $-\frac{1}{6}$

d. $-4(-9)(-3)$ -108

Now Try **Problems 31 and 37**

Teaching Example 3 Multiply:
a. $-3(-2)$ **b.** $5(-3.1)$
c. $\left(-\frac{2}{3}\right)\left(-\frac{9}{10}\right)$ **d.** $4(-3)(-2)$
Answers:
a. 6 **b.** -15.5 **c.** $\frac{3}{5}$ **d.** 24

When two numbers are divided, the result is their **quotient.** In the division $\frac{x}{y} = q$, the quotient q is a number such that $y \cdot q = x$. We can use this relationship to find rules for dividing real numbers.

$\frac{10}{2} = 5$, because $2(5) = 10$ $\qquad$ $\frac{-10}{-2} = 5$, because $-2(5) = -10$

$\frac{-10}{2} = -5$, because $2(-5) = -10$ $\qquad$ $\frac{10}{-2} = -5$, because $-2(-5) = 10$

These results suggest the following rules for dividing real numbers. Note that they are similar to those for multiplying real numbers.

Dividing Two Real Numbers

To **divide two real numbers,** divide their absolute values.

1. The quotient of two numbers with *like* signs is positive.
2. The quotient of two numbers with *unlike* signs is negative.

Self Check 4

Divide:

a. $\frac{55}{-5}$ -11

b. $\frac{-7.2}{-6}$ 1.2

Now Try **Problems 39 and 43**

Teaching Example 4 Divide:

a. $\frac{-35}{7}$ **b.** $\frac{-3.6}{-12}$

Answers:

a. -5 **b.** 0.3

EXAMPLE 4 Divide: **a.** $\dfrac{-44}{11}$ **b.** $\dfrac{-2.7}{-9}$

Strategy We will use the rules for dividing positive and negative real numbers.

WHY Each quotient involves signed numbers.

Solution

a. $\dfrac{-44}{11} = -4$ Divide the absolute values, 44 by 11, to get 4. Since the signs are unlike, make the final answer negative.

b. $\dfrac{-2.7}{-9} = 0.3$ Divide the absolute values, 2.7 by 9, to get 0.3. Since the signs are like, the final answer is positive.

To divide two fractions, we multiply the first fraction by the **reciprocal** of the second fraction. In symbols, if a, b, c, and d are real numbers, and no denominators are 0, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$ $\frac{d}{c}$ is the reciprocal of $\frac{c}{d}$.

Self Check 5

Divide:

a. $-\frac{7}{8} \div \frac{2}{3}$ $-\frac{21}{16}$

b. $-\frac{1}{10} \div (-5)$ $\frac{1}{50}$

Now Try **Problem 45**

Teaching Example 5 Divide:

a. $\frac{4}{3} \div \left(-\frac{8}{15}\right)$

b. $-\frac{3}{5} \div (-9)$

Answers:

a. $-\frac{5}{2}$ **b.** $\frac{1}{15}$

EXAMPLE 5 Divide: **a.** $\dfrac{2}{3} \div \left(-\dfrac{3}{5}\right)$ **b.** $-\dfrac{1}{2} \div (-6)$

Strategy We will multiply the first fraction by the reciprocal of the second.

WHY This is the rule for dividing two fractions.

Solution

a. $\dfrac{2}{3} \div \left(-\dfrac{3}{5}\right) = \dfrac{2}{3} \cdot \left(-\dfrac{5}{3}\right)$ Multiply by the reciprocal of $-\frac{3}{5}$, which is $-\frac{5}{3}$.

$= -\dfrac{10}{9}$ Since the factors have unlike signs, the final answer is negative.

b. $-\dfrac{1}{2} \div (-6) = -\dfrac{1}{2} \cdot \left(-\dfrac{1}{6}\right)$ Multiply by the reciprocal of -6, which is $-\frac{1}{6}$.

$= \dfrac{1}{12}$ Since the factors have like signs, the final answer is positive.

Students often confuse division problems such as $\frac{0}{4}$ and $\frac{4}{0}$. We know that $\frac{0}{4} = 0$ because $4 \cdot 0 = 0$. However, $\frac{4}{0}$ is undefined, because there is no real number q such that $0 \cdot q = 4$. In general, if $x \neq 0$, $\frac{0}{x} = 0$ and $\frac{x}{0}$ is undefined.

The Language of Algebra When we say a division by 0, such as $\frac{4}{0}$, is *undefined,* we mean it is not allowed or it is not defined. That is, $\frac{4}{0}$ does not represent a number.

3 Find powers and square roots of real numbers.

Exponents indicate repeated multiplication. For example,

$$3^2 = 3 \cdot 3$$ Read 3^2 as "3 to the second power" or "3 squared."

$$(-9.1)^3 = (-9.1)(-9.1)(-9.1)$$ Read $(-9.1)^3$ as "-9.1 to the third power" or "-9.1 cubed."

$$\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$$ Read $\left(\frac{2}{3}\right)^4$ as "$\frac{2}{3}$ to the fourth power."

These examples suggest the following definition.

Natural-Number Exponents

A natural-number exponent indicates how many times its base is to be used as a factor.

For any real number x and any natural number n,

$$x^n = \overbrace{x \cdot x \cdot x \cdot \cdots \cdot x}^{n \text{ factors of } x}$$

The exponential expression x^n is called a **power of *x*,** and we read it as "x to the nth power." In this expression, x is called the **base,** and n is called the **exponent.** A natural-number exponent indicates how many times the base of an exponential expression is to be used as a factor in a product.

Base $\rightarrow x^n \leftarrow$ Exponent

EXAMPLE 6

Find each power: **a.** $(-2)^4$ **b.** $\left(\frac{3}{4}\right)^2$ **c.** -0.1 cubed

Strategy We will write each exponential expression as a product of repeated factors, and then perform the multiplication. This requires that we identify the base and the exponent.

WHY The exponent indicates the number of times the base is to be written as a factor.

Solution

a. $(-2)^4 = (-2)(-2)(-2)(-2) = 16$ The base is -2. The exponent is 4.

b. $\left(\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{9}{16}$ The base is $\frac{3}{4}$. The exponent is 2.

c. -0.1 cubed means $(-0.1)^3$. The base is -0.1. The exponent is 3.

$$(-0.1)^3 = (-0.1)(-0.1)(-0.1) = -0.001$$

Self Check 6

Find each power:

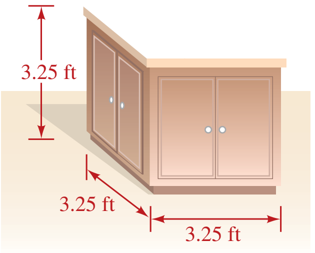
a. $(-3)^3$ -27
b. $(0.8)^2$ 0.64
c. 2^4 16
d. $\frac{7}{5}$ squared $\frac{49}{25}$

Now Try **Problems 49 and 53**

Teaching Example 6 Find each power:
a. $(-3)^4$ **b.** $\left(\frac{5}{7}\right)^2$
c. $(-0.2)^3$
Answers:
a. 81 **b.** $\frac{25}{49}$ **c.** -0.008

Success Tip When multiplying signed numbers, an odd number of negative factors gives a negative product. An even number of negative factors gives a positive product.

Using Your CALCULATOR **The Squaring and Exponential Keys**



A homeowner plans to install a cooking island in her kitchen. (See the figure.) To find the number of square feet of floor space that will be lost, we substitute 3.25 for s in the formula for the area of a square, $A = s^2$. Using the squaring key $\boxed{x^2}$ on a scientific calculator, we can evaluate $(3.25)^2$ as follows:

3.25 $\boxed{x^2}$ $\boxed{10.5625}$

On a graphing calculator, we have:

3.25 $\boxed{x^2}$ $\boxed{\text{ENTER}}$

3.25²
10.5625

About 10.6 square feet of floor space will be lost.

The number of cubic feet of storage space that the cooking island will add can be found by substituting 3.25 for s in the formula for the volume of a cube, $V = s^3$. Using the exponential key $\boxed{y^x}$ $\left(\boxed{x^y}\text{ on some calculators}\right)$, we can evaluate $(3.25)^3$ on a scientific calculator as follows.

3.25 $\boxed{y^x}$ 3 $\boxed{=}$ $\boxed{34.328125}$

On a graphing calculator or direct-entry scientific calculator, we have:

3.25 $\boxed{\wedge}$ 3 $\boxed{\text{ENTER}}$

3.25^3
34.328125

The cooking island will add about 34.3 cubic feet of storage space.

Although the expressions $(-3)^2$ and -3^2 look alike, they are not. In $(-3)^2$, the base is -3. In -3^2, the base is 3. The $-$ sign in front of 3^2 means the opposite of 3^2. When we evaluate them, we see that the results are different:

$$\begin{aligned}(-3)^2 &= (-3)(-3) \\ &= 9\end{aligned}$$

Read as "negative 3 squared."

$$\begin{aligned}-3^2 &= -(3 \cdot 3) \\ &= -9\end{aligned}$$

Read as "the opposite of the square of 3."

Different results

Using Your CALCULATOR **The Parentheses and Negative Keys**

To compute $(-3)^2$ with a reverse-entry scientific calculator, use the *parentheses* keys $\boxed{(}\,\boxed{)}$ and the *change of sign* key $\boxed{+/-}$. Notice that the change of sign key is different from the subtraction key $\boxed{-}$. To enter -3, press $\boxed{+/-}$ *after* entering 3.

$\boxed{(}$ 3 $\boxed{+/-}$ $\boxed{)}$ $\boxed{x^2}$ $\boxed{=}$ $\boxed{9}$

If a direct-entry or graphing calculator is used to find $(-3)^2$, press the negative key $\boxed{(-)}$ *before* entering 3.

$\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{)}$ $\boxed{x^2}$ $\boxed{\text{ENTER}}$

(−3)²
9

To compute -3^2 with a scientific calculator, think of the expression as $-1 \cdot 3^2$. First, find 3^2. Then press $\boxed{+/-}$, which is equivalent to multiplying 3^2 by -1.

3 $\boxed{x^2}$ $\boxed{+/-}$ $\boxed{-9}$

A graphing calculator recognizes -3^2 as $-1 \cdot 3^2$, so we can find -3^2 by entering the following:

$\boxed{(-)}$ 3 $\boxed{x^2}$ $\boxed{\text{ENTER}}$

-3²
-9

Since the product $3 \cdot 3$ can be denoted by the exponential expression 3^2, we say that 3 is squared. The opposite of squaring a number is called finding its **square root.**

All positive numbers have two square roots, one positive and one negative. For example, the two square roots of 9 are 3 and -3. The number 3 is a square root of 9, because $3^2 = 9$, and -3 is a square root of 9, because $(-3)^2 = 9$.

The symbol $\sqrt{}$, called a **radical symbol,** is used to represent the positive (or *principal*) square root of a number.

Principal Square Root

A number b is a square root of a if $b^2 = a$.

If $a > 0$, the expression $\sqrt{a}$ represents the **principal** (or positive) **square root** of a. The principal square root of 0 is 0: $\sqrt{0} = 0$.

The principal square root of a positive number is always positive. Although 3 and -3 are both square roots of 9, only 3 is the principal square root. The symbol $\sqrt{9}$ represents 3. To represent -3, we place a $-$ sign in front of the radical:

$$\sqrt{9} = 3 \qquad \text{and} \qquad -\sqrt{9} = -3$$

EXAMPLE 7

Find each square root:

a. $\sqrt{121}$ **b.** $-\sqrt{49}$ **c.** $\sqrt{\frac{1}{4}}$ **d.** $\sqrt{0.09}$

Strategy In each case, we will determine what positive number, when squared, produces the radicand.

WHY The symbol $\sqrt{}$ indicates that the positive square root of the number written under it should be found.

Solution

a. $\sqrt{121} = 11$, because $11^2 = 121$.

b. Since $\sqrt{49} = 7$, $-\sqrt{49} = -7$.

c. $\sqrt{\frac{1}{4}} = \frac{1}{2}$, because $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

d. $\sqrt{0.09} = 0.3$, because $(0.3)^2 = 0.09$.

Self Check 7

Find each square root:

a. $\sqrt{36}$ 6

b. $-\sqrt{100}$ -10

c. $\sqrt{\frac{4}{25}}$ $\frac{2}{5}$

d. $\sqrt{1}$ 1

e. $\sqrt{0.81}$ 0.9

f. $-\sqrt{400}$ -20

Now Try **Problems 55 and 59**

Teaching Example 7 Find each square root:

a. $\sqrt{144}$ **b.** $-\sqrt{81}$ **c.** $\sqrt{\frac{25}{49}}$ **d.** $\sqrt{0.25}$

Answers:

a. 12 **b.** -9 **c.** $\frac{5}{7}$ **d.** 0.5

4 Use the order of operations rule.

We will often have to evaluate expressions involving several operations. For example, consider the expression $3 + 2 \cdot 5$. To evaluate it, we can perform the addition first and then the multiplication. Or we can perform the multiplication first and then the addition. However, we get different results.

Method 1: Add first		*Method 2: Multiply first*	
~~$3 + 2 \cdot 5 = 5 \cdot 5$~~	~~Add 3 and 2 first.~~	$3 + 2 \cdot 5 = 3 + 10$	Multiply 2 and 5 first.
~~$= 25$~~	~~Multiply.~~	$= 13$	Add.

Different results

This example shows that we need to establish an order of operations. Otherwise, the same expression can have two different values. To guarantee that calculations will have one correct result, we will use the following set of priority rules.

Order of Operations Rule

1. Perform all calculations within parentheses and other grouping symbols, following the order listed in steps 2–4 and working from the innermost pair to the outermost pair.
2. Evaluate all exponential expressions (powers) and roots.
3. Perform all multiplications and divisions as they occur from left to right.
4. Perform all additions and subtractions as they occur from left to right.

When all grouping symbols have been removed, repeat steps 2–4 to complete the calculations.

If a fraction is present, evaluate the expression above the bar (the *numerator*) and the expression below the bar (the *denominator*) separately. Then simplify the fraction, if possible.

To evaluate $3 + 2 \cdot 5$ correctly, we follow steps 2, 3, and 4 of the order of operations rule. Since the expression does not contain any powers or roots, we perform the multiplication first, followed by the addition.

$$3 + 2 \cdot 5 = 3 + 10 \quad \text{Ignore the addition for now and multiply 2 and 5.}$$
$$= 13 \quad \text{Next, perform the addition.}$$

We see that the correct answer is 13.

Self Check 8

Evaluate:

a. $-9 + 2(-4)^2$

b. $20 \div (-5) - (-6)(-5) + (-12)$

Now Try **Problems 63 and 67**

Self Check 8 Answers

a. 23 **b.** -46

Teaching Example 8 Evaluate:

a. $-7 + 3(-5)^2$

b. $3 + 16 \div 2(6) - 7$

Answers:

a. 68 **b.** 44

EXAMPLE 8

Evaluate: **a.** $-5 + 4(-3)^2$ **b.** $-10 \div 5 - 5(3) + 6$

Strategy We will scan the expression to determine what operations need to be performed. Then we will perform those operations, one at a time, following the order of operations rule.

WHY If we don't follow the correct order of operations, we will not get the correct value.

Solution

a. Although the expression contains parentheses, there are no operations to perform within the parentheses. So we proceed with steps 2, 3, and 4 of the order of operations rule.

$$-5 + 4(-3)^2 = -5 + 4(9) \quad \text{First, evaluate the power: } (-3)^2 = 9.$$
$$= -5 + 36 \quad \text{Multiply.}$$
$$= 31 \quad \text{Add.}$$

b. Since the expression does not contain any powers, we perform the multiplications and divisions, working from left to right.

$$\begin{aligned} -10 \div 5 - 5(3) + 6 &= -2 - 5(3) + 6 && \text{Divide: } -10 \div 5 = -2. \\ &= -2 - 15 + 6 && \text{Multiply.} \\ &= -17 + 6 && \text{Working from left to right, subtract: } -2 - 15 = -17. \\ &= -11 && \text{Add.} \end{aligned}$$

Grouping symbols serve as mathematical punctuation marks. They help determine the order in which an expression is evaluated. Examples of grouping symbols are parentheses ( ), brackets [ ], absolute value bars | |, and the fraction bar —.

EXAMPLE 9

Evaluate:

a. $3 - (4 - 8)^2$ **b.** $2 + 3[-2 - 8(4 - 3^2)]$ **c.** $|-45 + 30|(2 - 7)$

Strategy We will perform all calculations within parentheses and other grouping symbols first.

WHY This is the first step of the order of operations rule.

Solution

a.

$$\begin{aligned} 3 - (4 - 8)^2 &= 3 - (-4)^2 && \text{Perform the subtraction: } 4 - 8 = -4. \\ &= 3 - 16 && \text{Evaluate the power: } (-4)^2 = 16. \\ &= -13 && \text{Subtract.} \end{aligned}$$

b. First, we work within the innermost grouping symbols, the parentheses.

$$\begin{aligned} 2 + 3[-2 - 8(4 - 3^2)] &= 2 + 3[-2 - 8(4 - 9)] && \text{Find the power: } 3^2 = 9. \\ &= 2 + 3[-2 - 8(-5)] && \text{Subtract: } 4 - 9 = -5. \end{aligned}$$

Next, we work within the outermost grouping symbols, the brackets.

$$\begin{aligned} &= 2 + 3[-2 - (-40)] && \text{Multiply: } 8(-5) = -40. \\ &= 2 + 3(-2 + 40) \end{aligned}$$

Since only one set of grouping symbols was needed, we wrote $-2 + 40$ within parentheses.

$$\begin{aligned} &= 2 + 3(38) && \text{Add: } -2 + 40 = 38. \\ &= 2 + 114 && \text{Multiply.} \\ &= 116 && \text{Add.} \end{aligned}$$

c. Since the absolute value bars are grouping symbols, we perform the operations within the absolute value bars and the parentheses first.

$$\begin{aligned} |-45 + 30|(2 - 7) &= |-15|(-5) \\ &= 15(-5) && \text{Find the absolute value: } |-15| = 15. \\ &= -75 && \text{Multiply.} \end{aligned}$$

Self Check 9

Evaluate:

a. $(5 - 3)^3 - 40$

b. $-3[-2(5^3 - 3) + 4] - 1$

c. $2|-25 - (-6)(3)|$

Now Try **Problems 71 and 79**

Self Check 9 Answers

a. -32 **b.** 719 **c.** 14

Teaching Example 9 Evaluate:

a. $(3 - 8)^2 - 4 + 2^2$

b. $4 + 3[5(3 - 1)^2 + 5]$

c. $|4 - 11| + 6(7) \div 21 + (-3)$

Answers:

a. 25 **b.** 79 **c.** 6

Using Your CALCULATOR Order of Operations

Scientific and graphing calculators are programmed to follow the rules for the order of operations. For example, when finding $3 + 2 \cdot 5$, both types of calculators give the correct answer, 13.

3 $\boxed{+}$ 2 $\boxed{\times}$ 5 $\boxed{=}$ | 13

3 $\boxed{+}$ 2 $\boxed{\times}$ 5 $\boxed{\text{ENTER}}$ | 3 + 2 * 5 13

Both types of calculators use parentheses keys $\boxed{(}\boxed{)}$ when grouping symbols are needed. To evaluate $3 - (4 - 8)^2$, we proceed as follows.

3 $\boxed{-}$ $\boxed{(}$ 4 $\boxed{-}$ 8 $\boxed{)}$ $\boxed{x^2}$ $\boxed{=}$ | −13

3 $\boxed{-}$ $\boxed{(}$ 4 $\boxed{-}$ 8 $\boxed{)}$ $\boxed{x^2}$ $\boxed{\text{ENTER}}$ | 3 − (4 − 8)² −13

Both types of calculators require that we group the terms in the numerator together and the terms in the denominator together when calculating the value of an expression such as $\frac{200 + 120}{20 - 16}$.

$\boxed{(}$ 200 $\boxed{+}$ 120 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 20 $\boxed{-}$ 16 $\boxed{)}$ $\boxed{=}$ | 80

$\boxed{(}$ 200 $\boxed{+}$ 120 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 20 $\boxed{-}$ 16 $\boxed{)}$ $\boxed{\text{ENTER}}$ | (200+120)/(20−16) 80

If parentheses aren't used when finding $\frac{200 + 120}{20 - 16}$, you will obtain an incorrect result of 190. That is because the calculator will interpret the entry as $200 + \frac{120}{20} - 16$.

5 Evaluate algebraic expressions.

Recall that an algebraic expression is a combination of variables and numbers with the operations of arithmetic. To *evaluate* these expressions, we substitute specific numbers for the variables and then apply the order of operations rule.

Self Check 10

If $r = 2$, $s = -5$, and $t = 3$, evaluate:

a. $-\frac{1}{3}s^3t$ 125

b. $\frac{\sqrt{-5s}}{(s + t)r^2}$ $-\frac{5}{8}$

Now Try **Problems 95 and 97**

Teaching Example 10 If $a = 16$, $b = -3$, and $c = -4$, evaluate:

a. $\frac{-2}{3}ab^2$ **b.** $\frac{c\sqrt{a} + b^2}{bc}$

Answers:

a. -96 **b.** $\frac{-7}{12}$

EXAMPLE 10 If $a = -2$, $b = 9$, and $c = -1$, evaluate:

a. $-\frac{1}{2}a^2$ **b.** $\frac{-a\sqrt{b} + 3c^3}{c(c - b)}$

Strategy We will replace each a, b, and c in the expression with the given value of the variable and evaluate the expression using the order of operations rule.

WHY To *evaluate an expression* means to find its numerical value, once we know the value of its variable(s).

Solution

a. We substitute -2 for a and use the order of operations rule.

$$-\frac{1}{2}a^2 = -\frac{1}{2}(-2)^2$$ Substitute −2 for *a*. Write parentheses around −2 so that it is squared.

$$= -\frac{1}{2}(4)$$ Evaluate the power: $(-2)^2 = 4$.

$$= -2$$ Multiply.

b. $$\frac{-a\sqrt{b} + 3c^3}{c(c - b)} = \frac{-(-2)\sqrt{9} + 3(-1)^3}{-1(-1 - 9)}$$ Substitute −2 for *a*, 9 for *b*, and −1 for *c*.

$$= \frac{-(-2)(3) + 3(-1)}{-1(-10)}$$ In the numerator, evaluate the square root and the power: $\sqrt{9} = 3$ and $(-1)^3 = -1$. In the denominator, subtract.

$$= \frac{2(3) + 3(-1)}{-1(-10)}$$ In the numerator, simplify: $-(-2) = 2$.

$$= \frac{6 + (-3)}{10}$$ In the numerator, multiply. In the denominator, multiply.

$$= \frac{3}{10}$$ In the numerator, add.

Using Your CALCULATOR Evaluating Algebraic Expressions

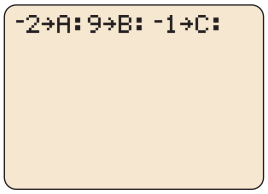
Graphing calculators can evaluate algebraic expressions. For example, to evaluate

$$\frac{-a\sqrt{b} + 3c^3}{c(c - b)}$$

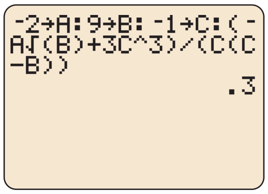
(Example 10, part b) using a TI-84 Plus calculator, we first enter the values of $a = -2$, $b = 9$, and $c = -1$, using the store key $\boxed{\text{STO}}$ and the $\boxed{\text{ALPHA}}$ key. See figure (a).

$\boxed{(-)}$ 2 $\boxed{\text{STO}}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{A}}$ $\boxed{\text{ALPHA}}$ $\boxed{:}$ This enters $a = -2$.
9 $\boxed{\text{STO}}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{B}}$ $\boxed{\text{ALPHA}}$ $\boxed{:}$ This enters $b = 9$.
$\boxed{(-)}$ 1 $\boxed{\text{STO}}$ $\boxed{\text{ALPHA}}$ $\boxed{\text{C}}$ $\boxed{\text{ALPHA}}$ $\boxed{:}$ This enters $c = -1$.

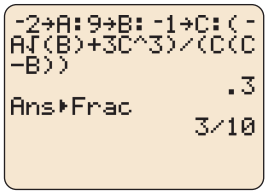
Next, enter the expression as shown in figure (b) and press $\boxed{\text{ENTER}}$ to find that the value of the expression is 0.3. To express the result as a fraction, press $\boxed{\text{MATH}}$, highlight ▶ Frac, and then press $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$. See figure (c).



(a)



(b)



(c)

ANSWERS TO SELF CHECKS

1. a. -9 **b.** -91.6 **c.** $\frac{1}{4}$ **d.** -4 **2. a.** -19 **b.** -4.5 **c.** $-\frac{2}{9}$ **d.** -6 **e.** 16
3. a. -30 **b.** 32.8 **c.** $-\frac{1}{6}$ **d.** -108 **4. a.** -11 **b.** 1.2 **5. a.** $-\frac{21}{16}$ **b.** $\frac{1}{50}$
6. a. -27 **b.** 0.64 **c.** 16 **d.** $\frac{49}{25}$ **7. a.** 6 **b.** -10 **c.** $\frac{2}{5}$ **d.** 1 **e.** 0.9 **f.** -20
8. a. 23 **b.** -46 **9. a.** -32 **b.** 719 **c.** 14 **10. a.** 125 **b.** $-\frac{5}{8}$

STUDY SKILLS CHECKLIST

Getting the Most from the Study Sets

The following checklist will help you become familiar with the Study Sets in this book. Place a check mark in each box after you answer the question.

- ☐ Answers to the odd-numbered **Study Set** problems are located in the appendix on page A-3. On what page do the answers to Study Set 1.6 appear?
- ☐ Each Study Set begins with **Vocabulary** problems. How many Vocabulary problems appear in Study Set 1.6, which begins on page 68?
- ☐ Following the Vocabulary problems, you will see **Concepts.** How many Concepts problems appear in Study Set 1.6?
- ☐ Following the Concepts problems, you will see **Notation** problems. How many Notation problems appear in Study Set 1.6?
- ☐ After the Notation problems, **Guided Practice** problems are given which are linked to similar examples within the section. How many Guided Practice problems appear in Study Set 1.6?
- ☐ After the Guided Practice problems, **Try It Yourself** problems are given and can be used to help you prepare for quizzes. How many Try It Yourself problems appear in Study Set 1.6?
- ☐ Following the Try It Yourself problems, you will see **Applications.** How many Applications problems appear in Study Set 1.6?
- ☐ After completing the Application problems, a few **Writing** problems are given. How many Writing problems appear in Study Set 1.6?
- ☐ Lastly, each Study Set ends with a few **Review** problems. How many Review problems appear in Study Set 1.6?

Answers: A-4, 4, 4, 2, 48, 20, 14, 4, 6

SECTION 1.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. When we add two numbers, the result is called the sum. When we subtract two numbers, the result is called the difference.

2. When we multiply two numbers, the result is called the product. When we divide two numbers, the result is called the quotient.

3. The reciprocal of $\frac{5}{9}$ is $\frac{9}{5}$.

4. In the exponential expression x^2, the base is x and 2 is the exponent.

5. 6^2 can be read as "six squared" and 6^3 can be read as "six cubed."

6. We read $\sqrt{25}$ as the square root of 25.

7. In the expression $9 + 6[22 - (6 - 1)]$, the parentheses are the innermost grouping symbols, and the brackets are the outermost grouping symbols.

8. To evaluate an algebraic expression, we substitute values for the variables and then apply the order of operations rule.

▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

9. Fill in the blanks.

a. An exponent indicates repeated multiplication.

b. Subtraction is the same as adding the opposite of the number being subtracted.

c. When multiplying signed numbers, an odd number of negative factors gives a negative product. An even number of negative factors gives a positive product.

▶ **10. a.** What is the related multiplication statement for the division statement $\frac{0}{6} = 0$? $6 \cdot 0 = 0$

b. Why isn't there a related multiplication statement for $\frac{6}{0}$? There is no number q such that $6 = 0 \cdot q$.

c. Fill in the blanks: if $x \neq 0$, $\frac{0}{x} =$ 0 and $\frac{x}{0}$ is undefined.

11. Consider the expression $6 + 3 \cdot 2$.

a. In what two different ways *might* we evaluate the given expression? addition first or multiplication first

b. Which result from part a is correct and why? 12; multiplication is performed before addition

12. In what order should the operations be performed to evaluate $60 - (-9)^2 + 5(-1)$? power, multiplication, subtraction, addition

NOTATION

13. Translate each expression into symbols, and then evaluate it.
   a. Negative four, squared $(-4)^2 = 16$
   b. The opposite of the square of 4 $-4^2 = -16$

14. What is the name of the symbol $\sqrt{}$? radical symbol or square root symbol

GUIDED PRACTICE

Perform the operations. See Examples 1 and 2.

15. $-3 + (-5)$ -8
16. $-2 + (-8)$ -10
17. $-7.1 + 2.8$ -4.3
18. $3.1 + (-5.2)$ -2.1
19. $-9 + (-8) + 4$ -13
20. $2 + (-6) + (-3)$ -7
21. $\frac{1}{2} + \left(-\frac{1}{3}\right)$ $\frac{1}{6}$
22. $-\frac{3}{4} + \left(-\frac{1}{5}\right)$ $-\frac{19}{20}$
23. $-3 - 4$ -7
24. $-11 - (-17)$ 6
25. $-3.3 - (-3.3)$ 0
26. $0.14 - (-0.13)$ 0.27
27. Subtract $-\frac{3}{5}$ from $\frac{1}{2}$ $\frac{11}{10}$
28. Subtract $\frac{11}{13}$ from $\frac{1}{26}$ $-\frac{21}{26}$
29. $-1 - 5 - (-4)$ -2
30. $5 - (-3) - 2$ 6

Perform the operations. See Examples 3–5.

31. $-2(6)$ -12
32. $-3(7)$ -21
33. $-0.3(5)$ -1.5
34. $-0.4(-0.6)$ 0.24
35. $-5(6)(-2)$ 60
36. $-9(-1)(-3)$ -27
37. $\left(-\frac{3}{5}\right)\left(\frac{10}{7}\right)$ $-\frac{6}{7}$
38. $\left(-\frac{6}{7}\right)\left(-\frac{5}{12}\right)$ $\frac{5}{14}$
39. $\frac{-8}{4}$ -2
40. $\frac{16}{-4}$ -4
41. $\frac{84}{-6}$ -14
42. $\frac{-78}{6}$ -13
43. $\frac{-10.8}{-1.2}$ 9
44. $\frac{-13.5}{-1.5}$ 9
45. $-\frac{16}{5} \div \left(-\frac{10}{3}\right)$ $\frac{24}{25}$
46. $-\frac{5}{24} \div \frac{10}{3}$ $-\frac{1}{16}$

Evaluate each expression. See Example 6.

47. 6^4 $1{,}296$
48. 2^5 32
49. $(-7.9)^2$ 62.41
50. $(-4.6)^2$ 21.16
51. -5^2 -25
52. -8^2 -64
53. $\left(-\frac{3}{5}\right)^3$ $-\frac{27}{125}$
54. $\left(-\frac{4}{3}\right)^3$ $-\frac{64}{27}$

Find each square root. See Example 7.

55. $\sqrt{64}$ 8
56. $\sqrt{121}$ 11
57. $-\sqrt{81}$ -9
58. $-\sqrt{36}$ -6
59. $-\sqrt{\frac{9}{16}}$ $-\frac{3}{4}$
60. $-\sqrt{\frac{81}{49}}$ $-\frac{9}{7}$
61. $\sqrt{0.04}$ 0.2
62. $\sqrt{0.64}$ 0.8

Evaluate each expression. See Example 8.

63. $3 - 5 \cdot 4$ -17
64. $12 - 2 \cdot 3$ 6
65. $4 \cdot 2^3$ 32
66. $4 \cdot 5^3$ 500
67. $-12 \div 3 \cdot 2$ -8
68. $-18 \div 6 \cdot 3$ -9
69. $7^2 - (-9)^2$ -32
70. $4^2 - (-8)^2$ -48

Evaluate each expression. See Example 9.

71. $(4 + 2 \cdot 3)^4$ $10{,}000$
72. $|9 - 5(1 - 8)|$ 44
73. $\left(-3 - \sqrt{25}\right)^2$ 64
74. $\left(-1 - \sqrt{144}\right)^2$ 169
75. $-2|4 - 8|$ -8
76. $\left|\sqrt{49} - 8(4 - 7)\right|$ 31
77. $2 + 3\left(\frac{25}{5}\right) + (-4)$ 13
78. $(-2)^3\left(\frac{-6}{-2}\right)(-1)$ 24
79. $30 + 6[-4 - 5(6 - 4)^2]$ -114
80. $7 - 12[7^2 - 4(2 - 5)^2]$ -149
81. $3 - [3^3 + (3 - 1)^3]$ -32
82. $8 - 4|-(3 \cdot 5 - 2 \cdot 6)^2|$ -28
83. $\frac{1}{3}\left(\frac{1}{6}\right) - \left(-\frac{1}{3}\right)^2$ $-\frac{1}{18}$
84. $\frac{1}{2}\left(\frac{1}{8}\right) + \left(-\frac{1}{4}\right)^2$ $\frac{1}{8}$
85. $\frac{-2 - 5}{-7 + (-7)}$ $\frac{1}{2}$
86. $\frac{-3 - (-1)}{-2 + (-2)}$ $\frac{1}{2}$
87. $\frac{|-25| - 2(-5)}{2^4 - 9}$ 5
88. $\frac{2[-4 - 2(3 - 1)]}{3(3)(2)}$ $-\frac{8}{9}$
89. $\frac{3[-9 + 2(7 - 3)]}{(8 - 5)(9 - 7)}$ $-\frac{1}{2}$
90. $\frac{(6 - 5)^4 + 21}{27 - \left(\sqrt{16}\right)^2}$ 2

Evaluate each expression for the given values. See Example 10.

91. $-\frac{2}{3}a^2$ for $a = -6$ -24
92. $\left(-\frac{2}{3}a\right)^2$ for $a = -6$ 16
93. $\frac{y_2 - y_1}{x_2 - x_1}$ for $x_1 = -3, x_2 = 5, y_1 = 12, y_2 = -4$ -2
94. $P_0\left(1 + \frac{r}{k}\right)^{kt}$ for $P_0 = 500, r = 4, k = 2, t = 3$ $364{,}500$
95. $(x + y)(x^2 - xy + y^2)$ for $x = -4, y = 5$ 61
96. $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ for $x = -3, y = -4, a = 5, b = -5$ 1
97. $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ for $a = 1, b = 2, c = -3$ 1
98. $\frac{n}{2}[2a_1 + (n - 1)d]$ for $n = 50, a_1 = -4, d = 5$ $5{,}925$

99. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for $x_1 = -2$, $x_2 = 4$, $y_1 = 4$, and $y_2 = -4$ 10

100. $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$ for $A = 3$, $B = 4$, $C = -5$, $x_0 = 2$, and $y_0 = -1$ $\frac{3}{5}$

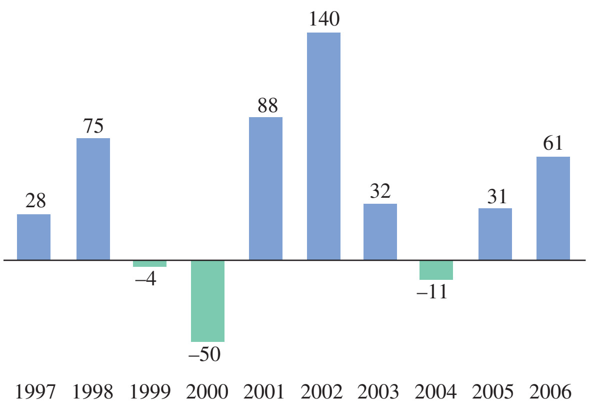
101. $-n(4n^2 - 27m^2)^3$ for $m = \frac{1}{3}$ and $n = \frac{1}{2}$ 4

102. $\dfrac{-s^2 + 1 + 16r^2}{3^2 - 2}$ for $s = -10$ and $r = \frac{1}{4}$ -14

APPLICATIONS

▶ **103.** INVESTMENT IN BONDS In the following graph, positive numbers represent new cash *inflow* into U.S. bond funds. Negative numbers represent cash *outflow* from bond funds. Was there a net inflow or outflow over the 10-year period? What was it?
There was a net inflow of $390 billion.

New Net Cash Flow to U.S. Bond Funds (in billions of dollars)



Source: Investment Company Institute

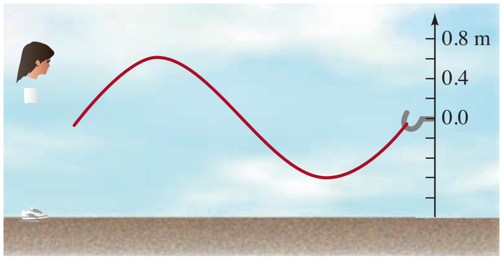
104. ACCOUNTING On a financial balance sheet, debts (negative numbers) are denoted within parentheses. Assets (positive numbers) are written without parentheses. What is the 2003 fund balance for the preschool whose financial records are shown in the table? $(967)

Community Care Preschool Balance Sheet, June 2003	
Fund balances	
Classroom supplies	$ 5,889
Emergency needs	927
Holiday program	(2,928)
Insurance	1,645
Janitorial	(894)
Licensing	715
Maintenance	(6,321)
BALANCE	?

▶ **105.** TEMPERATURE EXTREMES The highest and lowest temperatures ever recorded in several cities are shown in the table. List the cities in order, from the smallest to the largest range in temperature extremes. New York, Atlanta, Boise, Omaha, Helena

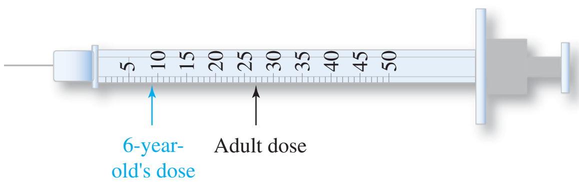
	Extreme temperatures	
City	**Highest**	**Lowest**
Atlanta, Georgia	105	−8
Boise, Idaho	111	−25
Helena, Montana	105	−42
New York, New York	107	−3
Omaha, Nebraska	114	−23

▶ **106.** PHYSICS Waves are motions that carry energy from one place to another. The illustration shows an example of a wave called a *standing wave*. What is the difference in the height of the crest of the wave and the depth of the trough of the wave? (m stands for meter.) 1.2 m



107. PEDIATRICS Young's rule, shown below, is used by some doctors to calculate the dose for infants and children.

$$\frac{\text{Age of child}}{\text{Age of child} + 12}\left(\begin{array}{c}\text{average}\\ \text{adult dose}\end{array}\right) = \text{child's dose}$$



The syringe shows the adult dose of a certain medication. Use Young's rule to determine the dose for a 6-year-old child. Then use an arrow to locate the dose on the calibration. 9

▶ **108.** DOSAGES The adult dosage of procaine penicillin is 300,000 units daily. Calculate the dosage for a 12-year-old child using Young's rule. (See Exercise 107.) 150,000 units

WRITING

109. Explain what the statement $x - y = x + (-y)$ means.

▶ **110.** Explain why the order of operations rule is necessary.

REVIEW

111. What two numbers are a distance of 5 away from -2 on the number line? -7 and 3

▶ **112.** Place the proper symbol ($>$ or $<$) in the blank: -4.6 $<$ -4.5

113. Write the set of integers. $\{\ \ldots\ , -2, -1, 0, 1, 2,\ \ldots\ \}$

114. Translate into mathematical symbols: ten less than twice x. $2x - 10$

115. True or false: The real numbers is the set of all decimals. true

116. True or false: Irrational numbers are nonterminating, nonrepeating decimals. true

SECTION 1.4
Simplifying Algebraic Expressions Using Properties of Real Numbers

Objectives

1. Identify terms, factors, and coefficients.
2. Identify and use properties of real numbers.
3. Simplify products.
4. Use the distributive property.
5. Combine like terms.

In algebra, we frequently replace one algebraic expression with another that is equivalent and simpler in form. That process, called *simplifying an expression,* often involves the use of one or more properties of real numbers.

1 Identify terms, factors, and coefficients.

Addition signs separate algebraic expressions into parts called *terms.* For example, the expression $3x^2 + x + 4$ has three terms: $3x^2$, x, and 4.

In general, a **term** is a product or quotient of numbers and/or variables. A single number or variable is also a term. Examples of terms are:

$$4, \qquad y, \qquad 6r, \qquad -w^3, \qquad 3.7x^5, \qquad \frac{3}{n}, \qquad -15ab^2$$

Since subtraction can be written as addition of the opposite, the expression $6a - 5b$ can be written in the equivalent form $6a + (-5b)$. We can then see that $6a - 5b$ contains two terms, $6a$ and $-5b$.

A term such as 9, that consists of a single number, is called a **constant term.**

The numerical factor of a term is called the **numerical coefficient** or simply the **coefficient** of the term. The coefficients of the terms of the expression $3x^2 + x + 4$ are 3, 1, and 4, respectively. The coefficients of the terms of $6a - 5b$ are 6 and -5, respectively.

It is important to be able to distinguish between the *terms* of an expression and the *factors* of a term.

EXAMPLE 1 Identify the coefficient of each term of $5n^3 + 10n^2 - n - 2$.

Strategy First, we will write each subtraction as addition of the opposite so that we can identify the terms of the expression. Then we will identify the coefficient of each term.

WHY Addition symbols separate algebraic expressions into terms.

Solution

If we write $5n^3 + 10n^2 - n - 2$ as $5n^3 + 10n^2 + (-n) + (-2)$, we see that it has four terms: $5n^3$, $10n^2$, $-n$, and -2. The numerical factor of each term is its coefficient.

Self Check 1

Identify the coefficient of each term of $m^3 - 2m^2 + 3m + 4$.

Now Try Problem 19

Self Check 1 Answers
1, -2, 3, 4

Teaching Example 1 Identify the coefficient of each term of $7a^4 - 10a^3 + 12a^2 - 5a + 4$
Answers:
7, -10, 12, -5, 4

The coefficient of $5n^3$ is **5** because $5n^3$ means $\mathbf{5} \cdot n^3$.
The coefficient of $10n^2$ is **10** because $10n^2$ means $\mathbf{10} \cdot n^2$.
The coefficient of $-n$ is **−1** because $-n$ means $\mathbf{-1} \cdot n$.
The coefficient of the constant term -2 is -2.

2 Identify and use properties of real numbers.

The following properties of real numbers are used to simplify algebraic expressions.

Properties of Real Numbers

If a, b, and c represent real numbers, we have

The commutative properties of addition and multiplication

$$a + b = b + a \qquad ab = ba$$

The associative properties of addition and multiplication

$$(a + b) + c = a + (b + c) \qquad (ab)c = a(bc)$$

The *commutative properties* enable us to add or multiply two numbers in either order and obtain the same result. Here are two examples.

$3 + (-5) = -2$ and $-5 + 3 = -2$
$-2.6(-8) = 20.8$ and $-8(-2.6) = 20.8$

The Language of Algebra *Commutative* is a form of the word *commute,* meaning to go back and forth. *Commuter* trains take people to and from work.

We can use the commutative properties (and other properties of real numbers) to write *equivalent expressions.* **Equivalent expressions** represent the same number. For example, $x + 3$ and $3 + x$ are equivalent expressions because for each value of x, they represent the same number. For instance, if $x = 6$, both expressions represent 9. If $x = -4$, both expressions represent -1, and so on.

***If x* = 6**

$$x + 3 = 6 + 3 = 9 \qquad 3 + x = 3 + 6 = 9$$

***If x* = −4**

$$x + 3 = -4 + 3 = -1 \qquad 3 + x = 3 + (-4) = -1$$

Subtraction and division are not commutative, because performing these operations in different orders will give different results. For example,

$8 - 4 = 4$ but $4 - 8 = -4$

$8 \div 4 = 2$ but $4 \div 8 = \frac{1}{2}$

The *associative properties* enable us to group the numbers in an addition or multiplication any way that we wish and get the same result. For example,

$(19 + 7) + 3 = 26 + 3 = 29$ and $19 + (7 + 3) = 19 + 10 = 29$
$(4 \cdot 2)6 = 8 \cdot 6 = 48$ and $4(2 \cdot 6) = 4 \cdot 12 = 48$

The Language of Algebra *Associative* is a form of the word *associate*, meaning to join a group. For example, the National Basketball *Association* (NBA) is a group of professional basketball teams.

Subtraction and division are not associative, because different groupings give different results. For example,

$$(8 - 4) - 2 = 4 - 2 = 2 \quad \text{but} \quad 8 - (4 - 2) = 8 - 2 = 6$$
$$(8 \div 4) \div 2 = 2 \div 2 = 1 \quad \text{but} \quad 8 \div (4 \div 2) = 8 \div 2 = 4$$

The real numbers 0 and 1 have important special properties.

Properties of 0 and 1

Additive identity: The sum of 0 and any number is the number itself.

$$0 + a = a + 0 = a$$

Multiplicative identity: The product of 1 and any number is the number itself.

$$1 \cdot a = a \cdot 1 = a$$

Multiplication property of 0: The product of any number and 0 is 0.

$$a \cdot 0 = 0 \cdot a = 0$$

For example,

$$7 + 0 = 7, \qquad 1(5.4) = 5.4, \qquad \left(-\frac{7}{3}\right)1 = -\frac{7}{3}, \qquad \text{and} \qquad -19(0) = 0$$

If the sum of two numbers is 0, they are called **additive inverses,** or **opposites** of each other. For example, 6 and -6 are additive inverses, because $6 + (-6) = 0$.

The Additive Inverse Property

For every real number a, there exists a real number $-a$ such that

$$a + (-a) = -a + a = 0$$

If the product of two numbers is 1, the numbers are called **multiplicative inverses** or **reciprocals** of each other.

The Multiplicative Inverse Property

For every nonzero real number a, there exists a real number $\frac{1}{a}$ such that

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

Some examples of reciprocals (multiplicative inverses) are

- 5 and $\frac{1}{5}$ are reciprocals, because $5\left(\frac{1}{5}\right) = 1$.
- $\frac{3}{2}$ and $\frac{2}{3}$ are reciprocals, because $\frac{3}{2}\left(\frac{2}{3}\right) = 1$.
- -0.25 and -4 are reciprocals, because $-0.25(-4) = 1$.

Caution! The reciprocal of 0 does not exist, because $\frac{1}{0}$ is undefined.

Self Check 2

Use the given property to complete each statement.

a. $-23 + \underline{23} = 0$ (Additive inverse property)

b. $-16\left(\frac{1}{2} \cdot 7\right) = \underline{\left(-16 \cdot \frac{1}{2}\right) \cdot 7}$ (Associative property of multiplication)

Now Try **Problems 23 and 27**

Teaching Example 2 Complete each statement so that the indicated property is illustrated.

a. $\frac{5}{6} \cdot \underline{\quad} = 1$ (Multiplicative inverse property)

b. $-3 + \underline{\quad} = 0$ (Additive inverse property)

c. $4 + x = \underline{\quad}$ (Commutative property of addition)

Answers:

a. $\frac{6}{5}$ **b.** 3 **c.** $x + 4$

EXAMPLE 2

Complete each statement so that the indicated property is illustrated.

a. $(14 + 92) + 8 = \underline{\qquad}$ (Associative property of addition)

b. $\frac{7}{6} \cdot \underline{\qquad} = \frac{7}{6}$ (Multiplicative identity property)

c. $x \cdot 5 = \underline{\qquad}$ (Commutative property of multiplication)

Strategy For problems like these, it is important to have memorized the properties of real numbers by name. To fill in each blank, we will determine in what way the indicated property enables us to write an equivalent expression.

WHY We should memorize the properties of real numbers by name because their names remind us how to use them.

Solution

a. To *associate* means to group together. The associative property of addition enables us to group the numbers in a different way. Thus, we have

$$(14 + 92) + 8 = \underline{14 + (92 + 8)} \quad \text{(Associative property of addition)}$$

Note that the *order* of the numbers on each side of the statement remains the same.

b. The word *identical* means to be exactly the same. The multiplicative identity property indicates that if we multiply $\frac{7}{6}$ by 1, it remains the same. Thus, we have

$$\frac{7}{6} \cdot \underline{1} = \frac{7}{6} \quad \text{(Multiplicative identity property)}$$

c. To *commute* means to go back and forth. The commutative property of multiplication enables us to change the order of the factors. Thus, we have

$$x \cdot 5 = \underline{5 \cdot x} \quad \text{(Commutative property of multiplication)}$$

Recall that when a number is divided by 1, the result is the number itself, and when a nonzero number is divided by itself, the result is 1.

Division Properties

Division by 1: If a represents any real number, then $\frac{a}{1} = a$.

Division of a number by itself: For any nonzero real number a, $\frac{a}{a} = 1$.

There are three possible cases to consider when discussing division involving 0.

Division with 0

Division of 0: For any nonzero real number a, $\frac{0}{a} = 0$.

Division by 0: For any nonzero real number a, $\frac{a}{0}$ is undefined.

Division of 0 *by* 0: $\frac{0}{0}$ is indeterminate.

To show that division of 0 by 0 is indeterminate, we consider $\frac{0}{0} = ?$ and its equivalent multiplication fact $0(?) = 0$.

Multiplication fact	***Division fact***
$0(?) = 0$	$\frac{0}{0} = ?$
Any number multiplied by 0 gives 0.	We cannot determine this—it could be any number.

3 Simplify products.

The commutative and associative properties of multiplication can be used to simplify certain products. For example, let's simplify $6(5x)$.

$$6(5x) = 6 \cdot (5 \cdot x) \quad \text{Rewrite } 5x \text{ as } 5 \cdot x.$$
$$= (6 \cdot 5) \cdot x \quad \text{Use the associative property of multiplication to group 5 with 6.}$$
$$= 30x \quad \text{Multiply within the parentheses.}$$

Since $6(5x) = 30x$, we say that $6(5x)$ simplifies to $30x$.

EXAMPLE 3 Simplify: **a.** $9(10t)$ **b.** $-5.3r(-2s)$ **c.** $-\frac{21}{2}a\left(\frac{1}{3}\right)$

Strategy We will use the commutative and associative properties of multiplication to reorder and regroup the factors in each expression.

WHY We want to group all of the numerical factors of an expression together so that we can find their product.

Solution

a. $9(10t) = (9 \cdot 10)t$ Use the associative property of multiplication to regroup the factors.

$= 90t$ Multiply within the parentheses: $9 \cdot 10 = 90$.

b. $-5.3r(-2s) = [-5.3(-2)](r \cdot s)$ Use the commutative and associative properties to group the numbers and group the variables.

$= 10.6rs$ Multiply within the brackets.

c. $-\frac{21}{2}a\left(\frac{1}{3}\right) = -\frac{21}{2}\left(\frac{1}{3}\right)a$ Use the commutative property of multiplication to change the order of the factors a and $\frac{1}{3}$.

$= -\frac{7}{2}a$ Multiply: $-\frac{21}{2} \cdot \frac{1}{3} = -\frac{21 \cdot 1}{2 \cdot 3} = -\frac{\overset{1}{\cancel{3}} \cdot 7 \cdot 1}{2 \cdot \underset{1}{\cancel{3}}} = -\frac{7}{2}$.

Self Check 3

Simplify:

a. $14 \cdot 3s$ $42s$

b. $-1.6b(3t)$ $-4.8bt$

c. $-\frac{2}{3}x(-9)$ $6x$

Now Try **Problems 35 and 39**

Teaching Example 3 Simplify:
a. $5x(9)$ **b.** $6(3a)$ **c.** $\left(\frac{5}{6}\right)b\left(\frac{-10}{3}\right)$
Answers:
a. $45x$ **b.** $18a$ **c.** $-\frac{25}{9}b$

4 Use the distributive property.

Another property that we can use to simplify algebraic expressions is the **distributive property.** To introduce it, we will evaluate $4(5 + 3)$, in two ways.

Method 1 ***Use the order of operations:***	***Method 2*** ***Distribute the multiplication:***
$4(5 + 3) = 4(8)$	$4(5 + 3) = 4(5) + 4(3)$
$= 32$	$= 20 + 12$
	$= 32$

The Language of Algebra To *distribute* means to give from one to several. You have probably *distributed* candy to children coming to your front door on Halloween.

Each method gives a result of 32. This observation suggests the following property.

The Distributive Property

The distributive property of multiplication over addition

If a, b, and c represent real numbers,

$$a(b + c) = ab + ac$$

Self Check 4

Multiply by using the distributive property to remove parentheses:

a. $9(r + 4)$ $9r + 36$

b. $-11(-3x - 5)$ $33x + 55$

c. $-(-27k + 15)$ $27k - 15$

Now Try **Problems 43 and 47**

Teaching Example 4 Multiply by using the distributive property to remove parentheses:

a. $9(x - 4)$ **b.** $-3(5x - 4)$

c. $-(5x - 3)$ **d.** $-6(2x + 3)$

Answers:

a. $9x - 36$ **b.** $-15x + 12$

c. $-5x + 3$ **d.** $-12x - 18$

EXAMPLE 4 Multiply by using the distributive property to remove parentheses: **a.** $6(a + 9)$ **b.** $-15(4b - 1)$ **c.** $-(-21 - 20m)$

Strategy We will distribute the multiplication by the factor outside the parentheses over each term within the parentheses.

WHY We cannot simplify the expression within the parentheses. To multiply, we must use the distributive property.

Solution

a. $6(a + 9) = 6 \cdot a + 6 \cdot 9$ Distribute the multiplication by 6.

$= 6a + 54$ Multiply.

b. $-15(4b - 1) = -15(4b) - (-15)(1)$ Distribute the multiplication by −15.

$= -60b + 15$ Multiply.

The Language of Algebra When we use the distributive property to write a product, such as $6(a + 9)$, as the sum, $6a + 54$, we say that we have *removed* or *cleared* parentheses.

c. To use the distributive property to simplify $-(-21 - 20m)$, we interpret the $-$ symbol as a factor of -1, and proceed as follows.

$-(-21 - 20m) = -1(-21 - 20m)$ Write the − sign in front of the parentheses as −1.

$= -1(-21) - (-1)(20m)$ Distribute the multiplication by −1.

$= 21 + 20m$ Multiply.

$= 20m + 21$ Write the variable term of the answer first.

A more general form of the distributive property is the **extended distributive property.**

$$a(b + c + d + e + \cdots) = ab + ac + ad + ae + \cdots$$

Since multiplication is commutative, we can write the distributive property in the following forms.

$$(b + c)a = ba + ca, \quad (b - c)a = ba - ca, \quad (b + c + d)a = ba + ca + da$$

EXAMPLE 5 Multiply: **a.** $-0.5(7 - 5y + 6z)$ **b.** $(8x - 3y)\frac{3}{2}$

Strategy We will multiply each term within the parentheses by the factor outside the parentheses.

WHY We cannot simplify the expression within the parentheses. To multiply, we must use an extension of the distributive property.

Solution

a. $-0.5(7 - 5y + 6z)$

$= -0.5(7) - (-0.5)(5y) + (-0.5)(6z)$ Distribute the multiplication by -0.5.

$= -3.5 + 2.5y - 3z$ Multiply.

$= 2.5y - 3z - 3.5$ Write the variable terms of the answer first.

b. $(8x - 3y)\frac{3}{2} = (8x)\frac{3}{2} - (3y)\frac{3}{2}$ Distribute the multiplication by $\frac{3}{2}$.

$= \frac{24x}{2} - \frac{9y}{2}$ Multiply.

$= 12x - \frac{9}{2}y$ Simplify.

Self Check 5

Multiply:

a. $0.8(-6t + 3s - 10)$

b. $(10a + 16b)\frac{3}{5}$

Now Try **Problems 55 and 59**

Self Check 5 Answers

a. $-4.8t + 2.4s - 8$

b. $6a + \frac{48}{5}b$

Teaching Example 5 Multiply:

a. $-0.3(8 - 4m - 2n)$

b. $(6x + 12y)\frac{2}{3}$

Answers:

a. $-2.4 + 1.2m + 0.6n$

b. $4x + 8y$

5 Combine like terms.

Before we can discuss methods for simplifying algebraic expressions involving addition and subtraction, we must define like and unlike terms.

Like Terms

Like terms are terms with exactly the same variables raised to exactly the same powers. Any constant terms in an expression are considered to be like terms. Terms that are not like terms are called **unlike terms.**

Here are some examples of like and unlike terms.

$5x$ and $6x$ are like terms.	$27x^2y^3$ and $-326x^2y^3$ are like terms.
$4x$ and $-17y$ are unlike terms, because they have different variables.	$15x^2y$ and $6xy^2$ are unlike terms, because the variables have different exponents.

If we are to add (or subtract) objects, they must have the same units. For example, we can add dollars to dollars and inches to inches, but we cannot add dollars to inches. The same is true when working with terms of an expression. They can be added or subtracted only when they are like terms.

Simplifying the sum or difference of like terms is called **combining like terms.** To simplify expressions containing like terms, we use the distributive property in reverse. For example,

$$5x + 6x = (5 + 6)x = 11x \quad \text{and} \quad 32y - 16y = (32 - 16)y = 16y$$

These examples suggest the following general rule.

Combining Like Terms

Like terms can be combined by adding or subtracting the coefficients of the terms and keeping the same variables with the same exponents.

Self Check 6

Simplify by combining like terms:

a. $5k + 8k$

b. $-600a^2 - (-800a^2) + 100a^2$

c. $\frac{2}{3}xy - \frac{3}{4}xy$

d. $c + 32d^2 - 19c - 20d^2$

Now Try **Problems 65, 67, and 71**

Self Check 6 Answers

a. $13k$ **b.** $300a^2$

c. $-\frac{1}{12}xy$ **d.** $12d^2 - 18c$

Teaching Example 6 Simplify by combining like terms:

a. $-14x + 12x$

b. $0.6r^2 + 1.2r^2 - 5.9r^2$

c. $-1.2x + (-3.4x) - (-2.1x)$

Answers:

a. $-2x$ **b.** $-4.1r^2$ **c.** $-2.5x$

EXAMPLE 6

Simplify by combining like terms: **a.** $-8f + 12f$

b. $0.6s^3 - 0.2s^3 - (-0.9s^3)$ **c.** $-\frac{1}{2}ab + \frac{1}{3}ab$ **d.** $16n + 8n^2 - 42n + 4n^2$

Strategy We will use the distributive property to add (or subtract) the coefficients of the like terms.

WHY To *combine like terms* means to add or subtract the like terms in an expression.

Solution

a. Since $-8f$ and $12f$ are like terms with the common variable f, we can combine them by adding the coefficients of the like terms and keeping the variable f.

$$-8f + 12f = 4f \quad \text{Think: } (-8 + 12)f = 4f.$$

b.
$$0.6s^3 - 0.2s^3 - (-0.9s^3) = 0.6s^3 - 0.2s^3 + 0.9s^3 \quad \text{Add the opposite of } -0.9s^3.$$
$$= 1.3s^3 \quad \text{Think: } (0.6 - 0.2 + 0.9)s^3 = 1.3s^3.$$

c.
$$-\frac{1}{2}ab + \frac{1}{3}ab = -\frac{1}{2}\cdot\frac{3}{3}ab + \frac{1}{3}\cdot\frac{2}{2}ab \quad \text{Build each fraction into an equivalent fraction that has the LCD 6 for its denominator.}$$
$$= -\frac{3}{6}ab + \frac{2}{6}ab \quad \text{Multiply the numerators. Multiply the denominators.}$$
$$= -\frac{1}{6}ab \quad \text{Think: } \left(-\tfrac{3}{6} + \tfrac{2}{6}\right)ab = -\tfrac{1}{6}ab.$$

d. We will combine the n-terms and combine the n^2-terms.

$$16n + 8n^2 - 42n + 4n^2 = -26n + 12n^2 \quad \text{Think: } (16 - 42)n = -26n \text{ and } (8 + 4)n^2 = 12n^2.$$
$$= 12n^2 - 26n \quad \text{Write the terms of the result in descending powers of } n.$$

Success Tip We can use the commutative property of addition to reorder the terms of the result. It is standard practice to write such answers in descending powers of the variable.

Self Check 7

Simplify:

a. $44a^3 - 2(10a^3 - a^2) - a^2$

b. $12\left(\frac{5}{6}r^2 + \frac{1}{4}r\right) + 12\left(\frac{2}{3}r\right)$

Now Try **Problems 79 and 85**

Self Check 7 Answers

a. $24a^3 + a^2$

b. $10r^2 + 11r$

EXAMPLE 7

Simplify each expression:

a. $20b^2 - 5(3b^2 + 1) + 8$ **b.** $6\left(\frac{3}{2}d - \frac{4}{3}\right) + 6\left(\frac{5}{6}d\right)$

Strategy We will use the distributive property to remove parentheses and then combine any like terms.

WHY Since we cannot simplify the expressions within the parentheses, we will perform the indicated multiplication.

Solution

a. $20b^2 - 5(3b^2 + 1) + 8 = 20b^2 - 15b^2 - 5 + 8$ Distribute the multiplication by -5.

$= 5b^2 + 3$ Combine like terms.

b. $6\left(\frac{3}{2}d - \frac{4}{3}\right) + 6\left(\frac{5}{6}d\right) = 6\left(\frac{3}{2}d\right) - 6\left(\frac{4}{3}\right) + 6\left(\frac{5}{6}d\right)$ Distribute the multiplication by 6.

$= \frac{18}{2}d - \frac{24}{3} + \frac{30}{6}d$ Multiply.

$= 9d - 8 + 5d$ Simplify each fraction.

$= 14d - 8$ Combine like terms.

Teaching Example 7 Simplify each expression:
a. $5(3x^3 + 2) - 4(2x^3 - 1) + 5(3)$
b. $7\left(\frac{1}{7}x - \frac{3}{7}\right) + 4\left(\frac{1}{2}x + 1\right)$
Answers:
a. $7x^3 + 29$
b. $3x + 1$

EXAMPLE 8 Simplify: $3x + 4[6x - 2(7x + 8)]$

Strategy We will simplify the expression by working from the innermost grouping symbols (the parentheses) to the outermost grouping symbols (the brackets).

WHY To simplify expressions, we follow the order of operations rule.

Solution

$3x + 4[6x - 2(7x + 8)] = 3x + 4[6x - 14x - 16]$ Remove the innermost parentheses by distributing the multiplication by -2.

$= 3x + 4[-8x - 16]$ Combine like terms within the brackets.

$= 3x - 32x - 64$ Remove the outermost brackets by distributing the multiplication by 4.

$= -29x - 64$ Combine like terms.

Self Check 8

Simplify:
$8t - 4[10t + 2(2t + 1) - 3]$

Now Try **Problem 87**

Self Check 8 Answer
$-48t + 4$

Teaching Example 8 Simplify:
$4r + 3[5r - 2(2r + 3)]$
Answer:
$7r - 18$

ANSWERS TO SELF CHECKS

1. 1, −2, 3, 4 **2. a.** 23 **b.** $\left(-16 \cdot \frac{1}{2}\right) \cdot 7$ **3. a.** $42s$ **b.** $-4.8bt$ **c.** $6x$ **4. a.** $9r + 36$ **b.** $33x + 55$ **c.** $27k - 15$ **5. a.** $-4.8t + 2.4s - 8$ **b.** $6a + \frac{48}{5}b$ **6. a.** $13k$ **b.** $300a^2$ **c.** $-\frac{1}{12}xy$ **d.** $12d^2 - 18c$ **7. a.** $24a^3 + a^2$ **b.** $10r^2 + 11r$ **8.** $-48t + 4$

STUDY SKILLS CHECKLIST

Get the Most from Your Textbook

The following checklist will help you become familiar with some useful features in this book. Place a check mark in each box after you answer the question.

☐ Locate the **Definition** for Algebraic Expressions on page 3 and the Order of Operations Rule on page 30. What color are these boxes?

☐ Find the **Caution** box on page 14, the **Success Tip** box on page 27, and the **Language of Algebra** box on page 38. What color is used to identify these boxes?

☐ Each chapter begins with **From Campus to Careers** (see page 215). Chapter 3 gives information on how to become a fashion designer. On what page does a related problem appear in Study Set 3.5?

☐ Locate the **Study Skills Workshop** at the beginning of your text beginning on page S-1. How many Objectives appear in the Study Skills Workshop?

Answers: green, red, 267, 7

SECTION 1.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. A term is a product or quotient of numbers and/or variables, such as $6r$, $-t^3$, and $\frac{44}{m}$.

2. The coefficient of the term $-8c$ is -8.

3. A term, such as 9, that consists of a single number is called a constant term.

4. To simplify expressions, we use properties of real numbers to write equivalent expressions in a less complicated form.

5. The commutative properties of real numbers involve changing *order* and the associative properties of real numbers involve changing *grouping*.

▶ **6.** We can use the distributive property to remove parentheses in the expression $2(x + 8)$.

7. Like terms are terms with exactly the same variables raised to exactly the same powers.

8. Simplifying the sum or difference of like terms is called combining like terms.

CONCEPTS

9. **a.** Using the variables x, y, and z, write the associative property of addition. $(x + y) + z = x + (y + z)$

b. Using the variables x and y, write the commutative property of multiplication. $xy = yx$

c. Using the variables r, s, and t, write the distributive property of multiplication over addition. $r(s + t) = rs + rt$

10. Complete each property of addition. Then give its name.

a. $a + (-a) = 0$ add. prop. of opp. (inv. prop. of add.)

b. $a + 0 = a$ ident. prop. of add.

c. $a + b = b + a$ comm. prop. of add.

d. $(a + b) + c = a + (b+c)$ assoc. prop. of add.

11. Complete each property of multiplication. Then give its name.

a. $a \cdot b = b \cdot a$ comm. prop. of mult.

b. $(ab)c = a(bc)$ assoc. prop. of mult.

c. $0 \cdot a = 0$ mult. prop. of 0

d. $1 \cdot a = a$ ident. prop. of mult.

e. $a\left(\frac{1}{a}\right) = 1$ mult. inv. prop.

12. Complete each property of division.

a. $\frac{a}{1} = a$ **b.** $\frac{a}{a} = 1$

▶ **c.** $\frac{0}{a} = 0$ ▶ **d.** $\frac{a}{0}$ is undefined

13. **a.** What is the additive identity? 0

b. What is the multiplicative identity? 1

c. What is the additive inverse (opposite) of x? $-x$

d. What is the multiplicative inverse (reciprocal) of x? $\frac{1}{x}$

14. What number should be

a. subtracted from 5 to obtain 0? 5

b. added to 5 to obtain 0? -5

▶ **15.** By what number should

a. 5 be divided to obtain 1? 5

b. 5 be multiplied to obtain 1? $\frac{1}{5}$

16. Are the terms listed here like terms? If they are, combine them.

▶ **a.** $2x$, $6x$ yes, $8x$ **b.** $-3x$, $5y$ no

c. $-5xy$, $-7yz$ no **d.** $24t^2$, $24t^3$ no

NOTATION

17. In $-(x - 7)$, what does the negative sign in front of the parentheses represent? multiplication by -1

▶ **18.** Does the distributive property apply?

a. $2(3)(5)$ no **b.** $2(3 \cdot 5)$ no

c. $2(3x)$ no **d.** $2(x - 3)$ yes

GUIDED PRACTICE

What are the terms of the expression? Give the coefficient of each term. **See Example 1.**

19. $3x^3 + 11x^2 - x + 9$ $3x^3, 11x^2, -x, 9; 3, 11, -1, 9$

20. $2y^4 - y^3 + 6y + 4$ $2y^4, -y^3, 6y, 4; 2, -1, 6, 4$

▶ **21.** $\frac{11}{12}a^4 - \frac{3}{4}b^2 + 25b$ $\frac{11}{12}a^4, -\frac{3}{4}b^2, 25b; \frac{11}{12}, -\frac{3}{4}, 25$

22. $0.78m^3 - 1.55n - 0.99$
$0.78m^3, -1.55n, -0.99; 0.78, -1.55, -0.99$

Complete each statement so that the indicated property is illustrated. **See Example 2.**

23. $3 + 7 = 7 + 3$ (Commutative property of addition)

▶ **24.** $2(5 \cdot 97) = (2 \cdot 5)97$ (Associative property of multiplication)

▶ Selected exercises available online at **www.webassign.net/brookscole**

▶ **25.** $3(2 + d) = \underline{3 \cdot 2 + 3d}$ (Distributive property)

26. $1 \cdot y = \underline{y \cdot 1}$ (Commutative property of multiplication)

27. $c + 0 = \underline{c}$ (Additive identity property)

▶ **28.** $-4(x - 2) = \underline{-4x + 8}$ (Distributive property and simplifying)

29. $25 \cdot \frac{1}{25} = \underline{1}$ (Multiplicative inverse property)

▶ **30.** $z + (9 - 27) = \underline{(9 - 27) + z}$ (Commutative property of addition)

31. $8 + (7 + a) = \underline{(8 + 7) + a}$ (Associative property of addition)

32. $\underline{1} \cdot 3 = 3$ (Multiplicative identity property)

33. $(x + y)2 = \underline{2(x + y)}$ (Commutative property of multiplication)

34. $h + (-h) = \underline{0}$ (Additive inverse property)

Multiply. See Example 3.

35. $9(8m)$ $72m$

36. $12n(4)$ $48n$

▶ **37.** $5(-9q)$ $-45q$

38. $-3(2t)$ $-6t$

▶ **39.** $\frac{7}{8}x(-56)$ $-49x$

40. $\frac{5}{9}r(-45)$ $-25r$

▶ **41.** $-4(8r)(-2y)$ $64ry$

42. $-6s(-4t)(-1)$ $-24st$

Multiply. See Example 4.

43. $9(9x + 2)$ $81x + 18$

44. $7(6y + 1)$ $42y + 7$

45. $-4(-3t + 3)$ $12t - 12$

46. $-4(-5y + 3)$ $20y - 12$

47. $-(24 - d)$ $-24 + d$ or $d - 24$

48. $-(19 - w)$ $-19 + w$ or $w - 19$

49. $\frac{2}{3}(3s^2 - 9)$ $2s^2 - 6$

50. $\frac{1}{5}(5b^3 - 15)$ $b^3 - 3$

51. $0.7(m + 2n)$ $0.7m + 1.4n$

▶ **52.** $2.5(6c - 8d)$ $15c - 20d$

53. $100(0.09x + 0.02y)$ $9x + 2y$

54. $100(8.36x - 2.75y)$ $836x - 275y$

Multiply. See Example 5.

55. $5(9t^2 - 12t - 3)$ $45t^2 - 60t - 15$

56. $25(2a^2 - 3a + 1)$ $50a^2 - 75a + 25$

57. $3\left(\frac{4}{3}x - \frac{5}{3}y + \frac{1}{3}\right)$ $4x - 5y + 1$

▶ **58.** $6\left(-\frac{4}{3} + \frac{7}{6}s + \frac{16}{3}t\right)$ $-8 + 7s + 32t$

59. $(16t + 24)\frac{1}{8}$ $2t + 3$

60. $(18q + 9)\frac{1}{9}$ $2q + 1$

61. $(y - 2)(-3)$ $-3y + 6$

62. $(2t + 5)(-2)$ $-4t - 10$

Simplify by combining like terms. See Example 6.

63. $3x + 15x$ $18x$

64. $12y - 17y$ $-5y$

65. $0.7h - 3.8h$ $-3.1h$

66. $-5.7m + 5.3m$ $-0.4m$

67. $1.8x^2 - 5.1x^2 + 4.1x^2$ $0.8x^2$

68. $3.7x^2 + 3.3x^2 - 1.1x^2$ $5.9x^2$

▶ **69.** $-8x + 5x - (-x)$ $-2x$

70. $-20y + 3y - (-6y)$ $-11y$

71. $\frac{2}{5}ab - \left(-\frac{1}{2}ab\right)$ $\frac{9}{10}ab$

▶ **72.** $-\frac{3}{4}st - \frac{1}{3}st$ $-\frac{13}{12}st$

73. $\frac{3}{5}t + \frac{1}{3}t$ $\frac{14}{15}t$

74. $\frac{3}{16}x - \frac{5}{4}x$ $-\frac{17}{16}x$

75. $-9a + 11ad - 35a + ad$ $12ad - 44a$

76. $-7a + 2ab - 7a + 12ab$ $14ab - 14a$

77. $4m - t - (-2m) + 3t$ $6m + 2t$

78. $14g + h - (-g) - 8h$ $15g - 7h$

Simplify. See Example 7.

79. $2x^2 + 4(3x - x^2) + 3x$ $-2x^2 + 15x$

▶ **80.** $3p^2 - 6(5p^2 + p) + p^2$ $-26p^2 - 6p$

81. $-3(p - 2) + 2(p + 3) - 5(p - 1)$ $-6p + 17$

82. $5(q + 7) - 3(q - 1) - (q + 2)$ $q + 36$

83. $36\left(\frac{2}{9}x - \frac{3}{4}\right) + 36\left(\frac{1}{2}\right)$ $8x - 9$

84. $40\left(\frac{3}{8}y - \frac{1}{4}\right) + 40\left(\frac{4}{5}\right)$ $15y + 22$

85. $24\left(\frac{5}{6}y - \frac{9}{8}\right) - 24\left(\frac{3}{24}y\right)$ $17y - 27$

86. $18\left(\frac{11}{18}w - \frac{7}{2}\right) - 18\left(\frac{1}{9}w\right)$ $9w - 63$

Simplify. See Example 8.

87. $3[2(x + 2)] - 5[3(x - 5)] + 5x$ $-4x + 87$

88. $-5[3(x - 4) - 2(x + 2)] - 7(x - 3)$ $-12x + 101$

▶ **89.** $2\left[6\left(\frac{1}{3}a + 2b\right) - 8\left(\frac{1}{4}a - 2b\right) + 3\right]$ $56b + 6$

90. $10\left[\frac{3}{5}(2s + 2t) - \frac{4}{5}(s - t) + 1\right]$ $4s + 20t + 10$

TRY IT YOURSELF

Simplify each expression.

91. $-(a + 2A + 1) - (a - A + 2)$ $-2a - A - 3$

▶ **92.** $3T - 2(t - T) + t$ $5T - t$

93. $8(2cd + 7c) - 2(cd - 3c)$ $14cd + 62c$

94. $2tz + 5(tz - 4) - 10(8 - tz)$ $17tz - 100$

95. $6.4a^2 + 11.8a - 9.2a + 5.7$ $6.4a^2 + 2.6a + 5.7$

96. $9.1m^2 - 6.1m + 12.3m - 4.9$ $9.1m^2 + 6.2m - 4.9$

97. $-\frac{7}{16}x - \frac{3}{4}x$ $-\frac{19}{16}x$

98. $-\frac{5}{9}y - \frac{7}{18}y$ $-\frac{17}{18}y$

99. $-2[4(z-9) - 6(3z-7)] - 7(2z-1)$ $14z - 5$

100. $9(m^3+3) - 5(3-m^3) - 8(-1-m^3)$ $22m^3 + 20$

101. $21\left(\frac{6}{7}h^2 - \frac{15}{21}h\right) + 21\left(\frac{1}{3}h\right)$ $18h^2 - 8h$

102. $\frac{1}{12}(y - 12x) - \frac{1}{3}(y - 3x)$ $-\frac{1}{4}y$

103. $4.3(y+9) - 8.1y$ $-3.8y + 38.7$

104. $2.1(4+5z) + 0.9z$ $11.4z + 8.4$

105. $3x^2 - (-2x^2) - 5x^2$ 0

106. $8x^3 - x^3 - (-2x^3)$ $9x^3$

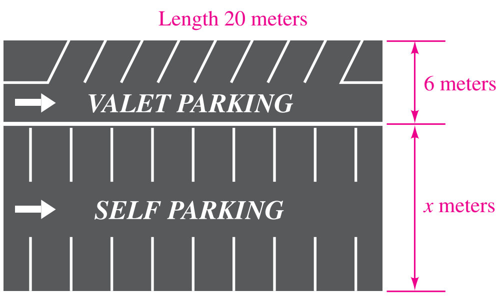
APPLICATIONS

107. PARKING AREAS Refer to the illustration below.

a. Express the area of the entire parking lot as the product of its length and width. $20(x+6)$ m^2

b. Express the area of the entire lot as the sum of the areas of the self-parking space and the valet parking space. $(20x + 120)$ m^2

c. Write an equation that shows that your answers to parts (a) and (b) are equal. What property of real numbers is illustrated by this example? $20(x+6) = 20x + 120$; distrib. prop.



108. CHECKING ACCOUNTS To find the total dollar amount of the checks entered in the register in the next column, we could add the check amounts in the order in which they are written: \$39 + \$75 + \$34 + \$25 + \$111 + \$16. Write an expression with the amounts reordered and grouped in such a way that the addition is easier. Then find the sum. What properties of real numbers did you use?
(\$39 + \$111) + (\$75 + \$25) + (\$34 + \$16) = \$150 + \$100 + \$50 = \$300; commutative and associative properties of addition

Number	Date	Description of Transaction	Payment/Debit	
101	3/6	DR. OKAMOTO, DDS	\$39	00
102	3/6	UNION OIL CO.	\$75	00
103	3/8	STATER BROS.	\$34	00
104	3/9	LITTLE LEAGUE	\$25	00
105	3/11	NORDSTROM	\$111	00
106	3/12	OFFICE MAX	\$16	00

WRITING

109. Explain why the distributive property does not apply when simplifying $6(2 \cdot x)$.

110. In each case, explain what you can conclude about one or both of the numbers.

a. When the two numbers are added, the result is 0.

b. When the two numbers are subtracted, the result is 0.

c. When the two numbers are multiplied, the result is 0.

d. When the two numbers are divided, the result is 0.

111. What are like terms?

▶ **112.** Use each of the words *commute, associate,* and *distribute* in a sentence in which the context is nonmathematical.

REVIEW

Evaluate each expression.

113. $\left(-\frac{3}{2}\right)\left(\frac{7}{12}\right)$ $-\frac{7}{8}$

114. $\frac{1}{2} - \left(-\frac{4}{5}\right)$ $\frac{13}{10}$

115. $-3|4-8| + (4 + 2\cdot 3)^3$ 988

116. $\left(\frac{-\sqrt{4^3} - 5^2}{2\cdot 2^2 - (1^9 - 4)}\right)^3$ -27

SECTION 1.5

Solving Linear Equations Using Properties of Equality

Objectives

1. Determine whether a number is a solution.
2. Use properties of equality to solve equations.
3. Simplify expressions to solve equations.
4. Clear equations of fractions and decimals.
5. Identify identities and contradictions.

One of the most useful concepts in algebra is the equation. Writing and then solving an equation is a powerful problem-solving strategy. In this section, we will review some fundamental properties that are used to solve equations.

1 Determine whether a number is a solution.

An **equation** is a statement indicating that two expressions are equal. All equations contain an equal symbol $=$. An example of an equation is $7x - 3 = 4$. The equal symbol separates the equation into two parts: The expression $7x - 3$ is the **left side** and 4 is the **right side.** The letter x is the **variable** (or the **unknown**). Since the sides of an equation can be reversed, we can write $7x - 3 = 4$ or $4 = 7x - 3$.

- An equation can be true: $6 + 3 = 9$.
- An equation can be false: $2 + 4 = 7$.
- An equation can be neither true nor false. For example, $7x - 3 = 4$ is neither true nor false because we don't know what number x represents.

An equation that contains a variable is made true or false by substituting a number for the variable. For example, if $x = 1$, then the equation $7x - 3 = 4$ is true.

$7x - 3 = 4$

$7(1) - 3 \stackrel{?}{=} 4$ Substitute 1 for x. At this stage, we don't know whether the left and right sides of the equation are equal, so we use an "is possibly equal to" symbol $\stackrel{?}{=}$.

$7 - 3 \stackrel{?}{=} 4$

$4 = 4$ We obtain a true statement.

A number that makes an equation true when substituted for the variable is called a **solution,** and it is said to *satisfy* the equation. Therefore, 1 is a solution of $7x - 3 = 4$. The **solution set** of an equation is the set of all numbers that make the equation true.

The Language of Algebra It is important to know the difference between an *equation* and an *expression.* An equation contains an $=$ symbol; an expression does not.

EXAMPLE 1 Determine whether 2 is a solution of $3x + 2 = 2x + 5$.

Strategy We will substitute 2 for each x in the equation and evaluate the expressions on the left side and the right side separately.

WHY If a true statement results, 2 is a solution of the equation. If we obtain a false statement, 2 is not a solution.

Solution

Evaluate the expression on the left side.

$3x + 2 = 2x + 5$	This is the original equation.
$3(2) + 2 \stackrel{?}{=} 2(2) + 5$	Substitute 2 for x.
$6 + 2 \stackrel{?}{=} 4 + 5$	
$8 = 9$	False

Evaluate the expression on the right side.

Since $8 = 9$ is a false statement, the number 2 does not satisfy the equation. It is not a solution of $3x + 2 = 2x + 5$.

Self Check 1

Is -5 a solution of $2x - 5 = 3x$? yes

Now Try Problem 15

Teaching Example 1 Determine whether -2 is a solution of $4(x + 3) - 5 = x + 1$
Answer:
yes

2 Use properties of equality to solve equations.

Usually, we do not know the solutions of an equation—we need to find them. In this text, we will discuss how to solve many different types of equations. The easiest equations to solve are *linear equations in one variable.*

Linear Equations

A **linear equation in one variable** can be written in the form

$ax + b = c$ where a, b, and c are real numbers, and $a \neq 0$.

Some examples of linear equations in one variable are

$$2x - 8 = 0, \qquad -\frac{3}{4}y = -7, \qquad \text{and} \qquad 4b - 7 + 2b = 1 + 2b + 8$$

Notice for these linear equations that the highest power on the variable is 1.

When solving linear equations, the objective is to *isolate* the variable on one side of the equation. This is achieved by undoing the operations performed on the variable. As we undo the operations, we produce a series of simpler equations, all having the same solutions. Such equations are called *equivalent equations.*

Equivalent Equations

Equations with the same solutions are called **equivalent equations.**

The solution of the equation $x = 2$ is obviously 2, because replacing x with 2 yields a true statement, $2 = 2$. The equation $x + 4 = 6$ also has a solution of 2. Since $x = 2$ and $x + 4 = 6$ have the same solution, they are equivalent equations.

We can use the following properties to write equivalent equations, in which we will isolate the variable on one side of the equation.

Properties of Equality

Adding the same number to, or subtracting the same number from, both sides of an equation does not change the solution.

If a, b, and c are real numbers and $a = b$,

$a + c = b + c$ **Addition property of equality**

$a - c = b - c$ **Subtraction property of equality**

Multiplying or dividing both sides of an equation by the same nonzero number does not change the solution.

If a, b, and c are real numbers with $c \neq 0$, and $a = b$,

$ca = cb$ **Multiplication property of equality**

$\dfrac{a}{c} = \dfrac{b}{c}$ **Division property of equality**

EXAMPLE 2 Solve: **a.** $2x - 8 = 0$ **b.** $-35.6 = 77.89 - x$

Strategy We will use a property of equality to isolate the *variable term* on one side of the equation and then use another property to isolate the *variable*.

WHY To solve the original equation, we want to find a simpler equivalent equation of the form $x =$ **a number,** whose solution is obvious.

Solution

a. We note that x is multiplied by 2, and then 8 is subtracted from that product. To isolate x on the left side of the equation, we use the order of operations rule in reverse.

- To undo the subtraction of 8, we add 8 to both sides.
- To undo the multiplication by 2, we divide both sides by 2.

$2x - 8 = 0$	This is the equation to solve.
$2x - 8 + 8 = 0 + 8$	Use the addition property of equality: Add 8 to both sides to isolate the variable term, 2x.
$2x = 8$	Simplify both sides of the equation.
$\frac{2x}{2} = \frac{8}{2}$	Use the division property of equality: Divide both sides by 2 to isolate x.
$x = 4$	Do the divisions.

Check: We substitute 4 for x to verify that it satisfies the original equation.

$2x - 8 = 0$	
$2(4) - 8 \stackrel{?}{=} 0$	Substitute 4 for x.
$8 - 8 \stackrel{?}{=} 0$	Multiply.
$0 = 0$	True

Since we obtain a true statement, 4 is the solution of $2x - 8 = 0$ and the solution set is $\{4\}$.

b.

$-35.6 = 77.89 - x$	This is the equation to solve.
$-35.6 - 77.89 = 77.89 - x - 77.89$	Use the subtraction property of equality: Subtract 77.89 from both sides to isolate the variable term, $-x$.
$-113.49 = -x$	Simplify each side of the equation.

The variable x is not yet isolated, because there is a $-$ sign in front of it. Since the term $-x$ has an understood coefficient of -1, we can write $-x$ as $-1x$. To isolate x, we can either multiply or divide both sides by -1.

$-113.49 = -1x$	Write $-x = -1x$.
$\frac{-113.49}{-1} = \frac{-1x}{-1}$	Use the division (or multiplication) property of equality: Divide (or multiply) both sides by -1 to isolate x.
$113.49 = x$	Simplify each side of the equation.
$x = 113.49$	Reverse the sides of the equation so that x is on the left.

Verify that 113.49 is the solution by checking it in the original equation.

Self Check 2

Solve:

a. $3a + 15 = 0$ -5

b. $-1.3 = -2.6 - x$ -1.3

Now Try **Problems 19 and 29**

Teaching Example 2 Solve:
a. $5x - 20 = 10$
b. $-4.14 = 6.7 - 2x$
Answers:
a. 6 **b.** 5.42

EXAMPLE 3 Solve: $\frac{3}{4}y = -7$

Strategy We will isolate y by multiplying both sides of the equation by $\frac{4}{3}$.

Self Check 3

Solve: $\frac{2}{3}b - 3 = -15$ -18

Now Try **Problem 35**

Teaching Example 3 Solve: $\frac{3}{7}x = -10$
Answer: $-\frac{70}{3}$

WHY On the left side, y is multiplied by $\frac{3}{4}$. We can undo the multiplication by dividing both sides by $\frac{3}{4}$. Since division by $\frac{3}{4}$ is equivalent to multiplication by its reciprocal, it is easier to isolate y by multiplying both sides by $\frac{4}{3}$.

Solution

$$\frac{3}{4}y = -7 \quad \text{This is the equation to solve.}$$

$$\frac{4}{3}\left(\frac{3}{4}y\right) = \frac{4}{3}(-7) \quad \text{Use the multiplication property of equality to isolate } y\text{. Multiply both sides by the reciprocal of } \tfrac{3}{4}\text{, which is } \tfrac{4}{3}.$$

$$\left(\frac{4}{3} \cdot \frac{3}{4}\right)y = \frac{4}{3}(-7) \quad \text{Use the associative property of multiplication to regroup.}$$

$$1y = \frac{4}{3}(-7) \quad \text{The product of a number and its reciprocal is 1: } \tfrac{4}{3} \cdot \tfrac{3}{4} = 1.$$

$$y = -\frac{28}{3} \quad \text{On the left side, } 1y = y\text{. On the right side, multiply.}$$

Check:

$$\frac{3}{4}y = -7 \quad \text{This is the original equation.}$$

$$\frac{3}{4}\left(-\frac{28}{3}\right) \stackrel{?}{=} -7 \quad \text{Substitute } -\tfrac{28}{3} \text{ for } y.$$

$$-\frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{4}} \cdot 7}{\underset{1}{\cancel{4}} \cdot \underset{1}{\cancel{3}}} \stackrel{?}{=} -7 \quad \text{Multiply the numerators and the denominators. Factor 28 as } 4 \cdot 7 \text{ and simplify.}$$

$$-7 = -7 \quad \text{True}$$

The solution is $-\frac{28}{3}$ and the solution set is $\left\{-\frac{28}{3}\right\}$.

Success Tip Variable terms with fractional coefficients can be written in two ways. For example, $\frac{3}{4}y = \frac{3y}{4}$ and $\frac{2}{3}b = \frac{2b}{3}$.

The equation in Example 3 can be solved using an alternate two-step approach.

$$\frac{3}{4}y = -7 \quad \text{This is the equation to solve.}$$

$$4\left(\frac{3}{4}y\right) = 4(-7) \quad \text{Multiply both sides by 4 to undo the division by 4.}$$

$$3y = -28 \quad \text{Simplify: } 4\left(\tfrac{3}{4}y\right) = \tfrac{4}{1}\left(\tfrac{3}{4}y\right) = \tfrac{\overset{1}{\cancel{4}} \cdot 3}{1 \cdot \underset{1}{\cancel{4}}}y = 3y.$$

$$\frac{3y}{3} = -\frac{28}{3} \quad \text{To isolate } y\text{, undo the multiplication by 3 by dividing both sides by 3.}$$

$$y = -\frac{28}{3}$$

3 Simplify expressions to solve equations.

To solve more complicated equations, we often need to combine like terms.

Self Check 4
Solve:
$-6t - 16 + 6t = 1 + 2t - 5$ -6

EXAMPLE 4 Solve: $4b - 7 + 2b = 1 + 2b + 8$

Strategy We will combine like terms on each side of the equation and then eliminate $2b$ from the right side by subtracting $2b$ from both sides.

WHY To solve for b, all the terms containing b must be on the same side of the equation.

Solution

$4b - 7 + 2b = 1 + 2b + 8$ This is the equation to solve.

$6b - 7 = 2b + 9$ Combine like terms: $4b + 2b = 6b$ and $1 + 8 = 9$.

We note that terms involving b appear on both sides of the equation. To isolate b on the left side, we need to eliminate $2b$ on the right side.

$6b - 7 = 2b + 9$

$6b - 7 - 2b = 2b + 9 - 2b$ Subtract $2b$ from both sides.

$4b - 7 = 9$ Combine like terms on each side: $6b - 2b = 4b$ and $2b - 2b = 0$.

$4b - 7 + 7 = 9 + 7$ To undo the subtraction of 7, add 7 to both sides.

$4b = 16$ Simplify each side of the equation.

$b = 4$ To isolate b, undo the multiplication by 4 by dividing both sides by 4.

Check: $4b - 7 + 2b = 1 + 2b + 8$ This is the original equation.

$4(4) - 7 + 2(4) \stackrel{?}{=} 1 + 2(4) + 8$ Substitute 4 for b.

$16 - 7 + 8 \stackrel{?}{=} 1 + 8 + 8$

$17 = 17$ True

The solution is 4 and the solution set is $\{4\}$.

Now Try **Problem 49**

Teaching Example 4 Solve: $5a - 6 + 2a = 3 - a + 5$
Answer: $\frac{7}{4}$

Caution! When checking solutions, always use the original equation.

EXAMPLE 5 Solve: **a.** $7(a - 2) = 8$ **b.** $d - 3(d - 7) = 2(4d + 10)$ **c.** $6[x - (2 - x)] = -4(8x + 3)$

Strategy We will use the distributive property to remove all sets of parentheses (and brackets), simplify each side of the equation by combining like terms, and isolate the variable.

WHY It's best to simplify each side of an equation before isolating the variable.

Solution

a. $7(a - 2) = 8$ This is the equation to solve.

$7a - 14 = 8$ Distribute the multiplication by 7.

$7a - 14 + 14 = 8 + 14$ To undo the subtraction of 14, add 14 to both sides.

$7a = 22$

$\frac{7a}{7} = \frac{22}{7}$ To isolate a, undo the multiplication by 7 by dividing both sides by 7.

$a = \frac{22}{7}$

Check: $7(a - 2) = 8$ This is the original equation.

$7\left(\frac{22}{7} - 2\right) \stackrel{?}{=} 8$ Substitute $\frac{22}{7}$ for a.

$7\left(\frac{22}{7} - \frac{14}{7}\right) \stackrel{?}{=} 8$ Within the parentheses, express 2 as a fraction that has the LCD for its denominator: $2 = \frac{14}{7}$.

Self Check 5

Solve:

a. $-2(x + 3) = 18$

b. $y + 9(y - 5) = 5(4y + 1)$

c. $10[h - (1 - 3h)] = -2(5h + 25)$

Now Try **Problems 55 and 61**

Self Check 5 Answers

a. -12 **b.** -5 **c.** $-\frac{4}{5}$

Teaching Example 5 Solve:
a. $6(a - 4) = 5$
b. $b - 2(b - 5) = 3(2b + 4)$
c. $5[x - (3 - x)] = -2(4x + 1) + 5$
Answers:
a. $\frac{29}{6}$ **b.** $-\frac{2}{7}$ **c.** 1

$$7\left(\frac{8}{7}\right) \stackrel{?}{=} 8 \quad \text{Subtract the fractions.}$$

$$8 = 8 \quad \text{True}$$

The solution is $\frac{22}{7}$ and the solution set is $\left\{\frac{22}{7}\right\}$.

b.

$d - 3(d - 7) = 2(4d + 10)$	This is the equation to solve.
$d - 3d + 21 = 8d + 20$	Distribute the multiplication by -3 and by 2.
$-2d + 21 = 8d + 20$	Combine like terms: $d - 3d = -2d$.
$-2d + 21 + \mathbf{2d} = 8d + 20 + \mathbf{2d}$	To eliminate the term $-2d$ on the left side, add $2d$ to both sides.
$21 = 10d + 20$	Combine like terms on both sides: $-2d + 2d = 0$.
$21 - \mathbf{20} = 10d + 20 - \mathbf{20}$	To undo the addition of 20, subtract 20 from both sides.
$1 = 10d$	Combine like terms on both sides: $20 - 20 = 0$.
$\frac{1}{\mathbf{10}} = \frac{10d}{\mathbf{10}}$	To isolate d, undo the multiplication by 10 by dividing both sides by 10.
$\frac{1}{10} = d$	
$d = \frac{1}{10}$	Reverse the sides of the equation so that d is on the left.

Check: To simplify the computations, we can use the decimal equivalent of $\frac{1}{10}$, which is 0.1, in the check.

$d - 3(d - 7) = 2(4d + 10)$	This is the original equation.
$\mathbf{0.1} - 3(\mathbf{0.1} - 7) \stackrel{?}{=} 2[4(\mathbf{0.1}) + 10]$	Substitute 0.1 for d.
$0.1 - 3(-6.9) \stackrel{?}{=} 2[0.4 + 10]$	
$0.1 + 20.7 \stackrel{?}{=} 2(10.4)$	
$20.8 = 20.8$	True

The solution is $\frac{1}{10}$ or 0.1.

c. To simplify the expression on the left side of the equation, we will work from the innermost grouping symbols (the parentheses) to the outermost grouping symbols (the brackets).

$6[x - (2 - x)] = -4(8x + 3)$	This is the equation to solve.
$6[x - 2 + x] = -32x - 12$	Within the parentheses, distribute the multiplication by -1. On the right side, distribute the multiplication by -4.
$6[2x - 2] = -32x - 12$	Combine like terms within the brackets.
$12x - 12 = -32x - 12$	Distribute the multiplication by 6.
$12x - 12 + \mathbf{32x} = -32x - 12 + \mathbf{32x}$	To eliminate the term $-32x$ on the right side, add $32x$ to both sides.
$44x - 12 = -12$	Combine like terms.
$44x - 12 + \mathbf{12} = -12 + \mathbf{12}$	To undo the subtraction of 12, add 12 to both sides.
$44x = 0$	Simplify each side of the equation.
$\frac{44x}{\mathbf{44}} = \frac{0}{\mathbf{44}}$	To isolate x, undo the multiplication by 44 by dividing both sides by 44.
$x = 0$	

Verify that 0 is the solution by checking it in the original equation.

In general, we will follow these steps to solve linear equations in one variable. Not every step is needed to solve every equation.

Strategy for Solving Linear Equations in One Variable

1. **Clear the equation of fractions or decimals:** Multiply both sides by the LCD to clear fractions or multiply both sides by a power of 10 to clear decimals.
2. **Simplify each side of the equation:** Use the distributive property to remove parentheses and combine like terms on each side.
3. **Isolate the variable term on one side:** Add (or subtract) to get the variable term on one side of the equation and a number on the other using the addition (or subtraction) property of equality.
4. **Isolate the variable:** Multiply (or divide) to isolate the variable using the multiplication (or division) property of equality.
5. **Check the result:** Substitute the proposed solution for the variable in the *original* equation to see if a true statement results.

4 Clear equations of fractions and decimals.

Since equations are often easier to solve when they don't contain fractions, we will use the multiplication property of equality to clear an equation of fractions before we solve it. To do so, we will multiply both sides of the equation by the least common denominator of the fractions contained within the equation.

EXAMPLE 6 Solve: $\frac{1}{3}(2x - 1) = \frac{5}{4}x + \frac{31}{12}$

Strategy We will follow the steps of the equation-solving strategy.

WHY This is the best way to solve a linear equation in one variable.

Solution

Step 1 We can clear the equation of fractions by multiplying both sides by the least common denominator (LCD) of $\frac{1}{3}$, $\frac{5}{4}$, and $\frac{31}{12}$, which is 12.

$$\frac{1}{3}(2x - 1) = \frac{5}{4}x + \frac{31}{12}$$ This is the equation to solve.

$$\mathbf{12}\left[\frac{1}{3}(2x - 1)\right] = \mathbf{12}\left[\frac{5}{4}x + \frac{31}{12}\right]$$ To eliminate the fractions, multiply both sides by the LCD, 12.

$$4(2x - 1) = \mathbf{12} \cdot \frac{5}{4}x + \mathbf{12} \cdot \frac{31}{12}$$ On the left side, multiply: $12 \cdot \frac{1}{3} = 4$. On the right side, distribute the multiplication by 12.

$$4(2x - 1) = 15x + 31$$ Perform the multiplications on the right side.

Step 2 We remove parentheses.

$$8x - 4 = 15x + 31$$ Distribute the multiplication by 4.

Step 3 To get the variable term on the right side and the constant on the left side, subtract $8x$ and 31 from both sides.

$$8x - 4 \mathbf{- 8x - 31} = 15x + 31 \mathbf{- 8x - 31}$$

$$-35 = 7x$$ Simplify each side of the equation.

Self Check 6

Solve:

$\frac{1}{6}(4x + 10) = \frac{1}{9}x - \frac{5}{3}$ -6

Now Try Problem 63

Teaching Example 6 Solve:
$\frac{1}{5}(3x + 2) = \frac{1}{3}x + \frac{7}{10}$
Answer:
$\frac{9}{8}$

Step 4 To isolate the variable, undo the multiplication by 7 by dividing both sides by 7.

$$\frac{-35}{\mathbf{7}} = \frac{7x}{\mathbf{7}} \quad \text{Divide both sides by 7.}$$

$$-5 = x$$

$$x = -5 \quad \text{Reverse the sides of the equation so that } x \text{ is on the left.}$$

Step 5 We check by substituting -5 for x in the original equation and simplifying:

$$\frac{1}{3}(2x - 1) = \frac{5}{4}x + \frac{31}{12}$$

$$\frac{1}{3}[2(-5) - 1] \stackrel{?}{=} \frac{5}{4}(-5) + \frac{31}{12}$$

$$\frac{1}{3}[-11] \stackrel{?}{=} -\frac{25}{4} + \frac{31}{12}$$

$$-\frac{11}{3} \stackrel{?}{=} -\frac{75}{12} + \frac{31}{12}$$

$$-\frac{11}{3} \stackrel{?}{=} -\frac{44}{12}$$

$$-\frac{11}{3} = -\frac{11}{3} \quad \text{True}$$

The solution is -5.

Self Check 7

Solve:
$\frac{a+3}{2} + 2a = \frac{3}{2} - \frac{a+27}{5}$ -2

Now Try **Problem 71**

Teaching Example 7 Solve:
$\frac{2x-3}{4} - 3x = \frac{7}{4} - \frac{x+1}{3}$
Answer:
-1

EXAMPLE 7

Solve: $\frac{x+2}{5} - 4x = \frac{8}{5} - \frac{x+9}{2}$

Strategy We will follow the steps of the equation-solving strategy.

WHY This is the best way to solve a linear equation in one variable.

Solution

Some of the steps used to solve an equation can be done in your head, as you will see in this example.

$$\frac{x+2}{5} - 4x = \frac{8}{5} - \frac{x+9}{2}$$

$$\mathbf{10}\left(\frac{x+2}{5} - 4x\right) = \mathbf{10}\left(\frac{8}{5} - \frac{x+9}{2}\right) \quad \text{To clear the equation of the fractions, multiply both sides by the LCD, 10.}$$

$$\mathbf{10} \cdot \frac{x+2}{5} - \mathbf{10} \cdot 4x = \mathbf{10} \cdot \frac{8}{5} - \mathbf{10} \cdot \frac{x+9}{2} \quad \text{On each side, distribute the 10.}$$

$$2(x + 2) - 40x = 2(8) - 5(x + 9) \quad \text{Perform each multiplication by 10.}$$

$$2x + 4 - 40x = 16 - 5x - 45 \quad \text{On each side, remove parentheses.}$$

$$-38x + 4 = -5x - 29 \quad \text{On each side, combine like terms.}$$

$$33 = 33x \quad \text{Add 38x and 29 to both sides. Since these steps can be done mentally, we don't show them.}$$

$$1 = x \quad \text{Divide both sides by 33. This step is also done mentally.}$$

$$x = 1$$

The solution is 1. Check by substituting it for x in the original equation.

For more complicated equations involving decimals, we can multiply both sides of the equation by a power of 10 to clear the equation of decimals.

EXAMPLE 8 Solve: $0.04(12) + 0.01x = 0.02(12 + x)$

Strategy To clear the equation of decimals, we will multiply both sides by a carefully chosen power of 10.

WHY It's easier to solve an equation that involves only integers.

Solution

The equation contains the decimals 0.04, 0.01, and 0.02. Multiplying both sides by $10^2 = 100$ changes the decimals in the equation to integers.

$$0.04(12) + 0.01x = 0.02(12 + x)$$

$$\mathbf{100}[0.04(12) + 0.01x] = \mathbf{100}[0.02(12 + x)]$$ To make 0.04, 0.01, and 0.02 integers, multiply both sides by 100.

$$\mathbf{100} \cdot 0.04(12) + \mathbf{100} \cdot 0.01x = \mathbf{100} \cdot 0.02(12 + x)$$ On the left side, distribute the multiplication by 100.

$$4(12) + 1x = 2(12 + x)$$ Perform each multiplication by 100.

$$48 + x = 24 + 2x$$ Remove parentheses.

$$48 + x - 24 - x = 24 + 2x - 24 - x$$ To isolate the variable term on the right side, subtract 24 and x from both sides.

$$24 = x$$ Simplify each side.

$$x = 24$$

Verify that 24 is the solution by substituting it for x in the original equation.

Success Tip When we write the decimals in the equation as fractions, it becomes more apparent why it is helpful to multiply both sides by the LCD, 100.

$$\frac{4}{100}(12) + \frac{1}{100}x = \frac{2}{100}(12 + x)$$

Self Check 8

Solve: $0.08x + 0.07(15{,}000 - x) = 1{,}110$ 6,000

Now Try **Problem 75**

Teaching Example 8 Solve: $.04(x) + .01(15 - x) = .02(15)$
Answer: 5

5 Identify identities and contradictions.

The equations discussed so far are called **conditional equations.** For these equations, some numbers satisfy the equation and others do not. An **identity** is an equation that is satisfied by every number for which both sides of the equation are defined.

EXAMPLE 9 Solve: $-2(x - 1) - 4 = -4(1 + x) + 2x + 2$

Strategy We will follow the steps of the equation-solving strategy.

WHY This is the best way to solve a linear equation in one variable.

Solution

$$-2(x - 1) - 4 = -4(1 + x) + 2x + 2$$

$$-2x + 2 - 4 = -4 - 4x + 2x + 2$$ Use the distributive property.

$$-2x - 2 = -2x - 2$$ On each side, combine like terms.

Self Check 9

Solve: $3(a + 4) + 5 = 2(a - 1) + a + 19$

Now Try **Problem 79**

Self Check 9 Answer
all real numbers, $\mathbb{R}$

Teaching Example 9 Solve: $-3(x + 2) - 5 = x - 7 - 4(x + 1)$
Answer: all real numbers, $\mathbb{R}$

$$-2x - 2 + 2x = -2x - 2 + 2x$$ To attempt to isolate the variable on one side of the equation, add 2x to both sides.

$$-2 = -2$$ True

The terms involving x drop out. The resulting true statement indicates that the original equation is true for every value of x. The solution set is the set of real numbers denoted $\mathbb{R}$. The equation is an identity.

A **contradiction** is an equation that is never true.

Self Check 10

Solve: $3(a + 4) + 2 = 2(a - 1) + a + 19$

Now Try **Problem 81**

Self Check 10 Answer

no solution, $\varnothing$

Teaching Example 10

Solve:

$-5(x + 7) - 2 = 3x - 8(x + 1)$

Answer:

no solution, $\varnothing$

EXAMPLE 10

Solve: $-6.2(-x - 1) - 4 = 4.2x - (-2x)$

Strategy We will follow the steps of the equation-solving strategy.

WHY This is the best way to solve a linear equation in one variable.

Solution

$$-6.2(-x - 1) - 4 = 4.2x - (-2x)$$

$$6.2x + 6.2 - 4 = 4.2x + 2x$$ On the left side, remove parentheses. On the right side, write the subtraction as addition of the opposite.

$$6.2x + 2.2 = 6.2x$$ On each side, combine like terms.

$$6.2x + 2.2 - 6.2x = 6.2x - 6.2x$$ To attempt to isolate the variable on one side of the equation, subtract 6.2x from both sides.

$$2.2 = 0$$ False

The terms involving x drop out. The resulting false statement indicates that no value for x makes the original equation true. The solution set contains no elements and can be denoted as the **empty set** $\{\ \}$ or the **null set** $\varnothing$. The equation is a contradiction.

The Language of Algebra *Contradiction* is a form of the word *contradict,* meaning conflicting ideas. During a trial, evidence might be introduced that *contradicts* the testimony of a witness.

ANSWERS TO SELF CHECKS

1. yes **2. a.** -5 **b.** -1.3 **3.** -18 **4.** -6 **5. a.** -12 **b.** -5 **c.** $-\frac{4}{5}$ **6.** -6 **7.** -2 **8.** 6,000 **9.** all real numbers, $\mathbb{R}$ **10.** no solution, $\varnothing$

SECTION 1.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. An equation is a statement that two expressions are equal.
2. $2x + 1 = 4$ and $5(y - 3) = 8$ are examples of linear equations in one variable.
3. If a number is substituted for a variable in an equation and the equation is true, we say that the number satisfies the equation.
4. If two equations have the same solution set, they are called equivalent equations.

Selected exercises available online at **www.webassign.net/brookscole**

5. An identity is an equation that is satisfied by every number for which both sides are defined.

6. An equation that is not true for any values of its variable is called a contradiction.

CONCEPTS

Fill in the blanks.

7. If $a = b$, then $a + c = b + c$ and $a - c = b - c$. Adding (or subtracting) the same number to (or from) both sides of an equation does not change the solution.

8. If $a = b$, then $ca = cb$ and $\frac{a}{c} = \frac{b}{c}$. Multiplying (or dividing) both sides of an equation by the same nonzero number does not change the solution.

9. Solve each equation mentally.

a. $x + 3 = 6$ 3 **b.** $x - 3 = 6$ 9

c. $3x = 6$ 2 **d.** $\frac{x}{3} = 6$ 18

▶ **10. a.** When solving $\frac{x+1}{3} - \frac{2}{15} = \frac{x-1}{5}$, why would we multiply both sides by 15?
It clears the equation of fractions.

b. When solving $1.45x - 0.5(1 - x) = 0.7x$, why would we multiply both sides by 100?
It clears the equation of decimals.

11. a. Suppose you solve a linear equation in one variable, the variable drops out, and you obtain $8 = 8$. What is the solution set? What symbol is used to represent the solution set? all real numbers, $\mathbb{R}$

b. Suppose you solve a linear equation in one variable, the variable drops out, and you obtain $8 = 7$. What is the solution set? What symbol is used to represent the solution set? no solution, $\varnothing$

12. a. Simplify: $5y + 2 - 3y$ $2y + 2$

b. Solve: $5y + 2 - 3y = 8$ 3

c. Evaluate $5y + 2 - 3y$ for $y = 8$. 18

d. Check: Is -1 a solution of $5y + 2 - 3y = 8$? no

NOTATION

Complete the solution to solve the equation. Then check the result.

13.
$$\begin{aligned} -2(x + 7) &= 20 \\ -2x - 14 &= 20 \\ -2x - 14 + 14 &= 20 + 14 \\ -2x &= 34 \\ \frac{-2x}{-2} &= \frac{34}{-2} \\ x &= -17 \end{aligned}$$

Check:
$$\begin{aligned} -2(x + 7) &= 20 \\ -2(-17 + 7) &\stackrel{?}{=} 20 \\ -2(-10) &\stackrel{?}{=} 20 \\ 20 &= 20 \end{aligned}$$

The solution is -17.

▶ **14.** Fill in the blanks to make the statements true.

a. $-x = -1x$ **b.** $\frac{2t}{3} = \frac{2}{3}t$

GUIDED PRACTICE

Determine whether 5 is a solution of each equation. See Example 1.

15. $3x + 2 = 17$ yes ▶ **16.** $7x - 2 = 53 - 5x$ no

17. $3(2m - 3) = 15$ no **18.** $\frac{3}{5}p - 5 = -2$ yes

Solve each equation. Check each result. See Example 2.

19. $2x - 12 = 0$
6

▶ **20.** $3x - 24 = 0$
8

▶ **21.** $8k - 2 = 13$
$\frac{15}{8}$

22. $3x + 1 = 3$
$\frac{2}{3}$

23. $\frac{x}{4} - 6 = 1$
28

24. $\frac{m}{3} + 10 = 8$
-6

25. $\frac{y}{6} - 7 = -12$
-30

26. $\frac{a}{8} + 1 = -10$
-88

27. $1.6a + (-4) = 0.032$
2.52

28. $5.51 = 0.05y + (-9)$
290.2

29. $0.7 - 4y = 1.7$
-0.25

30. $0.3 - 2x = -0.9$
0.6

31. $-x + 12 = -17$
29

▶ **32.** $6 = -x + 41$
35

33. $-6 - y = -13$
7

34. $-1 - h = -9$
8

Solve each equation. Check each result. See Example 3.

35. $\frac{2}{3}c = 10$
15

▶ **36.** $\frac{9}{7}d = 81$
63

37. $-\frac{4}{5}s = 2$
$-\frac{5}{2}$

38. $-\frac{9}{8}s = 3$
$-\frac{8}{3}$

39. $-\frac{7}{16}w - 26 = -19$
-16

40. $-\frac{5}{8}a - 20 = -10$
-16

▶ **41.** $\frac{5}{6}k - 7.5 = 7.5$
18

42. $\frac{2}{5}c - 12.2 = 1.8$
35

Solve each equation. Check each result. See Example 4.

43. $8m + 44 = 4m$ -11

44. $9n + 36 = 6n$ -12

45. $60t - 50 = 15t - 5$ 1

46. $100s - 75 = 50s + 75$ 3

47. $9.8 - 16r = -15.7 - r$ 1.7

▶ **48.** $15s + 8.1 - 2s = 8.1 - s$ 0

49. $8b - 2 + b = 5b + 15$ $\frac{17}{4}$

50. $w + 7 + 3w = 4 + 10w$ $\frac{1}{2}$

51. $a + 18 = 5a - 3 + a$ $\frac{21}{5}$

52. $4a - 21 - a = -2a - 7$ $\frac{14}{5}$

53. $8x = x$ 0

54. $-z = 5z$ 0

Solve each equation. Check each result. See Example 5.

55. $3(k - 4) = -36$ -8

▶ **56.** $4(x + 6) = 84$ 15

57. $2(a - 5) - (3a + 1) = 0$ -11

58. $8(3a - 5) - 4(2a + 3) = 12$ 4

59. $9(x - 2) = -6(4 - x) + 18$ 4

60. $3(x + 2) - 2 = -(5 + x) + x$ -3

61. $12 + 3(x - 4) - 21 = 5[5 - 4(4 - x)]$ 2

▶ **62.** $1 + 3[-2 + 6(4 - 2x)] = -(x + 3)$ 2

Solve each equation. Check each result. See Examples 6 and 7.

63. $\frac{1}{2}(a - 2) = \frac{2}{3}a - 6$ 30

64. $\frac{2}{3}(b + 3) = \frac{5}{4}b + \frac{17}{12}$ 1

65. $\frac{1}{2}(3y + 2) - \frac{5}{8} = \frac{3}{4}y$ $-\frac{1}{2}$

66. $-\frac{3}{4}(4c - 3) + \frac{7}{8}c = \frac{19}{16}$ $\frac{1}{2}$

▶ **67.** $\frac{3}{4}x - 5 = \frac{2}{3}x + \frac{1}{4}$ 63

68. $\frac{3}{5}x + \frac{7}{10} = x - \frac{4}{5}$ $\frac{15}{4}$

69. $\frac{1}{2}b - \frac{19}{6} = \frac{1}{3}b + \frac{5}{6}$ 24

70. $\frac{1}{2}w - \frac{7}{6} = \frac{53}{6} - \frac{1}{3}w$ 12

71. $\frac{a + 1}{3} + \frac{a - 1}{5} = \frac{2}{15}$ 0

▶ **72.** $\frac{2z + 3}{3} + \frac{3z - 4}{6} = \frac{z - 2}{2}$ -2

73. $\frac{3 + p}{3} - 4p = 1 - \frac{p + 7}{2}$ $\frac{21}{19}$

74. $\frac{4 - t}{2} - \frac{3t}{5} = 2 + \frac{t + 1}{3}$ $-\frac{10}{43}$

Solve each equation. Check each result. See Example 8.

75. $0.45 = 16.95 - 0.25(75 - 3x)$ 3

▶ **76.** $0.02x + 0.0175(15{,}000 - x) = 277.5$ $6{,}000$

77. $0.04(12) + 0.01t - 0.02(12 + t) = 0$ 24

▶ **78.** $0.25(t + 32) = 3.2 + t$ 6.4

Solve each equation. If the equation is an identity or a contradiction, so indicate. See Examples 9 and 10.

79. $8x + 3(2 - x) = 5x + 6$ all real numbers, $\mathbb{R}$; identity

80. $4(2 - 3t) + 6t = -6t + 8$ all real numbers, $\mathbb{R}$; identity

81. $2x - 6 = -2x + 4(x - 2)$ no solution, $\varnothing$; contradiction

82. $3(x - 4) + 6 = -2(x + 4) + 5x$ no solution, $\varnothing$; contradiction

▶ **83.** $2(x - 3) = \frac{3}{2}(x - 4) + \frac{x}{2}$ all real numbers, $\mathbb{R}$; identity

84. $y + \frac{1}{2} = \frac{5}{2}(0.2y + 1) - \frac{1}{2}(4 - y)$ all real numbers, $\mathbb{R}$; identity

▶ **85.** $-3x = -2x + 1 - (5 + x)$ no solution, $\varnothing$; contradiction

86. $5(y + 2) + 7 - 3y = 2(y + 9)$ no solution, $\varnothing$; contradiction

TRY IT YOURSELF

Solve each equation, if possible.

87. $2(2x + 1) = x + 15 + 2x$ 13

▶ **88.** $-2(x + 5) = x + 30 - 2x$ -40

89. $\frac{5}{2}a - 12 = \frac{1}{3}a + 1$ 6

90. $3(x - 2) + 4 = 3x - 2$ all real numbers, $\mathbb{R}$; identity

91. $\frac{4}{5}a = -12$ -15

92. $4j + 12.54 = 18.12$ 1.395

93. $0.06(a + 200) + 0.1a = 172$ $1{,}000$

94. $0.03x + 0.05(6{,}000 - x) = 280$ $1{,}000$

95. $-4[p - (3 - p)] = 3(6p - 2)$ $\frac{9}{13}$

96. $2[5(4 - a) + 2(a - 1)] = 3 - a$ $\frac{33}{5}$

97. $2(x - 2) = \frac{2}{3}(3x + 8) - 2$ no solution, $\varnothing$; contradiction

▶ **98.** $5 - \frac{x + 2}{3} = 7 - x$ 4

99. $13.5y + 16.2 = 0$ -1.2

100. $\frac{7}{3}y + 1 = 0$ $-\frac{3}{7}$

101. $\frac{4}{5}(x + 5) = \frac{7}{8}(3x + 23) - 7$ -5

102. $\frac{2}{3}(2x + 2) + 4 = \frac{1}{6}(5x + 29)$ -1

103. $\frac{t - 2}{5} + 5t = \frac{7}{5} - \frac{t - 2}{2}$ $\frac{28}{57}$

104. $\frac{2}{3}(3m - 2) = \frac{3}{4}m + \frac{11}{12}$ $\frac{9}{5}$

105. $5c - 8 + 3c = 10 + 2c - 3$ $\frac{5}{2}$

106. $6 + 4t - 1 = 6 - 15t + 12t - 8$ -1

WRITING

107. What does it mean to *solve an equation?*

108. Why doesn't the equation $x = x + 1$ have a real-number solution?

109. What is an identity? Give an example.

110. When solving a linear equation in one variable, the objective is to isolate the variable on one side of the equation. What does that mean?

REVIEW

Use variables to state each property of real numbers.

111. **a.** Commutative property of addition $a + b = b + a$

b. Associative property of multiplication $(ab)c = a(bc)$

c. Distributive property of multiplication over addition $a(b + c) = ab + ac$

112. **a.** Additive inverse property $a + (-a) = -a + a = 0$

b. Multiplicative inverse property $a \cdot \frac{1}{a} = 1 \quad (a \neq 0)$

113. **a.** Additive identity property $0 + a = a$

b. Multiplicative identity property $1 \cdot a = a$

114. **a.** Division of 0 $\frac{0}{a} = 0 \quad (a \neq 0)$

b. Division by 0 $\frac{a}{0}$ is undefined $(a \neq 0)$

SECTION 1.6
Solving Formulas; Geometry

Objectives

1. Find the perimeter, area, and volume of geometric figures.
2. Solve for a specified variable.
3. Solve application problems using formulas.

A **formula** is an equation that states a relationship between two or more variables. Formulas are used in business, science, banking, and many other fields. A large collection of formulas are associated with geometric figures such as squares, rectangles, circles, and cylinders.

1 Find the perimeter, area, and volume of geometric figures.

To find the **perimeter** of a plane (two-dimensional, flat) geometric figure, we find the distance around the figure by computing the sum of the lengths of the sides. Perimeter is measured in American units of inches, feet, yards, and in metric units such as millimeters, meters, and kilometers. Several perimeter formulas are shown below.

Perimeter Formulas

$P = 2l + 2w$ (rectangle)

$P = 4s$ (square)

$P = a + b + c$ (triangle)

Turn to the inside back cover for a complete list of geometric formulas.

The Language of Algebra When you hear the word perimeter, think of the distance around the "rim" of a flat figure.

EXAMPLE 1 ***Landscaping*** Find the number of feet of edging needed to outline a square flowerbed having sides that are 6.5 feet long.

© Gavin Hellier/Alamy

Strategy We will substitute the length of a side of the flowerbed into the formula for the perimeter of a square, $P = 4s$, and find P.

Self Check 1

LANDSCAPING Find the amount of fencing needed to enclose a rectangular lot that is 205.5 feet long and 165 feet wide. 741 ft

Now Try Problem 11

Teaching Example 1 LANDSCAPING Find the number of feet of fencing needed to enclose a rectangular garden that is 5 feet wide and 9 feet long.
Answer:
28 ft

WHY Since the edging outlines the flowerbed, the concept of perimeter applies.

Solution

$P = 4s$ This is the formula for the perimeter of a square.

$P = 4(6.5)$ Substitute 6.5 for s, the length of one side of the square.

$= 26$

26 feet of edging is needed to outline the flowerbed.

The **area** of a plane (two-dimensional, flat) geometric figure is the amount of surface that it encloses. Area is measured in square units, such as square inches, square feet, square yards, and square meters (written as in.^{2}, ft^2, yd^2, and m^2, respectively). Several area formulas are shown below.

Area Formulas

$A = lw$ (rectangle)

$A = s^2$ (square)

$A = \frac{1}{2}bh$ (triangle)

$A = \frac{1}{2}h(b_1 + b_2)$ (trapezoid)

Turn to the inside back cover for a complete list of geometric formulas.

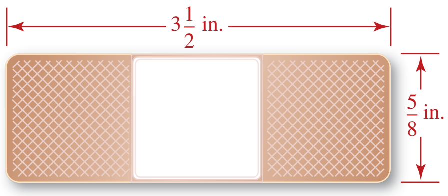
Self Check 2

SOLAR PANEL A solar panel is in the shape of a triangle. Its base is 79 centimeters long and the height is 54 centimeters. In square centimeters, how large a surface do the sun's rays strike? 2,133 cm^2

Now Try **Problem 15**

Teaching Example 2 FLOORING Find the amount of carpet needed to cover the floor of a room that is 15 feet long and 12 feet wide.
Answer:
180 ft^2

EXAMPLE 2 ***Band-aids*** Find the amount of skin covered by the rectangular-shaped bandage.



Strategy We will substitute the length and width of the bandage into the formula for the area of a rectangle, $A = lw$, and find A.

WHY The concept of area is suggested by the phrase *the amount of skin covered.*

Solution

$A = lw$ This is the formula for the area of a rectangle.

$A = 3\frac{1}{2}\left(\frac{5}{8}\right)$ Substitute $3\frac{1}{2}$ for l, the length of the bandage, and $\frac{5}{8}$ for w, the width.

$A = \frac{7}{2}\left(\frac{5}{8}\right)$ Write $3\frac{1}{2}$ as a fraction: $3\frac{1}{2} = \frac{7}{2}$.

$A = \frac{35}{16}$ Multiply the numerators. Multiply the denominators.

The bandage covers $\frac{35}{16}$ or $2\frac{3}{16}$ in.^{2} (square inches) of skin.

Caution! When finding area, remember to write the appropriate *square units* in the answer.

Circle Formulas

$D = 2r$ (diameter)

$r = \frac{1}{2}D$ (radius)

$C = 2\pi r = \pi D$ (circumference)

$A = \pi r^2$ (area)

Turn to the inside back cover for a complete list of geometric formulas.

EXAMPLE 3 **a.** Find the circumference of a circle with diameter 20 feet. Round to the nearest tenth of a foot. **b.** Find the area of the circle. Round to the nearest tenth of a square foot.

Strategy We will substitute the given values into the formulas $C = \pi D$ and $A = \pi r^2$ and find C and A.

WHY In the formulas, the variable C represents the circumference of the circle and A represents the area.

Solution

a. Recall that the circumference of a circle is the distance around it. To find the circumference C of a circle with diameter D equal to 20 ft, we proceed as follows.

$C = \pi D$	This is the formula for the circumference of a circle. πD means $\pi \cdot D$.
$C = \pi(20)$	Substitute 20 for D.
$= 20\pi$	The exact circumference of the circle is 20π ft.
≈ 62.83185307	To use a scientific calculator to approximate the circumference, enter π $\times$ 20 $=$. If you do not have a calculator, use 3.14 as an approximation of π. (Answers may vary slightly depending on which approximation of π is used.)

The circumference is exactly 20π ft. Rounded to the nearest tenth, this is 62.8 ft.

b. The radius r of the circle is one-half the diameter, or 10 feet. To find the area A of the circle, we proceed as follows.

$A = \pi r^2$	This is the formula for the area of a circle. πr^2 means $\pi \cdot r^2$.
$A = \pi(10)^2$	Substitute 10 for r.
$= 100\pi$	Evaluate the exponential expression. The exact area is 100π ft^2.
≈ 314.1592654	To use a scientific calculator to approximate the area, enter 100 $\times$ π $=$.

The area is exactly 100π ft^2. To the nearest tenth, the area is 314.2 ft^2.

Self Check 3

The diameter of a U.S. penny is 0.75 inch. Find the circumference and the area (of one side) of a penny. Round to the nearest hundredth. 2.36 in., 0.44 in.^{2}

Now Try **Problems 19 and 23**

Teaching Example 3

a. Find the circumference of a circle with diameter 8 feet. Round to the nearest thousandth of a foot.
b. Find the area of a circle with diameter 8 feet. Round to the nearest thousandth of a square foot.

Answers:
a. 25.133 ft **b.** 50.265 ft^2

Success Tip When an approximation of π is used in a calculation, it produces an approximate answer. Remember to use an *is approximately equal to symbol* $\approx$ in your solution to show that.

The **volume** of a three-dimensional geometric solid is the amount of space it encloses. Volume is measured in cubic units, such as cubic inches, cubic feet, and cubic meters (written as in.^{3}, ft^3, and m^3, respectively). Several volume formulas are shown below.

Volume Formulas

$V = lwh$ (rectangular solid) $\qquad V = s^3$ (cube)

$V = \frac{4}{3}\pi r^3$ (sphere) $\qquad V = \pi r^2 h$ (cylinder)

$V = \frac{1}{3}Bh^*$ (pyramid) $\qquad V = \frac{1}{3}\pi r^2 h$ (cone)

*B represents the area of the base. Turn to the inside back cover for a complete list of geometric formulas.

Self Check 4

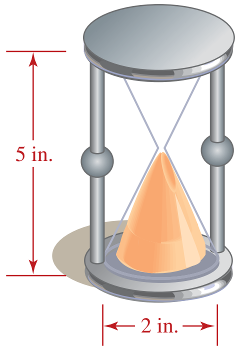
STRAWS Find the volume of a drinking straw that is 250 millimeters long with an inside diameter of 6 millimeters. Round to the nearest cubic millimeter.

7,069 mm^3

Now Try **Problem 27**

Teaching Example 4 SAND A pile of sand is in the shape of a cone whose radius is 10 feet and whose height is 6 feet. Find the amount of sand in the pile.
Answer:
200π ft^3

EXAMPLE 4 ***Timers*** Find the amount of sand in the hourglass.



Strategy We will substitute the given values into the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, and find V.

WHY To find the amount of sand in the hourglass, we need to find the amount of space that it occupies by finding a volume.

Solution
The sand is in the shape of a cone whose radius is one-half the diameter of the base of the hourglass and whose height is one-half the height of the hourglass. To find the amount of sand, we substitute 1 for r and 2.5 for h in the formula for the volume of a cone.

$V = \frac{1}{3}\pi r^2 h$ This is the formula for the volume of a cone.

$V = \frac{1}{3}\pi(1)^2(2.5)$ Substitute 1 for r, the radius of the circular base, and 2.5 for h, the height of the cone.

$V = \frac{2.5\pi}{3}$ Simplify. The exact volume of sand is $\frac{2.5\pi}{3}$ in.^{3}.

$V \approx 2.617993878$ Use a calculator.

There are exactly $\frac{2.5\pi}{3}$ in.^{3} of sand in the hourglass. Rounded to the nearest tenth, this is 2.6 in.^{3}.

Caution! When finding volume, remember to write the appropriate *cubic units* in your answer.

2 Solve for a specified variable.

Real-world applications sometimes call for a formula solved for one variable to be solved for a different variable. To **solve a formula for a specified variable** means to isolate that variable on one side of the equation, with all other variables and constants on the opposite side.

EXAMPLE 5 Solve each formula for the specified variable.

a. $V = lwh$ for w — The formula for the volume of a rectangular solid.

b. $A = \frac{1}{2}h(b_1 + b_2)$ for b_1 — The formula for the area of a trapezoid.

c. $v_f = v_i + at$ for t — A motion formula from physics.

Strategy To solve for a specified variable, we treat it as if it were the only variable in the equation. To isolate this variable, we will use the same strategy that we used to solve linear equations in one variable. (See page 55 if you need to review the strategy.)

WHY We can solve a formula as if it were an equation in one variable because all the other variables are treated as if they were numbers (constants).

Solution

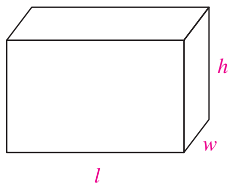
a. To solve for w, we will isolate w on this side of the equation.

$$V = lwh$$

$$\frac{V}{lh} = \frac{lwh}{lh}$$ To isolate w, undo the multiplication by l and h by dividing both sides by lh.

$$\frac{V}{lh} = w$$ On the right side, remove the common factors of l and h: $\frac{\overset{1}{\cancel{l}}w\overset{1}{\cancel{h}}}{\underset{1}{\cancel{l}}\underset{1}{\cancel{h}}} = w$.

$$w = \frac{V}{lh}$$ Reverse the sides of the equation so that w is on the left.



Rectangular solid

b. To solve for b_1, we will isolate b_1 on this side of the equation.

$$A = \frac{1}{2}h(b_1 + b_2)$$ Read b_1 as "b-sub-one" and b_2 as "b-sub-two."

$$2 \cdot A = 2 \cdot \frac{1}{2}h(b_1 + b_2)$$ Multiply both sides by 2 to clear the equation of the fraction.

$$2A = h(b_1 + b_2)$$ Simplify each side of the equation.

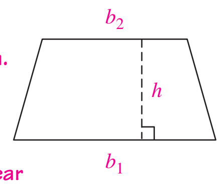
$$2A = hb_1 + hb_2$$ Distribute the multiplication by h.

$$2A - hb_2 = hb_1$$ Subtract hb_2 from both sides to isolate the variable term hb_1 on the right side. This step is done mentally.

$$\frac{2A - hb_2}{h} = \frac{hb_1}{h}$$ To isolate b_1, undo the multiplication by h by dividing both sides by h.

$$\frac{2A - hb_2}{h} = b_1$$ On the right side, remove the common factor of h: $\frac{\overset{1}{\cancel{h}}b_1}{\underset{1}{\cancel{h}}}$

$$b_1 = \frac{2A - hb_2}{h}$$ Reverse the sides of the equation so that b_1 is on the left.



Trapezoid

When solving formulas for a specified variable, there is often more than one way to express the result. In this case, we could perform the division by h on

Self Check 5

Solve each formula for the specified variable.

a. $I = Prt$ for r.

b. $S = \frac{n}{2}(f + l)$ for f.

c. $E = \frac{T_h - T_c}{T_h}$ for T_c.

Now Try **Problems 33, 39, and 43**

Self Check 5 Answers

a. $r = \frac{I}{Pt}$

b. $f = \frac{2S - nl}{n}$

c. $T_c = T_h - ET_h$

Teaching Example 5 Solve each formula for the specified variable.

a. $P = 2l + 2w$ for l

b. $V = \frac{1}{3}lwh$ for h

c. $A = \frac{1}{2}h(b_1 + b_2)$ for b_2

Answers:

a. $l = \frac{P - 2w}{2}$

b. $h = \frac{3V}{lw}$

c. $b_2 = \frac{2A - hb_1}{h}$

the right side term-by-term: $b_1 = \frac{2A}{h} - \frac{hb_2}{h}$. After removing the common factor of h in the numerator and denominator of the second fraction, we obtain the following equivalent form of the result: $b_1 = \frac{2A}{h} - b_2$.

Language of Algebra The 1 and 2 in b_1 and b_2 are called **subscripts.** This notation allows us to distinguish between the variables b_1 and b_2, while still showing that each represents the length of a base of the trapezoid.

Caution! Do not try to simplify the result in the following way. It is incorrect because h is not a factor of the entire numerator.

$$\cancel{b_1 = \frac{2A - \overset{1}{\cancel{h}}b_2}{\underset{1}{\cancel{h}}}}$$

c.

To solve for t, we will isolate t on this side of the equation.

$$v_f = v_i + at$$ Read v_f as "v-sub-f and v_i as "v-sub-i."

$$v_f - v_i = at$$ To isolate the term at, subtract v_i from both sides. This step is done mentally.

$$\frac{v_f - v_i}{a} = t$$ To isolate t, undo the multiplication by a by dividing both sides by a. This step is done mentally.

$$t = \frac{v_f - v_i}{a}$$ Reverse the sides of the equation so that t is on the left.

Notation Variables are also used as subscripts. For example, in physics, the symbol v_i is used to represent initial velocity and v_f final velocity.

Self Check 6

Solve $W = g(h - 3t^2)$ for g.

Now Try Problem 49

Self Check 6 Answer

$g = \frac{W}{h - 3t^2}$

Teaching Example 6 Solve $A = P(1 + rt)$ for r.

Answer:

$r = \frac{A - P}{Pt}$

EXAMPLE 6 ***Banking*** For simple interest, the formula $A = P(1 + rt)$ gives the amount of money in an account at the end of a specific time. The variable A represents the amount, P the principal, r the rate of interest, and t the time. Solve the formula for P.

Strategy To solve for P, we will treat it as if it were the only variable in the equation and use a property of equality to isolate P on one side of the equation.

WHY We can solve the formula as if it were an equation in one variable because all the other variables are treated as if they were numbers (constants).

Solution

To solve for P, we will isolate P on this side of the equation.

$$A = P(1 + rt)$$

$$\frac{A}{\mathbf{1 + r}} = \frac{P(1 + r)}{\mathbf{1 + r}}$$ To isolate P, undo the multiplication by $1 + r$ by dividing both sides by $1 + r$.

$$\frac{A}{1 + r} = P$$ On the right side, remove the common factor of $1 + r$: $\frac{P\overset{1}{\cancel{(1 + r)}}}{\underset{1}{\cancel{1 + r}}}$.

$$P = \frac{A}{1 + r}$$ Write the equation with P on the left side.

In Chapter 2, we will work with equations in two variables, such as $6x + 5y = 35$ and $3x - 4y = 20$. It is often necessary to solve such equations for y.

EXAMPLE 7 Solve the equation $3x - 4y = 20$ for y.

Strategy To solve for y, we will treat it as if it were the only variable in the equation and isolate y on one side of the equation.

WHY We can solve the formula as if it were an equation in one variable, y, because the other variable, x, is treated as if it were a number (constant).

Solution

To solve for y, we will isolate y on this side of the equation.

$$3x - 4y = 20$$

$$3x - 4y - 3x = 20 - 3x$$ To isolate $-4y$, subtract $3x$ from both sides.

$$-4y = 20 - 3x$$ Simplify each side of the equation.

$$\frac{-4y}{-4} = \frac{20 - 3x}{-4}$$ To isolate y, undo the multiplication by -4 by dividing both sides by -4.

$$y = \frac{20 - 3x}{-4}$$

If we reorder the terms of the numerator and perform the division by -4 term-by-term, we obtain the equivalent result

$$y = \frac{-3x}{-4} + \frac{20}{-4} \text{ which simplifies to } y = \frac{3}{4}x - 5$$

Self Check 7

Solve $6x + 5y = 35$ for y.

Now Try Problem 55

Self Check 7 Answer

$y = -\frac{6}{5}x + 7$

Teaching Example 7 Solve the equation $6x - 5y = 25$ for y.
Answer:
$y = \frac{6}{5}x - 5$

3 Solve application problems using formulas.

EXAMPLE 8 ***Melting Points for Metals*** The formula that relates a Fahrenheit temperature F to a Celsius temperature C is $F = \frac{9}{5}C + 32$. Calculate the melting points in degrees Celsius for three metals shown in the table.

Strategy First, we will solve $F = \frac{9}{5}C + 32$ for C. Then, we will substitute each Fahrenheit temperature for F in the formula that is solved for C and calculate its corresponding Celsius temperature.

Metal	Melting point
Gold	1,948°F
Nickel	2,647°F
Aluminum	1,221°F

WHY It would be time consuming to substitute each melting point temperature for F in the formula $F = \frac{9}{5}C + 32$ and then repeatedly solve for C. A quicker way is to solve the formula for C first, substitute values for F, and compute C directly.

Solution

$$F = \frac{9}{5}C + 32$$

$$F - 32 = \frac{9}{5}C$$ To isolate the variable term $\frac{9}{5}C$, subtract 32 from both sides. This step is done mentally.

$$\frac{5}{9}(F - 32) = \frac{5}{9}\left(\frac{9}{5}C\right)$$ To isolate C, multiply both sides by the reciprocal of $\frac{9}{5}$, which is $\frac{5}{9}$.

$$\frac{5}{9}(F - 32) = C$$ Simplify the right side: $\frac{5}{9} \cdot \frac{9}{5} = 1$.

$$C = \frac{5}{9}(F - 32)$$ Write the equation with C on the left side.

Self Check 8

BOILING WATER Water boils at 212°F. Find the boiling point temperature for water in degrees Celcius. 100°C

Now Try Problem 83

Teaching Example 8 FREEZING WATER Water freezes at 32°F. Find the freezing temperature for water in degrees Celsius.
Answer:
0°C

To convert each melting point temperature from degrees Fahrenheit to degrees Celsius, we substitute for F in the formula $C = \frac{5}{9}(F - 32)$ and find C.

Gold: **1,948°F**	***Nickel:*** **2,647°F**	***Aluminum:*** **1,221°F**
$C = \frac{5}{9}(\mathbf{1{,}948} - 32)$	$C = \frac{5}{9}(\mathbf{2{,}647} - 32)$	$C = \frac{5}{9}(\mathbf{1{,}221} - 32)$
$C = \frac{5}{9}(1{,}916)$	$C = \frac{5}{9}(2{,}615)$	$C = \frac{5}{9}(1{,}189)$
$C \approx 1{,}064$	$C \approx 1{,}453$	$C \approx 661$
Gold: **1,064°C**	***Nickel:*** **1,453°C**	***Aluminum:*** **661°C**

The Language of Algebra In 1724, Daniel Gabriel *Fahrenheit,* a German scientist, introduced the temperature scale that bears his name. The Celsius scale was invented in 1742 by Swedish astronomer Anders *Celsius.*

ANSWERS TO SELF CHECKS

1. 741 ft **2.** 2,133 cm^2 **3.** 2.36 in.; 0.44 in.^{2} **4.** 7,069 mm^3 **5. a.** $r = \frac{I}{Pt}$ **b.** $f = \frac{2S - nl}{n}$ **c.** $T_c = T_h - ET_h$ **6.** $g = \frac{W}{h - 3t^2}$ **7.** $y = -\frac{6}{5}x + 7$ **8.** 100°C

SECTION 1.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. A formula is an equation that states a relationship between two or more variables.

2. To find the perimeter of a plane geometric figure, such as a rectangle or triangle, we calculate the distance around the figure. The area of a plane geometric figure is the amount of surface that it encloses.

3. The volume of a three-dimensional geometric solid is the amount of space it encloses.

4. To solve a formula for a specified variable means to isolate that variable on one side of the equation, with all other variables and constants on the opposite side.

CONCEPTS

5. Determine which concept (perimeter, circumference, area, or volume) should be used to find each of the following situations. Then determine which unit of measurement, ft, ft^2, or ft^3, would be appropriate.

a. The amount of ground covered by a lawn area, ft^2

b. The amount of storage in a safe volume, ft^3

c. The distance traveled by a rider on a Ferris wheel circumference, ft

d. The distance around a tennis court perimeter, ft

6. The area of a circle is exactly 54π ft^2. Approximate the area to the nearest tenth of a square foot. 169.6 ft^2

7. When solving formulas for a specified variable, there can be more than one way to express the result. Fill in the blanks to express this result in an equivalent form:

$$d = \frac{4m - at}{t}$$

$$d = \frac{4m}{t} - \frac{at}{t}$$

$$d = \frac{4m}{t} - a$$

8. Fill in the blanks: To solve a formula for a specified variable, we treat it as if it were the only variable in the equation. We treat all other variables as if they were numbers (constants).

NOTATION

Complete the solution.

9. Solve $t = ad + bc$ for c.

$$t - ad = ad + bc - ad$$

$$t - ad = bc$$

$$\frac{t - ad}{b} = \frac{bc}{b}$$

$$\frac{t - ad}{b} = c$$

$$c = \frac{t - ad}{b}$$

10. Fill in the blanks: In the notation b_1 and v_f, the number 1 and the variable f are called subscripts.

GUIDED PRACTICE

Find the perimeter of each figure. See Example 1.

11. A square with sides 2 yd long 8 yd
12. A triangle with sides 1.8, 1.8, and 1.5 cm long 5.1 cm
13. A trapezoid with parallel sides 10 in. and 15 in. long and the other two sides each 6 in. long 37 in.
14. A parallelogram with two adjacent sides 50 m and 100 m long 300 m

Find the area of each figure. See Example 2.

15. A triangle with a base that is 2.4 ft long and height 8.5 ft 10.2 ft^2
16. A rectangle with sides that measure $8\frac{1}{4}$ ft and $5\frac{1}{2}$ ft $45\frac{3}{8}\text{ ft}^2$
17. A square with sides 17.2 mi long 295.84 mi^2
18. A trapezoid whose parallel sides measure 8 cm and 12 cm and whose height is 10.5 cm. 105 cm^2

Find the circumference of each circle to the nearest hundredth. (Answers may vary slightly depending on which approximation of π is used.) See Example 3.

19. A circle with diameter 7.5 in. 23.56 in.
20. A circle with diameter $6\frac{1}{4}$ m 19.63 m
21. A circle with radius $2\frac{1}{2}$ ft 15.71 ft
22. A circle with radius 12.3 yd 77.28 yd

Find the area of each circle to the nearest tenth. (Answers may vary slightly depending on which approximation of π is used.) See Example 3.

23. A circle with radius 5.7 in. 102.1 in.^2
24. A circle with radius $5\frac{3}{4}$ cm 103.9 cm^2
25. A circle with diameter $10\frac{1}{2}$ ft 86.6 ft^2
26. A circle with diameter 12.25 m 117.9 m^2

Find the volume of each figure to the nearest hundredth. (Answers may vary slightly depending on which approximation of π is used.) See Example 4.

27. A rectangular solid with dimensions 2.51 ft, 3.71 ft, and 10.21 ft 95.08 ft^3
28. A pyramid whose base is a square with each side measuring 2.57 cm and with a height of 12.32 cm 27.12 cm^3
29. A sphere with radius 5.78 meters 808.86 m^3
30. A cone whose base has a radius of 5.50 in. and whose height is 8.52 in. 269.89 in.^3

Solve each formula for the specified variable. See Example 5.

31. $d = rt$ for t $t = \frac{d}{r}$
32. $E = mc^2$ for m $m = \frac{E}{c^2}$
33. $V = lwh$ for h $h = \frac{V}{lw}$
34. $I = Prt$ for t $t = \frac{I}{Pr}$
35. $V = \frac{1}{3}\pi r^2 h$ for h $h = \frac{3V}{\pi r^2}$
36. $A = \frac{1}{2}bh$ for b $b = \frac{2A}{h}$
37. $T = W + ma$ for W $W = T - ma$
38. $V = \frac{1}{3}Bh$ for B $B = \frac{3V}{h}$
39. $h = 48t + \frac{1}{2}at^2$ for a $a = \frac{2h - 96t}{t^2}$ or $a = \frac{2(h - 48t)}{t^2}$
40. $H = 17 - \frac{A}{2}$ for A $A = 34 - 2H$
41. $A = \frac{1}{2}h(b_1 + b_2)$ for b_2 $b_2 = \frac{2A - b_1 h}{h}$ or $b_2 = \frac{2A}{h} - b_1$
42. $\bar{v} = \frac{1}{2}(v + v_0)$ for v_0 $v_0 = 2\bar{v} - v$
43. $l = a + (n - 1)d$ for n $n = \frac{l - a + d}{d}$ or $n = \frac{l - a}{d} + 1$
44. $P = 2(l + w)$ for l $l = \frac{P - 2w}{2}$ or $l = \frac{P}{2} - w$
45. $P = 2(w + h + l)$ for w $w = \frac{P - 2h - 2l}{2}$ or $w = \frac{P}{2} - h - l$
46. $P = 2(w + h + l)$ for h $h = \frac{P - 2w - 2l}{2}$ or $h = \frac{P}{2} - w - l$

Solve each formula for the indicated variable. See Example 6.

47. $\lambda = A(x + B)$ for A $A = \frac{\lambda}{x + B}$
(λ is a letter from the Greek alphabet.)
48. $S = C(1 - r)$ for C $C = \frac{S}{1 - r}$
49. $T_f = T_a(1 - F)$ for T_a $T_a = \frac{T_f}{1 - F}$
50. $S = \frac{n}{2}(f + l)$ for n $n = \frac{2S}{f + l}$
51. $l = a + (n - 1)d$ for d $d = \frac{l - a}{n - 1}$
52. $S = \frac{n(a + l)}{2}$ for n $n = \frac{2S}{a + l}$
53. $v = \frac{1}{t}(d_1 - d_2)$ for t $t = \frac{d_1 - d_2}{v}$
54. $A = \frac{1}{2}(b_1 + b_2)h$ for h $h = \frac{2A}{b_1 + b_2}$

Solve each equation for y. See Example 7.

55. $2x - 5y = 20$ $y = \frac{2}{5}x - 4$
56. $5x - 6y = 12$ $y = \frac{5}{6}x - 2$
57. $-4x = 12 + 3y$ $y = -\frac{4}{3}x - 4$
58. $7x - 21 = -3y$ $y = -\frac{7}{3}x + 7$

TRY IT YOURSELF

Solve for the specified variable.

59. $y = mx + b$ for x $x = \frac{y - b}{m}$

60. $P = 2l + 2w$ for l $l = \frac{P - 2w}{2}$

61. $L = 2d + 3.25(r + R)$ for R $R = \frac{L - 2d - 3.25r}{3.25}$

62. $l = \dfrac{a - S + Sr}{r}$ for a $a = lr + S - Sr$

63. $s = \frac{1}{2}gt^2 + vt$ for g $g = \frac{2(s - vt)}{t^2}$

64. $K = \dfrac{Mv_0^2 + Iw^2}{2}$ for I $I = \frac{2K - Mv_0^2}{w^2}$

65. $y - y_1 = m(x - x_1)$ for x $x = \frac{y - y_1 + mx_1}{m}$

66. $s = v_o t - 16t^2$ for v_o $v_0 = \frac{s + 16t^2}{t}$

67. $G = U - TS + pV$ for S $S = \frac{U + pV - G}{T}$

68. $F = \dfrac{Gm_1m_2}{r^2}$ for m_1 $m_1 = \frac{Fr^2}{Gm_2}$

69. $PV = nrt$ for r $r = \frac{PV}{nt}$

70. $P = s_1 + s_2 + b$ for s_2 $s_2 = P - s_1 - b$

71. $E = IR + Ir$ for R $R = \frac{E - Ir}{I}$ or $R = \frac{E}{I} - r$

72. $Ax + By = C$ for x $x = \frac{C - By}{A}$

73. $A = \frac{1}{3}(s_1 + s_2 + s_3)$ for s_3 $s_3 = 3A - s_1 - s_2$

74. $P_1V_1T_2 = P_2V_2T_1$ for V_2 $V_2 = \frac{P_1V_1T_2}{P_2T_1}$

75. $S = \frac{n}{2}[2a + (n - 1)d]$ for d $d = \frac{2S - 2an}{n(n - 1)}$

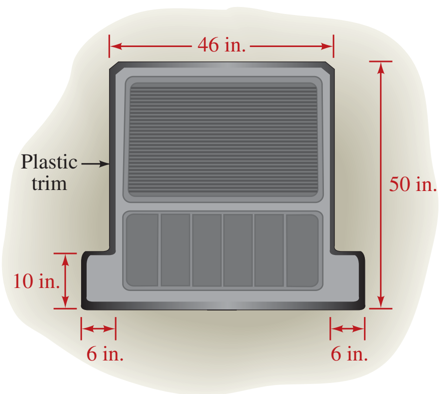
76. $x^2 = 4py$ for p $p = \frac{x^2}{4y}$

77. $d = \frac{4}{3}\pi h$ for h $h = \frac{3d}{4\pi}$

78. $I_Q = \dfrac{100M}{C}$ for C $C = \frac{100M}{I_Q}$

APPLICATIONS

79. FLOOR MATS What geometric concept applies when finding the length of the plastic trim around the cargo area floor mat? Estimate the amount of trim used. perimeter, 216 in.

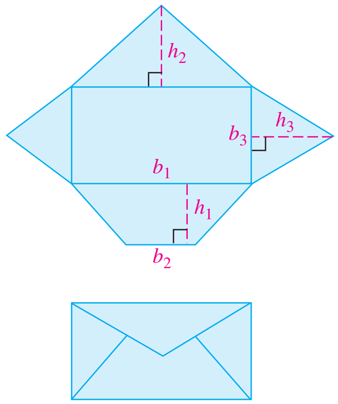


80. ALUMINUM FOIL Find the number of *square feet* of aluminum foil on a roll if the dimensions printed on the box are $8\frac{1}{3}$ yards $\times$ 12 inches. 25 ft^2

81. PAPER PRODUCTS When folded, the paper sheet shown in the illustration forms a rectangular-shaped envelope. The formula

$$A = \frac{1}{2}h_1(b_1 + b_2) + b_3h_3 + \frac{1}{2}b_1h_2 + b_1b_3$$

gives the amount of paper (in square units) used in the design. Explain what each of the four terms in the formula finds. Then evaluate the formula for $b_1 = 6$, $b_2 = 2$, $b_3 = 3$, $h_1 = 2$, $h_2 = 2.5$, and $h_3 = 3$. All dimensions are in inches.
1st term: area of bottom flap; 2nd term: area of left and right flaps; 3rd term: area of top flap; 4th term: area of face; 42.5 in.^{2}

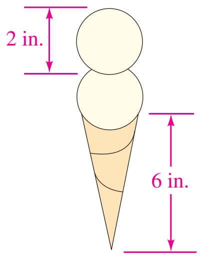


82. HOCKEY A goal is scored in hockey when the puck, a vulcanized rubber disk 2.5 cm (1 in.) thick and 7.6 cm (3 in.) in diameter, is driven into the opponent's goal. Find the volume of a puck in cubic centimeters and cubic inches. Round to the nearest tenth. 113.4 cm^3, 7.1 in.^{3}

83. CONVERTING TEMPERATURES In preparing an American almanac for release in Europe, editors need to convert temperature ranges for the planets from degrees Fahrenheit to degrees Celsius. Solve the formula $F = \frac{9}{5}C + 32$ for C. Then use your result to make the conversions for the data shown in the table. Round to the nearest degree.
$C = \frac{5}{9}(F - 32)$ or $C = \frac{5(F - 32)}{9}$

Planet	High °F	Low °F	High °C	Low °C
Mercury	810	−290	432	−179
Earth	136	−129	58	−89
Mars	63	−87	17	−66

84. ICE CREAM If the two equal-sized scoops of ice cream melt completely into the cone, will they overflow the cone? yes

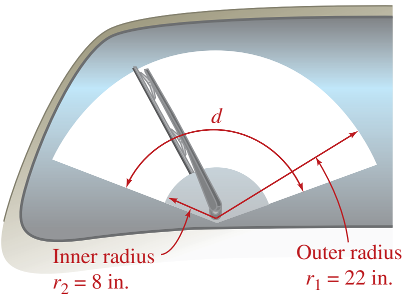


85. WIPER DESIGN The area cleaned by the windshield wiper assembly shown in the illustration is given by the formula

$$A = \frac{d\pi(r_1^2 - r_2^2)}{360}$$

Engineers have determined the amount of windshield area that needs to be cleaned by the wiper for two different vehicles. Solve the equation for d and use your result to find the number of degrees d the wiper arm must swing in each case. Round to the nearest degree. $d = \frac{360A}{\pi(r_1^2 - r_2^2)}$

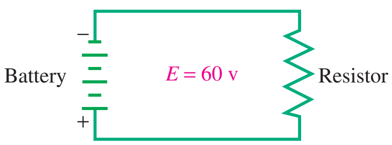
Vehicle	Area cleaned	d (deg)
Luxury car	513 in.^{2}	140
Sport utility vehicle	586 in.^{2}	160



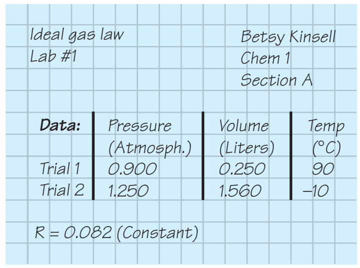
86. ELECTRONICS The illustration is a diagram of a resistor connected to a voltage source of 60 volts. As a result, the resistor dissipates power in the form of heat. The power P lost when a voltage E is placed across a resistance R (in ohms) is given by the formula

$$P = \frac{E^2}{R}$$

Solve for R. If P is 4.8 watts and E is 60 volts, find R. $R = \frac{E^2}{P}$, $R = 750$ ohms



87. CHEMISTRY In chemistry, the ideal gas law equation is $PV = nR(T + 273)$, where P is the pressure, V the volume, T the temperature, and n the number of moles of a gas. R is a constant, 0.082. Solve the equation for n. Then use your result and the data from the student lab notebook in the illustration to find the value of n to the nearest thousandth for trial 1 and trial 2. $n = \frac{PV}{R(T + 273)}$; 0.008, 0.090


Ideal gas law
Lab #1

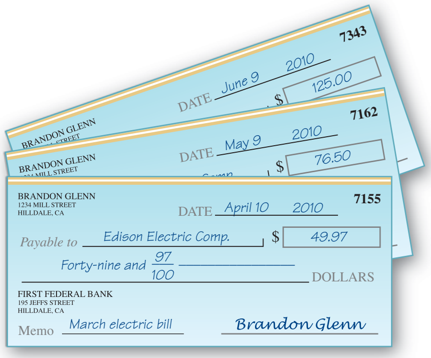
Betsy Kinsell
Chem 1
Section A

Data:	Pressure (Atmosph.)	Volume (Liters)	Temp (°C)
Trial 1	0.900	0.250	90
Trial 2	1.250	1.560	–10

R = 0.082 (Constant)

88. INVESTMENTS An amount P, invested at a simple interest rate r, will grow to an amount A in t years according to the formula $A = P(1 + rt)$. Solve for P. Suppose a man invested some money at 5.5%. If after 5 years, he had \$6,693.75 on deposit, what amount did he originally invest? $P = \frac{A}{1 + rt}$, \$5,250

89. COST OF ELECTRICITY The cost of electricity in a city is given by the formula $C = 0.07n + 6.50$, where C is the cost in dollars and n is the number of kilowatt hours used. Solve for n. Then find the number of kilowatt hours used each month by the homeowner whose checks to pay the monthly electric bills are shown in the illustration. $n = \frac{C - 6.50}{0.07}$; 621, 1,000, about 1,692.9 kwh

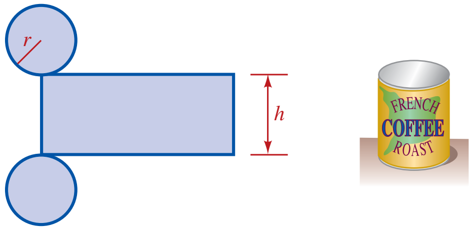


90. COST OF WATER A monthly water bill in a certain city is calculated by using the formula $n = \frac{5{,}000C - 17{,}500}{6}$, where n is the number of gallons used and C is the monthly cost in dollars. Solve for C and compute the bill for quantities of 500, 1,200, and 2,500 gallons. $C = \frac{6n + 17{,}500}{5{,}000}$; \$4.10, \$4.94, \$6.50

91. SURFACE AREA To find the amount of tin needed to make the coffee can shown in the illustration, we use the formula for the surface area of a right circular cylinder,

$$A = 2\pi r^2 + 2\pi rh$$

Solve the formula for h. $h = \frac{A - 2\pi r^2}{2\pi r}$



92. CARPENTRY A regular polygon has n equal sides and n equal angles. The measure a of an interior angle in degrees is given by $a = 180\left(1 - \frac{2}{n}\right)$. Solve for n. How many sides does the outdoor bandstand shown below have if the performance platform is a regular polygon with interior angles measuring 135°? $n = \frac{360}{180 - a}$, 8

WRITING

93. Explain the difference between what perimeter measures and what area measures.

94. After solving a formula for m, a student compared her answer with that at the back of the textbook. Could this problem have two different-looking answers? Explain why or why not.

$$\text{Student's answer: } m = \frac{5}{9}ar + 1$$

$$\text{Book's answer: } m = \frac{5ar + 9}{9}$$

95. Explain the error made below.

$$T = \frac{adx + \cancel{y}^{1}}{\cancel{y}_{1}} = adx + 1$$

96. A student solved $x + 5c = 3c + a$ for c. His answer was $c = \frac{3c + a - x}{5}$. Explain why the equation is not solved for c.

REVIEW

Simplify each expression.

97. $12(2r + 11t + 1) - 11 + 2r$ $26r + 132t + 1$

98. $(16b + 8)\left(\frac{5}{4}\right) - 8b$ $12b + 10$

99. $-7(a - 3) - 5[3(a - 4) - 2(a + 2)]$ $-12a + 101$

100. $0.9b^3 - 3.81b^3$ $-2.91b^3$

101. $-5.7pt - p + 5.1pt + 12p$ $-0.6pt + 11p$

102. $\frac{3}{5}t - \frac{2}{3}t$ $-\frac{1}{15}t$

SECTION 1.7
Using Equations to Solve Problems

Objectives

1. Apply the steps of a problem-solving strategy.
2. Solve number-value problems.
3. Solve geometry problems.
4. Use formulas to solve problems.

An objective of this course is to improve your problem-solving skills. In the next two sections, you will have the opportunity to do that as we discuss how to use equations to solve many different types of problems.

1 Apply the steps of a problem-solving strategy.

To become a good problem solver, you need a plan to follow, such as the following five-step strategy.

Problem Solving

1. ***Analyze the problem*** by reading it carefully. What information is given? What are you asked to find? What vocabulary is given? Often a diagram or table will help you visualize the facts of the problem.
2. ***Form an equation*** by picking a variable to represent the numerical value to be found. Then express all other unknown quantities as expressions involving that variable. Key words or phrases can be helpful. Finally, translate the words of the problem into an equation.
3. ***Solve the equation.***
4. ***State the conclusion*** using a complete sentence. Be sure to include the units (such as feet, seconds, or pounds) in your answer.
5. ***Check the result*** using the original wording of the problem, not the equation that was formed in step 2.

In order to solve problems, which are usually given in words, we must translate those words into mathematical symbols. In the next example, we use translation to write an equation that mathematically models the situation.

EXAMPLE 1 ***Leading Employers*** In 2007, Target and Home Depot were two of the nation's top employers. Their combined work forces totaled 658,000 people. If Home Depot employed 46,000 fewer people than Target, how many employees did each company have?

Analyze

- The phrase *combined work forces totaled* 658,000 suggests that if we add the number of employees of each company, the result will be 658,000.
- The phrase *Home Depot employed 46,000 fewer people than Target* suggests that the number of employees of Home Depot can be found by subtracting 46,000 from the number of employees of Target.
- We are to find the number of employees of each company.

Form If we let x = the number of employees of Target, then $x - 46{,}000$ = the number of employees of Home Depot. We can now translate the words of the problem into an equation.

The number of employees of Target	plus	the number of employees of Home Depot	is	658,000.
x	$+$	$x - 46{,}000$	$=$	658,000

Solve

$$x + x - 46{,}000 = 658{,}000$$
$$2x - 46{,}000 = 658{,}000 \quad \text{Combine like terms.}$$
$$2x = 704{,}000 \quad \text{Add 46,000 to both sides.}$$
$$x = 352{,}000 \quad \text{To isolate } x \text{, divide both sides by 2.}$$

Since x represents the number of employees of Target, we can find the number of employees of Home Depot by evaluating $x - 46{,}000$ for $x = 352{,}000$.

$$x - 46{,}000 = \mathbf{352{,}000} - 46{,}000$$
$$= 306{,}000$$

Self Check 1

iPODs The combined cost of an iPod and accessories is \$245. If the iPod costs \$55 more than the accessories, find the cost of the iPod. \$150

Now Try **Problem 13**

Teaching Example 1 GRADUATING CLASSES In a recent graduating class of 158 seniors, there were 26 more female students than male students. Find the number of male students in the class.
Answer:
66 male students

State In 2007, Target had 352,000 employees and Home Depot had 306,000 employees.

Check Since 352,000 + 306,000 = 658,000 and since 306,000 is 46,000 less than 352,000, the answers check.

Caution! For this problem, one common mistake is to let

x = the number of employees of each company

Since Target and Home Depot have different numbers of employees, x cannot represent both unknowns.

When solving problems, diagrams are often helpful, because they allow us to visualize the facts of the problem.

Self Check 2

DRIVING TRIPS A three-part drive consists of a first part that is 1 mile longer than the second, and a third part that is 1 mile longer than 4 times the second part. If the entire drive is 17 miles, find the length of each part. 3.5 mi, 2.5 mi, 11 mi

Now Try **Problem 15**

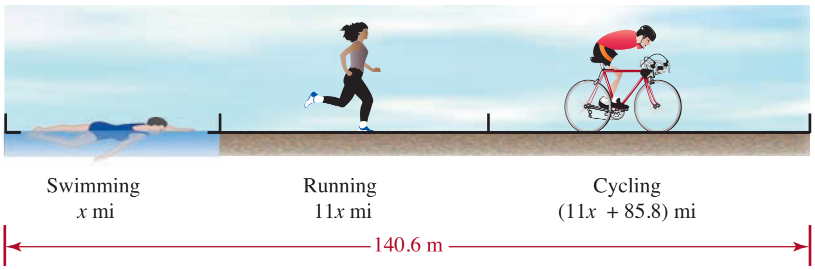
Teaching Example 2 TRIATHLONS To start training for a triathlon, an athlete runs 8 times longer than she swims, and cycles 45 miles longer than she runs. If she covers an overall distance of 70.5 miles, find the length of each part of her workout.
Answer:
swims 1.5 mi, runs 12 mi, cycles 57 mi

EXAMPLE 2 *Triathlons*

A triathlon includes swimming, long-distance running, and cycling. The long-distance run is 11 times longer than the distance the competitors swim. The distance they cycle is 85.8 miles longer than the run. Overall, the competition covers 140.6 miles. Find the length of each part of the triathlon and round each length to the nearest tenth of a mile.

Analyze The entire triathlon course covers a distance of 140.6 miles. We note that the distance the competitors run is related to the distance they swim, and the distance they cycle is related to the distance they run.

Form If x = the distance in miles that the competitors swim, then $11x$ = the length of the long-distance run, and $11x + 85.8$ = the distance they cycle. From the diagram below, we can see that the sum of the individual parts of the triathlon must equal the total distance covered.



We can now form the equation.

The distance they swim	plus	the distance they run	plus	the distance they cycle	equals	the total length of the course.
x	$+$	$11x$	$+$	$11x + 85.8$	$=$	140.6

Solve

$$x + 11x + 11x + 85.8 = 140.6$$
$$23x + 85.8 = 140.6 \quad \text{Combine like terms.}$$
$$23x = 54.8 \quad \text{Subtract 85.8 from both sides.}$$
$$x \approx 2.382608696 \quad \text{To isolate } x \text{, divide both sides by 23.}$$

State To the nearest tenth, the distance the competitors swim is 2.4 miles. The distance they run is $11x$, or approximately $11(2.382608696) = 26.20869565$ miles. To the nearest tenth, that is 26.2 miles. The distance they cycle is $11x + 85.8$, or

approximately $26.20869565 + 85.8 = 112.0086957$ miles. To the nearest tenth, that is 112.0 miles.

Check If we add the lengths of the three parts of the triathlon and round to the nearest tenth, we get 140.6 miles. The answers check.

The wording of a problem doesn't always contain key phrases that translate directly to an equation. In such cases, an analysis of the problem will give clues that help us write an equation.

EXAMPLE 3 *Travel Promotions*

The price of a 7-day Alaskan cruise, normally \$2,752 per person, is reduced by \$1.75 per person for large groups traveling together. How large a group is needed for the price to be \$2,500 per person?

© Ron Niebrugge/Alamy

Analyze For a group of 20 people, the cost is reduced by \$1.75 for each person and the \$2,752 price is reduced by 20(\$1.75) = \$35.

The per-person price of the cruise = \$2,752 − 20(\$1.75)

For a group of 30 people, the \$2,752 cost is reduced by 30(\$1.75) = \$52.50.

The per-person price of the cruise = \$2,752 − 30(\$1.75)

Form If we let x = the group size necessary for the price of the cruise to be \$2,500 per person, we can form the following equation:

The price of the cruise	is	\$2,752	minus	the number of people in the group	times	\$1.75.
2,500	=	2,752	−	x	$\cdot$	1.75

Solve

$$2{,}500 = 2{,}752 - 1.75x$$

$$2{,}500 - 2{,}752 = 2{,}752 - 1.75x - 2{,}752 \quad \text{Subtract 2,752 from both sides.}$$

$$-252 = -1.75x \quad \text{Simplify each side.}$$

$$144 = x \quad \text{To isolate } x \text{, divide both sides by } -1.75.$$

State If 144 people travel together, the price will be \$2,500 per person.

Check For 144 people, the cruise cost of \$2,752 will be reduced by 144(\$1.75) = \$252. If we subtract, \$2,752 − \$252 = \$2,500. The answer checks.

Self Check 3

BROADWAY SHOWS Tickets to a Broadway show normally sell for \$90 per person. This price is reduced by \$0.50 per person for large groups. How large a group is needed for the price to be \$75 per person? 30 people

Now Try **Problem 21**

Teaching Example 3 COMPANY ADVERTISING A trucking company had their logo embroidered on the front of baseball caps. The caps are normally \$20 each but will be reduced by \$0.05 per hat for large orders. How large an order is needed for the price per hat to be \$15 per hat?
Answer:
100 hats

2 Solve number-value problems.

Some problems deal with quantities that have a value. In these problems, we must distinguish between the *number of* and the *value of* the unknown quantity. For problems such as these, we will use the relationship

Number · value = total value

EXAMPLE 4 *Portfolio Analysis*

A college foundation owns stock in Kodak (selling at \$26 per share), Coca-Cola (selling at \$52 per share), and IBM (selling at \$103 per share). The foundation owns an equal number of shares of Kodak and Coca-Cola stock, but five times as many shares of IBM stock. If this portfolio (collection of stocks) is worth \$415,100, how many shares of each stock does the foundation own?

Self Check 4

RESTAURANT SUPPLIES A restaurant owner needs to purchase some tables, chairs, and dinner plates for the dining area of her establishment. She plans to buy 4 chairs and 4 plates for each new table. If the total number of items ordered is 180, how many of each did she buy?

Now Try **Problem 25**

Self Check 4 Answer
20 tables, 80 chairs, 80 plates

Teaching Example 4 CLASSROOM SUPPLIES An elementary teacher needs to purchase a small whiteboard, an eraser, and 3 dry-erase markers for each student in her class. If the total number of items on her order is 115, how many whiteboards, erasers, and dry-erase markers were ordered?
Answer:
23 whiteboards, 23 erasers, 69 markers

Analyze The value of the Kodak stock plus the value of the Coca-Cola stock plus the value of the IBM stock must equal $415,100. We need to find the number of shares of each of these stocks held by the foundation.

Form If we let $x =$ the number of shares of Kodak stock, then $x =$ the number of shares of Coca-Cola stock. Since the foundation owns five times as many shares of IBM stock as Kodak or Coca-Cola stock, $5x =$ the number of shares of IBM. The value of the shares of each stock is the *product* of the number of shares of that stock and its per-share value. See the table.

Stock	Number of shares ·	Value per share =	Total value of the stock
Kodak	x	26	$26x$
Coca-Cola	x	52	$52x$
IBM	$5x$	103	$103(5x)$
			Total: $415,100

Use the information in this column to form an equation.

We can now form the equation.

The value of Kodak stock	plus	the value of Coca-Cola stock	plus	the value of IBM stock	is	the total value of all of the stock.
$26x$	$+$	$52x$	$+$	$103(5x)$	$=$	415,100

Solve

$$26x + 52x + 515x = 415{,}100$$

$$593x = 415{,}100 \quad \text{Combine like terms on the left side.}$$

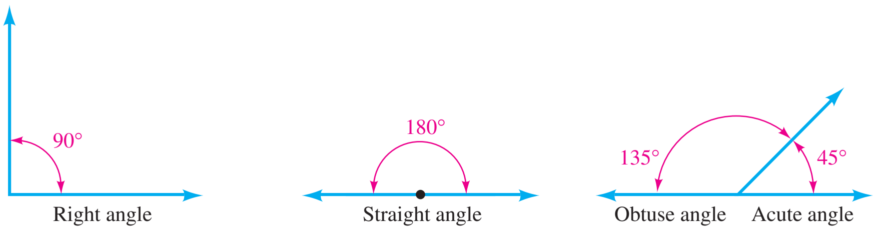
$$x = 700 \quad \text{To isolate } x \text{, divide both sides by 593.}$$

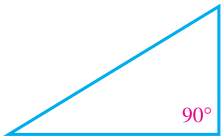
State The foundation owns 700 shares of Kodak, 700 shares of Coca-Cola, and $5(700) = 3{,}500$ shares of IBM.

Check The value of 700 shares of Kodak stock is 700($26) = $18,200. The value of 700 shares of Coca-Cola is 700($52) = $36,400. The value of 3,500 shares of IBM is 3,500($103) = $360,500. The sum is $18,200 + $36,400 + $360,500 = $415,100. The answers check.

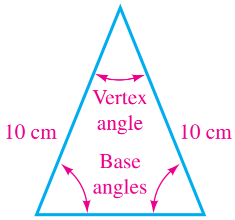
3 Solve geometry problems.

Sometimes we can use a geometric fact or formula to solve a problem. The following illustrations show two important types of geometric figures: angles and triangles. A **right angle** is an angle whose measure is 90°. A **straight angle** is an angle whose measure is 180°. An **acute angle** is an angle whose measure is greater than 0° and less than 90°. An angle whose measure is greater than 90° and less than 180° is called an **obtuse angle.**

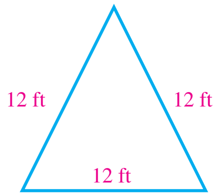




Right triangle



Isosceles triangle



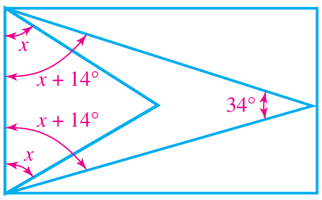
Equilateral triangle

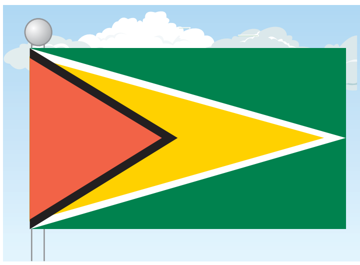
If the sum of two angles equals 90°, the angles are called **complementary,** and each angle is called the **complement** of the other. If the sum of two angles equals 180°, the angles are called **supplementary,** and each angle is the **supplement** of the other.

A **right triangle** is a triangle with one right angle. An **isosceles triangle** is a triangle with two sides of equal measure that meet to form the **vertex angle.** The angles opposite the equal sides, called the **base angles,** are also equal. An **equilateral triangle** is a triangle with three equal sides and three equal angles. It is important to note that *the sum of the angle measures of any triangle is 180°*.

EXAMPLE 5

Flags The flag of Guyana, a republic on the northern coast of South America, is one isosceles triangle superimposed over another on a field of green, as shown. The measure of a base angle of the larger triangle is 14° more than the measure of a base angle of the smaller triangle. The measure of the vertex angle of the larger triangle is 34°. Find the measure of each base angle of the smaller triangle.





Analyze We are working with isosceles triangles. Therefore, the base angles of the smaller triangle have the same measure, and the base angles of the larger triangle have the same measure.

Form If we let $x =$ the measure in degrees of one base angle of the smaller isosceles triangle, then the measure of its other base angle is also x. (See the figure.)

The measure of a base angle of the larger isosceles triangle is $x + 14°$, since its measure is 14° more than the measure of a base angle of the smaller triangle. We are given that the vertex angle of the larger triangle measures 34°.

The sum of the measures of the angles of any triangle (in this case, the larger triangle) is 180°.

We can now form the equation.

The measure of one base angle	plus	the measure of the other base angle	plus	the measure of the vertex angle	is	180°.
$x + 14$	$+$	$x + 14$	$+$	34	$=$	180

Self Check 5

ANGLE MEASURE The largest angle of a triangle is three times the smallest angle, and the midsized angle is 20° more than the smallest. Find the measure of the three angles. 32°, 52°, 96°

Now Try **Problem 33**

Teaching Example 5 ANGLE MEASURE The measure of ∠1 of a triangle is 25° more than that of ∠2. The measure of ∠3 is 5° more than three times the measure of ∠2. Find the measure of each angle.
Answer:
55°, 30°, 95°

Solve

$$x + 14 + x + 14 + 34 = 180$$

$$2x + 62 = 180 \quad \text{Combine like terms.}$$

$$2x = 118 \quad \text{Subtract 62 from both sides.}$$

$$x = 59 \quad \text{To isolate } x\text{, divide both sides by 2.}$$

State The measure of each base angle of the smaller triangle is 59°.

Check If $x = 59$, then $x + 14 = 73$. The sum of the measures of each base angle and the vertex angle of the larger triangle is $73° + 73° + 34° = 180°$. The answers checks.

4 Use formulas to solve problems.

When preparing to write an equation to solve a problem, the given facts of the problem often suggest a formula that we can use to model the situation mathematically.

Self Check 6

CRIME SCENE Police used 400 feet of yellow tape to fence off a rectangular-shaped lot for an investigation. If the width is 50 feet less than the length, find the dimensions of the lot.

75 ft by 125 ft

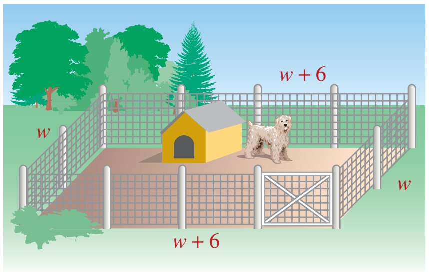
Now Try **Problem 39**

Teaching Example 6 REAL ESTATE
The perimeter of a rectangular lot is 460 ft. If the length is 20 ft longer than the width, find the dimensions of the lot.
Answer:
105 ft by 125 ft

EXAMPLE 6 ***Kennels*** A man has a 50-foot roll of fencing to make a rectangular kennel. If he wants the kennel to be 6 feet longer than it is wide, find its dimensions.

Analyze The perimeter P of the rectangular kennel is 50 feet. Recall that the formula for the perimeter of a rectangle is $P = 2l + 2w$. We need to find its length and width.

Form We let $w =$ the width in feet of the kennel shown below. Then the length, which is 6 feet more than the width, is represented by the expression $w + 6$.



We can now form the equation by substituting 50 for P and $w + 6$ for the length in the formula for the perimeter of a rectangle.

$$P = 2l + 2w$$

$$50 = 2(w + 6) + 2w$$

Solve

$$50 = 2(w + 6) + 2w$$

$$50 = 2w + 12 + 2w \quad \text{Distribute the multiplication by 2.}$$

$$50 = 4w + 12 \quad \text{Combine like terms.}$$

$$38 = 4w \quad \text{Subtract 12 from both sides.}$$

$$9.5 = w \quad \text{To isolate } w\text{, divide both sides by 4.}$$

State The width of the kennel is 9.5 feet. The length is 6 feet more than this, or 15.5 feet.

Check If a rectangle has a width of 9.5 feet and a length of 15.5 feet, its length is 6 feet more than its width, and the perimeter is 2(9.5) feet + 2(15.5) feet = 50 feet. The answers check.

ANSWERS TO SELF CHECKS

1. \$150 **2.** 3.5 mi, 2.5 mi, 11 mi **3.** 30 people **4.** 20 tables, 80 chairs, 80 plates
5. 32°, 52°, 96° **6.** 75 ft by 125 ft

SECTION 1.7 STUDY SET

VOCABULARY

Fill in the blanks.

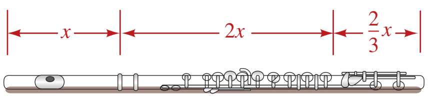
1. An acute angle has a measure of more than 0° and less than 90°.
2. A right angle is an angle whose measure is 90°.
3. If the sum of the measures of two angles equals 90°, the angles are called complementary angles.
4. If the sum of the measures of two angles equals 180°, the angles are called supplementary angles.
5. If a triangle has a right angle, it is called a right triangle.
6. If a triangle has two sides with equal measures, it is called an isosceles triangle.
7. The sum of the measures of the angles of a triangle is 180°.
8. An equilateral triangle has three sides of equal length and three angles of equal measure.

CONCEPTS

9. The unit used to measure the intensity of sound is called the *decibel*. In the table, translate the comments in the right column into mathematical symbols to complete the decibels column.

Activity	Decibels	Compared to conversation
Conversation	d	—
Vacuum cleaner	$d + 15$	15 decibels more
Circular saw	$2d - 10$	10 decibels less than twice
Jet takeoff	$2d + 20$	20 decibels more than twice
Whispering	$\frac{d}{2} - 10$	10 decibels less than half
Rock band	$2d$	Twice the decibel level

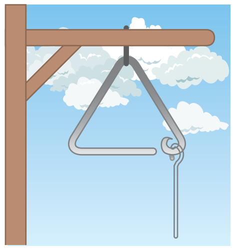
10. INSTRUMENTS The flute consists of three pieces. Write an algebraic expression that represents
    a. the length of the shortest piece. $\frac{2}{3}x$
    b. the length of the longest piece. $2x$
    c. the length of the flute. $x + 2x + \frac{2}{3}x$



11. The following table shows the four types of problems an instructor put on a history test.
    a. Complete the table.
    b. Write an algebraic expression that represents the total number of points on the test.
    $5x + 6x + 10(x - 2) + 5x$
    c. Write an equation that could be used to find x.
    $5x + 6x + 10(x - 2) + 5x = 110$

Type of question	Number ·	Value =	Total value
Multiple choice	x	5	$5x$
True/false	$3x$	2	$6x$
Essay	$x - 2$	10	$10(x - 2)$
Fill-in	x	5	$5x$
			Total: 110 points

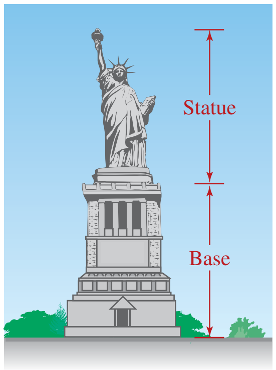
12. For each picture shown, what geometric concept studied in this section is illustrated?
    an equilateral triangle and a right angle



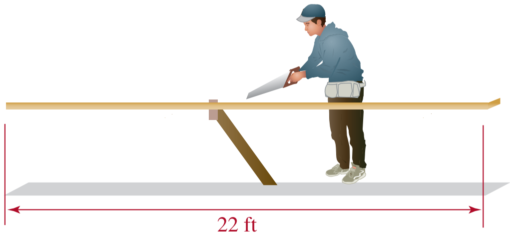
APPLICATIONS

13. CEREAL SALES In 2005, two of the top-selling cereals in the United States were General Mills' Cheerios and Kellogg's Frosted Flakes, with combined sales of \$939 million. Frosted Flakes sales were \$439 million less than sales of Cheerios. What were the 2005 sales for each brand?
    Cheerios: \$689 million, Frosted Flakes: \$250 million
14. FILMS Denzel Washington's three top domestic grossing films, *Remember the Titans, The Pelican Brief,* and *Crimson Tide,* have earned \$307.8 million. If *Remember the Titans* earned \$14.8 million more than *The Pelican Brief,* and if *The Pelican Brief* earned \$9.4 million more than *Crimson Tide,* how much did each film earn as of that date?
    Titans: \$115.6 million, *Pelican:* \$100.8 million, *Tide:* \$91.4 million

▶ **15.** STATUE OF LIBERTY From the foundation of the large pedestal on which it sits to the top of the torch, the Statue of Liberty National Monument measures 305 feet. The pedestal is 3 feet taller than the statue. Find the height of the pedestal and the height of the statue.
pedestal: 154 ft, statue: 151 ft



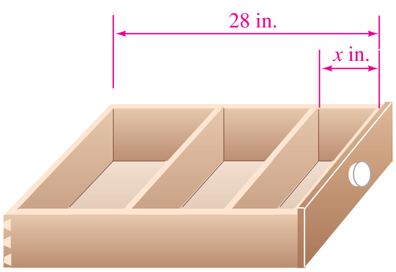
16. WOODWORKING The carpenter saws a board that is 22 feet long into two pieces. One piece is to be 1 foot longer than twice the length of the shorter piece. Find the length of each piece. 7 ft, 15 ft



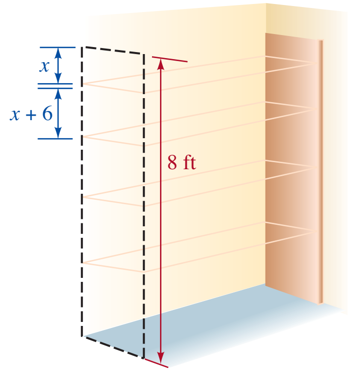
17. TV COMMERCIALS For the typical "one-hour" prime-time television slot, the number of minutes of commercials is $\frac{3}{7}$ of the number of minutes of the actual program. Determine how many minutes of the program are shown in that one hour.
42 min

18. CONCERT TOURING A rock group plans to travel for a total of 23 weeks, visiting three countries. They will be in Germany for 3 weeks longer than they will be in France. Their stay in Great Britain will be 1 week less than that in France. How many weeks will they be in each country?
Germany: 10 wk, Great Britain: 6 wk, France: 7 wk

19. MAKING FURNITURE A woodworker wants to put two partitions crosswise in a drawer that is 28 inches deep, as shown in the illustration. He wants to place the partitions so that the spaces created increase by 3 inches from front to back. If the thickness of each partition is $\frac{1}{2}$ inch, how far from the front end should he place the first partition? 6 in.



▶ **20.** BUILDING SHELVES A carpenter wants to put four shelves on an 8-foot wall so that the five spaces created decrease by 6 inches as we move up the wall. If the thickness of each shelf is $\frac{3}{4}$ inch, how far will the bottom shelf be from the floor? See the illustration. $30\frac{3}{5}$ in.



21. SPRING TOURS A group of junior high students will be touring Washington, D.C. Their chaperons will have the $1,810 cost of the tour reduced by $15.50 for each student they personally supervise. How many students will a chaperon have to supervise so that his or her cost to take the tour will be $1,500?
20

▶ **22.** MACHINING Each pass through a lumber plane shaves off 0.015 inch of thickness from a board. How many times must a board, originally 0.875 inch thick, be run through the planer if a board of thickness 0.74 inch is desired?
9

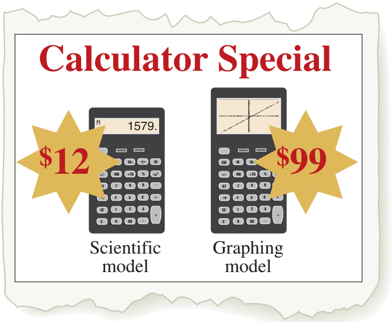
23. MOVING EXPENSES To help move his furniture, a man rents a truck for $41.50 per day plus 35¢ per mile. If he has budgeted $150 for transportation expenses, how many miles will he be able to drive the truck if the move takes 1 day?
310 mi

▶ **24.** COMPUTING SALARIES A student working for a delivery company earns $57.50 per day plus $4.75 for each package she delivers. How many deliveries must she make each day to earn $200 a day?
30

▶ **25.** ASSETS OF A PENSION FUND A pension fund owns 2,000 fewer shares in mutual stock funds than mutual bond funds. Currently, the stock funds sell for $12 per share, and the bond funds sell for $15 per share. How many shares of each does the pension fund own if the value of the securities is $165,000?
5,000 shares of stock funds, 7,000 shares of bond funds

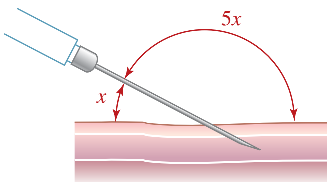
26. VALUE OF AN IRA In an Individual Retirement Account (IRA) valued at \$53,900, a couple has 500 shares of stock, some in Big Bank Corporation and some in Safe Savings and Loan. If Big Bank sells for \$115 per share and Safe Savings sells for \$97 per share, how many shares of each does the couple own? 300 shares of BB, 200 shares of SS

27. SELLING CALCULATORS Last month, a bookstore ran the following ad. Sales of \$5,370 were generated, with 15 more graphing calculators sold than scientific calculators. How many of each type of calculator did the bookstore sell? 35 \$12 calculators, 50 \$99 calculators

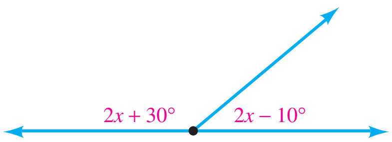


28. SELLING SEED A seed company sells two grades of grass seed. A 100-pound bag of a mixture of rye and Kentucky bluegrass sells for \$245, and a 100-pound bag of bluegrass sells for \$347. How many bags of each are sold in a week when the receipts for 19 bags are \$5,369? 12 bags of mixture, 7 bags of bluegrass

29. NURSING The illustration shows the angle a needle should make with the skin when administering a certain type of injection. Find the measure of both angles labeled. $30°$, $150°$



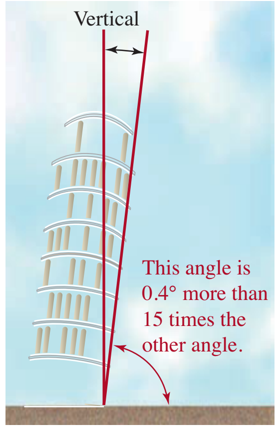
30. SUPPLEMENTARY ANGLES Refer to the illustration and find x. $40°$



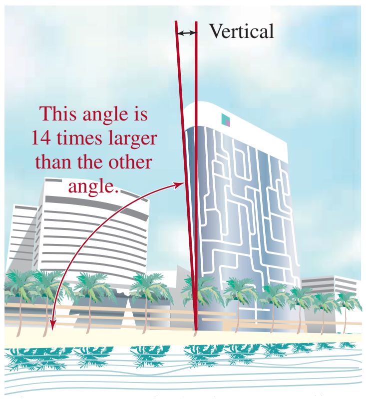
31. THE LEANING TOWER OF PISA Refer to the illustration in the next column. Because of soft soil and a shallow foundation, the Leaning Tower of Pisa in Italy is not vertical. Engineers predict that if the indicated angle gets to be 7°, the walls will not be able to support the structure and it will come crumbling down.

a. How many degrees from vertical is the tower now? $5.6°$

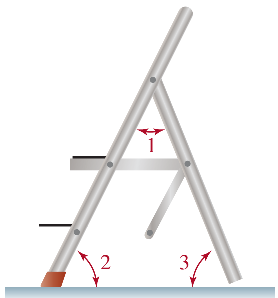
b. How many more degrees of lean must occur to cause the predicted collapse? $1.4°$



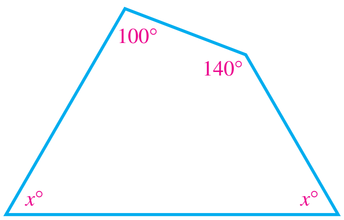
32. THE iPAD BUILDING The sleek design of Apple's iPod has inspired a 23-floor residential and office building in Dubai, called the iPad. It closely resembles a gigantic iPod sitting at an angle in its charger as shown below. How many degrees from the vertical is the building? $6°$



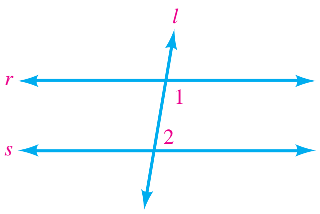
33. STEPSTOOLS The sum of the measures of the three angles of any triangle is 180°. In the illustration, the measure of $\angle 2$ (angle 2) is 10° larger than the measure of $\angle 1$. The measure of $\angle 3$ is 10° larger than the measure of $\angle 2$. Find each angle measure. $\angle 1$: $50°$, $\angle 2$: $60°$, $\angle 3$: $70°$



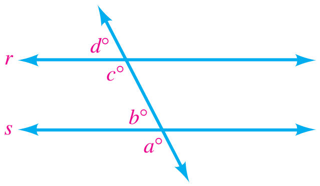
34. ANGLES OF A QUADRILATERAL The sum of the angles of any four-sided figure (called a *quadrilateral*) is 360°. The quadrilateral shown has two equal base angles. Find x. $60°$



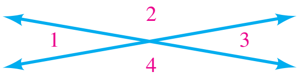
35. GEOMETRY In the illustration, lines r and s are cut by a third line l to form $\angle 1$ (angle 1) and $\angle 2$. When lines r and s are parallel, $\angle 1$ and $\angle 2$ are supplementary. If $\angle 1 = x + 50°$, $\angle 2 = 2x - 20°$, and lines r and s are parallel, find x. 50°



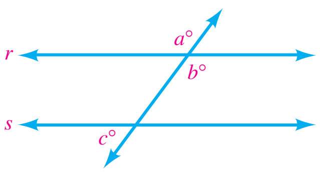
▶ **36.** GEOMETRY In the illustration, $r \parallel s$ (read as "line r is parallel to line s"), and $a = 103$. Find b, c, and d. (*Hint:* See Problem 35.)
$b = 77, c = 103, d = 77$



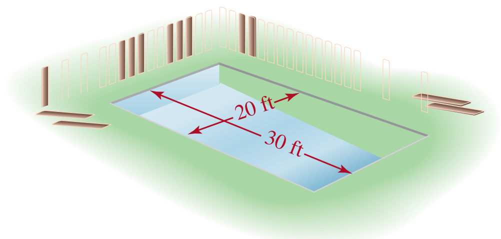
37. VERTICAL ANGLES When two lines intersect, four angles are formed. Angles that are side-by-side, such as $\angle 1$ (angle 1) and $\angle 2$, are called **adjacent angles.** Angles that are nonadjacent, such as $\angle 1$ and $\angle 3$ or $\angle 2$ and $\angle 4$, are called **vertical angles.** From geometry, we know that if two lines intersect, vertical angles have the same measure. If $\angle 1 = 3x + 10°$ and $\angle 3 = 5x - 10°$, find x. 10°



▶ **38.** GEOMETRY In the illustration, $r \parallel s$ (read as "line r is parallel to line s"), and $b = 137$. Find a and c. (*Hint:* See Problems 35 and 37.)
$a = 137, c = 43$

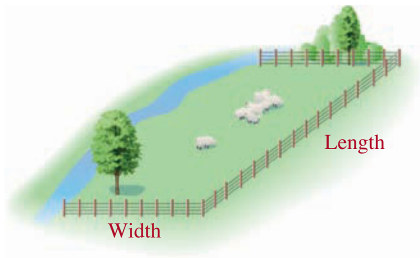


39. SWIMMING POOLS A woman wants to enclose the pool shown and have a walkway of uniform width all the way around. How wide will the walkway be if the woman uses 180 feet of fencing? 10 ft

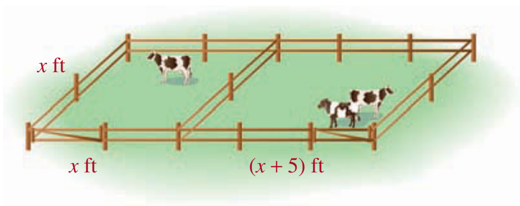


▶ **40.** QUILTING Throughout history, most artists and designers have felt that *golden rectangles* with a length 1.618 times as long as their width have the most visually attractive shape. A woman is planning to make a quilt in the shape of a golden rectangle. She has exactly 22 feet of a special lace that she plans to sew around the edge of the quilt. What should the length and width of the quilt be? Round both answers up to the nearest hundredth. 4.20 ft by 6.80 ft

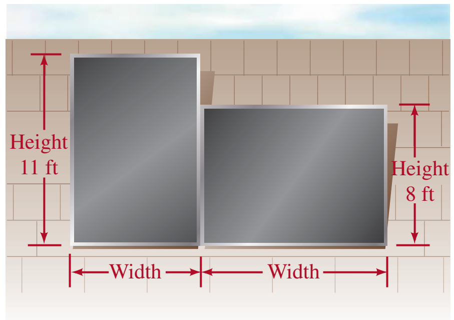
41. RANCHING A farmer has 624 feet of fencing to enclose a pasture. Because a river runs along one side, fencing will be needed on only three sides. Find the dimensions of the pasture if its length is double its width. 156 ft by 312 ft



▶ **42.** FENCING A man has 150 feet of fencing to build the two-part pen shown in the illustration. If one part is a square and the other a rectangle, find the outside dimensions of the pen. 20 ft by 45 ft



▶ **43.** SOLAR HEATING One solar panel in the illustration is to be 3 feet wider than the other. To be equally efficient, they must have the same area. Find the width of each. 8 ft, 11 ft



▶ **44.** HEIGHT OF A TRIANGLE If the height of a triangle with a base of 8 inches is tripled, its area is increased by 96 square inches. Find the height of the triangle. 12 in.

WRITING

45. Briefly explain what should be accomplished in each of the steps (*analyze, form, solve, state,* and *check*) of the problem-solving strategy used in this section.

46. Write a problem that can be represented by the following verbal model.

The measure of 1st angle	plus	the measure of 2nd angle	plus	the measure of 3rd angle	is	180°.
x	$+$	$2x$	$+$	$x + 10$	$=$	180

REVIEW

47. When expressed as a decimal, is $\frac{7}{9}$ a terminating or repeating decimal? repeating

48. Solve: $x + 20 = 4x - 1 + 2x$ $\frac{21}{5}$

49. Write the set of integers. $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

50. Solve: $2x + 2 = \frac{2}{3}x - 2$ -3

51. Evaluate $2x^2 + 5x - 3$ for $x = -3$. 0

52. Solve $T - R = ma$ for R. $R = T - ma$

SECTION 1.8 More about Problem Solving

Objectives

1. Solve percent problems.
2. Find the mean, median, and mode.
3. Solve investment problems.
4. Solve uniform motion problems.
5. Solve mixture problems.

In this section, we will again use equations as we solve a variety of problems.

1 Solve percent problems.

Percents are often used to present numeric information. **Percent** means parts per one hundred. One method to solve percent problems is to use the given facts to write a **percent sentence** of the form:

___	is	___	% of	___	?

We enter the appropriate numbers in two of the blanks and the word "what" in the remaining blank. Then we translate the sentence to mathematical symbols and solve the resulting equation.

The Language of Algebra The names of the parts of a percent sentence are:

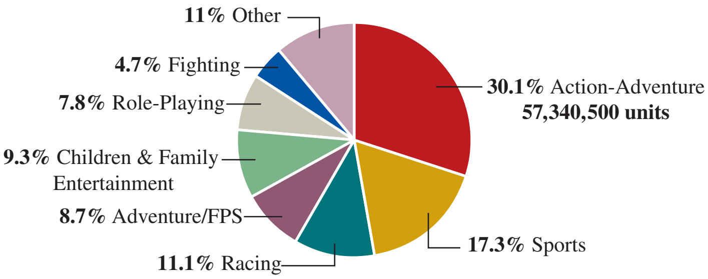
5	is	50%	of	10.
amount		percent		base

They are related by the formula:

Amount = percent · base

EXAMPLE 1 ***Video Games*** Refer to the following graph. Determine the number of video games sold in the United States in 2005.

BEST-SELLING VIDEO GAMES IN THE UNITED STATES, 2005



Source: The NPD Group/Point of Sale Information

Self Check 1

VIDEO GAMES Refer to the graph from Example 1. Determine the number of sports video games sold in the United States in 2005. 32,956,500

Now Try **Problem 21**

Teaching Example 1 VIDEO GAMES Refer to the graph from Example 1. Determine the number of children and family entertainment video games in the United States in 2005.
Answer:
17,716,500

Analyze In the **circle graph** on the previous page, we see that the 57,340,500 action-adventure video games that were sold in the United States in 2005 were 30.1% of the total number of video games sold.

Form Let x = the total number of video games sold in the United States in 2005. First, we write a percent sentence using the given data. Then we translate to form an equation.

57,340,500	is	30.1%	of	what?
↓	↓	↓	↓	↓
57,340,500	=	30.1%	·	x

The amount is 57,340,500, the percent is 30.1%, and the base is x.

Solve To find the total number of video games sold in the United States in 2005, we solve for x.

$$57{,}340{,}500 = 0.301x \qquad \text{Write 30.1\% as a decimal: 30.1\% = 0.301.}$$

$$\frac{57{,}340{,}500}{\mathbf{0.301}} = \frac{0.301x}{\mathbf{0.301}} \qquad \text{To isolate } x \text{, divide both sides by 0.301.}$$

$$190{,}500{,}000 = x \qquad \text{Do the division using a calculator.}$$

State There were 190,500,000 video games sold in the United States in 2005.

Check If a total of 190,500,000 video games were sold, then the 57,340,500 action-adventure video games sold were $\frac{57{,}340{,}500}{190{,}500{,}000} = 0.301$ or 30.1% of the units sold. The answer checks.

When the regular price of merchandise is reduced, the amount of reduction is called **markdown** (or discount).

Sale price = regular price − markdown

Usually, the markdown is expressed as a percent of the regular price.

Markdown = percent of markdown · regular price

Self Check 2

SEASONAL MARKDOWNS At an after-season sale, a winter coat that normally sells for \$279 is on sale for \$112. Find the percent of markdown. ≈ 60%

Now Try **Problem 25**

Teaching Example 2 GRAND OPENINGS During a grand-opening sale, a jacket that normally sells for \$139 is on sale for \$97. Find the percent of markdown.
Answer:
≈ 30%

EXAMPLE 2 ***Wedding Gowns*** At a bridal shop, a wedding gown that normally sells for \$397.98 is on sale for \$265.32. Find the percent of markdown.

Analyze In this case, \$265.32 is the sale price, \$397.98 is the regular price, and the markdown is the *product* of \$397.98 and the percent of markdown.

Form We let r = the percent of markdown, expressed as a decimal. We then substitute \$265.32 for the sale price and \$397.98 for the regular price in the formula.

Sale price	is	regular price	minus	markdown.
265.32	=	397.98	−	$r \cdot 397.98$

Markdown = percent of markdown · regular price.

Solve

$$265.32 = 397.98 - r \cdot 397.98$$

$$265.32 = 397.98 - 397.98r \qquad \text{Rewrite } r \cdot 397.98 \text{ as } 397.98r.$$

$$-132.66 = -397.98r \qquad \text{Subtract 397.98 from both sides.}$$

$$\frac{-132.66}{-397.98} = r \qquad \text{To isolate } r \text{, divide both sides by } -397.98.$$

$0.333333\ldots = r$ Do the division using a calculator.

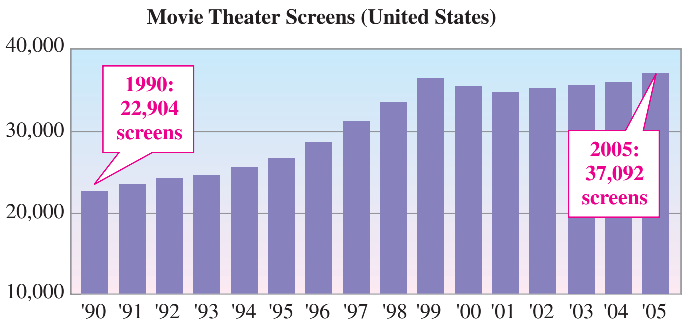
$33.3333\ldots\% = r$ Write the decimal as a percent. Multiply $0.333333\ldots$ by 100 by moving the decimal point two places to the right and insert a % sign.

State The percent of markdown on the wedding gown is $33.3333\ldots\%$ or $33\frac{1}{3}\%$.

Check The markdown is $33\frac{1}{3}\%$ of \$397.98, or \$132.66. The sale price is \$397.98 − \$132.66, or \$265.32. The answer checks.

Percents are often used to describe how a quantity has changed. To describe such changes, we use **percent of increase** or **percent of decrease.**

EXAMPLE 3 ***Entertainment*** Use the following data to determine the percent of increase in the number of indoor movie theater screens in the United States from 1990 to 2005. Round to the nearest one percent.



Source: National Association of Theater Owners

Analyze To find the percent of increase, we first find the *amount of increase* by subtracting the number of screens in 1990 from the number in 2005.

$37{,}092 - 22{,}904 = 14{,}188$

Form Next, we find what percent of the original 22,904 screens the 14,188 increase represents. We let $x =$ the unknown percent and translate the words into an equation.

14,188	is	what percent	of	22,904?
14,188	=	x	·	22,904

The amount is 14,188, the percent is x, and the base is 22,904.

Solve

$$14{,}188 = x \cdot 22{,}904$$

$$14{,}188 = 22{,}904x$$

$$\frac{14{,}188}{\mathbf{22{,}904}} = \frac{22{,}904x}{\mathbf{22{,}904}}$$ To isolate x, divide both sides by 22,904.

$$0.619455117\ldots \approx x$$ Do the division using a calculator.

$$61.9455117\ldots\% \approx x$$ Write the decimal as a percent.

$$62\% \approx x$$ Round to the nearest one percent.

Self Check 3

IDENTITY THEFT In 2001, there were 86,000 complaints of identity theft. In 2005, there were 256,000 complaints. Find the percent of increase in the number of complaints from 2001 to 2005. $\approx 197.7\%$

Now Try **Problem 31**

Teaching Example 3 IDENTITY THEFT In 2004, there were 247,000 complaints of identity theft, and in 2005 there were 256,000 complaints. Find the percent of increase from 2004 to 2005.
Answer:
$\approx 4\%$

State There was a 62% increase in the number of movie screens in the United States from 1990 to 2005.

Check A 50% increase from 22,904 screens would be approximately 11,000 additional screens. It seems reasonable that 14,188 more screens would be a 62% increase.

Caution! Always find the percent of increase (or decrease) with respect to the *original* amount.

THINK IT THROUGH *Fastest Growing Occupations*

"With the Baby Boomer generation aging, more medical professionals will be required to manage the health needs of the elderly."

Jonathan Stanewick, Compensation Analyst, Salary.com, 2004

The table below shows predictions by the U.S. Department of Labor, Bureau of Labor Statistics, of the fastest growing occupations for the years 2002–2012. Which occupation has the greatest percent of increase? medical assistant, about 59%

Occupation	Number of jobs in 2002	Predicted number of jobs in 2012	Education or training required
Home health aide	580,000	859,000	On-the-job training/ AA degree
Medical assistant	365,000	579,000	On-the-job training/ AA degree
Physician assistant	63,000	94,000	Bachelor's degree/ Masters degree
Social service assistant	305,000	454,000	On-the-job training/ AA degree
Systems, data analyst	186,000	292,000	Bachelor's degree

2 Find the mean, median, and mode.

Statistics is a branch of mathematics that deals with analysis of numerical data. Three types of averages are commonly used in statistics as measures of central tendency of a collection of data: the **mean,** the **median,** and the **mode.**

Mean, Median, and Mode

The **mean** $\overline{x}$ of a collection of values is the sum S of those values divided by the number of values n.

$$\overline{x} = \frac{S}{n}$$ Read $\overline{x}$ as "x bar."

The **median** of a collection of values is the middle value. To find the median,

1. Arrange the values in increasing order.
2. If there are an odd number of values, choose the middle value.
3. If there are an even number of values, add the middle two values and divide by 2.

The **mode** of a collection of values is the value that occurs most often.

EXAMPLE 4 *Reaction Time*

As a project for a science class, a student measured ten people's reaction times. The times, in seconds, are listed below. Using the collection of data, find: **a.** the mean **b.** the median **c.** the mode

0.29, 0.22, 0.19, 0.36, 0.28, 0.23,
0.16, 0.28, 0.33, 0.26

© Steve Morse photo, MU Cooperative Media Group

Strategy We will follow the steps in the definition box for the mean, medium, and mode.

WHY For each of the parts, a specific set of steps must be followed.

Solution

a. To find the mean, we add the values and divide by the number of values, which is 10.

$$\bar{x} = \frac{0.29 + 0.22 + 0.19 + 0.36 + 0.28 + 0.23 + 0.16 + 0.28 + 0.33 + 0.26}{10}$$

$$= 0.26 \text{ second}$$

b. To find the median, we first arrange the values in increasing order:

0.16, 0.19, 0.22, 0.23, 0.26, 0.28, 0.28, 0.29, 0.33, 0.36

Least Greatest

Because there is an even number of measurements, the median will be the sum of the middle two values, 0.26 and 0.28, divided by 2.

$$\text{Median} = \frac{0.26 + 0.28}{2} = 0.27 \text{ second}$$

c. Since the time 0.28 second occurs most often, it is the mode.

0.16, 0.19, 0.22, 0.23, 0.26, 0.28, 0.28, 0.29, 0.33, 0.36

The Language of Algebra In statistics, the mean, median, and mode are classified as types of *averages*. In daily life, when the word *average* is used, it most often is referring to the mean.

Self Check 4

Using the collection of data, find:
a. the mean ≈ 4.61
b. the median 4.265
c. the mode 3.99

3.25 4.17 6.52 5.19
4.36 3.99 5.44 3.99

Now Try **Problem 33**

Teaching Example 4 Using the collection of data, find:
a. the mean
b. the median
c. the mode
1.27 3.55 4.16 5.44 6.15
5.36 4.89 3.14 5.19 1.27
Answers:
a. 4.042 **b.** 4.525 **c.** 1.27

EXAMPLE 5 *Bank Service Charges*

When the average (mean) daily balance of a customer's checking account falls below $500 in any week, the bank assesses a $15 service charge. What minimum balance will the account shown need to have on Friday to avoid the service charge?

Security Savings
☐ Weekly Statement ☐
Acct: 201-234-002 Type: checking

Day	Date	Daily balance	Comments
Mon	3/11	$730.70	
Tue	3/12	$350.19	
Wed	3/13	–$50.19	overdrawn
Thu	3/14	$275.55	
Fri	3/15		

Self Check 5

GRADES To earn a grade of at least a B, a student must have an average of 80 or more. If he earned grades of 78, 75, 84, 79 on the first 4 exams, what is the lowest score on the fifth exam required to earn a B for the course? 84

Now Try **Problem 35**

Teaching Example 5 GRADES To earn an A in the course, a student must have an average of 90 or more for the 5 exams. If she earned grades of 92, 81, 85, 93 on the first 4 exams, what is the lowest score on the fifth exam is required to earn a grade of A for the course?
Answer:
99

Analyze We can find the average (mean) daily balance for the week by adding the daily balances and dividing by 5. We want the mean to be $500 so that there is no service charge.

Form We will let $x =$ the minimum balance needed on Friday. Then we translate the words into mathematical symbols.

The sum of the five daily balances	divided by	5	is	$500.

$$\frac{730.70 + 350.19 + (-50.19) + 275.55 + x}{5} = 500$$

Solve

$$\frac{730.70 + 350.19 + (-50.19) + 275.55 + x}{5} = 500$$

$$\frac{1{,}306.25 + x}{5} = 500$$ Combine like terms in the numerator.

$$5\left(\frac{1{,}306.25 + x}{5}\right) = 5(500)$$ To clear the equation of the fraction, multiply both sides by 5.

$$1{,}306.25 + x = 2{,}500$$

$$x = 1{,}193.75$$ To isolate x, subtract 1,306.25 from both sides.

State On Friday, the account balance needs to be $1,193.75 to avoid a service charge.

Check Check the result by adding the five daily balances and dividing by 5.

3 Solve investment problems.

The money an investment earns is called *interest.* **Simple interest** is computed by the formula $I = Prt$, where I is the interest earned, P is the principal (amount invested), r is the annual interest rate, and t is the length of time the principal is invested.

Self Check 6

INTEREST INCOME A college student invested a total of $12,000, some at 6% and the rest at 9%. If the total interest earned in 1 year is $945, find the amount invested at each rate. $4,500 at 6%, $7,500 at 9%

Now Try **Problem 37**

Teaching Example 6 INTEREST INCOME A man invested a total of $6,000, some at 4% and the rest at 6%. If the total interest earned in 1 year is $280, how much did he invest in each account?
Answer:
$4,000 at 4%, $2,000 at 6%

EXAMPLE 6 ***Interest Income*** To protect against a major loss, a financial analyst suggested the following plan for a client who has $50,000 to invest for 1 year.

1. Alco Development, Inc. Builds mini-malls. High yield: 12% per year. Risky!
2. Certificate of deposit (CD). Insured, safe. Low yield: 4.5% annual interest.

If the client puts some money in each investment and wants to earn $3,600 in interest, how much should be invested at each rate?

Analyze In this case, we are working with two investments made at two different rates for 1 year. If we add the interest from the two investments, the sum should equal $3,600.

Form If we let $x =$ the number of dollars invested at 12%, the interest earned is $I = Prt = \$x(12\%)(1) = \$0.12x$. If $\$x$ is invested at 12%, there is $\$(50{,}000 - x)$ to invest at 4.5%, which will earn $\$0.045(50{,}000 - x)$ in interest. These facts are listed in the table.

	P	$\cdot\ r$	$\cdot\ t =$	I
Alco Development, Inc.	x	0.12	1	$0.12x$
Certificate of deposit	$50{,}000 - x$	0.045	1	$0.045(50{,}000 - x)$
				Total: $3,600

Enter this information first.

Use the information in this column to form an equation.

The sum of the two amounts of interest should equal $3,600. We now translate the words into an equation.

The interest earned at 12%	plus	the interest earned at 4.5%	equals	the total interest earned.
$0.12x$	$+$	$0.045(50{,}000 - x)$	$=$	$3{,}600$

Solve

$$0.12x + 0.045(50{,}000 - x) = 3{,}600$$

$$1{,}000[0.12x + 0.045(50{,}000 - x)] = 1{,}000(3{,}600)$$ To eliminate the decimals, multiply both sides by 1,000.

$$120x + 45(50{,}000 - x) = 3{,}600{,}000$$ Distribute the 1,000 and simplify both sides.

$$120x + 2{,}250{,}000 - 45x = 3{,}600{,}000$$ Remove parentheses.

$$75x + 2{,}250{,}000 = 3{,}600{,}000$$ Combine like terms.

$$75x = 1{,}350{,}000$$ Subtract 2,250,000 from both sides.

$$x = 18{,}000$$ To isolate x, divide both sides by 75.

State $18,000 should be invested at 12% and $(50,000 − 18,000) = $32,000 should be invested at 4.5%.

Check The annual interest on $18,000 is 0.12($18,000) = $2,160. The interest earned on $32,000 is 0.045($32,000) = $1,440. The total interest is $2,160 + $1,440 = $3,600. The answers check.

4 Solve uniform motion problems.

Problems that involve an object traveling at a constant rate for a specified period of time over a certain distance are called **uniform motion** problems. To solve these problems, we use the formula $d = rt$, where d is distance, r is rate, and t is time.

EXAMPLE 7 *Travel Time* After a stay on her grandparents' farm, a girl is to return home, 385 miles away. To split up the drive, the parents and grandparents start at the same time and drive toward each other, planning to meet somewhere along the way. If the parents travel at an average rate of 60 mph and the grandparents at 50 mph, how long will it take them to meet?

Analyze The vehicles are traveling toward each other as shown in the following figure. We know the rates the cars are traveling (60 mph and 50 mph). We also know that they will travel for the same amount of time.

Form We can let t = the time in hours that each vehicle travels. Then the distance traveled by the parents is $60t$ miles, and the distance traveled by the grandparents

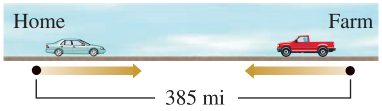
Self Check 7

TRAVEL TIME Two cars leave at the same time from two towns that are 345 miles apart. The first car is traveling at a rate of 65 mph, and the other car is traveling at 50 mph. If the cars are traveling toward each other, how long will it take the two cars to meet? 3 hr

Now Try **Problem 43**

Teaching Example 7 TRAVEL TIME
Two cars leave at the same time from two towns that are 420 miles apart. The first car is traveling at 50 mph, and the other car is traveling at 55 mph. If the cars are traveling toward each other, how long will it take for them to meet?
Answer:
4 hr

is $50t$ miles. The total distance traveled by the parents and grandparents is 385 miles. This information is organized in the table.



	r	$\cdot\ t$ =	d
Parents	60	t	$60t$
Grandparents	50	t	$50t$
			Total: 385 mi.

Enter this information first.

Use the information in this column to form an equation.

We now translate the words of the problem into an equation.

The distance the parents travel	plus	the distance the grandparents travel	equals	the distance between the child's home and the grandparent's farm.
$60t$	+	$50t$	=	385

Solve

$$60t + 50t = 385$$
$$110t = 385 \quad \text{Combine like terms.}$$
$$t = 3.5 \quad \text{To isolate } t\text{, divide both sides by 110.}$$

State The parents and grandparents will meet in $3\frac{1}{2}$ hours.

Check The parents travel $3.5(60) = 210$ miles. The grandparents travel $3.5(50) = 175$ miles. The total distance traveled is $210 + 175 = 385$ miles. The answer checks.

> ***Caution!*** When using $d = rt$, make sure the units are consistent. For example, if the rate is given in miles per hour, the time must be expressed in hours.

5 Solve mixture problems.

We now discuss two types of mixture problems. In the first example, a *dry mixture* of a specified value is created from two differently priced components. The value of each of its ingredients and the value of the dry mixture is given by

Amount · price = total value

Self Check 8

COFFEE BLENDS A store sells regular coffee for \$8 a pound and gourmet coffee for \$14 a pound. To get rid of 40 pounds of the gourmet coffee, a shopkeeper makes a blend to put on sale at \$10 a pound. How many pounds of the regular coffee should he use? 80 lb

Now Try **Problem 51**

EXAMPLE 8 ***Mixing Nuts*** The owner of a produce store notices that 20 pounds of gourmet cashews did not sell because of their high price of \$12 per pound. The owner decides to mix peanuts with the cashews to lower the price per pound. If peanuts sell for \$3 per pound, how many pounds of peanuts must be mixed with the cashews to make a mixture that could be sold for \$6 per pound?

Analyze We need to determine how many pounds of peanuts to mix with 20 pounds of cashews to obtain a mixture that could be sold for \$6 per pound.

Form We can let x = the number of pounds of peanuts to be used. Then $20 + x$ = the number of pounds in the mixture. We enter the known information in the following table.

	Amount	· Price	= Total value
Cashews	20	12	240
Peanuts	x	3	$3x$
Mixture	$20 + x$	6	$6(20 + x)$

Enter this information first.

Use the information in this column to form an equation.

We can now form the equation.

The value of the cashews	plus	the value of the peanuts	equals	the value of the mixture.
240	+	$3x$	=	$6(20 + x)$

Solve

$240 + 3x = 6(20 + x)$

$240 + 3x = 120 + 6x$ Use the distributive property to remove parentheses.

$120 = 3x$ Subtract $3x$ and 120 from both sides.

$40 = x$ To isolate x, divide both sides by 3.

State The owner should mix 40 pounds of peanuts with the 20 pounds of cashews.

Check The cashews are valued at $\$12(20) = \$240$, and the peanuts are valued at $\$3(40) = \$120$. The mixture is valued at $\$6(60) = \$360$. Since the value of the cashews plus the value of the peanuts equals the value of the mixture, the answer checks.

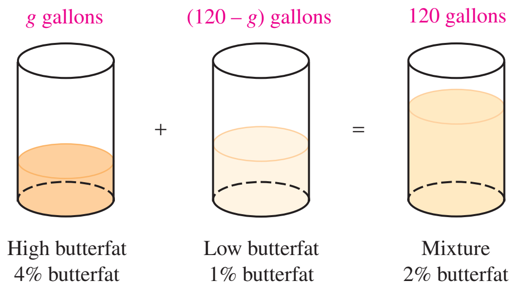
Caution! Check the result using the original wording of the problem, not by substituting it into the equation. Why? The equation may have been solved correctly, but the danger is that you may have formed it incorrectly.

In the next example, a *liquid mixture* of a desired strength is to be made from two solutions with different strengths (concentrations).

EXAMPLE 9 ***Milk Production*** Owners of a dairy find that milk with a 2% butterfat content is their best seller. Suppose the dairy has large quantities of whole milk having a 4% butterfat content and milk having a 1% butterfat content. How much of each type of milk should be mixed to obtain 120 gallons of milk that is 2% butterfat?

Analyze We are to find the amount of 4% milk to mix with 1% milk to get 120 gallons of a milk that has a 2% butterfat content. In the figure, if we let g = the number of gallons of the 4% milk used in the mixture, then $120 - g$ = the number of gallons of the 1% milk needed to obtain the desired concentration. The amount of pure butterfat in each solution is given by

Amount of solution · strength of the solution = amount of pure butterfat



Teaching Example 8 MIXING CANDY Chocolate-covered peanuts selling at \$10 per pound are mixed with 5 pounds of chocolate-covered caramels selling at \$16 per pound. If the mixture is to sell for \$12 per pound, how many pounds of mixture was made?
Answer:
15 lb

Self Check 9

MIXING ANTIFREEZE A 50% antifreeze solution is mixed with a 25% antifreeze solution to obtain 30 liters of a 30% solution. How much of each concentration should be used?

Now Try **Problem 59**

Self Check 9 Answer
6 liters of 50%, 24 liters of 25%

Teaching Example 9 MIXING BLEACH-WATER A chemical lab needs to make 100 gallons of a 6% chlorine bleach–water solution. If they have 3% and 15% chlorine in stock, how much of each should be used to create the desired solution?
Answer:
75 gal of 3%, 25 gal of 15%

	Amount	· Strength	= Amount of pure butterfat
High butterfat	g	0.04	$0.04g$
Low butterfat	$120 - g$	0.01	$0.01(120 - g)$
Mixture	120	0.02	$0.02(120)$

Enter this information first.

Use the information in this column to form an equation.

Form The amount of butterfat in a tank is the *product* of the percent butterfat and the number of gallons of milk in the tank. In the first tank shown in the figure, 4% of the g gallons, or $0.04g$ gallons, is butterfat. In the second tank, 1% of the $(120 - g)$ gallons, or $0.01(120 - g)$ gallons, is butterfat. Upon mixing, the third tank will have 0.02(120) gallons of butterfat in it. These results are recorded in the last column of the table.

We now translate the words of the problem into an equation.

The amount of butterfat in g gallons of 4% milk	plus	the amount of butterfat in $(120 - g)$ gallons of 1% milk	equals	the amount of butterfat in 120 gallons of the mixture.
$0.04(g)$	$+$	$0.01(120 - g)$	$=$	$0.02(120)$

Solve

$$0.04(g) + 0.01(120 - g) = 0.02(120)$$

$$4(g) + 1(120 - g) = 2(120)$$ Multiply both sides by 100 to clear the equation of decimals. This is done mentally.

$$4g + 120 - g = 240$$

$$3g + 120 = 240$$ Combine like terms.

$$3g = 120$$ Subtract 120 from both sides.

$$g = 40$$ To isolate g, divide both sides by 3.

State 40 gallons of 4% milk and $120 - 40 = 80$ gallons of 1% milk should be mixed to get 120 gallons of milk with a 2% butterfat content.

Check The 40 gallons of 4% milk contains $0.04(40) = 1.6$ gallons of butterfat, and 80 gallons of 1% milk contains $0.01(80) = 0.8$ gallons of butterfat—a total of $1.6 + 0.8 = 2.4$ gallons of butterfat. The 120 gallons of the 2% mixture contains 2.4 gallons of butterfat. The answers check.

ANSWERS TO SELF CHECKS

1. 32,956,500 **2.** ≈60% **3.** ≈197.7% **4. a.** ≈4.61 **b.** 4.265 **c.** 3.99 **5.** 84 **6.** \$4,500 at 6%, \$7,500 at 9% **7.** 3 hr **8.** 80 lb **9.** 6 liters of 50%, 24 liters of 25%

SECTION 1.8 STUDY SET

VOCABULARY

Fill in the blanks.

1. In the statement, "10 is 20% of 50," 10 is the amount, and 50 is the base.

2. When the regular price of an item is reduced, the amount of reduction is called the markdown.

3. For a collection of data, the mean is the sum of the values divided by the number of values, the value that occurs the most is called the mode, and the middle value is called the median.

▶ Selected exercises available online at **www.webassign.net/brookscole**

4. When an investment is made, the amount of money invested is called the principal.

CONCEPTS

5. One method to solve applied percent problems is to use the given facts to write a percent sentence. What is the basic form of a percent sentence?

_____ is _____ % of _____ ?

6. Fill in the blanks:

Sale price = regular price − markdown

Markdown = percent of markdown · regular price

7.

Total Paid Circulation
People Magazine
2005: 3,734,536 **2006:** 3,786,360

a. Find the *amount* of increase in circulation of *People* magazine. 51,824

b. Fill in the percent sentence that can be used to find the percent of increase in circulation.

51,824 is what % of 3,734,536 ?

8. For each collection of values, give the median.

a. 8, 9, 11, 15, 17 11 **b.** 1, 3, 8, 16, 21, 44 12

9. a. Complete the following table for each 1-year investment.

b. Write an equation that could be used to solve this investment problem.
$0.055x + 0.07(10{,}850 - x) = 1{,}205$

	Principal ·	Rate ·	Time =	Interest
Bonds	x	0.055	1	$0.055x$
Stocks	$10{,}850 - x$	0.07	1	$0.07(10{,}850 - x)$
				Total: $1,205

10. The following table shows how a retired teacher invested a total of $8,000 in two accounts for 1 year. The investments earned $290 in interest.

a. Complete the table.

b. Write an equation that could be used to solve this investment problem. $0.03x + 0.04(8{,}000 - x) = 290$

	P ·	r ·	t =	I
S & L	x	0.03	1	$0.03x$
Credit Union	$8{,}000 - x$	0.04	1	$0.04(8{,}000 - x)$
				Total: $290

11. A banker invested the *same* amount in two money-making opportunities for 1 year and earned $3,300 in interest.

a. Complete the table.

b. Write an equation that could be used to solve this investment problem. $0.15x + 0.18x = 3{,}300$

	P ·	r ·	t =	I
Cattle futures	x	0.15	1	$0.15x$
Soybeans	x	0.18	1	$0.18x$
				Total: $3,300

12. A husband and wife drive in opposite directions to work. Their drives last the same amount of time and their workplaces are 40 miles apart.

a. Complete the table.

b. Write an equation that could be used to solve this uniform motion problem. $35t + 45t = 40$

	r ·	t =	d
Husband	35	t	$35t$
Wife	45	t	$45t$
			Total: 40 mi

13. a. A 50-pound mixture of plain and peanut M&M's is to be made. Complete the table.

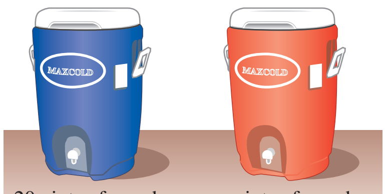
b. Write an equation that could be used to solve this mixture problem. $7.45p + 8.25(50 - p) = 7.75(50)$

	Pounds ·	Price =	Total Value
M&M's plain	p	7.45	$7.45p$
M&M's peanut	$50 - p$	8.25	$8.25(50 - p)$
Mixture	50	7.75	$7.75(50)$

14. a. Complete the following table that could be used to solve this problem: How many pints of punch from the orange cooler must be mixed with the entire contents of the blue cooler to get a 12% punch mixture?

b. Write an equation that could be used to solve this liquid mixture problem. $4 + 0.10x = 0.12(x + 20)$

	Amount ·	Strength =	Pure concentrate
Too strong	20	0.20	4
Too weak	x	0.10	$0.10x$
Mixture	$x + 20$	0.12	$0.12(x + 20)$



20 pints of punch, 20% concentrate

x pints of punch, 10% concentrate

NOTATION

15. **a.** Write 2.5% as a decimal. 0.025

b. Write 0.06 as a percent. 6%

16. What formula is used to find the mean of a collection of values? $\bar{x} = \frac{S}{n}$

GUIDED PRACTICE

Translate each statement into mathematical symbols. **Do not solve. See Example 1.**

17. What number is 5% of 10.56? $x = 0.05 \cdot 10.56$

18. 16 is what percent of 55? $16 = x \cdot 55$

19. 32.5 is 74% of what number? $32.5 = 0.74x$

20. What is 83.5% of 245? $x = 0.835 \cdot 245$

APPLICATIONS

21. ENERGY In 2005, the United States alone accounted for 23.5% of the world's total energy consumption, using 105.3 quadrillion British thermal units (Btu). What was the world's energy consumption in 2005? Round to the nearest quadrillion. 448 quadrillion Btu

22. COMPUTERS In 2007, 83 million, or 73.4%, of U.S. households had personal computers. How many U.S. households were there in 2007? Round to the nearest million. 113 million

23. BOATING ACCIDENTS According to the Insurance Information Institute, 12% of the 4,969 recreational boating accidents in 2005 involved alcohol use. How many boating accidents were alcohol-related? Round up to the nearest one accident. 597

24. BASKETBALL The following data is for the 2006–2007 NCAA men's college basketball season. What was the average number of three-point shots made per game? Round to the nearest tenth. 6.6

THREE-POINT SHOTS

Average number of attempts per game: 18.9

Percent made: 35.0%

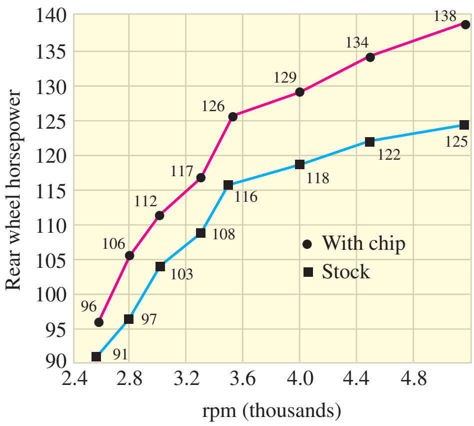
25. BUYING APPLIANCES Use the following ad to find the percent of markdown of the sale. 20%

26. BUYING FURNITURE A bedroom set regularly sells for \$983. If it is on sale for \$737.25, what is the percent of markdown? 25%

27. FLEA MARKETS A vendor sells tool chests at a flea market for \$65. If she makes a profit of 30% on each unit sold, what does she pay the manufacturer for each tool chest? (*Hint:* The retail price = the wholesale price + the markup.) \$50

28. BOOKSTORES A bookstore sells a textbook for \$39.20. If the bookstore makes a profit of 40% on each sale, what does the bookstore pay the publisher for each book? (*Hint:* The retail price = the wholesale price + the markup.) \$28

29. IMPROVING HORSEPOWER The following graph shows how the installation of a special computer chip increases the horsepower of a truck. Find the percent of increase in horsepower for the engine running at 4,000 revolutions per minute (rpm) and round to the nearest tenth of one percent. 9.3%



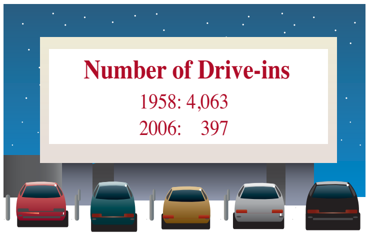
30. GREENHOUSE GASES The U.S. energy-related carbon dioxide emissions in 2006 were 5,877 million metric tons. In 2005, that figure was 5,955 million metric tons. Find the percent of decrease and round to the nearest tenth of one percent. 1.3%

31. BROADWAY SHOWS Complete the table to find the percent of increase or decrease in attendance at Broadway shows for each season compared with the previous season. Round to the nearest tenth of one percent.

Season	Broadway attendance	% of increase or decrease
2003–04	11.61 million	—
2004–05	11.53 million	−0.7%
2005–06	12.00 million	4.1%

Source: LiveBroadway.com

▶ **32.** DRIVE-INS The number of drive-in movie theaters in the United States peaked in 1958. Since then, the numbers have steadily declined. Determine the percent of decrease in the number of drive-ins. Round to the nearest one percent. 90% decrease



Source: United Drive-in Theater Owners Association

33. FUEL EFFICIENCY The ten most fuel-efficient cars in 2007, based on manufacturer's estimated city and highway average miles per gallon (mpg), are shown in the table. Find the mean, median, and mode of both sets of data.
city: mean 37.2, median 32.5, mode 32; hwy: mean 41, median 40, mode 40

Model	mpg city/hwy
Toyota Prius	60/51
Honda Civic Hybrid	49/51
Toyota Camry Hybrid	40/38
Toyota Yaris	34/40
Honda Fit	33/38
Toyota Corolla	32/41
Mini Cooper	32/40
Hyundai Accent	32/35
Honda Civic	30/40
Nissan Versa	30/36

Source: edmonds.com

34. SPORT FISHING The report shown below lists the fishing conditions at Pyramid Lake for a Saturday in January. Find the median and the mode of the weights of the striped bass caught at the lake. 5, 4

> **Pyramid Lake**—Some striped bass are biting but are on the small side. Striking jigs and plastic worms. Water is cold: 38°. Weights of fish caught (lb): 6, 9, 4, 7, 4, 3, 3, 5, 6, 9, 4, 5, 8, 13, 4, 5, 4, 6, 9

35. JOB TESTING To be accepted into a police training program, a recruit must have an average score of 85 on a battery of four tests. If a candidate scored 76 on the oral test, 87 on the physical fitness test, and 83 on the psychological test, what is the lowest score she can obtain on the written test and still be accepted into the training program? 94

36. WNBA CHAMPIONS The results of each 2006 playoff game for the Detroit Shock are shown below. The two Detroit scores that are missing, one in a win and one in a loss, are the same number. If they averaged 74.8 points per game in the playoffs, find the missing scores. 68

First Round
Detroit XX, Indiana 56
Detroit 98, Indiana 83

Conference Finals
Detroit 70, Connecticut 59
Connecticut 77, Detroit XX
Detroit 79, Connecticut 59

WNBA Finals
Sacramento 95, Detroit 71
Detroit 73, Sacramento 63
Sacramento 89, Detroit 69
Detroit 72, Sacramento 52
Detroit 80, Sacramento 75

37. HIGHEST RATES Based on the information in the table, a woman invested \$12,000, some in an account paying the highest rate and the rest in an account paying the second highest rate. How much was invested in each account if the interest from both investments is \$1,060 per year? CD: \$10,000, money market: \$2,000

First Republic Savings and Loan	
Account	**Rate**
NOW	5.5%
Savings	7.5%
Money market	8.0%
Checking	4.0%
5-year CD	9.0%

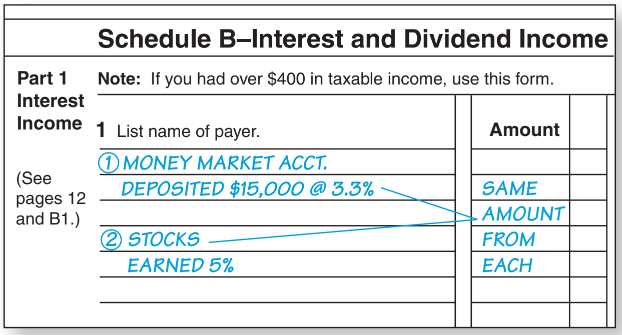
▶ **38.** ENTREPRENEURS Last year, a women's professional organization made two small-business loans totaling \$28,000 to young women beginning their own businesses. The money was lent at 7% and 10% simple interest rates. If the annual income the organization received from these loans was \$2,560, what was each loan amount?
\$8,000 at 7%, \$20,000 at 10%

39. INHERITANCES Paula used some of the money that she received from an inheritance to invest in a certificate of deposit paying 7% annual interest and the rest of the money in a promising biotech company offering an annual return of 10%. She invested twice as much in the 10% investment as she did in the 7% investment. Her combined annual income from the two investments was \$4,050.

a. How much did she invest in each account?
\$15,000 at 7%, \$30,000 at 10%

b. How much did she inherit?
\$45,000

40. TAX RETURNS On a federal income tax form, Schedule B, a taxpayer forgot to write in the amount of interest income he earned for the year. From what is written on the form, determine the amount of interest earned from each investment and the amount he invested in stocks. $495, $495; $9,900


Schedule B–Interest and Dividend Income

Part 1 Interest Income

Note: If you had over $400 in taxable income, use this form.

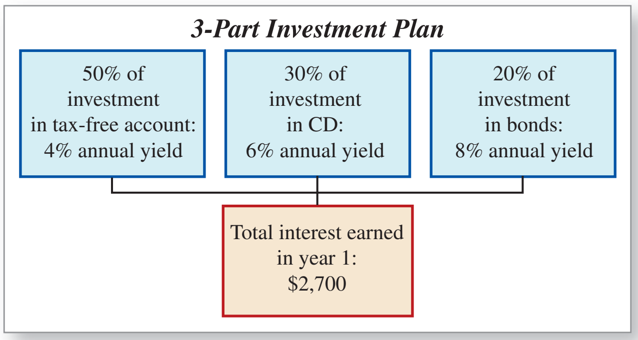
(See pages 12 and B1.)

1 List name of payer.	Amount
① MONEY MARKET ACCT.	
DEPOSITED $15,000 @ 3.3%	SAME
	AMOUNT
② STOCKS	FROM
EARNED 5%	EACH

41. MONEY-LAUNDERING Use the evidence compiled by investigators to determine how much money a suspect deposited in the Cayman Islands bank. $100,000

- On 6/1/06, the suspect electronically transferred $300,000 to a Swiss bank account paying an 8% annual yield.
- That same day, the suspect opened another account in a Cayman Islands bank that offered a 5% annual yield.
- A document dated 6/3/07 was seized during a raid of the suspect's home. It stated, "The total interest earned in one year from the two overseas accounts was 7.25% of the total amount deposited."

42. FINANCIAL PRESENTATIONS A financial planner showed her client the following investment plan. Find the total amount the client will have to invest to earn $2,700 in interest. $50,000



43. TRAVEL TIMES A man called his wife to tell her that they needed to switch vehicles so he could use the family van to pick up some building materials after work. The wife left their home, traveling toward his office in their van at 35 mph. At the same time, the husband left his office in his car, traveling toward their home at 45 mph. If his office is 20 miles from their home, how long will it take them to meet so they can switch vehicles? $\frac{1}{4}$ hr = 15 min

44. AIR TRAFFIC CONTROL An airplane leaves Los Angeles bound for Caracas, Venezuela, flying at an average rate of 500 mph. At the same time, another airplane leaves Caracas bound for Los Angeles, averaging 550 mph. If the airports are 3,675 miles apart, when will the air traffic controllers have to make the pilots aware that the planes are passing each other? 3.5 hr into the flights

45. CYCLING A cyclist leaves his training base for a morning workout, riding at the rate of 18 mph. One hour later, his support staff leaves the base in a car going 45 mph in the same direction. How long will it take the support staff to catch up with the cyclist? $\frac{2}{3}$ hr

46. MARATHONS Two marathon runners leave the starting gate, one running 12 mph and the other 10 mph. If they maintain the pace, how long will it take for them to be one-quarter of a mile apart? $\frac{1}{8}$ hr

47. RADIO COMMUNICATIONS At 2 P.M., two military convoys leave Eagle River, Wisconsin, one headed north and one headed south. The convoy headed north averages 50 mph, and the convoy headed south averages 40 mph. They will lose radio contact when the distance between them is more than 135 miles. When will this occur? 3:30 P.M.

48. SEARCH AND RESCUE Two search-and-rescue teams leave base camp at the same time, looking for a lost child. The first team, on horseback, heads north at 3 mph, and the other team, on foot, heads south at 1.5 mph. How long will it take them to search a distance of 18 miles between them? 4 hr

49. JET SKIING A jet ski can go 12 mph in still water. If a rider goes upstream for 3 hours against a current of 4 mph, how long will it take the rider to return? (*Hint:* Upstream speed is (12–4) mph; how far can the rider go in 3 hours?) $1\frac{1}{2}$ hr

50. PHYSICAL FITNESS For her workout, Sarah walks north at the rate of 3 mph and returns at the rate of 4 mph. How many miles does she walk if the round trip takes 3.5 hours? 12 mi

51. MIXING CANDY How many pounds of red licorice bits that sell for $1.90 per pound should be mixed with 5 pounds of lemon gumdrops that sell for $2.20 per pound to make a candy mixture that could be sold for $2 per pound? 10 lb

52. COFFEE BLENDS A store sells regular coffee for \$8 a pound and gourmet coffee for \$14 a pound. To get rid of 40 pounds of the gourmet coffee, a shopkeeper makes a blend to put on sale for \$10 a pound. How many pounds of regular coffee should he use? 80 lb

▶ **53.** HEALTH FOODS A pound of dried pineapple bits sells for \$6.19, a pound of dried banana chips sells for \$4.19, and a pound of raisins sells for \$2.39 a pound. Two pounds of raisins are to be mixed with equal amounts of pineapple and banana to create a trail mix that will sell for \$4.19 a pound. How many pounds of pineapple and banana chips should be used? 1.8 lb of each

▶ **54.** METALLURGY A 1-ounce commemorative coin is to be made of a combination of pure gold, costing \$380 an ounce, and a gold alloy that costs \$140 an ounce. If the cost of the coin is to be \$200, and 500 are to be minted, how many ounces of gold and gold alloy are needed to make the coins? 125 oz pure gold, 375 oz gold alloy

55. GARDENING A wholesaler of premium organic planting mix notices that the retail garden centers are not buying her product because of its high price of \$1.57 per cubic foot. She decides to mix sawdust with the planting mix to lower the price per cubic foot. If the wholesaler can buy the sawdust for \$0.10 per cubic foot, how many cubic feet of each must be mixed to have 6,000 cubic feet of planting mix that could be sold to retailers for \$1.08 per cubic foot? 4,000 ft^3 of the premium mix, 2,000 ft^3 of sawdust

56. BRONZE A pound of tin is worth \$1 more than a pound of copper. Four pounds of tin are mixed with 6 pounds of copper to make bronze that sells for \$3.65 per pound. How much is a pound of tin worth? \$4.25

57. Suppose, as registered dietician for a school district, you must make sure that only extra lean ground beef (16% fat) is served in the cafeteria. Further suppose that the kitchen has 8 pounds of regular ground beef (30% fat) on hand. How many pounds of super lean ground beef (12% fat) must be purchased and added to the regular ground beef to obtain a mixture that has the correct fat content? 28 lb

from Campus to Careers

Registered Dietician

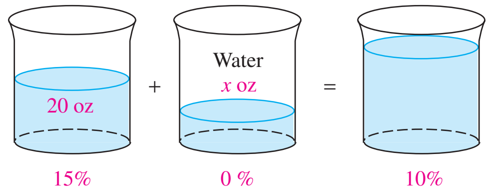
© Baerbel Schmidt/Getty Images

58. PHARMACISTS How many liters of a 1% glucose solution should a pharmacist mix with 0.5 liter of a 5% glucose solution to obtain a 2% glucose solution? 1.5 L

59. DAIRY FOODS Cream is approximately 22% butterfat. How many gallons of cream must be mixed with milk testing at 2% butterfat to get 20 gallons of milk containing 4% butterfat? 2 gal

60. FLOOD DAMAGE One website recommends a 6% chlorine bleach–water solution to remove mildew. A chemical lab has 3% and 15% chlorine bleach–water solutions in stock. How many gallons of each should be mixed to obtain 100 gallons of the mildew spray? 75 gallons of 3%, 25 gallons of 15%

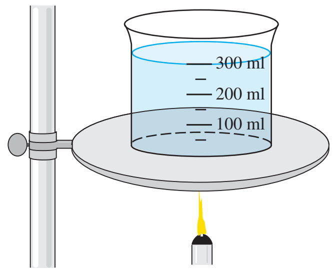
61. DILUTING SOLUTIONS How much water should be added to 20 ounces of a 15% solution of alcohol to dilute it to a 10% alcohol solution? 10 oz



62. EVAPORATION The beaker shown below contains a 2% saltwater solution.

a. How much water must be boiled away to increase the concentration of the salt solution from 2% to 3%? 100 ml

b. Where on the beaker would the new water level be? (ml means milliliter.) 200-ml mark



WRITING

63. If a car travels at 60 mph for 30 minutes, explain why the distance traveled is not $60 \cdot 30 = 1{,}800$ miles.

▶ **64.** If a mixture is to be made from solutions with concentrations of 12% and 30%, can the mixture have a concentration less than 12%? Can the mixture have a concentration greater than 30%? Explain.

65. Write a mixture problem that can be represented by the following verbal model and equation.

The value of the regular coffee	plus	the value of the gourmet coffee	equals	the value of the blend.
$4x$	$+$	$7(40 - x)$	$=$	$5(40)$

66. Write a uniform motion problem using the facts entered in the table.

	r	$\cdot\ t$	$= \quad d$
West	8	t	$8t$
East	6	t	$6t$
			Total: 24 mi.

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

REVIEW

Solve each equation.

67. $9x = 6x$ 0

68. $7a + 2 = 12 - 4(a - 3)$ 2

69. $\dfrac{8(y - 5)}{3} = 2(y - 4)$ 8

70. $\dfrac{t - 1}{3} = \dfrac{t + 2}{6} + 2$ 16

CHAPTER 1 SUMMARY AND REVIEW

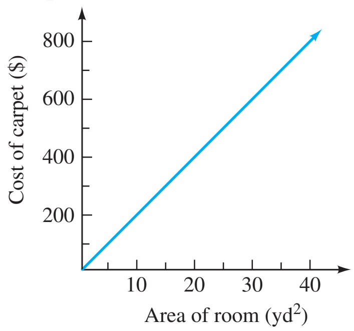
SECTION 1.1 The Language of Algebra

DEFINITIONS AND CONCEPTS	EXAMPLES
In this course, we will mathematically **model** real-life situations using verbal sentences, mathematical statements (in symbols), tables, and graphs. For more examples of **tables, bar graphs** and **line graphs** see pages 5–6.	Suppose we want to purchase carpeting for a room and it costs \$20 a square yard. *Verbal model:* The cost c (in dollars) to carpet the room is \$20 times the area a of the room (in square yards). *Mathematical model:* $c = 20a$

Table model

a	c
10	200
20	400
30	600
40	800

Graphical model



A **variable** is a letter (or symbol) that stands for a number.	Variables: x, t, a, and m
Algebraic expressions contain numbers, variables, and mathematical operations such as addition, subtraction, multiplication, division, powers, or roots.	Expressions: $5y + 7$, $\frac{12 - x}{5}$, and $8a(b - 3)$
An **equation** is a mathematical sentence that contains an $=$ symbol. The $=$ symbol indicates that the expressions on either side of it have the same value.	Equations: $3x + 4 = 8$, $\frac{t}{9} = 12$, and $I = Prt$

REVIEW EXERCISES

1. Translate each verbal model into a mathematical model.

a. The cost C (in dollars) to rent t tables is \$15 more than the product of \$2 and t. $C = 2t + 15$

b. A rectangle has an area of 25 in.^{2}. The length of the rectangle is the quotient of its area and its width. $l = \frac{25}{w}$

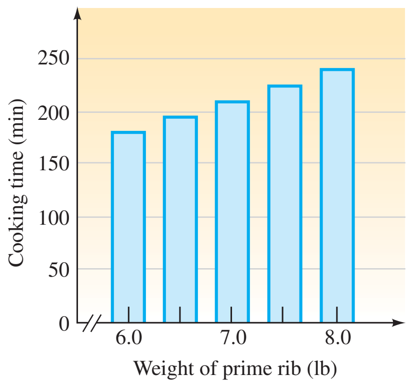
c. The waiting period for a business license is now 3 weeks less than it used to be. $P = u - 3$

2. To determine the cooking time for prime rib, a cookbook suggests using the equation $T = 30p$, where T is the cooking time in minutes and p is the weight of the prime rib in pounds. Use this equation to complete the table.

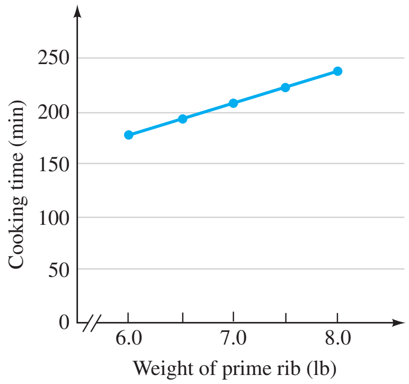
p	T
6.0	180
6.5	195
7.0	210
7.5	225
8.0	240

3. Use the data from the table in Exercise 2 to draw each type of graph.

a. Bar graph

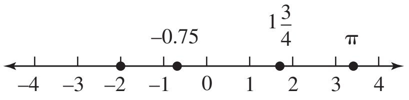


b. Line graph



4. Explain the difference between an expression and an equation. Give examples.
An equation such as $2x + 3 = 1$ contains an equal symbol. An expression such as $2x + 3$ does not.

SECTION 1.2 The Real Numbers

DEFINITIONS AND CONCEPTS	EXAMPLES
Important sets of numbers	
The set of **natural numbers:**	$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
The set of **whole numbers:**	$\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
The set of **integers:**	$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$
The set of **prime numbers:**	$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$
The set of **composite numbers:**	$\{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \ldots\}$
The set of integers divisible by 2 are **even integers.**	$\{\ldots, -8, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$
The set of integers not divisible by 2 are **odd integers.**	$\{\ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots\}$
The set of **rational numbers,** denoted by the symbol $\mathbb{Q}$, contains the numbers that can be written as fractions in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$.	Rational numbers: $-6, \quad -3.1, \quad 0, \quad \frac{11}{12}, \quad 9\frac{4}{5}, \quad \text{and} \quad 87$
Terminating and **repeating decimals** can be expressed as fractions and are therefore rational numbers.	Rational numbers: $0.25 = \frac{1}{4} \quad \text{and} \quad -0.\overline{6} = -\frac{2}{3}$
The set of **irrational numbers,** denoted by the symbol $\mathbb{H}$, contains the nonterminating, nonrepeating decimals. An irrational number cannot be expressed as a fraction with an integer numerator and a nonzero integer denominator.	Irrational numbers: $\sqrt{2}, \quad \pi, \quad -\frac{\sqrt{11}}{4}, \quad 73.050050005\ldots, \quad \text{and} \quad -32\pi$
The set of **real numbers,** denoted by the symbol $\mathbb{R}$, contains the numbers that are either a rational or an irrational number.	All of the numbers previously listed are real numbers.
Every real number corresponds to a point on the number line, and every point on the **number line** corresponds to exactly one real number.	Graph the set $\left\{-2, -0.75, 1\frac{3}{4}, \pi\right\}$ on a number line. 
Symbols used with sets	
The symbol $\in$ is used to indicate that an **element** belongs to a set.	$6 \in \mathbb{N}$ *6 is an element of the set of natural numbers.* $3 \notin \mathbb{H}$ *3 is not an element of the set of irrational numbers.*
When all the members of one set are also members of a second set, we say the first set is a **subset** of the second set.	
The symbol $\subseteq$ is used to indicate that one set is a **subset** of another set.	$\mathbb{N} \subseteq \mathbb{W}$ *The set of natural numbers is a subset of the set of whole numbers.* $\mathbb{Q} \not\subseteq \mathbb{Z}$ *The set of rational numbers is not a subset of the set of integers.*

Inequality symbols	
$\neq$ means "is not equal to"	$3 \neq 5$
$<$ means "is less than"	$0.23 < 0.24$
$\leq$ means " is less than or equal to"	$8 \leq 12$ and $-9 \leq -9$
$>$ means "is greater than"	$\frac{3}{4} > \frac{2}{3}$
$\geq$ means "is greater than or equal to"	$-5 \geq -6$ and $8 \geq 8$
Two numbers are called **opposites** if they are the same distance from 0 on the number line but are on opposite sides of it.	Opposites: 3 and -3
The **opposite of a number** a is $-a$. $-(-a) = a$	$-(-11) = 11$ and $-(-20) = 20$
The **absolute value** of a number is the distance on the number line between the number and 0. For any real number x: $\begin{cases} \text{If } a \geq 0, \text{ then } \lvert a \rvert = a \\ \text{If } a < 0, \text{ then } \lvert a \rvert = -a \end{cases}$	Find the value of each expression: $\lvert 7.2 \rvert = 7.2$ $\quad \lvert -1 \rvert = 1$ $\quad -\left\lvert -\frac{13}{5} \right\rvert = -\frac{13}{5}$

REVIEW EXERCISES

List the numbers in $\left\{-5, 0, -\sqrt{3}, 2.4, 7, -\frac{2}{3}, -3.\overline{6}, \pi, \frac{15}{4}, 0.13242368\ldots\right\}$ ***that belong to the following sets.***

5. a. Natural numbers 7 **b.** Whole numbers 0, 7

6. a. Integers $-5, 0, 7$ **b.** Rational numbers $-5, 0, 2.4, 7, -\frac{2}{3}, -3.\overline{6}, \frac{15}{4}$

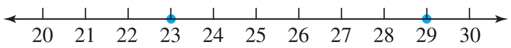
7. a. Irrational numbers $-\sqrt{3}, \pi, 0.13242368\ldots$ **b.** Real numbers all

8. a. Negative numbers $-5, -\sqrt{3}, -\frac{2}{3}, -3.\overline{6}$ **b.** Positive numbers $2.4, 7, \pi, \frac{15}{4}, 0.13242368\ldots$

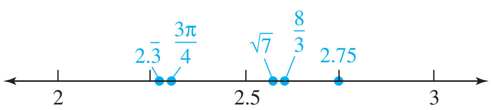
9. a. Prime numbers 7 **b.** Composite numbers none

10. a. Even integers 0 **b.** Odd integers $-5, 7$

11. Graph the prime numbers between 20 and 30 on the number line.



12. Graph the set $\left\{2.75, 2.\overline{3}, \sqrt{7}, \frac{8}{3}, \frac{3\pi}{4}\right\}$ on the number line.



Determine whether each statement is true or false.

13. a. $0 \in \mathbb{N}$ false **b.** $\mathbb{W} \subseteq \mathbb{Z}$ true

14. a. $-5 \notin \mathbb{R}$ false **b.** $\mathbb{Q} \not\subseteq \mathbb{H}$ true

15. Use one of the symbols $>$ or $<$ to make each statement true.

a. $-16 \; > \; -17$ **b.** $-(-1.8) \; < \; 2\frac{1}{2}$

16. Determine whether each statement is true or false.

a. $23.000001 \geq 23.1$ false **b.** $-11 \leq -11$ true

Find the value of each expression.

17. $\lvert -18 \rvert$ 18 **18.** $-\lvert -6.26 \rvert$ -6.26

SECTION 1.3 Operations with Real Numbers

DEFINITIONS AND CONCEPTS	EXAMPLES
Adding real numbers:	
To add two positive numbers, add them in the usual way. The final answer is positive.	Add: $3 + 5 = 8$ $5.2 + 7.3 = 12.5$
To add two negative numbers, add their absolute values and make the final answer negative.	Add: $-3 + (-5) = -8$ $-5.2 + (-7.3) = -12.5$
To add a positive number and a negative number, subtract the smaller absolute value from the larger.	Add: $5 + (-3) = 2$ $-5.2 + 7 = 1.8$
1. If the positive number has the larger absolute value, make the final answer positive. **2.** If the negative number has the larger absolute value, make the final answer negative.	Add: $6 + (-20) = -14$ $-9 + 3 = -6$
To **subtract two real numbers,** add the first to the opposite of the number to be subtracted. For any real numbers a and b, $a - b = a + (-b)$	Subtract: $4 - 7 = 4 + (-7) = -3$ $6 - (-8) = 6 + 8 = 14$ $-1 - (-2) = -1 + 2 = 1$ $-3 - 4 = -3 + (-4) = -7$
Multiplying and dividing real numbers: With **unlike signs,** multiply (or divide) their absolute values and make the final answer negative.	Multiply: $-5(7) = -35$ Divide: $\frac{-12}{4} = -3$
With **like signs,** multiply (or divide) their absolute values and make the final answer positive.	Multiply: $-6(-8) = 48$ Divide: $\frac{-25}{-5} = 5$
x^n is a *power of x*. x is the **base,** and n is the **exponent.** An exponent represents repeated multiplication. $x^n = \overbrace{x \cdot x \cdot x \cdot \cdots \cdot x}^{n \text{ factors of } x}$	In 4^3, the base is 4 and 3 is the exponent. $4^3 = 4 \cdot 4 \cdot 4$ $(-3)^4 = (-3)(-3)(-3)(-3)$
A number b is a **square root** of a if $b^2 = a$. If $a > 0$, $\sqrt{a}$ represents the **principal** (positive) **square root** of a.	Find each square root: $\sqrt{16} = 4$ Because $4^2 = 16$. $\sqrt{144} = 12$ Because $12^2 = 144$.

Order of operations rule: **1.** Perform all calculations within parentheses in the order listed in steps 2–4, working from the innermost pair to the outermost pair. **2.** Evaluate all powers and roots. **3.** Perform all multiplications and divisions, working from left to right. **4.** Perform all additions and subtractions, working from left to right. **5.** When all grouping symbols have been removed, repeat steps 2–4 to finish the calculation. If a fraction is present, evaluate the numerator and denominator separately, and then simplify the fraction, if possible.	Evaluate: $3 + 2[-4 - 7(5 - 2)^2]$ $= 3 + 2[-4 - 7(3)^2]$ Work within the parentheses. $= 3 + 2[-4 - 7(9)]$ Evaluate the power. $= 3 + 2(-4 - 63)$ Do the multiplication. $= 3 + 2(-67)$ Do the subtraction with parentheses. $= 3 - 134$ Do the multiplication. $= -131$ Do the subtraction.
To **evaluate an algebraic expression,** substitute the values for the variables and then use the order of operations rule.	Evaluate $\frac{a+b}{2(b-a)}$ for $a = 3$ and $b = -5$. $\frac{a+b}{2(b-a)} = \frac{3+(-5)}{2(-5-3)}$ Substitute 3 for *a* and −5 for *b*. $= \frac{-2}{2(-8)}$ Evaluate numerator and denominator separately. $= \frac{-2}{-16}$ $= \frac{1}{8}$ Simplify the fraction.

REVIEW EXERCISES

Perform the operations.

19. $-13 + (-14)$ -27

20. $-70.5 + 80.6$ 10.1

21. $-\frac{1}{2} - \frac{1}{4}$ $-\frac{3}{4}$

22. $-6 - (-8)$ 2

23. $(-4.2)(-3.0)$ 12.6

24. $-\frac{1}{10} \cdot \frac{5}{16}$ $-\frac{1}{32}$

25. $\frac{-2.2}{-11}$ 0.2

26. $-\frac{9}{8} \div 21$ $-\frac{3}{56}$

27. $15 - 25 - 23$ -33

28. $-3.5 + (-7.1) + 4.9$ -5.7

29. $-(-5)(-8)$ -40

30. $-1(-1)(-1)(-1)$ 1

Evaluate each expression.

31. $(-3)^5$ -243

32. $\left(-\frac{2}{9}\right)^2$ $\frac{4}{81}$

33. 0.4 cubed 0.064

34. -5^2 -25

Evaluate each expression.

35. $\sqrt{4}$ 2

36. $-\sqrt{100}$ -10

37. $\sqrt{\frac{9}{25}}$ $\frac{3}{5}$

38. $\sqrt{0.64}$ 0.8

Evaluate each expression.

39. $-6 + 2(-5)^2$ 44

40. $\frac{-20}{4} - (-3)(-2)\left(-\sqrt{1}\right)$ 1

41. $4 - (5 - 9)^2$ -12

42. $4 + 6[-1 - 5(25 - 3^3)]$ 58

43. $2|-1.3 + (-2.7)|$ 8 **44.** $\dfrac{(7-6)^4 + 32}{36 - (\sqrt{16} + 1)^2}$ 3

45. $(-10)^3\left(\dfrac{-6}{-2}\right)(-1)$ 3,000 **46.** $-(-2 \cdot 4)^2 \div 8 \cdot 2$ -16

Evaluate the algebraic expression for the given values of the variables.

47. $(x + y)(x^2 - xy + y^2)$ for $x = -2$ and $y = 4$ 56

48. $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ for $a = 2, b = -3$, and $c = -2$ $-\frac{1}{2}$

SECTION 1.4 Simplifying Algebraic Expressions Using Properties of Real Numbers

DEFINITIONS AND CONCEPTS	EXAMPLES
A **term** is a product or quotient of numbers and/or variables. A single number or variable is also a term. A term that is a number called is a **constant term.**	Terms: $z, \quad -7t, \quad 3.7x^7, \quad \frac{3}{5}y, \quad 25ab^2c^3, \quad$ and $\quad 6$ (a constant)
The numerical factor of a term is called its **coefficient.**	The coefficients of the previous terms are $1, \quad -7, \quad 3.7, \quad \frac{3}{5}, \quad 25, \quad$ and $\quad 6$
Properties of real numbers The **commutative properties** enable us to add or multiply two numbers in either order and obtain the same result.	
$a + b = b + a$ Commutative property of addition	$3 + (-5) = -5 + 3$ Reorder the addends.
$ab = ba$ Commutative property of multiplication	$3(-5) = -5(3)$ Reorder the factors.
The **associative properties** enable us to group the numbers in an addition or multiplication any way that we wish and get the same result.	
$(a + b) + c = a + (b + c)$ Associative property of addition	$(-5 + 7) + 4 = -5 + (7 + 4)$ Regroup the addends.
$(ab)c = a(bc)$ Associative property of multiplication	$(-5 \cdot 7)4 = -5(7 \cdot 4)$ Regroup the factors.
0 is the **additive identity:** $a + 0 = a$ and $0 + a = a$	$5 + 0 = 5$ The sum of a real number and 0 is the number.
1 is the **multiplicative identity:** $1 \cdot a = a$ and $a \cdot 1 = a$	$1 \cdot 22 = 22$ The product of any real number and 1 is the number.
Multiplication property of 0: $a \cdot 0 = 0$ and $0 \cdot a = 0$	$-1.78 \cdot 0 = 0$ The product of any real number and 0 is 0.
The **additive inverse property:** $a + (-a) = 0$ and $-a + a = 0$	$9 + (-9) = 0$ The sum of a real number and its opposite is 0.
The **multiplicative inverse property:** $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1 \quad (a \neq 0)$	$7 \cdot \frac{1}{7} = 1$ The product of a real number and its reciprocal is 1.

Division properties of real numbers: $\frac{a}{1} = a \quad \frac{a}{a} = 1 \quad \frac{0}{a} = 0$ $\frac{a}{0}$ is undefined $\quad \frac{0}{0}$ is indeterminate	Divide, if possible: $\frac{6.8}{1} = 6.8 \quad \frac{3}{3} = 1$ $\frac{0}{15} = 0 \quad \frac{-2.03}{0}$ is undefined
The **distributive property** and its related forms can be used to remove parentheses. $a(b + c) = ab + ac \quad a(b - c) = ab - ac$ $(b + c)a = ba + ca \quad (b - c)a = ba - ca$ The **extended distributive property:** $a(b + c + d + e + \cdots) =$ $ab + ac + ad + ae + \cdots$	Multiply: $3(a + 4) = 3a + 3 \cdot 4$ $= 3a + 12$ $(x - 5)x = x \cdot x - 5 \cdot x$ $= x^2 - 5x$ $-4(2a + b - 7) = -4(2a) + (-4)b - (-4)(7)$ $= -8a - 4b + 28$
Terms with exactly the same variables raised to exactly the same powers are called **like terms.**	Like terms: $6x$ and $-7x \quad -4a^2b$ and $8a^2b$ Unlike terms: $6x$ and $-7y \quad -4a^2b$ and $8a^3b$
Simplifying the sum or difference of like terms is called **combining like terms.** Like terms can be combined by adding or subtracting the coefficients of the terms and keeping the same variables with the same exponents.	Simplify: $4a + 2a = 6a$ Think: $(4 + 2)a = 6a$. $5p^2 + p - p^2 - 9p = 4p^2 - 8p$ Think: $(5 - 1)p^2 = 4p^2$ and $(1 - 9)p = -8p$. $2(k - 1) - 3(k + 2) = 2k - 2 - 3k - 6$ Distribute. $= -k - 8$ Think: $(2 - 3)k = -k$ and $(-2 - 6) = -8$.

REVIEW EXERCISES

Fill in the blanks by applying the indicated property of the real numbers.

49. $3(x + 7) = \underline{3x + 21}$ (Distributive property and simplify)

50. $t \cdot 5 = \underline{5t}$ (Commutative property of multiplication)

51. $-x + x = \underline{0}$ (Additive inverse property)

52. $(27 + 1) + 99 = \underline{27 + (1 + 99)}$ (Associative property of addition)

53. $\frac{1}{8} \cdot 8 = \underline{1}$ (Multiplicative inverse property)

54. $0 + m = \underline{m}$ (Additive identity property)

55. $\underline{1} \cdot 9.87 = 9.87$ (Multiplicative identity property)

56. $5(-9)(0)(2{,}345) = \underline{0}$ (Multiplication property of 0)

57. $(-3 \cdot 5)2 = \underline{-3(5 \cdot 2)}$ (Associative property of multiplication)

58. $(t + z) \cdot t = \underline{(z + t) \cdot t}$ (Commutative property of addition)

59. Perform each division.

a. $\frac{102}{102}$ 1 **b.** $\frac{-25}{1}$ -25

60. Perform each division, if possible.

a. $\frac{0}{6}$ 0 **b.** $\frac{5.88}{0}$ undefined

Multiply.

61. $8(9x + 6)$ $72x + 48$

62. $-(6y - 2)$ $-6y + 2$

63. $(3x - 2y)(1.2)$ $3.6x - 2.4y$

64. $\frac{3}{4}(8c^2 - 4c + 1)$ $6c^2 - 3c + \frac{3}{4}$

Simplify each expression.

65. $8(6k)$ $48k$

66. $(-7.5x)(-10y)$ $75xy$

67. $-9(-3p)(-7)$ $-189p$

68. $15a + 7 + 30a + 9$ $45a + 16$

69. $3g^2 + g^2 - 3g^2 - g^2$ 0

70. $-m + 4(m - 12n) - (-8n)$ $3m - 40n$

71. $\frac{7}{5}x - \frac{3}{4}x$ $\frac{13}{20}x$

72. $21.45l + (-45.99l)$ $-24.54l$

73. $4[-2(a^3 - a^2) - 2(3a^2 - 6a^3)]$ $40a^3 - 16a^2$

74. $\frac{3}{4}(2h + 9) - \frac{5}{4}(h - 1)$ $\frac{1}{4}h + 8$

SECTION 1.5 Solving Linear Equations Using Properties of Equality

DEFINITIONS AND CONCEPTS	EXAMPLES
A **linear equation in one variable** is an equation that can be written in the form $ax + b = c$, where $a \neq 0$.	Linear equations: $5y - 2 = 12$ $\quad \frac{11}{6}t = 7$ $\quad 4b - 7 + 2b = 1 + 2b + 8$ Notice that the highest power on the variable is 1.
A number that makes an equation a true statement when substituted for the variable is called a **solution** of the equation. The **solution set** of an equation is the set of all numbers that make the equation true.	Determine whether 2 is a solution of $x + 4 = 3x$. ***Check:*** $x + 4 = 3x$ $2 + 4 \stackrel{?}{=} 3(2)$ Substitute 2 for each x. $6 = 6$ True Since the resulting statement is true, 2 is a solution of $x + 4 = 3x$.
Equations with the same solution set are called **equivalent equations.**	$x + 1 = 9$ and $x = 8$ are equivalent equations because they have the same solution, 8.
To **solve an equation,** isolate the variable on one side of the equation by undoing the operations performed on it using properties of equality. **Addition (subtraction) property of equality:** If the same number is added to (or subtracted from) both sides of an equation, the result is an equivalent equation. **Multiplication (division) property of equality:** If both sides of an equation are multiplied (or divided) by the same nonzero number, the result is an equivalent equation.	If $a = b$, then $a + c = b + c$ $\qquad a - c = b - c$ $ca = cb \quad (c \neq 0)$ $\qquad \frac{a}{c} = \frac{b}{c} \quad (c \neq 0)$
Strategy for Solving Linear Equations in One Variable **1.** Clear the equation of fractions or decimals. **2.** Simplify each side of the equation by removing all sets of parentheses and combining like terms. **3.** Isolate the variable term on one side of the equation. **4.** Isolate the variable. **5.** Check the result in the original equation.	Solve: $\frac{x-1}{6} + x = \frac{2}{3} - \frac{x+2}{6}$ $6\left(\frac{x-1}{6} + x\right) = 6\left(\frac{2}{3} - \frac{x+2}{6}\right)$ Multiply both sides by 6 to clear the fractions. $x - 1 + 6x = 4 - (x + 2)$ Simplify. Don't forget the parentheses. $x - 1 + 6x = 4 - x - 2$ Remove parentheses. $7x - 1 = 2 - x$ Combine like terms on each side. $7x - 1 + x = 2 - x + x$ To eliminate $-x$ on the right side, add x to both sides. $8x - 1 = 2$ Combine like terms on each side. $8x - 1 + 1 = 2 + 1$ To isolate the variable term $8x$, add 1 to both sides. $8x = 3$ Simplify each side.

	$\frac{8x}{8} = \frac{3}{8}$ Isolate the variable x by dividing both sides by 8. $x = \frac{3}{8}$ The solution is $\frac{3}{8}$ and the solution set is $\left\{\frac{3}{8}\right\}$. Check this result to verify that it satisfies the *original* equation.
An **identity** is an equation that is satisfied by every number for which both sides are defined.	When we solve $x + 5 + x = 2x + 5$, the variables drop out and we obtain a true statement $5 = 5$. All real numbers are solutions. The solution set is the set of real numbers denoted $\mathbb{R}$.
A **contradiction** is an equation that is never true.	When we solve $y + 2 = y$, the variables drop out and we obtain a false statement $2 = 0$. The equation has no solutions. The solution set contains no elements and can be denoted as the **empty set** $\{\ \}$ or the **null set** $\varnothing$.

REVIEW EXERCISES

Determine whether −6 is a solution of each equation.

75. $6 - x = 2x + 24$ yes

76. $\frac{5}{3}(x - 3) = -12$ no

Solve each equation and check the result.

77. $\frac{x}{5} = -45$ -225

78. $t - 3.67 = 4.23$ 7.9

79. $0.0035 = 0.25g$ 0.014

80. $0 = x + 4$ -4

81. $11 - 5x = -1$ $\frac{12}{5}$

82. $-3x - 7 + x = 6x + 20 - 5x$ -9

83. $-4(y - 1) + (-3) = 25$ -6

84. $5 + 3[2 - 13(x - 1)] = 17 - 18x$ $\frac{11}{7}$

85. $\frac{8}{3}(x - 5) = \frac{2}{5}(x - 4)$ $\frac{88}{17}$

86. $\frac{3y}{4} - 14 = -\frac{y}{3} - 1$ 12

87. $-k = -0.06$ 0.06

88. $\frac{5}{4}p = -10$ -8

89. $\frac{4t + 1}{3} - \frac{t + 5}{6} = \frac{t - 3}{6}$ 0

90. $33.9 - 0.5(75 - 3x) = 0.9$ 3

Solve each equation. If the equation is an identity or a contradiction, so state.

91. $2(x - 6) = 10 + 2x$ no solution, $\varnothing$; contradiction

92. $-5x + 2x - 1 = -(3x + 1)$ all real numbers, $\mathbb{R}$; identity

SECTION 1.6 Solving Formulas; Geometry

DEFINITIONS AND CONCEPTS	EXAMPLES
A **formula** is an equation that states a relationship between two or more variables.	Turn to the inside back cover for a complete list of geometric formulas.
The **perimeter** of a plane geometric figure is the distance around it. The **area** of a plane geometric figure is the amount of surface that it encloses. The **volume** of a three-dimensional geometric figure is the amount of space it encloses.	Find the volume of a cone whose base has a radius of 6 cm and whose height is 8 cm. $V = \frac{1}{3}\pi r^2 h$ This is the formula for the volume of a cone. $= \frac{1}{3}\pi(6)^2(8)$ Substitute 6 for r and 8 for h. $= \frac{1}{3}\pi(288)$ Evaluate $(6)^2(8)$. $= 96\pi$ Multiply $\frac{1}{3}$ and 288. The exact volume is 96π cm^3. Rounded to the nearest tenth of a cubic centimeter, the approximate volume is 301.6 cm^3.

To **solve a formula for a specified variable** means to isolate that variable on one side of the equation, with all other variables and constants on the opposite side.

Solve $F = \frac{mMg}{r^2}$ for M.

$Fr^2 = mMg$ Multiply both sides by r^2.

$\frac{Fr^2}{mg} = M$ To isolate M, divide both sides by mg.

$M = \frac{Fr^2}{mg}$ Write M on the left side.

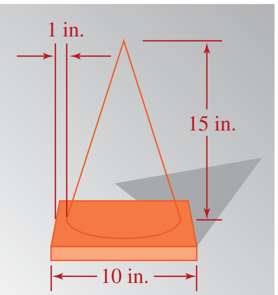
REVIEW EXERCISES

93. Find the perimeter of a trapezoid whose parallel sides measure 10.5 feet and 12.5 feet and whose nonparallel sides measure 3.5 feet and 4.5 feet. 31 ft

94. Find the circumference and the area of a circle with a diameter of 17 centimeters. Round to the nearest hundredth. 53.41 cm, 226.98 cm^2

95. Find the volume of a sphere with a radius of 7.5 meters. Round to the nearest hundredth. 1,767.15 m^3

96. HIGHWAY SAFETY CONES



a. Find the area covered by the square rubber base if its sides are 10 inches long. 100 in.^2

b. The safety cone is centered atop the base, as shown. Give its exact volume and its approximate volume, rounded to the nearest tenth. $80\pi \approx 251.3 \text{ in.}^3$

Solve each formula for the specified variable.

97. $V = \frac{1}{3}\pi r^2 h$ for h $h = \frac{3V}{\pi r^2}$

98. $K = \frac{Mv_0^2 + Iw^2}{2}$ for M $M = \frac{2K - Iw^2}{v_0^2}$

99. $l = a + (n - 1)d$ for d $d = \frac{l - a}{n - 1}$

100. $9x - 5y = 35$ for y $y = \frac{9}{5}x - 7$

SECTION 1.7 Using Equations to Solve Problems

DEFINITIONS AND CONCEPTS

Problem-solving strategy:

1. Analyze the problem.
2. Form an equation.
3. Solve the equation.
4. State the conclusion.
5. Check the result.

Some problem-solving hints

Diagrams are often helpful in solving application problems. See page 74.

For problems that deal with quantities that have a value (see page 75), use the relationship:

Number · value = total value

EXAMPLES

BILLIONAIRES In 2007, *Forbes* magazine ranked Bill Gates and Warren Buffet as the two richest Americans. Their combined net worth was estimated to be \$108.4 billion, with Gates the wealthier by \$3.6 billion. Find the net worth of each man.

Analyze The phrase *combined net worth* suggests addition. If Gates was the wealthier, then his net worth was \$3.6 billion *more than* Buffet's.

Form Let x = Buffet's net worth in billions of dollars. Since Gates is the wealthier, $x + 3.6$ = Gates's net worth. We can use the words of the problem to form an equation.

Buffet's net worth	plus	Gate's net worth	equals	\$108.4 billion.
x	$+$	$x + 3.6$	$=$	108.4

It is often helpful to list the facts of a number-value problem in a **table.**

Sometimes a **geometric fact** or formula can be used to solve a problem. See pages 76–77.

The given facts of a problem often suggest a formula that can be used to model the situation mathematically. See page 78.

Solve

$$2x + 3.6 = 108.4 \quad \text{Combine like terms.}$$
$$2x = 104.8 \quad \text{Subtract 3.6 from both sides.}$$
$$x = 52.4 \quad \text{Divide both sides by 2.}$$

To find Gates's net worth we evaluate $x + 3.6$ for $x = 52.4$.

$$x + 3.6 = \mathbf{52.4} + 3.6 = 56$$

State In 2007, Warren Buffet's net worth was \$52.4 billion and Bill Gates's net worth was \$56 billion.

Check The sum of \$52.4 billion and \$56 billion is \$108.4 billion, and \$56 billion is \$3.6 billion more than \$52.4 billion. The answers check.

REVIEW EXERCISES

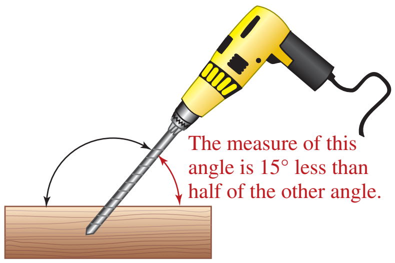
101. AIRPORTS The world's two busiest airports are Hartsfield Atlanta International and Chicago O'Hare International. Together they served 161 million passengers in 2006, with Atlanta handling 8.6 million more than O'Hare. How many passengers did each airport serve?
O'Hare: 76.2 million, Atlanta: 84.8 million

102. TUITION A private school reduces the monthly tuition cost of \$245 by \$5 per child if a family has more than one child attending the school. Write an algebraic expression that gives the monthly tuition cost per child for a family having c children.
$245 - 5c$

103. WAREHOUSING A large warehouse stores 150 more computers than printers. The monthly storage cost for a computer is \$2.50 and a printer is \$1.50. If storage for the computers and printers is \$2,775 per month, how many printers are in the warehouse? 600

104. CABLE TV A 186-foot television cable is to be cut into four pieces. Find the length of each piece if each successive piece is 3 feet longer than the previous one.
42 ft, 45 ft, 48 ft, 51 ft

105. TOOLING The illustration shows the angle at which a drill is to be held when drilling a hole into a piece of aluminum. Find the measures of both labeled angles.
50°, 130°



106. COLLECTIBLES In North America, most new movie releases are advertised using a poster size commonly referred to as a *one-sheet.* A one-sheet movie poster is rectangular, has a perimeter of 134 inches, and its length is 13 inches longer than its width. Find the dimensions of a one-sheet.
27 in. by 40 in.

SECTION 1.8 More about Problem Solving

DEFINITIONS AND CONCEPTS	EXAMPLES
Percent means parts per one hundred.	$7\% = \frac{7}{100} = 0.07$ $\quad 125\% = \frac{125}{100} = 1.25$
One method to solve percent problems is to use the given facts to write a **percent sentence** of the form: $\boxed{}$ is $\boxed{}$ % of $\boxed{}$? Then we translate the sentence to mathematical symbols and solve the resulting equation. Always find the **percent of increase** (or decrease) with respect to the *original* amount.	TAXES In Texas, \$31.25 in state sales tax is charged on a \$500 purchase. What is the Texas state sales tax rate? $\boxed{31.25}$ is $\boxed{\text{what percent}}$ of $\boxed{500?}$ $31.25 \;=\; x \;\cdot\; 500$ $31.25 = 500x$ $\frac{31.25}{500} = x$ To isolate x, divide both sides by 500. $0.0625 = x$ Do the division. $6.25\% = x$ Change the decimal to a percent. The Texas state sales tax rate is 6.25%.
Three types of averages are commonly used in statistics: $\textbf{Mean} = \frac{\text{sum of the values}}{\text{number of values}}$ The **median** is the middle value after the values have been arranged in increasing order. The **mode** is the value that occurs most often.	Consider the values: 3, 5, 5, 6, 7, 8, 8, 8, 9, 10. The mean is: $\frac{3+5+5+6+7+8+8+8+9+10}{10} = 6.9$ The median is the average of the middle terms: $\frac{7+8}{2} = \frac{15}{2} = 7.5$ Since 8 occurs most often, 8 is the mode.
To solve **investment** problems involving simple interest, use the formula $I = Prt$ Interest = principal · rate · time	See page 88 for an example.
To solve **uniform motion** problems, use the formula $d = rt$ Distance = rate · time	See page 89 for an example.
To solve problems where a **dry mixture** of a specified value is created from two differently priced components, use **Amount · price = total value**	See page 90 for an example.
To solve a **liquid mixture** problem, where a desired strength solution is to be made from two solutions with different strengths (concentrations), use **Amount of solution · strength of the solution = amount of pure ingredient**	See page 91 for an example.

REVIEW EXERCISES

107. GROUNDHOG DAY According to groundhog.org, the weather-predicting groundhog has emerged from his burrow and seen his shadow 96 times, which is 80% of the years on record. How many groundhog days does this website have on record? 120

108. EARLY REGISTRATION An early-bird discount lowers the registration fee for a financial seminar from \$550 to \$375. Find the percent of markdown. Round to the nearest percent. 32%

109. CAR SALES

a. Determine the percent of increase in the number of Toyota Camrys sold in 2006 compared with 2005. Round to the nearest tenth of one percent. 3.4%

b. Determine the percent of decrease in the number of Honda Accords sold in 2006 compared with 2005. Round to the nearest tenth of one percent. 4.0%

The Two Top-Selling Passenger Cars in the U.S.

2006	
1. Toyota Camry	448,445
2. Honda Accord	354,441
2005	
1. Toyota Camry	433,703
2. Honda Accord	369,293

Source: *MSN Autos*

110. HURRICANES The following table gives the number of hurricanes that made landfall in the United States for each of the years 1993–2006. Find the mean, median, and mode. 2, 1.5, 0

1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
1	0	3	2	1	3	2	0	0	1	2	6	7	0

Source: Insurance Information Institute

111. INVESTMENTS Sally has \$25,000 to invest. She invests some money at 10% interest and the rest at 9%. If her total annual income from these two investments is \$2,430, how much does she invest at each rate? \$18,000 at 10%, \$7,000 at 9%

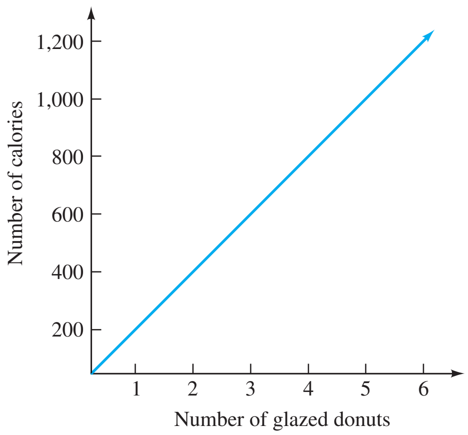
112. PAPARAZZI A celebrity leaves a nightclub in his car and travels at 1 mile per minute (60 mph) trying to avoid a tabloid photographer. One minute later, the photographer leaves the nightclub on his motorcycle, traveling at 1.5 miles per minute (90 mph) in pursuit of the celebrity. How long will it take the photographer to catch up with the celebrity? 2 min after the photographer leaves

113. PEST CONTROL How much of a 4% pesticide solution must be added to 20 gallons of a 12% pesticide solution to dilute it to a 10% solution? $6\frac{2}{3}$ gal

114. COFFEE Mild coffee that sells for \$7.50 per pound is to be mixed with a robust coffee that sells for \$8.40 per pound to make 90 pounds of a mixture that will be sold for \$7.90 per pound. How many pounds of each type of coffee should be used? mild: 50 lb, robust: 40 lb

CHAPTER 1 TEST

1. Fill in the blanks.
   a. For any nonzero real number a, $\frac{a}{0}$ is ___undefined___.
   b. $>, \geq, <$, and $\leq$ are called ___inequality___ symbols.
   c. $9x^2$ and $7x^2$ are ___like___ ___terms___ because they have the same variable raised to exactly the same power.
   d. To ___solve___ an equation means to find all of the values of the variable that make the equation true.
   e. The ___addition___ property of ___equality___ says that adding the same number to both sides of an equation does not change the solution.
2. Translate each verbal model into a mathematical model.
   a. Each test score T was increased by 10 points to give a new adjusted test score s. $s = T + 10$
   b. The area A of a triangle is the product of one-half the length of the base b and the height h. $A = \frac{1}{2}bh$
3. COUNTING CALORIES Refer to the graph below that gives the number of calories in a given number of glazed donuts.
   a. What units are used to scale the vertical axis? 200 calories
   b. How many calories are in one-half dozen glazed donuts? 1,200
   c. How many glazed donuts did a person eat if his calorie intake from the donuts was 700? $3\frac{1}{2}$

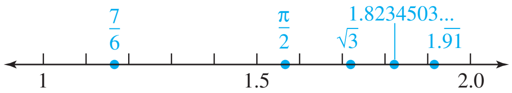


4. Consider the set: $\left\{-2, \pi, 0, -3\frac{3}{4}, 9.2, \frac{14}{5}, 5, -\sqrt{7}\right\}$
   a. Which numbers are integers? $-2, 0, 5$
   b. Which numbers are rational numbers? $-2, 0, -3\frac{3}{4}, 9.2, \frac{14}{5}, 5$
   c. Which numbers are irrational numbers? $\pi, -\sqrt{7}$
   d. Which numbers are real numbers? all

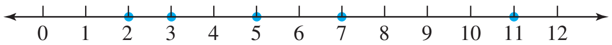
5. Determine whether each statement is true or false.
   a. $-6 \in \mathbb{Z}$ true
   b. $76 \notin \mathbb{N}$ false
   c. $\mathbb{W} \subseteq \mathbb{R}$ true
   d. $\mathbb{H} \not\subseteq \mathbb{Q}$ true

Graph each set on the number line.

6. $\left\{\frac{7}{6}, \frac{\pi}{2}, 1.8234503\ldots, \sqrt{3}, 1.9\overline{1}\right\}$



7. The set of prime numbers less than 12



8. Determine whether each statement is true or false.
   a. $-|-2.78| > -(-2.71)$ false
   b. $(-3)^2 \leq -3^2$ false

Evaluate each expression.

9. $-\dfrac{5}{3} \div \left(-\dfrac{25}{4}\right)$ $\frac{4}{15}$

10. $\dfrac{2|-4 - 2(3 - 1)|}{-3(\sqrt{9})(-2)}$ $\frac{8}{9}$

11. $10 - 3[5^2 - 6(-1 - 1)^3]$ -209

12. Evaluate the expression for $a = 2$, $b = -3$, and $c = 4$.
$$\frac{(-3b + c)^2 - 17a}{-b + a^2bc}$$ -3

13. PEDIATRICS Some doctors use Young's rule in calculating the dose for infants and children.
$$\frac{\text{Age of child}}{\text{Age of child} + 12}\left(\begin{matrix}\text{average}\\ \text{adult dose}\end{matrix}\right) = \text{child's dose}$$
The adult dose of Achromycin is 250 milligrams (mg). What is the dose for an 8-year-old child? 100 mg

Determine which property of real numbers justifies each statement.

14. a. $3 + 5 = 5 + 3$ commutative property of addition
    b. $x(yz) = (xy)z$ associative property of multiplication
    c. $-17 + 17 = 0$ additive inverse property
    d. $\dfrac{1}{2} \cdot 1 = \dfrac{1}{2}$ multiplicative identity property

Simplify each expression.

15. $11.1n^2 - 7.3n + 15.1n - 9.8$ $11.1n^2 - 7.8n - 9.8$

16. $-5(9s)(-2t)$ $90st$

17. $-7(c - 4) - 5[3(c - 4) - 2(c + 2)]$ $-12c + 108$

18. $\frac{2}{9}(xy + 45x) - \frac{1}{4}(xy - 24x)$ $-\frac{1}{36}xy + 16x$

Solve each equation.

19. $9(x + 4) + 4 - 8x = 4(x - 5) + x$ 15

20. $\frac{m - 1}{5} = \frac{2m - 3}{3} - 2$ 6

21. $6 - (x - 3) - 5x = 3[1 - 2(x + 2)]$
no solution, $\varnothing$; contradiction

22. $\frac{1}{2}r - \frac{7}{6} = -\frac{1}{3}r + \frac{53}{6}$ 12

23. Use a check to determine whether 6.7 is a solution of $1.6y + (-3) = y + 1.02$. yes

24. Solve $P = L + \frac{s}{f}i$ for i. $i = \frac{f(P - L)}{s}$

25. Solve $y - y_1 = m(x - x_1)$ for x_1. $x_1 = \frac{y_1 + mx - y}{m}$

26. CROP CIRCLES In 1992, two Hungarian high school students were charged for the damage that they caused in creating a 36-meter diameter crop circle in a wheat field. Find the area covered by the crop circle. Round to the nearest square meter. 1,018 m^2

27. HAND TOOLS With each pass that a craftsman makes with a sander over a piece of fiberglass, he removes 0.03125 inch of thickness. If the fiberglass was originally 0.9375 inch thick, how many passes are needed to obtain the desired thickness of 0.6875 inch? 8

28. RENTALS The owners of an apartment building rent equal numbers of 1- and 2-bedroom units. The monthly rent for a 1-bedroom is $950, and a 2-bedroom is $1,200. If the total monthly income is $53,750, how many of each type of unit are there? 25

29. ISOSCELES TRIANGLES The measure of a base angle of an isosceles triangle is 5° more than eight times the measure of the vertex angle. Find the measure of each angle of the triangle. 85°, 85°, 10°

30. CALCULATORS The viewing window of a calculator has a perimeter of 26 centimeters and is 5 centimeters longer than it is wide. Find the dimensions of the window. 4 cm by 9 cm

31. FUEL EFFICIENCY Use the data below to determine the percent of increase in U.S. sales of hybrid vehicles from 2005 to 2006. Round to the nearest percent. 28%

U.S. Gas–Electric Hybrid Vehicle Sales
2005: 199,148
2006: 254,545

Source: MSNBC.com

32. AIRLINE ACCIDENTS Refer to the data in the table.

a. Find the mean. Round to the nearest tenth. 2.2

b. Find the median. 2

c. Find the mode. 2

Number of Major Accidents for U.S. Air Carriers (1997–2006)

1997	1998	1999	2000	2001
2	0	2	3	5
2002	**2003**	**2004**	**2005**	**2006**
1	2	4	2	1

Source: National Transportation Safety Board

33. INVESTING An investment club invested part of $10,000 at 9% annual interest and the rest at 8%. If the annual income from these investments was $860, how much was invested at 8%? $4,000

34. RENTAL CARS While waiting for his car to be repaired, a man rents a car for $17 per day and 33 cents per mile. His insurance company will pay up to $200 of the rental fee. If he needs the car for four days, how many miles of driving will his policy cover? 400 mi

35. MIXING ALLOYS How many ounces of a 40% copper alloy must be mixed with 10 ounces of a 10% copper alloy to obtain an alloy that is 25% copper? 10 oz

36. MEN'S COLOGNE How many ounces of Skin Soother men's cologne (unit price: $2.40 per ounce) must be mixed with Cool Sport men's cologne (unit price: $1.60 per ounce) to make 8 ounces of a mixture having a unit price of $1.90 per ounce?
Skin Soother: 3 oz, Cool Sport: 5 oz

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2

Graphs, Equations of Lines, and Functions

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from Campus to Careers

Certified Fitness Instructor

Because of our busy schedules, many of us have difficulty making exercise a part of our daily routine. A certified fitness instructor can often provide the motivation, discipline, and instruction that a person needs to get and stay in shape. Fitness instructors plan and lead classes, weigh and measure clients, analyze records and graphs, and perform assessment and testing related to weight training and cardiovascular exercise.

Fitness instructors stress the importance of healthy eating and regular exercise. In **problem 93** of **Study Set 2.4,** you will see how a graph can be used to show the relationship between the time spent exercising and the number of calories burned.

JOB TITLE:
Certified Fitness Instructor
EDUCATION: An increasing number of employers are requiring a bachelor's degree in a health-related field. Also, some level of training certification is often required.
JOB OUTLOOK: Good because of rapid growth in the fitness industry.
ANNUAL EARNINGS: $44,088 is the median annual salary.
FOR MORE INFORMATION:
www.bls.gov/oco/ocos296.htm

Objectives

1. Plot ordered pairs and determine the coordinates of a point.
2. Graph paired data.
3. Read graphs.
4. Find the midpoint of a line segment.

SECTION 2.1 The Rectangular Coordinate System

It is often said that a picture is worth a thousand words. In this chapter, we will show how numerical relationships can be described by mathematical pictures called *graphs*.

1 Plot ordered pairs and determine the coordinates of a point.

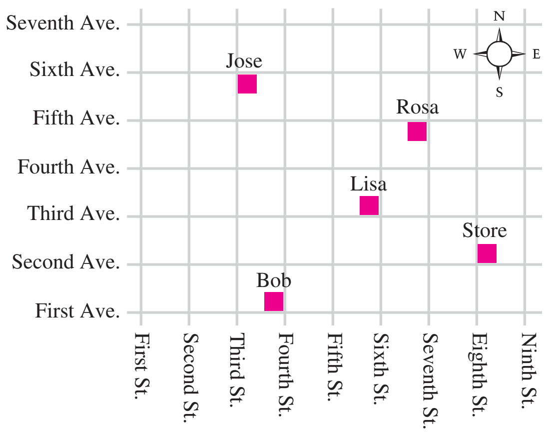
Many cities are laid out on a rectangular grid as shown below. For example, on the east side of Rockford, Illinois, all streets run north and south, and all avenues run east and west. If we agree to list the street numbers first, every address can be identified by using an ordered pair of numbers. If Jose Montoya lives on the corner of Third Street and Sixth Avenue, his address is given by the ordered pair (3, 6).

This is the street. This is the avenue.

(3, 6)

If Lisa Kumar has an address of (6, 3), we know that she lives on the corner of Sixth Street and Third Avenue. From the figure, we can see that

- Bob Anderson's address is (4, 1).
- Rosa Vang's address is (7, 5).
- The address of the store is (8, 2).



The idea of associating an ordered pair of numbers with points on a grid is attributed to the 17th-century French mathematician René Descartes. The grid is called a **rectangular coordinate system,** or **Cartesian coordinate system** after its inventor.

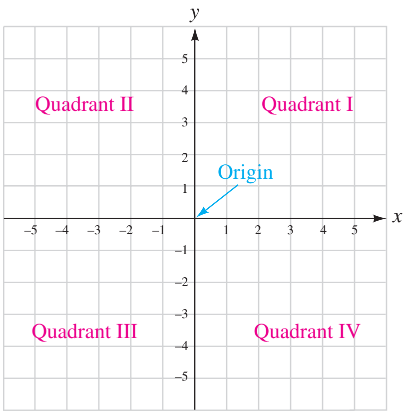
In general, a rectangular coordinate system is formed by two intersecting perpendicular number lines, as shown in the figure on the next page.

- The horizontal number line is usually called the ***x*-axis.**
- The vertical number line is usually called the ***y*-axis.**

The positive direction on the x-axis is to the right, and the positive direction on the y-axis is upward.

The point where the axes cross is called the **origin.** This is the 0 point on each axis. The two axes form a **coordinate plane** and divide it into four regions called **quadrants,** which are numbered using Roman numerals.

Every point on a coordinate plane can be identified by a pair of real numbers x and y, written as (x, y). The first number in the pair is the ***x*-coordinate,** and the second number is the ***y*-coordinate.** The numbers are called the **coordinates** of the point. Some examples of such pairs are $(-4, 6)$, $(2, 3)$, and $(6, -4)$.

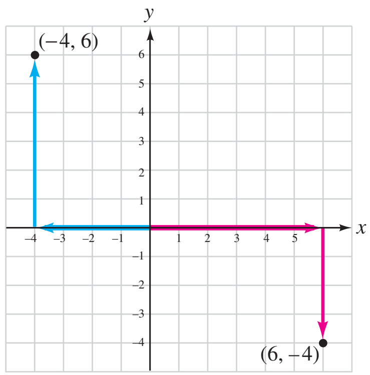


$(6, -4)$

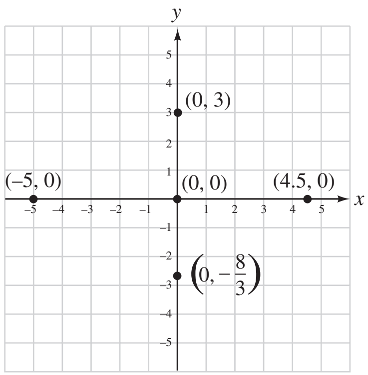
The x-coordinate is listed first. The y-coordinate is listed second.

The process of locating a point in the coordinate plane is called **graphing** or **plotting** the point. Below, we use red arrows to graph the point with coordinates $(6, -4)$. Since the x-coordinate, 6, is positive, we start at the origin and move 6 units to the *right* along the x-axis. Since the y-coordinate, -4, is negative, we then move *down* 4 units, and draw a dot. This locates the point $(6, -4)$, which lies in quadrant IV.

In figure (a), blue arrows are used to show how to plot $(-4, 6)$. We start at the origin, move 4 units to the *left* along the x-axis, and then 6 units *up* and draw a dot. This locates the point $(-4, 6)$, which lies in quadrant II.



(a)



(b)

Caution! Note that the point $(-4, 6)$ is not the same as the point $(6, -4)$. This illustrates that the order of the coordinates of a point is important. This is why we call the pairs **ordered pairs.**

In figure (b), we see that the points $(-5, 0)$, $(0, 0)$, and $(4.5, 0)$ all lie on the x-axis. In fact, every point with a y-coordinate of 0 lies on the x-axis. Note that the coordinates of the origin are $(0, 0)$.

We also see that the points $\left(0, -\frac{8}{3}\right)$, $(0, 0)$, and $(0, 3)$ lie on the y-axis. In fact, every point with an x-coordinate of 0 lies on the y-axis.

A point may lie in one of the four quadrants or it may lie on one of the axes, in which case the point is not considered to be in any quadrant. For points in quadrant I,

the x- and y-coordinates are positive. Points in quadrant II have a negative x-coordinate and a positive y-coordinate. In quadrant III, both coordinates are negative. In quadrant IV, the x-coordinate is positive and the y-coordinate is negative.

Success Tip If no scale is indicated on the axes, we assume that they are scaled in units of 1.

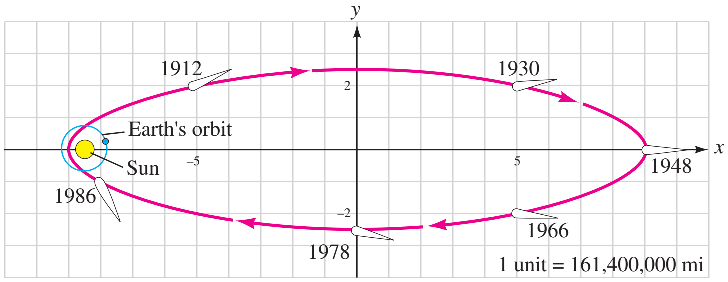
Self Check 1

HALLEY'S COMET Determine the comet's position for 1978. $(0, -2.5)$

Now Try **Problems 30, 33, and 34**

Teaching Example 1 HALLEY'S COMET Use the graph to determine the comet's position for the most recent time it passed by Earth in 1986.
Answer:
$(-8, -1)$

EXAMPLE 1 ***Halley's Comet*** Halley's comet passes Earth every 76 years as it travels in an elliptical orbit about the sun. Use the graph to determine the comet's position for the years 1912, 1930, 1948, and 1966.



Strategy We will start at the origin and count to the left or right on the x-axis, and then up or down to reach the point.

WHY The movement left or right from the origin gives the x-coordinate of the ordered pair and the movement up or down from the origin gives the y-coordinate.

Solution

Year	Position of comet on graph	Coordinates
1912	5 units to the *left*, then 2 units *up*	$(-5, 2)$
1930	5 units to the *right*, then 2 units *up*	$(5, 2)$
1948	9 units to the *right*, no units *up* or *down*	$(9, 0)$
1966	5 units to the *right*, then 2 units *down*	$(5, -2)$

2 Graph paired data.

Every day, we deal with quantities that are related.

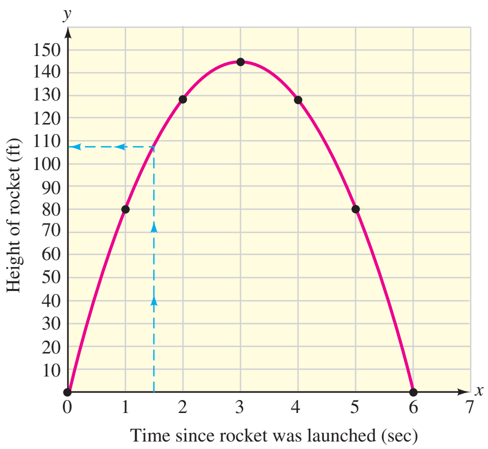
- The distance we travel depends on how fast we are going.
- Your test score depends on the amount of time you study.
- The height of a toy rocket depends on the time since it was shot into the air.

We can use graphs to visualize relationships between two quantities. For example, suppose that we know the height of a toy rocket at 1-second intervals from 0 to 6 seconds. We can list this information in the table shown in the figure on the next page.

Each entry in the table represents an ordered pair, where the x-coordinate is the time (in seconds) since the rocket was shot into the air, and the y-coordinate is the height of the rocket (in feet).

Time (in seconds)	Height of rocket (in feet)	
0	0	→ (0, 0)
1	80	→ (1, 80)
2	128	→ (2, 128)
3	144	→ (3, 144)
4	128	→ (4, 128)
5	80	→ (5, 80)
6	0	→ (6, 0)
↑ x-coordinate	↑ y-coordinate	↑ The data in the table can be expressed as ordered pairs.

The information in the table can be used to construct a graph that shows the relationship between the height of the rocket and the time since it was shot into the air. To get the graph in the figure below, we plot the seven ordered pairs shown in the table and draw a smooth curve through the data points.



This graph shows the height of the rocket in relation to the time since it was shot into the air. It does not show the path of the rocket.

From the graph, we can see that the height of the rocket increases as the time increases from 0 second to 3 seconds. Then the height decreases until the rocket hits the ground in 6 seconds. We can also use the graph to make observations about the height of the rocket at other times. For example, the dashed blue lines on the graph show that in 1.5 seconds, the height of the rocket will be approximately 108 feet.

3 Read graphs.

Since graphs are becoming an increasingly popular way to present information, the ability to read and interpret them is becoming ever more important.

Self Check 2

WATER MANAGEMENT Refer to the graph in Example 2 to answer the following questions.

a. When was the water at the normal level?

b. By how many feet did the water level rise during the storm?

c. After the storm ended and the water level began to fall, how long did it take for the water level to return to normal?

Now Try **Problem 38**

Self Check 2 Answer

a. 4 days before the storm began and 1 day and 9 days after the storm began

b. 6 ft **c.** 4 days

Teaching Example 2 WATER MANAGEMENT Refer to the graph in Example 2 to answer the following questions.

a. How long did the storm last?

b. Before the storm started, when was the water level one foot below normal?

Answers:

a. 3 days

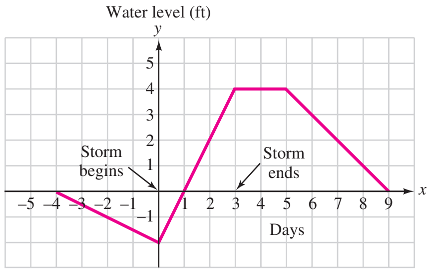
b. 2 days before the storm

EXAMPLE 2 ***Water Management*** The graph below shows the water level of a reservoir before, during, and after a storm. On the x-axis, zero represents the day the storm began. On the y-axis, zero represents the normal water level that operators try to maintain.

a. In anticipation of the storm, operators released water to lower the level of the reservoir. By how many feet was the water lowered prior to the storm?

b. After the storm ended, on what day did the water level begin to fall?

c. When was the water level 2 feet above normal?

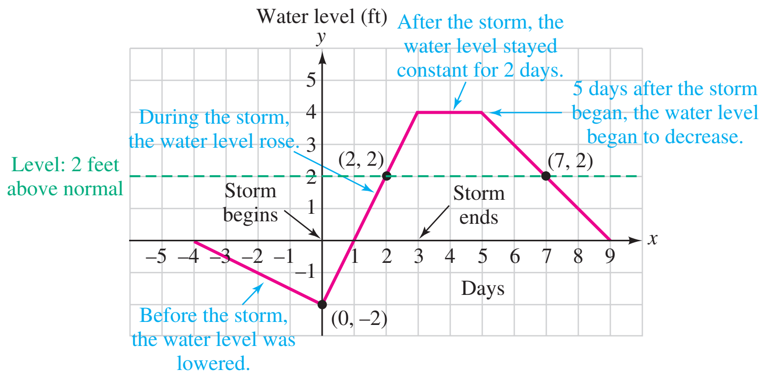


Strategy We will use ordered pairs to describe the situations mentioned in parts a, b, and c.

WHY The coordinates of specific points on the graph can be used to answer the given questions.

Solution

a. The graph starts at the point $(-4, 0)$. This means that four days before the storm began, the water level was at the normal level. If we look below zero on the y-axis, we see that the point $(0, -2)$ is on the graph. So the day the storm began, the water level had been lowered 2 feet.



b. If we look at the x-axis, we see that the storm lasted 3 days. From the third to the fifth day, the water level remained constant, 4 feet above normal. The graph does not begin to decrease until day 5.

c. We can draw a horizontal line passing through 2 on the y-axis. This line intersects the graph in two places—at the points $(2, 2)$ and $(7, 2)$. This means that 2 days and 7 days after the storm began, the water level was 2 feet above normal.

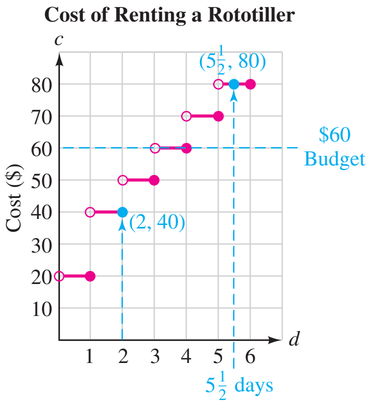
EXAMPLE 3 ***Rental Costs*** Use the graph to answer the following questions.

a. Find the cost of renting the rototiller for 2 days.

b. Find the cost of renting the rototiller for $5\frac{1}{2}$ days.

c. How long can you rent the rototiller if you have budgeted $60 for the rental?

d. Is the cost of renting the rototiller the same each day?



Strategy To answer questions about the rental costs, we will scan from the horizontal axis, up and over, to the vertical axis. To answer questions about length of time the equipment can be rented, we will scan from the vertical axis, over and down, to the horizontal axis.

WHY The scale on the vertical axis gives the cost to rent the equipment. The scale on the horizontal axis gives the length of time the equipment is rented.

Solution

a. We locate 2 on the d-axis and move up to locate the point on the graph directly above the 2. Since that point has coordinates (2, 40), a two-day rental costs $40.

b. We locate $5\frac{1}{2}$ on the d-axis and move straight up to locate the point on the graph with coordinates $\left(5\frac{1}{2}, 80\right)$, which indicates that a $5\frac{1}{2}$-day rental would cost $80.

c. We draw a horizontal line through the point labeled 60 on the c-axis. Since this line intersects one of the steps of the graph, we can look down to the d-axis to find the d-values that correspond to a c-value of 60. We see that the rototiller can be rented for more than 3 and up to 4 days for $60.

d. The cost each day is not the same. If we look at how the c-coordinates change, we see that the first-day rental fee is $20. The second day, the cost jumps another $20. The third day, and all subsequent days, the cost jumps $10.

Self Check 3

RENTAL COSTS Use the graph in Example 3 to find the cost of renting the rototiller for $2\frac{1}{2}$ days. $50

Now Try **Problem 42**

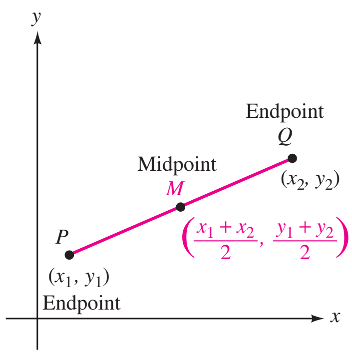
Teaching Example 3 RENTAL COSTS Use the graph in Example 3 to find the cost of renting the rototiller for $3\frac{1}{2}$ days.
Answer:
$60

We have seen that points in the coordinate plane can be identified using ordered pairs of real numbers. There is a formula we can use to find the coordinates of the midpoint of a line segment joining two points in the plane.

4 Find the midpoint of a line segment.

If point M in the figure to the right lies midway between point P and point Q, it is called the **midpoint** of line segment PQ. We call the points P and Q, the **endpoints** of the segment.

To distinguish between the coordinates of the endpoints of a line segment, we can use *subscript notation.* In the figure, the point P with coordinates (x_1, y_1) is read as "point P with coordinates x sub 1 and y sub 1," and the point Q with coordinates (x_2, y_2) is read as "point Q with coordinates x sub 2 and y sub 2."



To find the coordinates of point M, we find the average of the x-coordinates and the average of the y-coordinates of points P and Q. Using subscript notation, we can write the midpoint formula in the following way.

The Midpoint Formula

The **midpoint** of a line segment with endpoints (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The Language of Algebra The prefix *sub* means below or beneath, as in submarine or subway. In x_2, the *subscript* 2 is written lower than the variable. It is not an exponent.

Caution! Note the difference between x^2 and x_2. In the notation x^2, the exponent 2 is a *superscript,* used to represent repeated multiplication: $x^2 = x \cdot x$. In the notation x_2, the 2 is a *subscript.* In this section, it is used to represent the x-coordinate of a point.

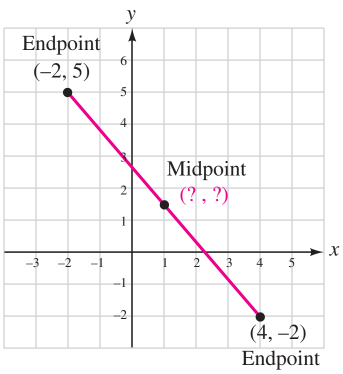
Self Check 4

Find the midpoint of the line segment with endpoints $(-1, 8)$ and $(5, 2)$. $(2, 5)$

Now Try Problem 46

Teaching Example 4 Find the midpoint of the line segment with endpoints $(7, -3)$ and $(5, -7)$.
Answer:
$(6, -5)$

EXAMPLE 4 Find the midpoint of the line segment with endpoints $(-2, 5)$ and $(4, -2)$.



Strategy To find the coordinates of the midpoint, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.

WHY This is what is called for by the expressions $\frac{x_1 + x_2}{2}$ and $\frac{y_1 + y_2}{2}$ of the midpoint formula.

Solution

We can let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (4, -2)$. After substituting these values into the expressions for the x- and y-coordinates in the midpoint formula, we will evaluate each expression to find the coordinates of the midpoint.

$$\frac{x_1 + x_2}{2} = \frac{-2 + 4}{2} = \frac{2}{2} = 1 \qquad \frac{y_1 + y_2}{2} = \frac{5 + (-2)}{2} = \frac{3}{2}$$

Thus, the midpoint is $\left(1, \frac{3}{2}\right)$.

Self Check 5

If the midpoint of a segment PQ is $M(-2, 5)$ and one endpoint is $Q(6, -2)$, find the coordinates of point P. $(-10, 12)$

EXAMPLE 5 The midpoint of the line segment joining $P(-5, -3)$ and $Q(x_2, y_2)$ is the point $M(-1, 2)$. Find the coordinates of point Q.

Strategy As in Example 4, we will use the midpoint formula to find the unknown coordinates. However, this time, we need to find x_2 and y_2.

WHY We want to find the coordinates of one of the endpoints.

Solution

We can let $P(x_1, y_1) = P(-5, -3)$ and $M(x_M, y_M) = M(-1, 2)$, where x_M represents the x-coordinate of point M, and y_M represents the y-coordinate of point M. We can then find the coordinates of point Q using the midpoint formula.

$$x_M = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad y_M = \frac{y_1 + y_2}{2}$$

$$-1 = \frac{-5 + x_2}{2} \qquad\qquad 2 = \frac{-3 + y_2}{2}$$

$$-2 = -5 + x_2 \qquad\qquad 4 = -3 + y_2 \qquad \text{Multiply both sides by 2.}$$

$$3 = x_2 \qquad\qquad 7 = y_2$$

Since $x_2 = 3$ and $y_2 = 7$, the coordinates of point Q are (3, 7).

Now Try **Problem 57**

Teaching Example 5 If the midpoint of a segment PQ is $M(-3, 7)$ and one endpoint is $Q(5, 1)$, find the coordinates of P.
Answer:
$(-11, 13)$

ANSWERS TO SELF CHECKS

1. $(0, -2.5)$ **2. a.** 4 days before the storm began and 1 day and 9 days after the storm began **b.** 6 ft **c.** 4 days **3.** \$50 **4.** (2,5) **5.** $(-10, 12)$

SECTION 2.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. The pair of numbers $(6, -2)$ is called an ordered pair.
2. In the ordered pair $(-2, -9)$, -2 is called the x-coordinate and -9 is called the y-coordinate.
3. Ordered pairs of numbers can be graphed on a rectangular coordinate system.
4. The x- and y-axes divide the coordinate plane into four regions called quadrants.
5. The point with coordinates (0, 0) is the origin.
6. The process of locating a point on a coordinate plane is called graphing or plotting the point.
7. If a point is midway between two points P and Q, it is called the midpoint of segment PQ.
8. If a line segment joins points P and Q, points P and Q are called endpoints of the segment.

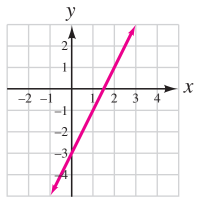
CONCEPTS

Fill in the blanks.

9. To plot the point $(6, -3.5)$, we start at the origin and move 6 units to the right and then 3.5 units down.
10. To plot the point $\left(-6, \frac{3}{2}\right)$, we start at the origin and move 6 units to the left and then $\frac{3}{2}$ units up.
11. In which quadrant do points with a negative x-coordinate and a positive y-coordinate lie? II
12. In which quadrant do points with a positive x-coordinate and a negative y-coordinate lie? IV

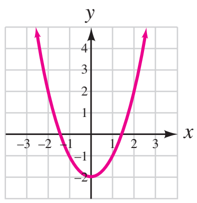
13. Use the graph to complete the table.

x	y
0	−3
1	−1
2	1
3	3



14. Use the graph to complete the table.

x	y
−2	2
−1	−1
0	−2
1	−1
2	2



15. How many midpoints does a segment have? one
16. The x-coordinate of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) is $\frac{x_1 + x_2}{2}$, and the y-coordinate is $\frac{y_1 + y_2}{2}$.

NOTATION

17. For the ordered pair (t, d), which variable is associated with the horizontal axis? t
18. Do these ordered pairs name the same point?
$\left(5.25, -\frac{3}{2}\right), \left(5\frac{1}{4}, -1.5\right), \left(\frac{21}{4}, -1\frac{1}{2}\right)$ yes

Selected exercises available online at
www.webassign.net/brookscole

19. How do you read the expression "x_1"? x sub 1

20. Explain the difference between x^2 and x_2.
$x^2 = x \cdot x$; x_2 represents the x-coordinate of a point.

GUIDED PRACTICE

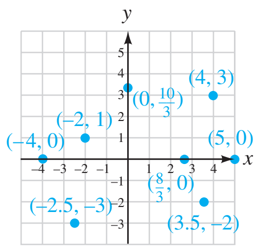
Plot each point on the rectangular coordinate system. **See Objective 1.**

21. (4, 3) **22.** (−2, 1)

23. (3.5, −2) ▶ **24.** (−2.5, −3)

25. (5, 0) ▶ **26.** (−4, 0)

27. $\left(\frac{8}{3}, 0\right)$ **28.** $\left(0, \frac{10}{3}\right)$



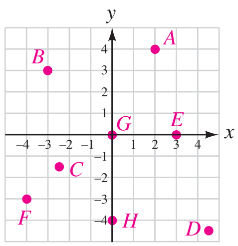
Give the coordinates of each point shown. **See Example 1.**

29. A (2, 4) ▶ **30.** B (−3, 3)

31. C (−2.5, −1.5) **32.** D (4.5, −4.5)

33. E (3, 0) **34.** F (−4, −3)

35. G (0, 0) **36.** H (0, −4)



Use each graph to answer each question. **See Example 2.**

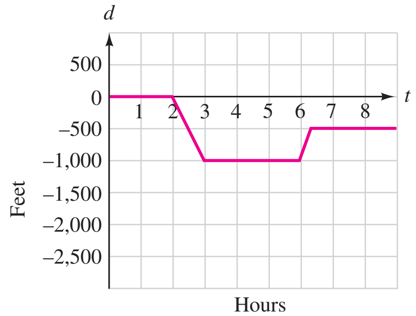
▶ **37.** SUBMARINES The graph in the next column shows the depths of a submarine at certain times.

a. Where is the sub when $t = 2$? on the surface

b. What is the sub doing as t increases from $t = 2$ to $t = 3$? diving

c. How deep is the sub when $t = 4$? 1,000 ft

d. How large an ascent does the sub begin to make when $t = 6$? 500 ft



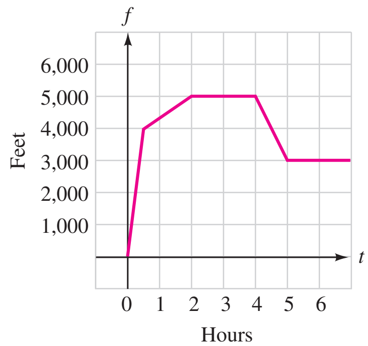
38. AIRPLANES The following graph shows the altitudes of a plane at certain times.

a. Where is the plane when $t = 0$? on the ground

b. What is the plane doing as t increases from $t = 1$ to $t = 2$? gaining altitude

c. What is the altitude of the plane when $t = 2$? 5,000 ft

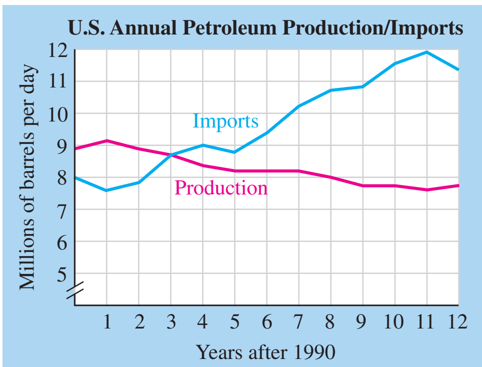
d. How much of a descent does the plane begin to make when $t = 4$? 2,000 ft



39. PETROLEUM Refer to the following graph.

a. When did imports first surpass production? 1993

b. Estimate the difference in U.S. petroleum imports and production for 2002.
Imports exceeded production by about 3.5 million barrels per day.



Source: United States Department of Energy

40. TRACK AND FIELD Refer to the following graph.

a. Which runner ran faster at the start of the race? 1

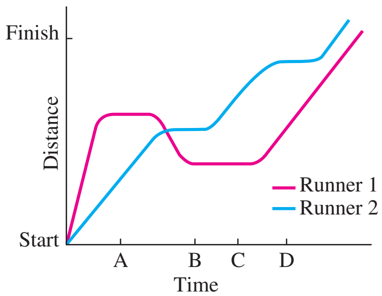
b. Which runner stopped to rest first? 1

c. Which runner dropped the baton and had to go back and get it? 1

d. At what times was runner 1 stopped and runner 2 running? A and C

e. Describe what was happening at time D. 1 was running, 2 was stopped.

f. Which runner won the race? 2



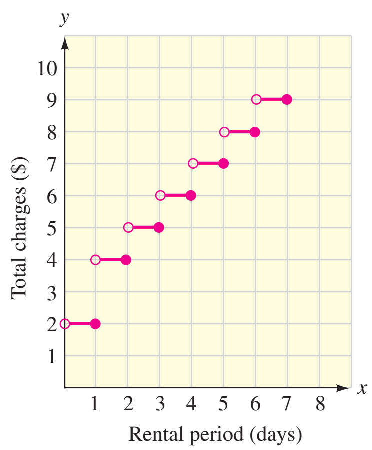
Use each graph to answer each question. See Example 3.

41. VIDEO RENTALS The charges for renting a video are shown in the following graph.

a. Find the 1-day rental charge. \$2

b. Find the charge if the video is kept for 1 week. \$9

c. How long can you keep a video if you budgeted \$5 for the rental? 3 days

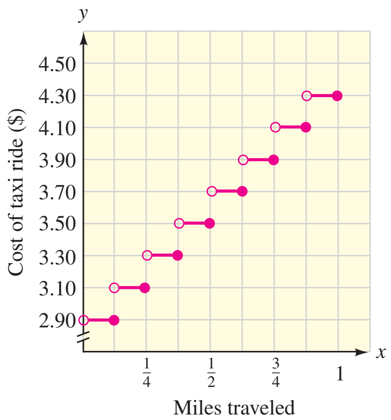


42. TAXIS The graph in the next column gives the fares charged for taxi rides up to 1 mile. In the graph, the symbol ⫲ indicates a break in the labeling of the vertical axis. This break enables us to omit a portion of the grid that would not be used.

a. Find the fare for a $\frac{1}{2}$-mile ride. \$3.50

b. Is the fare the same for each $\frac{1}{8}$-mile traveled? no

c. Find the fare for a $\frac{7}{10}$-mile ride. \$3.90



Find the midpoint of line segment PQ. See Example 4.

43. $P(0, 0), Q(6, 8)$ $(3, 4)$

44. $P(10, 12), Q(0, 0)$ $(5, 6)$

45. $P(6, 8), Q(12, 16)$ $(9, 12)$

46. $P(10, 4), Q(2, -2)$ $(6, 1)$

47. $P(2, 4), Q(5, 8)$ $\left(\frac{7}{2}, 6\right)$

48. $P(5, 9), Q(8, 13)$ $\left(\frac{13}{2}, 11\right)$

49. $P(-2, -8), Q(3, 4)$ $\left(\frac{1}{2}, -2\right)$

50. $P(-5, -2), Q(7, 3)$ $\left(1, \frac{1}{2}\right)$

51. $Q(-3, 5), P(-5, -5)$ $(-4, 0)$

52. $Q(2, -3), P(4, -8)$ $\left(3, -\frac{11}{2}\right)$

53. $Q(7, 1), P(-10, 4)$ $\left(-\frac{3}{2}, \frac{5}{2}\right)$

54. $Q(-4, -3), P(4, -8)$ $\left(0, -\frac{11}{2}\right)$

Solve each problem. See Example 5.

55. If $M(-2, 3)$ is the midpoint of segment PQ and the coordinates of P are $(-8, 5)$, find the coordinates of Q. $(4, 1)$

56. If $M(6, -5)$ is the midpoint of segment PQ and the coordinates of Q are $(-5, -8)$, find the coordinates of P. $(17, -2)$

57. If $M(-7, -3)$ is the midpoint of segment PQ and the coordinates of Q are $(6, -3)$, find the coordinates of P. $(-20, -3)$

58. If $M\left(\frac{1}{2}, -2\right)$ is the midpoint of segment PQ and the coordinates of P are $\left(-\frac{5}{2}, 5\right)$, find the coordinates of Q. $\left(\frac{7}{2}, -9\right)$

APPLICATIONS

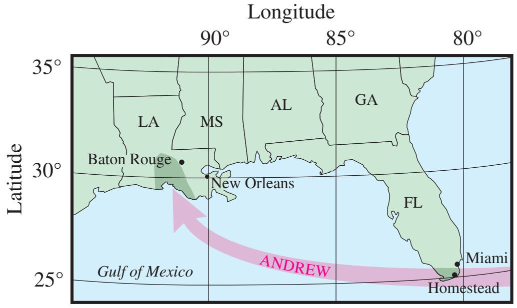
59. ROAD MAPS Road maps have a built-in coordinate system to help locate cities. Use the map below to find the coordinates of these cities in South Carolina: Jonesville, Easley, Hodges, and Union. Express each answer in the form (number, letter).
Jonesville (5, B), Easley (1, B), Hodges (2, E), Union (6, C)

60. HURRICANES A coordinate system that designates the location of places on the surface of Earth uses a series of latitude and longitude lines, as shown below.

a. If we agree to list longitude first, what are the coordinates of New Orleans, expressed as an ordered pair? (90, 30)

b. In August 1992, Hurricane Andrew destroyed Homestead, Florida. Estimate the coordinates of Homestead. (81, 25)

c. Estimate the coordinates of where the hurricane hit Louisiana. (92, 29)

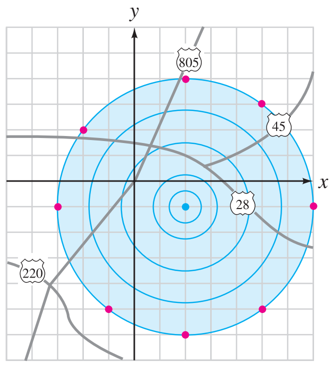


61. EARTHQUAKES The map in the next column shows the area where damage was caused by an earthquake.

a. Find the coordinates of the epicenter (the source of the quake). (2, −1)

b. Was damage done at the point (4, 5)? no

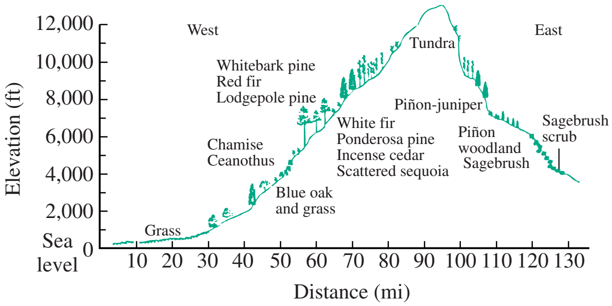
c. Was damage done at the point (−1, −4)? yes



62. GEOGRAPHY The illustration shows a cross-sectional profile of the Sierra Nevada mountain range in California.

a. Estimate the coordinates of blue oak, sagebrush scrub, and tundra using an ordered pair of the form (distance, elevation).
(50, 3,000), (128, 4,000), (100, 11,000)

b. The *tree line* is the highest elevation at which trees grow. Estimate the tree line for this mountain range.
about 10,000 ft



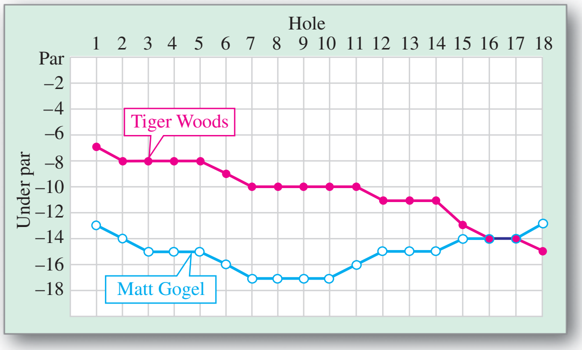
63. GOLF In one of the greatest comebacks in golf, Tiger Woods rallied in the final round of the 2000 AT&T Pebble Beach National Pro-Am golf tournament to overtake Matt Gogel. (In golf, the player with the score that is the farthest *under* par is the winner.) Refer to the graph on the next page and answer the following questions.

a. At the beginning of the final round, by how large was Gogel's lead? 6 strokes

b. In the final round, what was Gogel's largest lead? 7 strokes

c. On what hole did Woods tie up the match? 16th

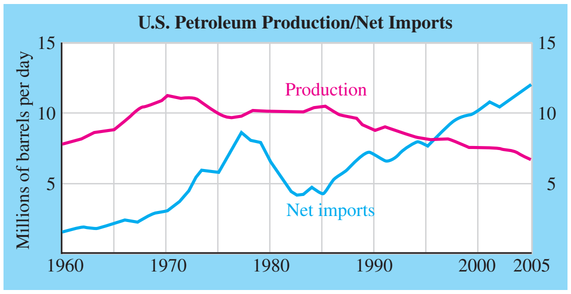
d. On what hole did Woods take the lead? 18th



64. PETROLEUM Refer to the graph below.

a. When did the U.S. net petroleum imports first surpass production? 1996

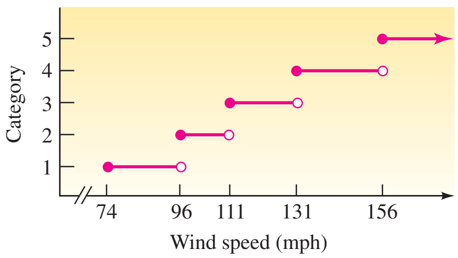
b. Estimate the difference in U.S. petroleum net imports and production for 2005.
Net imports exceeded production by about 5 million barrels per day.



Source: Energy Information Administration

65. HURRICANES The following graph shows the scale used to rate the intensity of hurricanes. Use the graph to categorize the hurricanes listed in the table below.

Saffir-Simpson Hurricane Scale



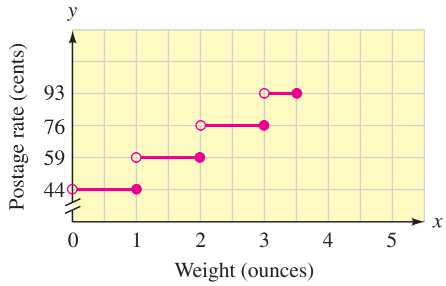
Name/Year	Location	Wind Speed	Category
Luis/1995	Leeward Isl.	138 mph	4
Fran/1996	North Carolina	115 mph	3
Danny/1997	Alabama	80 mph	1
Mitch/1998	Caribbean	178 mph	5
Bonnie/1998	North Carolina	110 mph	2

66. POSTAGE RATES The graph shows the first-class postage rates for mailing letters weighing up to 3.5 ounces.

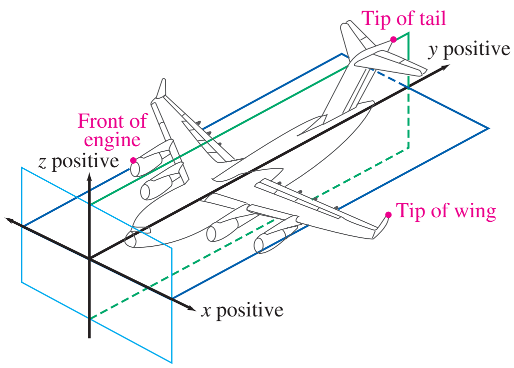
a. Find the cost to mail a letter that weighs 1-oz. 44¢

b. Find the difference in postage for a 0.75-oz letter and a 2.75-oz letter. 32¢

c. What is the heaviest letter that can be mailed first class for 59¢? 2 oz

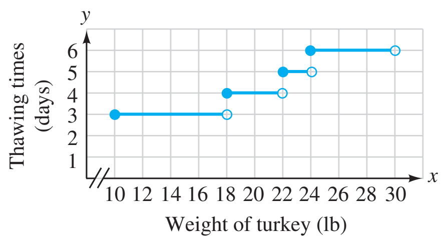


67. AIRPLANE DESIGN Engineers use a coordinate system with three axes when designing airplanes. As shown in the illustration, the x-axis is used to describe left/right on the airplane, the y-axis forward/backward, and the z-axis up/down. Any point on the airplane can be described by an *ordered triple* of the form (x, y, z). The coordinates of three points on the plane are $(0, 181, 56)$, $(-46, 48, 19)$, and $(84, 94, 24)$. Which highlighted part of the plane corresponds with which ordered triple?
tip of tail, front of engine, tip of wing



▶ **68.** ROAST TURKEY The thawing guidelines that appear on the label of a frozen turkey are listed in the following table. In the illustration, draw a step graph that represents these instructions.

Size	Refrigerator thawing
10 lb to just under 18 lb	3 days
18 lb to just under 22 lb	4 days
22 lb to just under 24 lb	5 days
24 lb to just under 30 lb	6 days



WRITING

69. Explain how to plot the point with coordinates of $(-2, 5)$.

▶ **70.** Explain why the coordinates of the origin are $(0, 0)$.

REVIEW

Evaluate each expression.

71. $-5 - 5(-5)$ 20

72. $(-5)^2 + (-5)$ 20

73. $\dfrac{-3 + 5(2)}{9 + 5}$ $\frac{1}{2}$

74. $|-1 - 9|$ 10

75. $\dfrac{|-25| - 2(-5)}{2^4 - 9}$ 5

76. $\dfrac{3[-9 + 2(7 - 3)]}{(8 - 5)(9 - 7)}$ $-\frac{1}{2}$

77. Solve: $-4x + 0.7 = -2.1$ 0.7

78. Solve $P = 2l + 2w$ for w. $w = \dfrac{P - 2l}{2}$

SECTION 2.2
Graphing Linear Equations

Objectives

1. Determine whether an ordered pair is a solution of an equation.
2. Find a solution of an equation in two variables.
3. Graph linear equations by plotting points.
4. Graph linear equations by finding intercepts.
5. Graph horizontal and vertical lines.
6. Use linear models to solve applied problems.

In this section, we will discuss equations that contain two variables. Such equations are used to describe algebraic relationships between two quantities. To see a picture of these relationships, we can construct graphs of their equations.

1 Determine whether an ordered pair is a solution of an equation.

We will now extend our equation-solving skills to find solutions of equations in two variables. To begin, let's consider $y = -\frac{1}{2}x + 4$, an equation that contains the variables x and y. The solutions of this equation can be written as ordered pairs of real numbers. For example, the ordered pair $(-4, 6)$ is a solution, because the equation is satisfied when $x = -4$ and $y = 6$.

$$y = -\frac{1}{2}x + 4$$

$$6 \stackrel{?}{=} -\frac{1}{2}(-4) + 4 \quad \text{Substitute } -4 \text{ for } x \text{ and } 6 \text{ for } y.$$

$$6 \stackrel{?}{=} 2 + 4 \quad \text{Perform the multiplication: } -\tfrac{1}{2}(-4) = 2.$$

$$6 = 6 \quad \text{This is a true statement.}$$

Since the result is a true statement, $(-4, 6)$ is a **solution** of $y = -\frac{1}{2}x + 4$. We say that $(-4, 6)$ satisfies the equation.

2 Find a solution of an equation in two variables.

To find a solution of an equation in two variables, we can select a number for one of the variables and find the corresponding value of the other variable. For example, to find a solution of $y = -\frac{1}{2}x + 4$, we can select a value for x, say -2, and find the corresponding value of y.

$$y = -\frac{1}{2}x + 4$$

$$y = -\frac{1}{2}(-2) + 4 \quad \text{Substitute } -2 \text{ for } x.$$

$$y = 1 + 4 \quad \text{Evaluate the right side.}$$

$$y = 5 \quad \text{This is the } y\text{-coordinate of the point.}$$

Thus $(-2, 5)$ is a solution of $y = -\frac{1}{2}x + 4$.

Since we can choose any real number for x, and since any choice for x will give a corresponding value y, the equation $y = -\frac{1}{2}x + 4$ has infinitely many solutions. Because it is impossible to list all of the solutions, we can draw a mathematical picture of the solutions, called the *graph of the equation.*

3 Graph linear equations by plotting points.

EXAMPLE 1

Graph: $y = -\frac{1}{2}x + 4$

Strategy We will find three solutions of the equation, plot them on a rectangular coordinate system, and then draw a straight line passing through the points.

WHY To *graph* an equation in two variables means to make a drawing that represents all of its solutions.

Solution

To find three solutions of this equation, we select three values for x that will make the computations easy. Then we find each corresponding value of y. For example, if x is 2, we have

$$y = -\frac{1}{2}x + 4$$

$$y = -\frac{1}{2}(2) + 4 \quad \text{Substitute 2 for } x.$$

$$y = -1 + 4 \quad \text{Evaluate.}$$

$$y = 3$$

Thus $(2, 3)$ is a solution. In a similar manner, we find corresponding y-values for x-values of 0 and 4 and enter the solutions in the table.

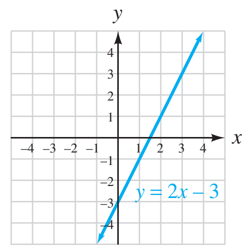
To graph the equation, we plot the three ordered pairs listed in the table. These points lie on the straight line shown. In fact, if we were to plot more pairs that satisfied the equation, it would become obvious that the resulting points will all lie on the line.

When we say that the graph of an equation is a line, we imply two things:

1. Every point with coordinates that satisfy the equation will lie on the line.
2. Any point on the line will have coordinates that satisfy the equation.

Self Check 1

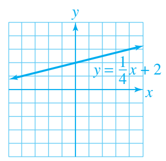
Graph: $y = 2x - 3$



Now Try **Problems 20 and 24**

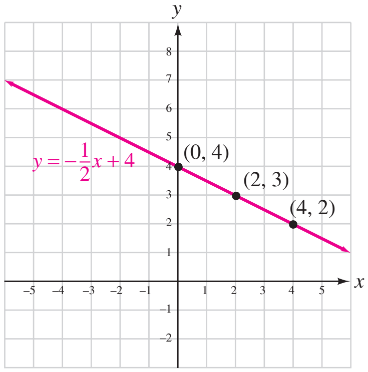
Teaching Example 1 Graph: $y = \frac{1}{4}x + 2$

Answer:



$y = -\frac{1}{2}x + 4$		
x	y	(x, y)
0	4	(0, 4)
2	3	(2, 3)
4	2	(4, 2)

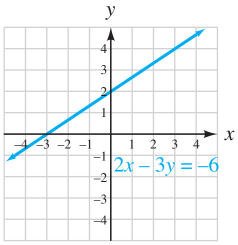
Choose values for x. Compute each y-value. Write each solution as an ordered pair.



When the graph of an equation is a straight line, we call the equation a **linear equation.** Linear equations can be written in the form $Ax + By = C$, called **standard (general) form,** where A, B, and C represent numbers (called **constants**) and x and y are variables.

Self Check 2

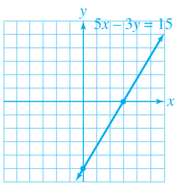
Graph: $2x - 3y = -6$



Now Try **Problem 28**

Teaching Example 2 Graph: $5x - 3y = 15$

Answer:



EXAMPLE 2

Graph: $3x + 2y = 12$

Strategy We will find three solutions of the equation, plot them on a rectangular coordinate system, and then draw a straight line passing through the points.

WHY To *graph* an equation in two variables means to make a drawing that represents all of its solutions.

Solution

We can pick values for either x or y, substitute them into the equation, and solve for the other variable. For example, if $x = 2$,

$$3x + 2y = 12$$
$$3(2) + 2y = 12 \quad \text{Substitute 2 for } x.$$
$$6 + 2y = 12 \quad \text{Perform the multiplication.}$$
$$2y = 6 \quad \text{Subtract 6 from both sides.}$$
$$y = 3 \quad \text{Divide both sides by 2.}$$

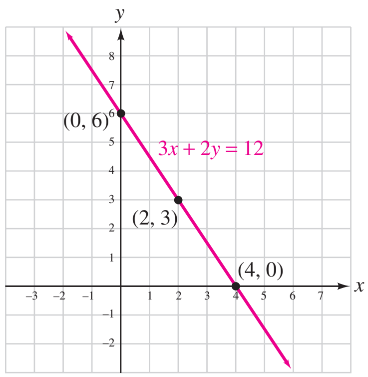
The ordered pair (2, 3) satisfies the equation. If $y = 6$,

$$3x + 2y = 12$$
$$3x + 2(6) = 12 \quad \text{Substitute 6 for } y.$$
$$3x + 12 = 12 \quad \text{Perform the multiplication.}$$
$$3x = 0 \quad \text{Subtract 12 from both sides.}$$
$$x = 0 \quad \text{Divide both sides by 3.}$$

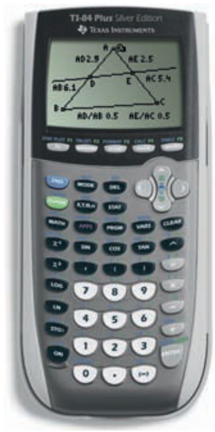
A second ordered pair that satisfies the equation is (0, 6).

These pairs and one other that satisfy the equation are shown in the table. After we plot each pair, we see that they all lie on a line. The graph of the equation is the line shown in the figure.

$3x + 2y = 12$		
x	y	(x, y)
0	6	(0, 6)
2	3	(2, 3)
4	0	(4, 0)



Using Your CALCULATOR Generating Tables of Solutions


Courtesy of Texas Instruments

If an equation in x and y is solved for y, we can use a graphing calculator to generate tables of solutions like the one shown in the figure to the right. Several brands of graphing calculators are available, and each one has its own sequence of keystrokes to make tables. The instructions in this discussion are for a TI-84 Plus graphing calculator. For specific details about other brands, please consult the owner's manual.

To construct a table of solutions for $3x + 2y = 12$, we first solve for y.

$$3x + 2y = 12$$

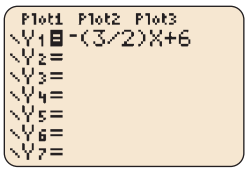
$$2y = -3x + 12 \quad \text{Subtract } 3x \text{ from both sides.}$$

$$y = -\frac{3}{2}x + 6 \quad \text{Divide both sides by 2 and simplify.}$$

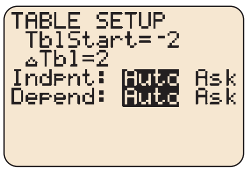
To enter $y = -\frac{3}{2}x + 6$, we press $\boxed{Y =}$ and enter $-(3/2)x + 6$, as shown in figure (a). (Ignore the subscript 1 on y; it is not relevant at this time.)

To enter the x-values that are to appear in the table, we press $\boxed{\text{2nd}}$ $\boxed{\text{TBLSET}}$ and enter the first value for x on the line labeled TblStart =. In figure (b), -2 has been entered on this line. Other values for x that are to appear in the table are determined by setting an **increment value** on the line labeled $\triangle$Tbl =. Figure (b) shows that an increment of 2 was entered. This means that each x-value in the table will be 2 units larger than the previous x-value.

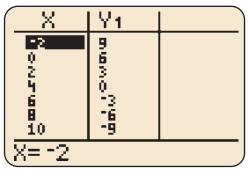
The final step is to press the keys $\boxed{\text{2nd}}$ $\boxed{\text{TABLE}}$. This will display a table of solutions, as shown in figure (c). The table contains all of the entries that we obtained in Example 2, as well as four additional solutions: $(-2, 9)$, $(6, -3)$ $(8, -6)$ and $(10, -9)$.



(a)



(b)



(c)

4 Graph linear equations by finding intercepts.

In Example 2, the graph intersected the y-axis at the point with coordinates (0, 6) (called the ***y*-intercept**) and intersected the x-axis at the point with coordinates (4, 0) (called the ***x*-intercept**). In general, we have the following definitions.

Intercepts of a Line

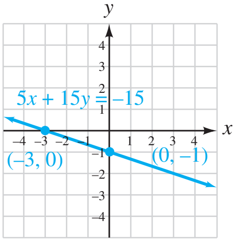
The ***y*-intercept** of a line is the point $(0, b)$, where the line intersects the y-axis. To find b, substitute 0 for x in the equation of the line and solve for y.

The ***x*-intercept** of a line is the point $(a, 0)$, where the line intersects the x-axis. To find a, substitute 0 for y in the equation of the line and solve for x.

Plotting the x- and y-intercepts of a graph and drawing a line through them is called the intercept method of graphing a line. This method is useful when graphing linear equations written in the standard (general) form $Ax + By = C$.

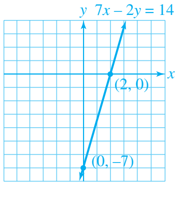
Self Check 3

Find the x- and y-intercepts and graph $5x + 15y = -15$.



Now Try **Problem 32**

Teaching Example 3 Find the x- and y-intercepts and graph $7x - 2y = 14$.
Answer:



EXAMPLE 3 Use the x- and y-intercepts to graph $2x - 5y = 10$.

Strategy We will substitute 0 for x to find the y-intercept and then substitute 0 for y to find the x-intercept.

WHY Since two points determine a line, the y-intercept and the x-intercept are enough information to graph this linear equation.

Solution

To find the y-intercept, we substitute 0 for x and solve for y:

$$\begin{aligned} 2x - 5y &= 10 && \text{This is the equation to graph.} \\ 2(0) - 5y &= 10 && \text{Substitute 0 for } x. \\ -5y &= 10 && \text{Perform the multiplication: } 2(0) = 0. \\ y &= -2 && \text{Divide both sides by } -5. \end{aligned}$$

The y-intercept is the point $(0, -2)$. To find the x-intercept, we substitute 0 for y and solve for x:

$$\begin{aligned} 2x - 5y &= 10 && \text{This is the equation to graph.} \\ 2x - 5(0) &= 10 && \text{Substitute 0 for } y. \\ 2x &= 10 && \text{Perform the multiplication: } 5(0) = 0. \\ x &= 5 && \text{Divide both sides by 2.} \end{aligned}$$

The x-intercept is the point (5, 0).

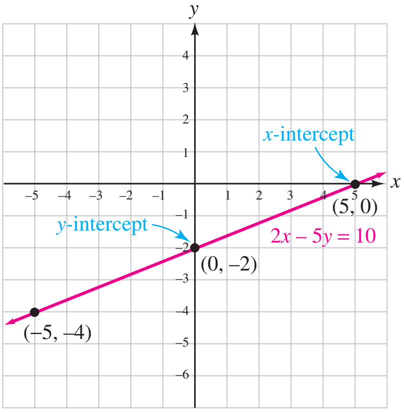
Although two points are enough to draw the line, it is a good idea to find and plot a third point as a check. To find the coordinates of a third point, we can substitute any convenient number (such as -5) for x and solve for y:

$$\begin{aligned} 2x - 5y &= 10 && \text{This is the equation to graph.} \\ 2(-5) - 5y &= 10 && \text{Substitute } -5 \text{ for } x. \\ -10 - 5y &= 10 && \text{Perform the multiplication.} \\ -5y &= 20 && \text{Add 10 to both sides.} \\ y &= -4 && \text{Divide both sides by } -5. \end{aligned}$$

The line will also pass through the point $(-5, -4)$.

A table of solutions and the graph of $2x - 5y = 10$ are shown.

$2x - 5y = 10$		
x	y	(x, y)
0	-2	$(0, -2)$
5	0	$(5, 0)$
-5	-4	$(-5, -4)$



5 Graph horizontal and vertical lines.

Equations such as $y = 3$ and $x = -2$ are linear equations, because they can be written in the form $Ax + By = C$.

$y = 3$ is equivalent to $0x + 1y = 3$

$x = -2$ is equivalent to $1x + 0y = -2$

EXAMPLE 4

Graph: **a.** $y = 3$ **b.** $x = -2$

Strategy To find three ordered-pair solutions of $y = 3$, we will select three values for x and use 3 for y each time. To find three ordered-pair solutions of $x = -2$, we will select three values for y and use -2 for x each time.

WHY The first equation requires that $y = 3$ and the second equation requires that $x = -2$.

Solution

a. Since the equation $y = 3$ does not contain x, the numbers chosen for x have no effect on y. The value of y is always 3. (See the table below on the left.)

After plotting the ordered pairs shown in the table on the left, we see that the graph is a horizontal line, parallel to the x-axis, with a y-intercept of $(0, 3)$. The line has no x-intercept.

b. Since the equation $x = -2$ does not contain y, the value of y can be any number. After plotting the ordered pairs shown in the table on the right below, we see that the graph is a vertical line, parallel to the y-axis, with an x-intercept of $(-2, 0)$. The line has no y-intercept.

$y = 3$		
x	y	(x, y)
-3	3	$(-3, 3)$
0	3	$(0, 3)$
2	3	$(2, 3)$
4	3	$(4, 3)$

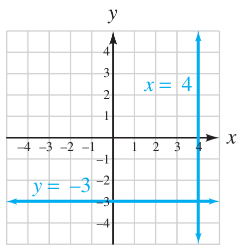
The value of x can be any number.

$x = -2$		
x	y	(x, y)
-2	-2	$(-2, -2)$
-2	0	$(-2, 0)$
-2	2	$(-2, 2)$
-2	6	$(-2, 6)$

The value of y can be any number.

Self Check 4

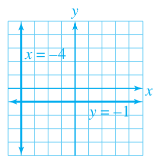
Graph $x = 4$ and $y = -3$ on one set of coordinate axes.

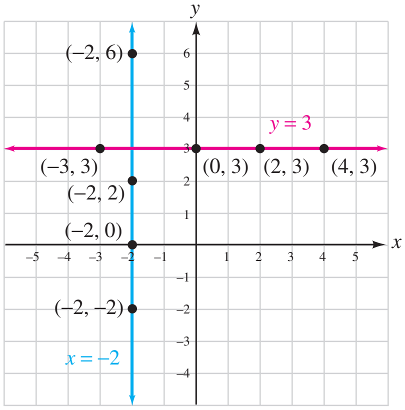


Now Try **Problems 36 and 38**

Teaching Example 4 Graph $x = -4$ and $y = -1$ on one set of coordinate axes.

Answer:





The results of Example 4 suggest the following facts.

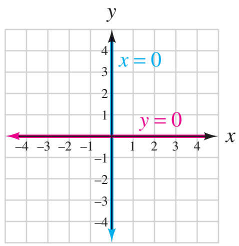
Horizontal and Vertical Lines

If a and b represent real numbers, then

The graph of the equation $x = a$ is a vertical line with x-intercept at $(a, 0)$.

The graph of the equation $y = b$ is a horizontal line with y-intercept at $(0, b)$.

The graph of the equation $y = 0$ has special significance; it is the x-axis. Similarly, the graph of the equation $x = 0$ is the y-axis.



6 Use linear models to solve applied problems.

In the next examples, we will see how linear equations can model real-life situations. In each case, the equations describe a *linear relationship* between two quantities; when they are graphed, the result is a line. We can make observations about what has happened in the past and what might take place in the future by carefully inspecting the graph.

Self Check 5

BOTTLED WATER Use the graph obtained in Example 5 to estimate what the annual per capita consumption of bottled water was in 2007. 30 gal

Now Try **Problem 58**

Teaching Example 5 BOTTLED WATER Use the graph obtained in Example 5 to estimate what the annual per capita consumption of bottled water will be in the year 2014.
Answer:
≈ 43 gal

EXAMPLE 5 ***Bottled Water*** The increasing popularity of bottled water in the United States can be modeled by the linear equation $w = 1.8t + 17.3$, where t represents the number of years after 2000 and w represents the annual per capita consumption in gallons. (Source: Beverage Marketing Corporation)

a. Graph the equation. **b.** Suppose the current trend continues. Use the graph to estimate what the annual per capita consumption of bottled water will be in the year 2012.

Strategy We will find three solutions of the equation, plot them on a rectangular coordinate system, and then draw a straight line passing through the points.

WHY To *graph* a linear equation in two variables means to make a drawing that represents all of its solutions.

Solution

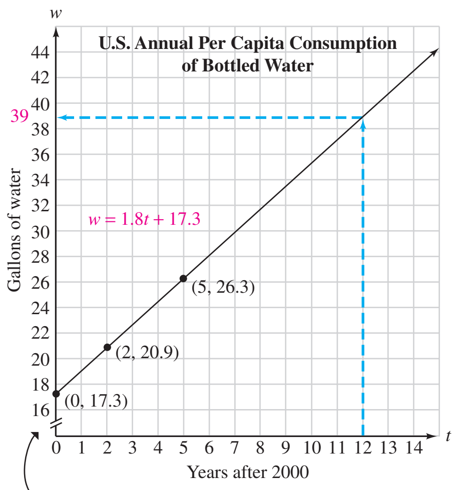
a. The variables t and w are used in the equation. If we associate t with the horizontal axis and w with the vertical axis, then the ordered pairs have the form (t, w).

To graph the equation, we pick three values for t, substitute them into the equation, and find each corresponding value of w. Since t represents the number of years *after* 2000, we will not select any negative values for t. The results are listed in the following table.

For $t = 0$ (The year 2000)	**For $t = 2$** (The year 2002)	**For $t = 5$** (The year 2005)
$w = 1.8t + 17.3$	$w = 1.8t + 17.3$	$w = 1.8t + 17.3$
$w = 1.8(0) + 17.3$	$w = 1.8(2) + 17.3$	$w = 1.8(5) + 17.3$
$w = 17.3$	$w = 3.6 + 17.3$	$w = 9 + 17.3$
	$w = 20.9$	$w = 26.3$

The pairs (0, 17.3), (2, 20.9), and (5, 26.3) are plotted, and a straight line is drawn through them to give the graph of the equation.

$w = 1.8t + 17.3$		
t	w	(t, w)
0	17.3	(0, 17.3)
2	20.9	(2, 20.9)
5	26.3	(5, 26.3)



Success Tip In the graph, the symbol $\ddagger$ indicates a break in the labeling of the vertical axis. The break enables us to omit a large portion of the grid that would not be used.

b. To estimate the per capita consumption in 2012 (which is 12 years after 2000), we locate 12 on the horizontal axis. Then we move upward and over (as shown in blue) to estimate a reading of 39 on the vertical axis. This means that if the current trend continues, in the year 2012, the annual per capita consumption of bottled water in the United States will be approximately 39 gallons.

For tax purposes, many businesses use the equation of a line to find the declining value of aging equipment. This method is called **straight-line depreciation.**

EXAMPLE 6 ***Depreciation of a Copier*** A copy machine that was purchased for \$6,750 is expected to depreciate according to the straight-line depreciation equation $y = -950x + 6{,}750$, where y is the value of the copier after x years of use. When will the copier have no value?

Strategy To find when the copier will have no value, we will substitute 0 for y in the equation $y = -950x + 6{,}750$ and solve for x.

Self Check 6

DEPRECIATION OF A COPIER

a. Use the equation $y = -950x + 6{,}750$ to determine when the copier will be worth \$3,900. 3 yr

b. Use the graph in Example 6 to determine when the copier will be worth \$2,000. 5 yr

Now Try **Problem 59**

Teaching Example 6
DEPRECIATION OF A COPIER

a. Use the equation $y = -950x + 6{,}750$ to determine when the copier will be worth \$4,850.

b. Use the graph in Example 6 to determine when the copier will be worth \$1,000.

Answers:
a. 2 yr **b.** 6 yr

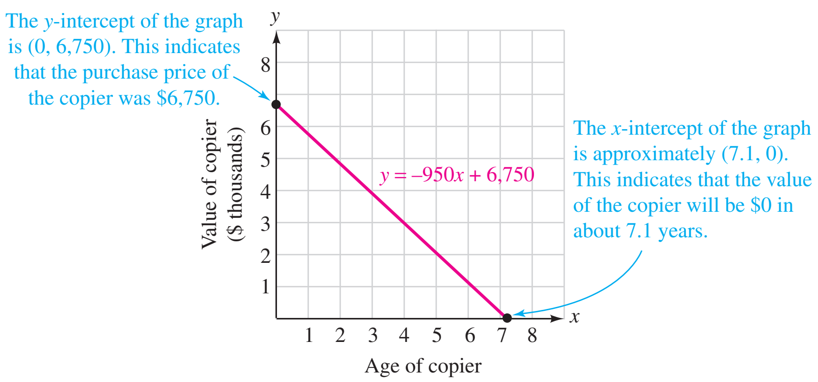
WHY The variable y represents the value of the computer. When the copier has no value, y will be equal to 0.

Solution

$$y = -950x + 6{,}750 \quad \text{This is the straight-line depreciation model.}$$
$$0 = -950x + 6{,}750 \quad \text{Substitute 0 for } y.$$
$$-6{,}750 = -950x \quad \text{Subtract 6,750 from both sides.}$$
$$7.105263158 \approx x \quad \text{To isolate } x \text{, divide both sides by } -950.$$

The copier will have no value after it has been in use for approximately 7.1 years.

The equation $y = -950x + 6{,}750$ is graphed below. Important information can be obtained from the intercepts of the graph.



Success Tip When the copier is new, it has been in use 0 years. In that case, x is 0. When the copier has no value, y is 0.

Using Your CALCULATOR Graphing Lines

We have graphed linear equations by finding solutions, plotting points, and drawing lines through those points. Graphing is often easier using a graphing calculator.

Window settings

Graphing calculators have a window to display graphs. To see the proper picture of a graph, we must decide on the minimum and maximum values for the x- and y-coordinates. A window with standard settings of

$$\text{Xmin} = -10 \qquad \text{Xmax} = 10 \qquad \text{Ymin} = -10 \qquad \text{Ymax} = 10$$

will produce a graph where the values of x and the values of y are between -10 and 10, inclusive. We can use the notation $[-10, 10]$ to describe such intervals.

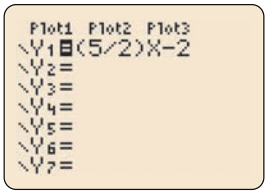
Graphing lines

To graph $5x - 2y = 4$, we must first solve the equation for y.

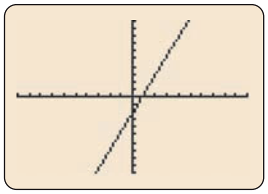
$$y = \frac{5}{2}x - 2 \quad \text{Subtract } 5x \text{ from both sides and then divide both sides by } -2.$$

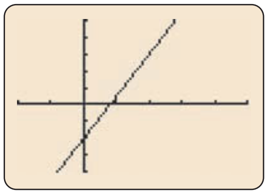
Next, we press $\boxed{\text{Y} =}$ and enter the right side of the equation after the symbol $\text{Y}_1 =$. See figure (a). We then press $\boxed{\text{GRAPH}}$ key to get the graph shown in figure (b). To show more detail, we can change the window settings to

[−2, 5] for x and [−4, 5] for y by pressing WINDOW and entering −2 for Xmin, 5 for Xmax, −4 for Ymin, and 5 for Ymax. See figure (c).

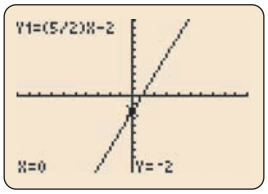


(a)

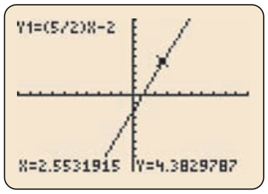

(b)


(c)

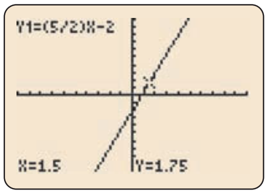
Finding the coordinates of a point on the graph
If we reenter the standard window settings of [−10, 10] for x and for y, press GRAPH, and press the TRACE key, we get the display shown in figure (d). The y-intercept of the graph is highlighted by the flashing cursor, and the x- and y-coordinates of that point are given at the bottom of the screen. We can use the ▶ and ◀ keys to move the cursor along the line to find the coordinates of any point on the line. After pressing the ▶ key 12 times, we will get the display in figure (e).



(d)



(e)



(f)

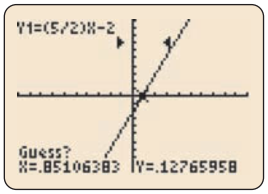
To find the y-coordinate of any point on the line, given its x-coordinate, we press 2nd CALC and select the value option. We enter the x-coordinate of the point and press ENTER. The y-coordinate is then displayed. In figure (f), 1.5 was entered for the x-coordinate, and its corresponding y-coordinate, 1.75, was found.

The table feature, discussed on page 131, gives us a third way of finding the coordinates of a point on the line.

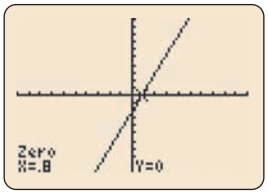
Determining the x-intercepts of a graph
To determine the x-intercept of the graph of $y = \frac{5}{2}x - 2$, we can use the zero option, found under the CALC menu. (Be sure to reenter the standard window settings for x and y before using CALC.) After we enter left and right bounds and a guess, as shown in figure (g) on the next page, the cursor automatically moves to the x-intercept of the graph when we press ENTER. Figure (h) on the next page shows how the coordinates of the x-intercept are then displayed at the bottom of the screen.

We can also use the trace and zoom features to determine the x-intercept of the graph of $y = \frac{5}{2}x - 2$. After graphing the equation using the standard window settings, we press TRACE. Then we move the cursor along the line toward the x-intercept until we arrive at a point with the coordinates shown in figure (i) on the next page. To get better results, we press ZOOM, select the zoom in option, and press ENTER to get a magnified picture. We press

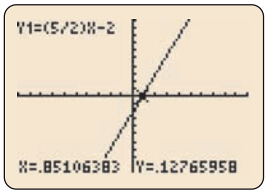
TRACE again and move the cursor to the point with coordinates shown in figure (j). Since the y-coordinate is nearly 0, this point is nearly the x-intercept. We can achieve better results with more zooms and traces.



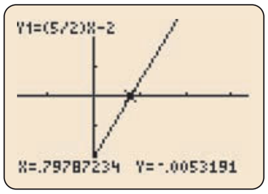
(g)



(h)

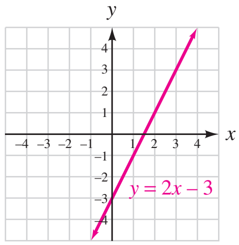


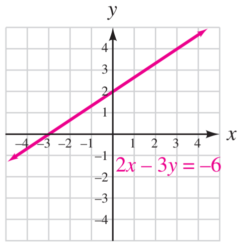
(i)

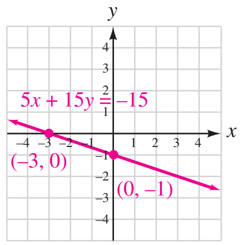


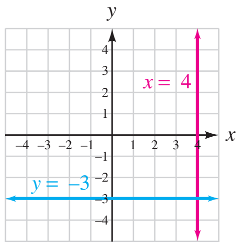
(j)

ANSWERS TO SELF CHECKS

1.


2.


3.


4.


5. 30 gal **6. a.** 3 yr **b.** 5 yr

SECTION 2.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. A solution of an equation in two variables is an ordered pair of numbers that make a true statement when substituted into the equation.
2. The graph of an equation is the graph of all points (x, y) on the rectangular coordinate system whose coordinates satisfy the equation.
3. Any equation whose graph is a straight line is called a linear equation.
4. The point where a graph intersects the x-axis is called the x-intercept. The point where a graph intersects the y-axis is called the y-intercept.
5. The graph of any equation of the form $x = a$ is a vertical line.
6. The graph of any equation of the form $y = b$ is a horizontal line.

Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

7. Determine whether the given ordered pair is a solution of $y = -5x - 2$.

a. $(-1, 3)$ yes **b.** $(3, -13)$ no

8. Consider the linear equation $6x - 4y = -12$.

a. Find the x-intercept of its graph. $(-2, 0)$

b. Find the y-intercept of its graph. $(0, 3)$

c. Does its graph pass through $(2, 6)$? yes

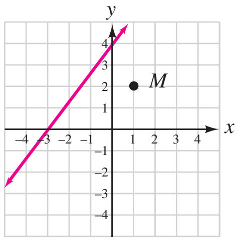
9. A table of solutions for a linear equation is given below. From the table, determine the x-intercept and the y-intercept of the graph of the equation.
x-intercept: $(-6, 0)$, y-intercept: $(0, 3)$

x	y	(x, y)
-6	0	$(-6, 0)$
-4	1	$(-4, 1)$
-2	2	$(-2, 2)$
0	3	$(0, 3)$
2	4	$(2, 4)$

10. Assume that $a \neq 0$ and $b \neq 0$. On which axis does each point lie?

a. $(0, b)$ y-axis **b.** $(a, 0)$ x-axis

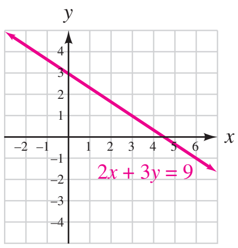
11. See the graph on the right.



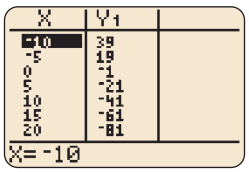
a. What is the x-intercept and what is the y-intercept of the line? $(-3, 0)$, $(0, 4)$

b. If the coordinates of point M are substituted into the equation of the line that is graphed here, will a true or a false statement result? false

12. Use the graph shown on the right to determine three solutions of $2x + 3y = 9$.
$(0, 3)$, $(3, 1)$, $(6, -1)$



13. A graphing calculator display is shown here. It is a table of solutions for which one of the following linear equations? $y = -4x - 1$



$y = -2x - 1$

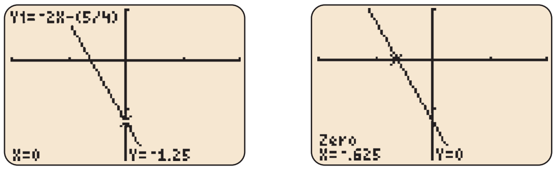
$y = -3x - 1$

$y = -4x - 1$

14. The graphing calculator displays below show the graph of $y = -2x - \frac{5}{4}$.

a. In illustration (a), what important feature of the line is highlighted by the cursor? the y-intercept

b. In illustration (b), what important feature of the line is highlighted by the cursor? the x-intercept



NOTATION

Complete each solution.

15. Verify that $(-3, -1)$ is a solution of $2x + 2y = -8$.

$$2x + 2y = -8$$
$$2(\boxed{-3}) + 2(-1) \stackrel{?}{=} -8$$
$$-6 + (\boxed{-2}) \stackrel{?}{=} -8$$
$$-8 = -8$$

16. To find the coordinates of a point on the graph of $5x + 2y = 10$, choose $x = 1$ and find y.

$$5x + 2y = 10$$
$$5(\boxed{1}) + 2y = 10$$
$$5 + \boxed{2y} = 10$$
$$2y = \boxed{5}$$
$$y = \frac{5}{2}$$

The point $(\boxed{1, \frac{5}{2}})$ is on the graph of $5x + 2y = 10$.

17. The graph of the equation $x = 0$ is which axis? the y-axis

18. The graph of the equation $y = 0$ is which axis? the x-axis

GUIDED PRACTICE

Complete each table. See Example 1.

19. $y = -x + 4$

x	y
-1	5
0	4
2	2

20. $y = x - 2$

x	y
-2	-4
0	-2
4	2

21. $y = -\frac{1}{3}x - 1$

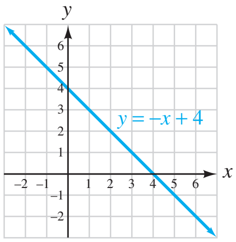
x	y
-3	0
0	-1
3	-2

▶ **22.** $y = -\frac{1}{2}x + \frac{5}{2}$

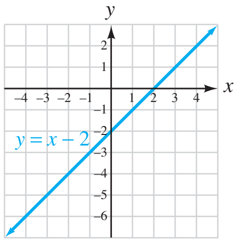
x	y
-1	3
3	1
5	0

Use the results from Problems 19–22 to graph each equation. See Example 1.

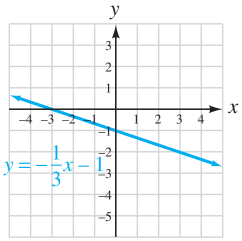
23. $y = -x + 4$



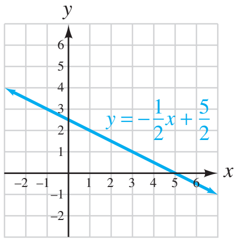
▶ **24.** $y = x - 2$



25. $y = -\frac{1}{3}x - 1$

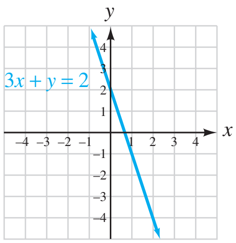


▶ **26.** $y = -\frac{1}{2}x + \frac{5}{2}$

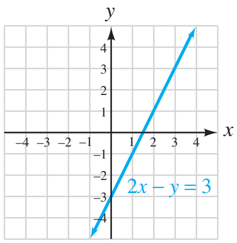


Construct a table of solutions and graph each equation. See Example 2.

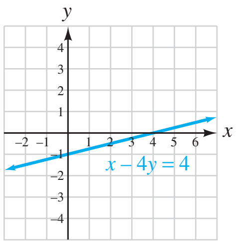
27. $3x + y = 2$



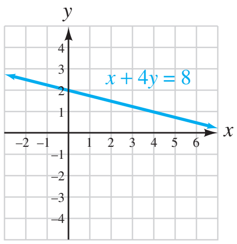
28. $2x - y = 3$



29. $x - 4y = 4$

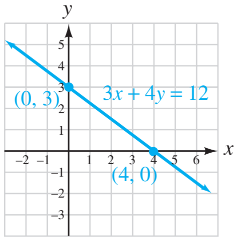


▶ **30.** $x + 4y = 8$

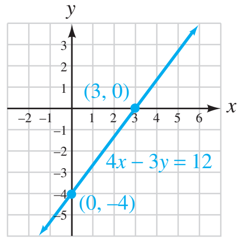


Graph each equation using the intercept method. Label the intercepts on each graph. See Example 3.

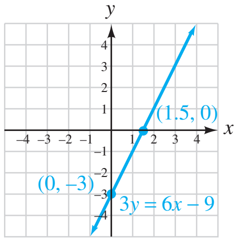
31. $3x + 4y = 12$



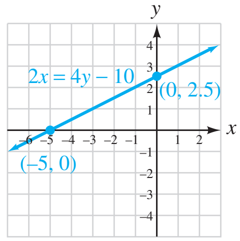
32. $4x - 3y = 12$



33. $3y = 6x - 9$

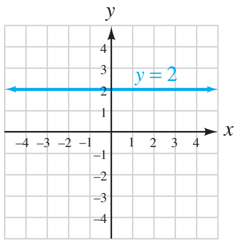


▶ **34.** $2x = 4y - 10$

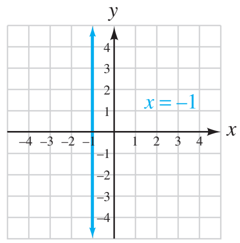


Write each equation in $y = b$ or $x = a$ form and graph it. See Example 4.

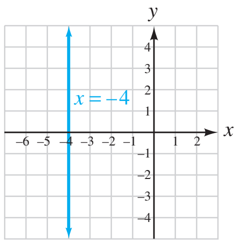
35. $y - 2 = 0$



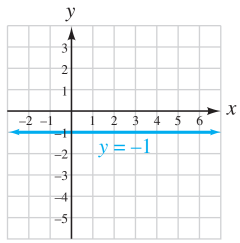
▶ **36.** $x + 1 = 0$



▶ **37.** $-2x + 3 = 11$



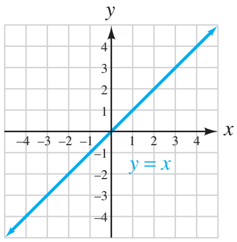
38. $-3y + 2 = 5$



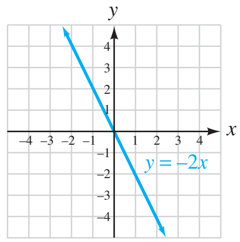
TRY IT YOURSELF

Graph each equation.

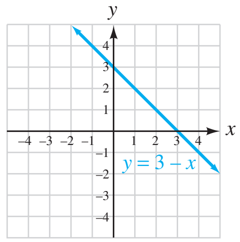
39. $y = x$



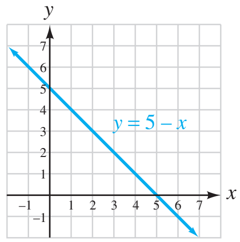
40. $y = -2x$



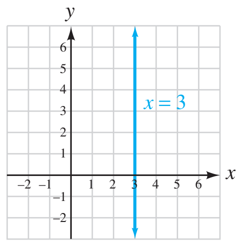
41. $y = 3 - x$



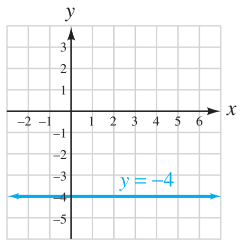
42. $y = 5 - x$



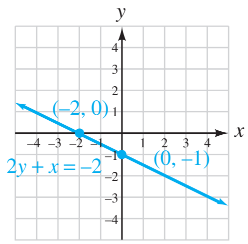
43. $x = 3$



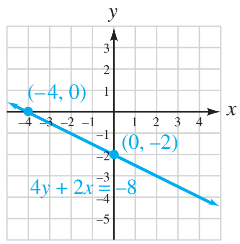
44. $y = -4$



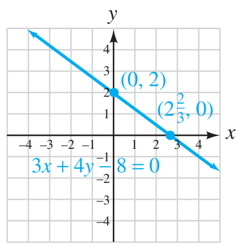
45. $2y + x = -2$



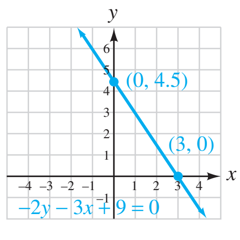
46. $4y + 2x = -8$



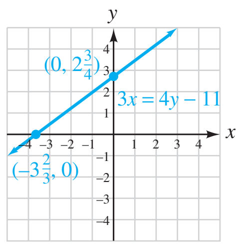
47. $3x + 4y - 8 = 0$



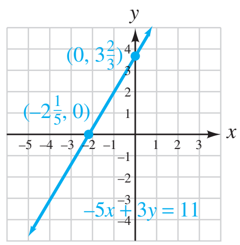
48. $-2y - 3x + 9 = 0$



49. $3x = 4y - 11$



50. $-5x + 3y = 11$



Use a graphing calculator to graph each equation, and find the x-coordinate of the x-intercept to the nearest hundredth.

51. $y = 3.7x - 4.5$ 1.22

52. $y = \frac{3}{5}x + \frac{5}{4}$ −2.08

53. $1.5x - 3y = 7$ 4.67

54. $0.3x + y = 7.5$ 25.00

APPLICATIONS

55. HOURLY WAGES This table gives the amount y (in dollars) that a student can earn for working x hours. Plot the ordered pairs and estimate how much the student will earn for working 8 hours. $60

x	2	4	6
y	15	30	45

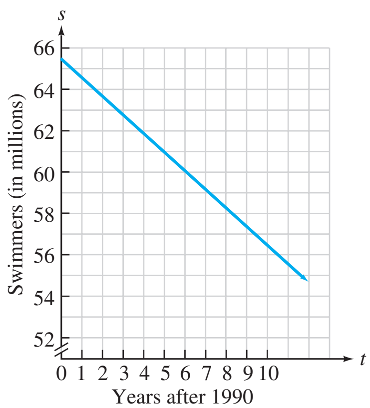
56. VALUE OF A CAR This table shows the value y (in dollars) of a car that is x years old. Plot the ordered pairs and estimate the value of the car when it is 4 years old. $3,000

x	0	1	3
y	15,000	12,000	6,000

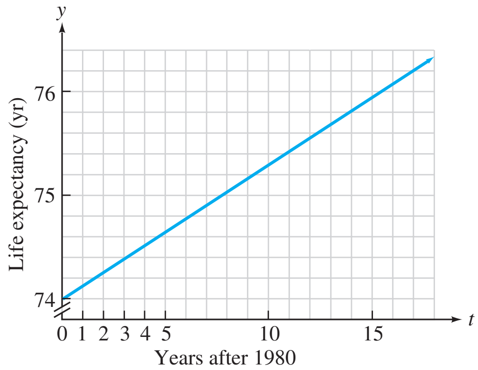
57. SWIMMING Surveys done by the National Sporting Goods Association have found that interest in swimming has been declining. The equation $s = -0.9t + 65.5$ is a linear model that gives the approximate number of people who went swimming during a given year. s is the annual number of swimmers (in millions), and t is the number of years since 1990. Graph the equation below.

a. What information about the number of swimmers can be obtained from the s-intercept of the graph? In 1990, there were 65.5 million swimmers.

b. From the graph, estimate the number of swimmers in 1998. about 58.2 million



58. LIVING LONGER According to the National Center for Health Statistics, life expectancy in the United States is increasing. The equation $y = 0.13t + 74$ is a linear model that approximates life expectancy; y is the number of years of life expected for a child born t years after 1980. Graph the equation below. From the graph, estimate the life expectancy of someone born in 1998. 76.3 yr



59. HOUSE APPRECIATION A house purchased for \$125,000 is expected to appreciate (gain value) according to the formula $y = 7{,}500x + 125{,}000$, where y is the value of the house after x years. Find the value of the house 5 years later. \$162,500

60. DEMAND EQUATIONS The number of television sets that consumers buy depends on price. The higher the price, the fewer TVs people will buy. The equation that relates price to the number of TVs sold at that price is called a **demand equation.** If the demand equation for a 13-inch TV is $p = -\frac{1}{10}q + 170$, where p is the price and q is the number of TVs sold at that price, how many TVs will be sold at a price of \$150? 200

▶ **61.** CAR DEPRECIATION A car purchased for \$17,000 is expected to depreciate (lose value) according to the formula $y = -1{,}360x + 17{,}000$. When will the car have no value? 12.5 yr

▶ **62.** SUPPLY EQUATIONS The number of television sets that manufacturers produce depends on price. The higher the price, the more TVs manufacturers will produce. The equation that relates price to the number of TVs produced at that price is called a **supply equation.** If the supply equation for a 13-inch TV is $p = \frac{1}{10}q + 130$, where p is the price and q is the number of TVs produced for sale at that price, how many TVs will be produced if the price is \$150? 200

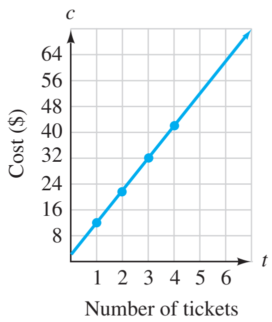
63. BUYING TICKETS Tickets to a circus cost \$10 each from Ticketron plus a \$2 service fee for each block of tickets.

a. Write a linear equation that gives the cost c when t tickets are purchased. $c = 10t + 2$

b. Complete the table and graph the equation.

c. Use the graph to estimate the cost of buying 6 tickets. \$62

t	c
1	12
2	22
3	32
4	42



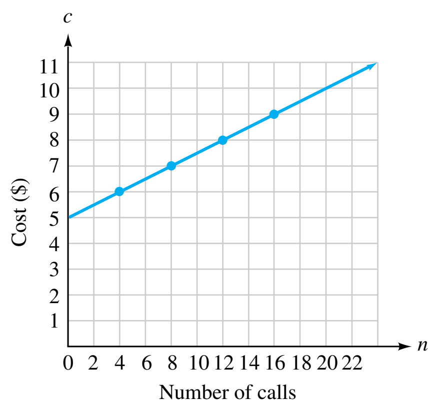
▶ **64.** TELEPHONE COSTS In a community, the monthly cost of local telephone service is \$5 per month, plus 25¢ per call.

a. Write a linear equation that gives the cost c for a person making n calls. $c = 0.25n + 5$

b. Complete the table and graph the equation.

c. Use the graph to estimate the cost of service in a month when 20 calls were made. \$10

n	c
4	6
8	7
12	8
16	9



WRITING

65. Explain how to graph a line using the intercept method.

▶ **66.** When graphing a line by plotting points, why is it a good practice to find three solutions instead of two?

REVIEW

67. List the prime numbers between 10 and 30.
11, 13, 17, 19, 23, 29

68. Write the first ten composite numbers.
4, 6, 8, 9, 10, 12, 14, 15, 16, 18

69. In what quadrant does the point $(-2, -3)$ lie? III

70. What is the formula that gives the area of a circle? $A = \pi r^2$

71. Simplify: $-4(-20s)$ $80s$

72. Approximate π to the nearest thousandth. 3.142

73. Simplify: $-(-3x - 8)$ $3x + 8$

▶ **74.** Simplify: $\frac{1}{3}b + \frac{1}{3}b + \frac{1}{3}b$ b

SECTION 2.3
Rate of Change and the Slope of a Line

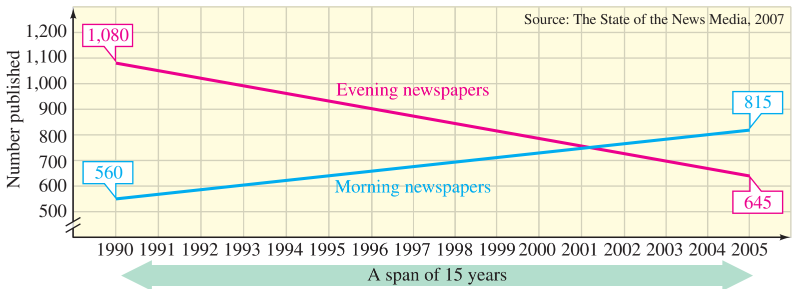
Objectives

1. Calculate the average rate of change.
2. Find the slope of a line from its graph.
3. Find the slope of a line given two points.
4. Find the slope of horizontal and vertical lines.
5. Solve applications of slope.
6. Determine whether lines are parallel or perpendicular using slope.

Our world is one of constant change. In this section, we will show how to describe the amount of change in one quantity in relation to the amount of change in another by finding an *average rate of change.*

1 Calculate an average rate of change.

The following line graphs model the approximate the number of daily morning newspapers and the number of evening newspapers published in the United States for the years 1990–2005. From the graph, we can see that the number of morning newspapers increased and the number of evening newspapers decreased over this span of time.



If we want to know the rate at which the number of morning newspapers increased or the rate at which the number of evening newspapers decreased over this period, we can do so by finding an **average rate of change.** To find an average rate of change, we compare the change in the number of newspapers published to the length of time in which that change took place, using a **ratio.**

Ratios and Rates

A **ratio** is a comparison of two numbers or two quantities with the same units. In symbols, if a and b are two numbers, the ratio of a to b is $\frac{a}{b}$. Ratios that are used to compare quantities with different units are called **rates.**

In the graph, we see that in 1990, the number of morning newspapers published was 560. In 2005, the number grew to 815. This is a change of $815 - 560$ or 255 over a 15-year time span. So we have

$$\text{Average rate of change} = \frac{\text{change in number of morning newspapers}}{\text{change in time}}$$

$$= \frac{255 \text{ newspapers}}{15 \text{ years}}$$

The rate of change is a ratio that involves units.

$$= \frac{\cancel{15}^{1} \cdot 17 \text{ newspapers}}{\cancel{15}_{1} \text{ years}}$$

Factor 255 as $15 \cdot 17$ and simplify: $\frac{15}{15} = 1$. We could also simply divide: $255 \div 15 = 17$

$$= \frac{17 \text{ newspapers}}{1 \text{ year}}$$

The number of morning newspapers published in the United States increased, on average, at a rate of 17 newspapers per year (written as 17 newspapers/year) from 1990 through 2005.

Self Check 1

NEWSPAPERS In 1992, there were approximately 888 Sunday edition newspapers being published in the United States. By 2005, that number had risen to 914. Find the average rate at which the number of Sunday edition newspapers increased from 1992 through 2005.

Now Try **Problem 61**

Self Check 1 Answer

2 Sunday edition newspapers/year

Teaching Example 1 NEWSPAPERS

In 1996 the number of Norwegian online newspapers was 72. In 2006 the number of Norwegian newspapers was 225. Find the average rate of change from 1996 to 2006. (*Source:* www.medienorge.uib.no/english)

Answer:

15.3 online newspapers/yr

EXAMPLE 1 ***Evening Newspapers*** Refer to the graph on page 143. Find the average rate at which the number of evening newspapers published in the United States decreased from 1990 through 2005.

Strategy We will write the ratio of the change in the number of evening newspapers to the change in time and attach the appropriate units. Then we will simplify the result, if possible.

WHY An average rate of change compares the amount of change in one quantity with respect to the amount of change in another quantity using a ratio with units.

Solution

From the graph, we see that in 1990 the number of evening newspapers published was 1,080. In 2005, the number fell to 645. To find the change, we subtract: $645 - 1{,}080 = -435$. The negative result indicates a decline in the number of evening newspapers over the 15-year time span. So we have

$$\text{Average rate of change} = \frac{\text{change in number of evening newspapers}}{\text{change in time}}$$

A rate of change is a ratio that includes units.

$$= \frac{-435 \text{ newspapers}}{15 \text{ years}}$$

$$= \frac{\cancel{-15}^{-1} \cdot 29 \text{ newspapers}}{\cancel{15}_{1} \text{ years}}$$

Factor -435 as $-15 \cdot 29$ and simplify: $\frac{-15}{15} = -1$. We could also just simply divide: $-435 \div 15 = -29$.

$$= \frac{-29 \text{ newspapers}}{1 \text{ year}}$$

The number of evening newspapers being published changed at a rate of -29 newspapers/year. That is, on average, there were 29 fewer per year, every year, from 1990 through 2005.

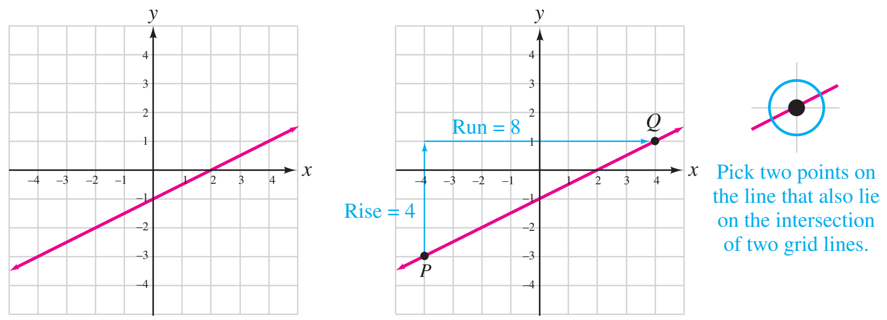
2 Find the slope of a line from its graph.

In the newspaper example, we measured the steepness of the lines in the graph to determine the average rates of change. In doing so, we found the *slope* of each line. The **slope of a line** is a ratio that compares the vertical change to the corresponding horizontal change as we move along the line from one point to another.

To determine the slope of a line (usually denoted by the letter m) from its graph, we first pick two points on the line. Then we write the ratio of the vertical change, called the **rise,** to the corresponding horizontal change, called the **run,** as we move from one point to the other.

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

EXAMPLE 2 Find the slope of the line graphed in figure (a) below.



(a) (b)

Strategy We will pick two points on the line, construct a slope triangle, and find the rise and run. Then we will write the ratio of rise to run and simplify the result, if possible.

WHY The slope of a line is the ratio of the rise to the run.

Solution

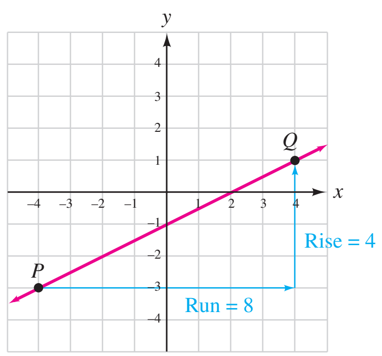
We begin by choosing two points on the line, P and Q, as shown in figure (b). One way to move from P to Q is to start at P, move upward, a rise of 4 grid squares, and then to the right, a run of 8 grid squares, to reach Q. These steps create a right triangle called a **slope triangle.**

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{8} = \frac{1}{2}$$ Simplify the fraction. The result is positive.

The slope of the line is $\frac{1}{2}$.

The two-step process to move from P to Q can be reversed. Starting at P, we can move to the right, a run of 8; and then upward, a rise of 4, to reach Q. With this approach, the slope triangle is below the line. When we form the ratio to find the slope, we get the same result as before:

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{8} = \frac{1}{2}$$



Caution! Slopes are normally written as fractions, sometimes as decimals, but never as mixed numbers.

As with any fractional answer, always express slope in simplified form (lowest terms).

The identical answers from Example 2 and its Self Check illustrate that *the same value will be obtained no matter which two points on a line are used to find its slope.*

Self Check 2

Find the slope of the line shown in figure (a) using two points different from those used in the solution of Example 2. $\frac{1}{2}$

Now Try **Problems 17 and 21**

Teaching Example 2 Find the slope of the line shown in Example 2 using $(-2, -2)$ and $(0, -1)$.

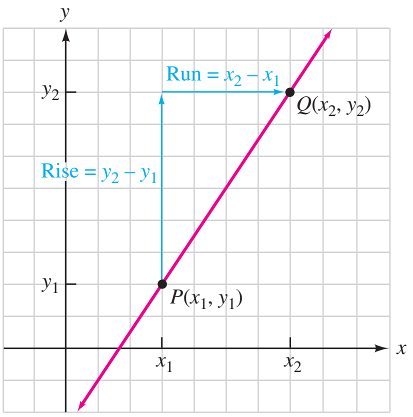
Answer:

$m = \frac{\text{rise}}{\text{run}} = \frac{1}{2}$

3 Find the slope of a line given two points.

We can use the graphic method for finding slope to develop a slope formula. To begin, we select points P and Q on the line shown in the figure on the right. To distinguish between the coordinates of these points, we use **subscript notation.** Point P has coordinates (x_1, y_1) and point Q has coordinates (x_2, y_2).

As we move from point P to point Q, the rise is the difference of the y-coordinates: $y_2 - y_1$. The run is the difference of the x-coordinates: $x_2 - x_1$. Since the slope is the ratio $\frac{\text{rise}}{\text{run}}$, we have the following formula for calculating slope.



Slope of a Line

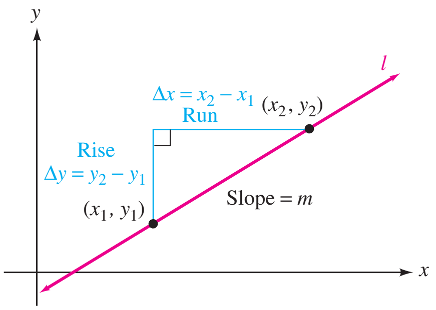
The **slope** of a line passing through points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \text{ if } x_2 \neq x_1$$

Another notation that we use to define slope involves the symbol Δ, which is the letter *delta* from the Greek alphabet. If the change in y is represented by Δy (read as "delta y") and the change in x is represented by Δx (read as "delta x"), then:

$$m = \frac{\Delta y}{\Delta x} \qquad \text{where } \Delta x \neq 0$$

The graph on the right shows all of the notation associated with the concept of slope of a line.



Self Check 3

Find the slope of the line passing through the points $(-3, 6)$ and $(4, -8)$. -2

Now Try **Problem 28**

Teaching Example 3 Find the slope of the line passing through $(9, -1)$ and $(-2, 6)$.
Answer:
$-\frac{7}{11}$

EXAMPLE 3 Find the slope of the line shown in the figure on the next page, passing through $(-2, 4)$ and $(3, -4)$.

Strategy We will use the slope formula to find the slope.

WHY We know the coordinates of two points on the line.

Solution

We can let $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (3, -4)$. Then

$$x_1 = -2 \qquad y_1 = 4 \qquad \text{and} \qquad x_2 = 3 \qquad y_2 = -4$$

$$m = \frac{\text{change in } y}{\text{change in } x}$$

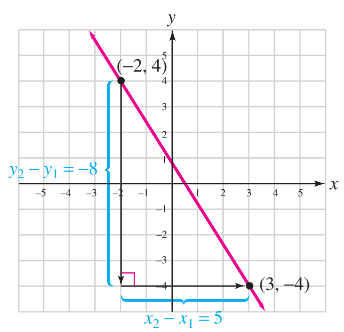
$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{This is the slope formula.}$$

$$= \frac{-4 - 4}{3 - (-2)}$$ Substitute -4 for y_2, 4 for y_1, 3 for x_2, and -2 for x_1.

$$= \frac{-8}{5}$$

$$= -\frac{8}{5}$$

The slope of the line is $-\frac{8}{5}$.



When calculating slope, it does not matter which point is (x_1, y_1) and which point is (x_2, y_2). We would have obtained the same result in Example 1 if we had let $(x_1, y_1) = (3, -4)$ and $(x_2, y_2) = (-2, 4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-4)}{-2 - 3} = \frac{8}{-5} = -\frac{8}{5}$$

Caution! When using the slope formula, be careful to subtract the y-coordinates and the x-coordinates in the same order. For example, in Example 3 with $(x_1, y_1) = (-2, 4)$ and $(x_2, y_2) = (3, -4)$, it would be incorrect to write either of the following.

This is $y_2 - y_1$.

$$\cancel{m = \frac{-4 - 4}{-2 - 3}}$$

This is $x_1 - x_2$. The subtraction is not in the same order.

This is $y_1 - y_2$. The subtraction is not in the same order.

$$\cancel{m = \frac{4 - (-4)}{3 - (-2)}}$$

This is $x_2 - x_1$.

4 Find the slope of horizontal and vertical lines.

If (x_1, y_1) and (x_2, y_2) are points on a horizontal line as shown in figure (a), then $y_1 = y_2$, and the numerator of the fraction

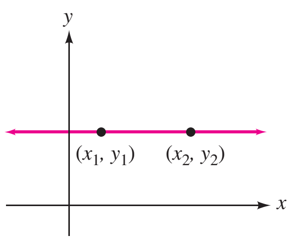
$$\frac{y_2 - y_1}{x_2 - x_1}$$ On a horizontal line, $x_2 \neq x_1$.

is 0. Thus, the value of the fraction is 0, and the slope of the horizontal line is 0.

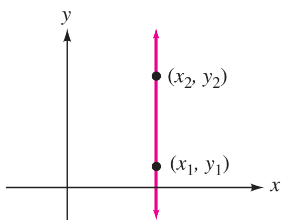
If (x_1, y_1) and (x_2, y_2) are two points on a vertical line as shown in figure (b), then $x_1 = x_2$, and the denominator of the fraction

$$\frac{y_2 - y_1}{x_2 - x_1}$$ On a vertical line, $y_2 \neq y_1$.

is 0. Since the denominator of a fraction cannot be 0, a vertical line has no defined slope.



(a)



(b)

Slopes of Horizontal and Vertical Lines

Horizontal lines (lines with equations of the form $y = b$) have a slope of 0.

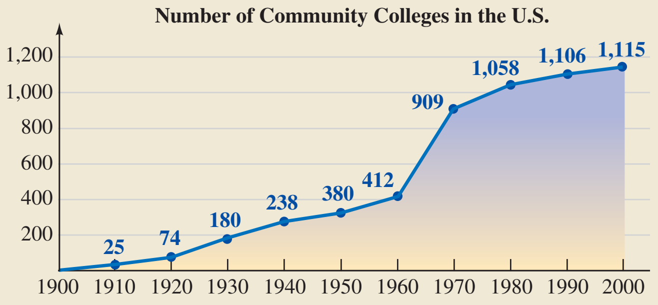
Vertical lines (lines with equations of the form $x = a$) have no defined slope.

THINK IT THROUGH *Community Colleges*

The community college has maintained a unique role as a vital component of postsecondary education in America. And now that role is on the ascent.

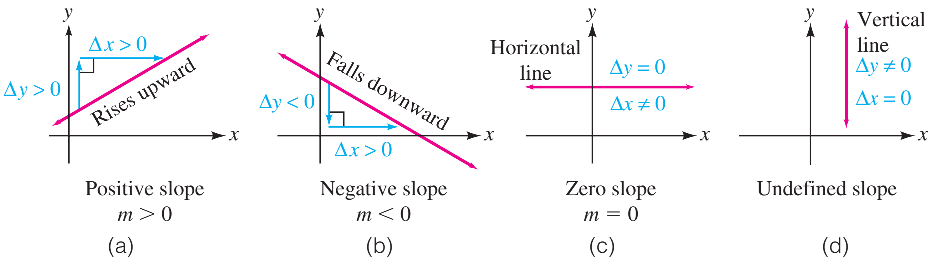
Arthur M. Cohen, Director Educational Resources Information Center, UCLA, 2002

The graph below shows that the number of community colleges in the United States has steadily increased since 1900. In which decade was the rate of increase in community colleges the greatest? Find that rate of increase.
1960–1970, an increase of about 50 community colleges per yr



Source: American Association of Community Colleges

If a line rises as we follow it from left to right, as in figure (a), its slope is positive. If a line drops as we follow it from left to right, as in figure (b), its slope is negative. If a line is horizontal, as in figure (c), its slope is 0. If a line is vertical, as in figure (d), it has no defined slope.

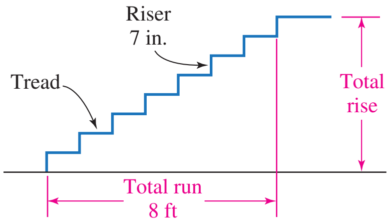


5 Solve applications of slope.

For applied problems, slope can be thought of as the average rate of change in one quantity per unit change in another quantity.

EXAMPLE 4 *Building Stairs*

The slope of a staircase is defined to be the ratio of the total rise to the total run, as shown in the illustration. What is the slope of the staircase?



Strategy We will express the total rise and the total run in terms of the same units and form their ratio.

WHY Here, the slope is a ratio that compares two quantities with the same units.

Solution

Since the design has eight 7-inch risers, the total rise is $8 \cdot 7 = 56$ inches. The total run is 8 feet, or 96 inches. With these quantities expressed in the same units, we can now form their ratio.

$$m = \frac{\text{total rise}}{\text{total run}}$$

$$= \frac{56}{96}$$

$$= \frac{7}{12} \qquad \text{Simplify the fraction: } \frac{56}{96} = \frac{\overset{1}{\cancel{8}} \cdot 7}{\underset{1}{\cancel{8}} \cdot 12} = \frac{7}{12}.$$

The slope of the staircase is $\frac{7}{12}$.

Self Check 4

BUILDING STAIRS Find the slope of the staircase if the riser height is changed to 6.5 inches. $\frac{13}{24}$

Now Try **Problem 63**

Teaching Example 4 BUILDING STAIRS Find the slope of the staircase if the total run is changed to 6 feet 8 inches.
Answer:
$\frac{7}{10}$

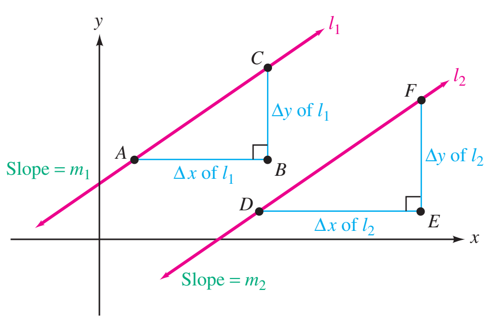
6 Determine whether lines are parallel or perpendicular using slope.

To see a relationship between parallel lines and their slopes, we refer to the parallel lines l_1 and l_2 shown in the figure below, with slopes of m_1 and m_2, respectively. Because right triangles ABC and DEF are similar, it follows that

$$m_1 = \frac{\Delta y \text{ of } l_1}{\Delta x \text{ of } l_1} \qquad \text{Read } l_1 \text{ as "line } l \text{ sub 1."}$$

$$= \frac{\Delta y \text{ of } l_2}{\Delta x \text{ of } l_2} \qquad \text{Since the triangles are similar, corresponding sides of } \Delta ABC \text{ and } \Delta DEF \text{ are proportional: } \frac{CB}{BA} = \frac{FE}{ED}.$$

$$= m_2$$



Thus, if two nonvertical lines are parallel, they have the same slope. It is also true that when two lines have the same slope, they are parallel.

Slopes of Parallel Lines

Nonvertical parallel lines have the same slope, and different lines having the same slope are parallel.

Since vertical lines are parallel, lines with no defined slope are parallel.

Self Check 5

Determine whether the line that passes through $(4, -8)$ and $(1, -2)$ is parallel to a line with a slope of 2.

Now Try **Problem 45**

Self Check 5 Answer
They are not parallel.

Teaching Example 5 Determine whether the line that passes through (5, 7) and (3, 1) is parallel to a line with a slope of 3.
Answer:
They are parallel.

EXAMPLE 5 Determine whether the line that passes through $(-6, 2)$ and $(3, -1)$ is parallel to a line with a slope of $-\frac{1}{3}$.

Strategy We will compare the slopes of the lines.

WHY If the slopes are equal, the lines are parallel. If the slopes are not equal, they are not parallel.

Solution

We can use the slope formula to find the slope of the line that passes through $(-6, 2)$ and $(3, -1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1 - 2}{3 - (-6)} \quad \text{Substitute } -1 \text{ for } y_2\text{, 2 for } y_1\text{, 3 for } x_2\text{, and } -6 \text{ for } x_1.$$

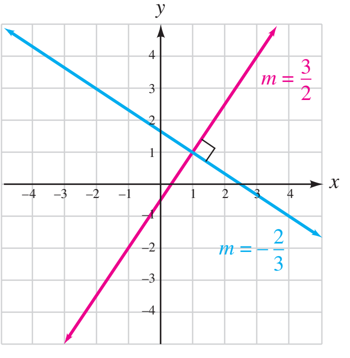
$$= \frac{-3}{9}$$

$$= -\frac{1}{3}$$

Both lines have a slope of $-\frac{1}{3}$, and therefore they are parallel.

The two lines shown in the figure to the right meet at right angles and are called **perpendicular lines.** In the figure, the symbol ⏋ is used to denote a right angle. Each of the four angles that are formed has a measure of 90°.

The product of the slopes of two (nonvertical) perpendicular lines is -1. For example, the perpendicular lines shown in the figure have slopes of $\frac{3}{2}$ and $-\frac{2}{3}$. If we find the product of their slopes, we have



$$\frac{3}{2}\left(-\frac{2}{3}\right) = -\frac{6}{6} = -1$$

Two numbers whose product is -1, such as $\frac{3}{2}$ and $-\frac{2}{3}$, are called **negative reciprocals.**

Slopes of Perpendicular Lines

If two nonvertical lines are perpendicular, their slopes are negative reciprocals.

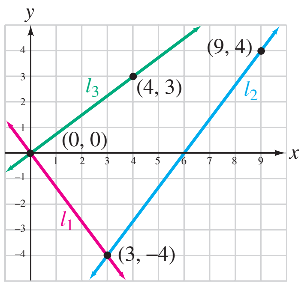
If the slopes of two lines are negative reciprocals, the lines are perpendicular.

We can also state the fact symbolically: If the slopes of two nonvertical lines are m_1 and m_2, then the lines are perpendicular if

$$m_1 \cdot m_2 = -1 \qquad \text{or} \qquad m_2 = -\frac{1}{m_1}$$

Because a horizontal line is perpendicular to a vertical line, a line with a slope of 0 is perpendicular to a line with no defined slope.

EXAMPLE 6 Are the lines l_1 and l_2 shown in the figure to the right perpendicular?



Strategy We will compare the slopes of the lines.

WHY If the slopes are negative reciprocals, the lines are perpendicular. If the slopes are not negative reciprocals, the lines are not perpendicular.

Solution

We find the slopes of the lines and see whether they are negative reciprocals.

$$\text{Slope of line } l_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 0}{3 - 0} = -\frac{4}{3}$$

$$\text{Slope of line } l_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-4)}{9 - 3} = \frac{8}{6} = \frac{4}{3}$$

Since their slopes are not negative reciprocals $\left(-\frac{4}{3} \cdot \frac{4}{3} \neq -1\right)$, the lines are not perpendicular.

Self Check 6

Are the lines l_1 and l_3 shown in the figure perpendicular? yes

Now Try **Problem 46**

Teaching Example 6 Is the line passing through (10, 4) and (3, −4) perpendicular to l_1?
Answer:
no

ANSWERS TO SELF CHECKS

1. 2 Sunday edition newspapers/year **2.** $\frac{1}{2}$ **3.** −2 **4.** $\frac{13}{24}$ **5.** They are not parallel. **6.** yes

SECTION 2.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. Slope is defined as the change in y divided by the change in x.

2. A slope is an average rate of change.

3. The change in x (denoted as Δx) is the horizontal run of the line between two points on the line.

4. The change in y (denoted as Δy) is the vertical rise of the line between two points on the line.

5. $\frac{7}{8}$ and $-\frac{8}{7}$ are negative reciprocals.

▶ **6.** Parallel lines have the same slope. The slopes of perpendicular lines are negative reciprocals.

▶ Selected exercises available online at
www.webassign.net/brookscole

CONCEPTS

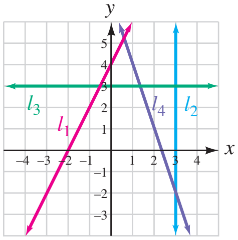
7. See the graph to the right.

a. Which line is horizontal? What is its slope? l_3, 0

b. Which line is vertical? What is its slope? l_2, undefined

c. Which line has a positive slope? What is it? l_1, 2

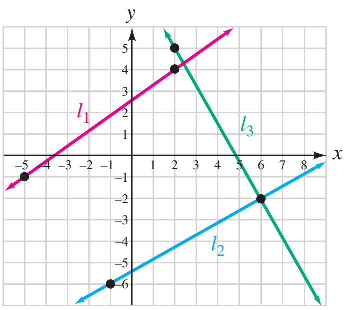
d. Which line has a negative slope? What is it? l_4, -3



8. See the graph.

a. Find the slopes of lines l_1 and l_2. Are they parallel? $\frac{5}{7}$, $\frac{4}{7}$, no

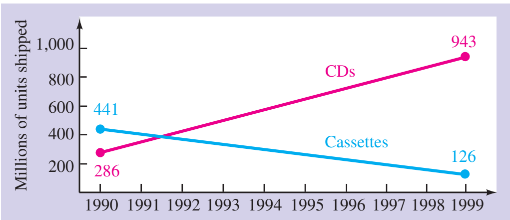
b. Find the slopes of lines l_2 and l_3. Are they perpendicular? $\frac{4}{7}$, $-\frac{7}{4}$, yes



9. THE RECORDING INDUSTRY The following graphs are models that approximate the number of CDs and cassettes that were shipped for sale from 1990 to 1999.

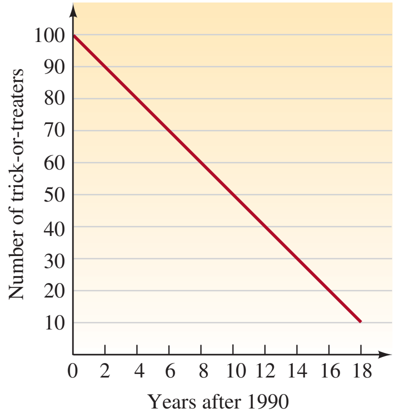
a. What was the rate of increase in the number of CDs shipped? an increase of 73 million units/yr

b. What was the rate of decrease in the number of cassettes shipped? a decrease of 35 million units/yr

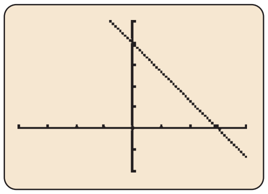


Source: *Statistical Abstract of the United States* (2003)

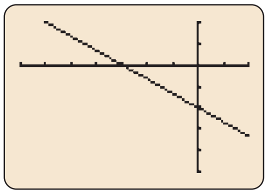
10. HALLOWEEN See the graph in the next column. A couple kept records of the number of trick-or-treaters who came to their door on Halloween night. Find the slope of the line. What information does the slope give?
-5; the number of trick-or-treaters has decreased over the years by 5 per year



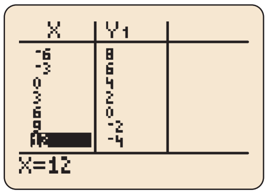
11. a. Determine the slope of the line shown. $-\frac{4}{3}$



b. Determine the slope of the line shown. $-\frac{2}{3}$



12. A table of solutions for a linear equation is shown. Find the slope of the graph of the equation. $-\frac{2}{3}$



NOTATION

13. What formula is used to find the slope of a line? $m = \frac{y_2 - y_1}{x_2 - x_1}$

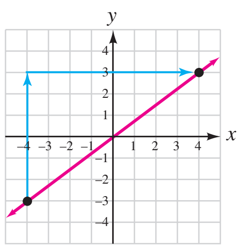
14. Explain the difference between x^2 and x_2.
x^2 is x squared and means $x \cdot x$. The symbol x_2 is x sub 2 and represents the x-coordinate of a point.

15. See the illustration.

a. Find Δy. 6

b. Find Δx. 8

c. Find $\frac{\Delta y}{\Delta x}$. $\frac{3}{4}$

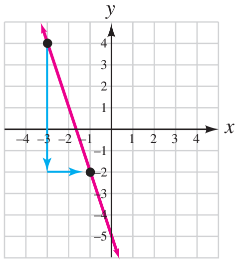


16. See the illustration.

a. Find Δy. -6

b. Find Δx. 2

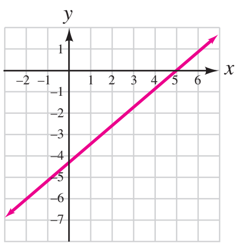
c. Find $\dfrac{\Delta y}{\Delta x}$. -3



GUIDED PRACTICE

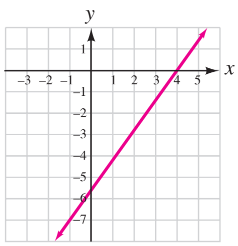
Find the slope of each line. See Example 2.

17.



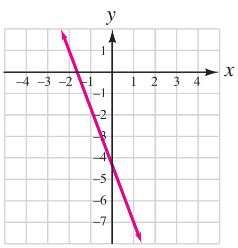
$\frac{6}{7}$

18.



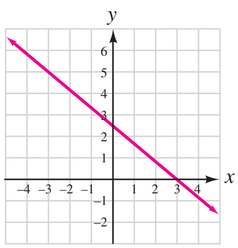
$\frac{7}{5}$

19.



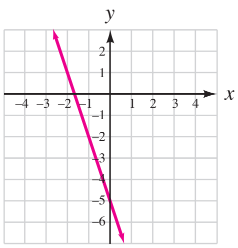
$-\frac{8}{3}$

20.



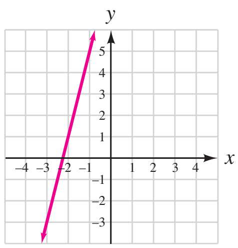
$-\frac{5}{6}$

21.



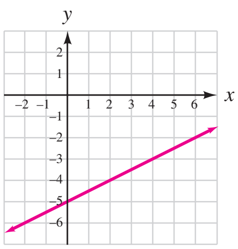
-3

22.



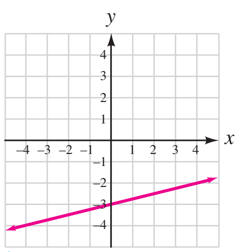
4

23.



$\frac{1}{2}$

24.



$\frac{1}{4}$

Find the slope of the line that passes through the given points, if possible. See Example 3.

25. $(0, 0), (3, 9)$ 3

26. $(9, 6), (0, 0)$ $\frac{2}{3}$

27. $(-1, 8), (6, 1)$ -1

28. $(-5, -8), (3, 8)$ 2

29. $(3, -1), (-6, 2)$ $-\frac{1}{3}$

30. $(0, -8), (-5, 0)$ $-\frac{8}{5}$

31. $(7, 5), (-9, 5)$ 0

32. $(2, -8), (3, -8)$ 0

33. $(-7, -5), (-7, -2)$ undefined

34. $(3, -5), (3, 14)$ undefined

35. $(a, b), (b, a)$ -1

36. $(a, b), (-b, -a)$ 1

Determine whether distinct lines with the given slopes are parallel, perpendicular, or neither. See Objective 6.

37. $m_1 = 3, m_2 = \dfrac{6}{2}$ parallel

38. $m_1 = \dfrac{1}{4}, m_2 = -4$ perpendicular

39. $m_1 = 4, m_2 = 0.25$ neither

40. $m_1 = -5, m_2 = -\dfrac{1}{0.2}$ parallel

41. $m_1 = 3, m_2 = -\dfrac{1}{3}$ perpendicualr

42. $m_1 = \dfrac{1}{4}, m_2 = 0.25$ parallel

43. $m_1 = \dfrac{1}{a}, m_2 = a$ neither

44. $m_1 = a, m_2 = -\dfrac{1}{a}$ perpendicular

Determine whether the line that passes through the two given points is parallel or perpendicular (or neither) to a line with a slope of -2. See Examples 5–6.

45. $(3, 4), (4, 2)$ parallel

46. $(6, 4), (8, 5)$ perpendicular

47. $(-2, 1), (6, 5)$ perpendicular

48. $(3, 4), (-3, -5)$ neither

49. $(5, 4), (6, 6)$ neither

50. $(-2, 3), (4, -9)$ parallel

51. $(3.2, 12.3), (6.2, 6.3)$ parallel

52. $\left(\dfrac{2}{3}, \dfrac{3}{4}\right), \left(\dfrac{1}{2}, \dfrac{2}{3}\right)$ perpendicular

TRY IT YOURSELF

Find two points that satisfy each equation and then find the slope of the line graph determined by the equation.

53. $3x + 2y = 12$ $-\frac{3}{2}$

54. $2x - y = 6$ 2

55. $3x = 4y - 2$ $\frac{3}{4}$

56. $x = y$ 1

57. $y = \dfrac{x - 4}{2}$ $\frac{1}{2}$

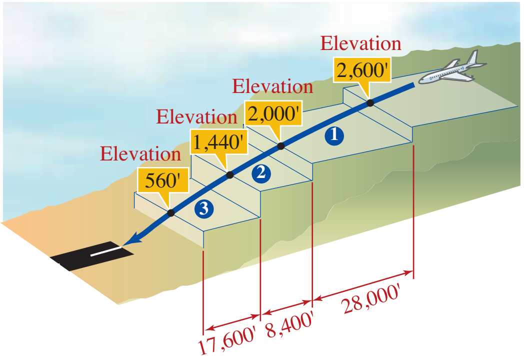
58. $x = \dfrac{3 - y}{4}$ -4

59. $4Y = 3(Y + 2)$ 0

60. $x + y = \dfrac{2 - 3y}{3}$ $-\frac{1}{2}$

APPLICATIONS

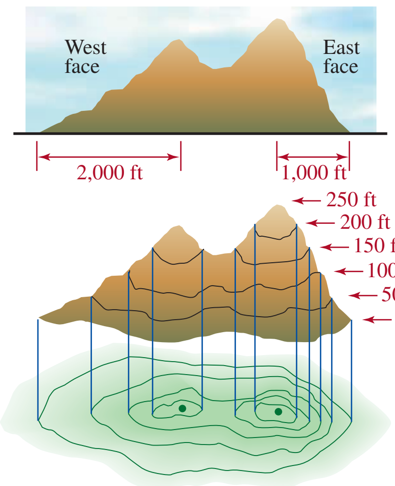
61. LANDING PLANES A jet descends in a stair-step pattern, as shown below. The required elevations of the plane's path are given. Find the slope of the descent in each of the three parts of its landing that are labeled. Which part is the steepest? $\frac{3}{140}, \frac{1}{15}, \frac{1}{20}$, part 2



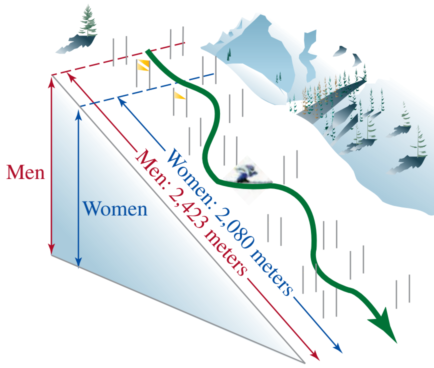
Based on data from *Los Angeles Times* (August 7, 1997), p. A8

▶ **62.** DROP IN PRICES The price of computers has been dropping for the past ten years. If a desktop PC cost \$5,700 ten years ago, and the same computing power cost \$400 two years ago, find the rate of decrease per year. (Assume a straight-line model.) \$662.50

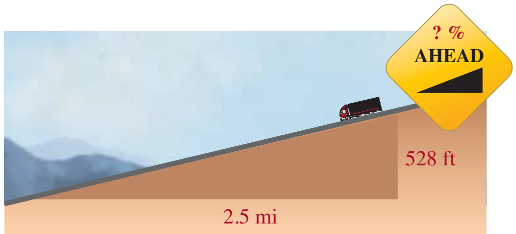
63. MAPS Topographic maps have contour lines that connect points of equal elevation on a mountain. The vertical distance between contour lines in the illustration is 50 feet. Find the slope of the west face and the east face of the mountain peak. $\frac{1}{10}, \frac{1}{4}$



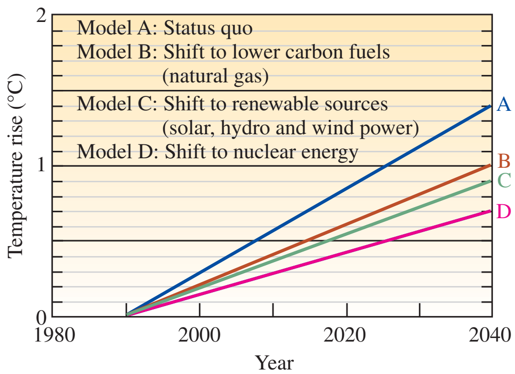
▶ **64.** SKIING The men's giant slalom course shown in the next column is longer than the women's course. Does this mean that the men's course is steeper? Use the concept of the slope of a line to explain.
No, they are equally steep.



65. ROAD SIGNS Find the slope of the road shown below. Use this information to complete the road warning sign for truckers by expressing the slope as a percent. (*Hint:* 1 mi = 5,280 ft.) $\frac{1}{25}$, 4%

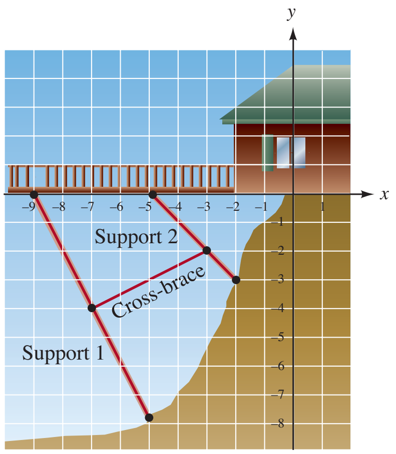


▶ **66.** GLOBAL TEMPERATURES The following graphs are estimates of future average global temperature rise due to the greenhouse effect. Assume that the models are straight lines. Estimate the average rate of change of each model. Express your answers as fractions. A: $\frac{7}{250}$ °/yr, B: $\frac{1}{50}$ °/yr, C: $\frac{9}{50}$ °/yr, D: $\frac{7}{50}$ °/yr

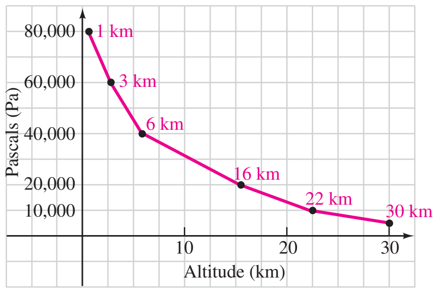


Based on data from *The Blue Planet* (Wiley, 1995)

67. DECK DESIGN See the illustration below. Find the slopes of the cross-brace and the supports. Is the cross-brace perpendicular to either support?
brace: $\frac{1}{2}$, support 1: -2, support 2: -1; yes, to support 1



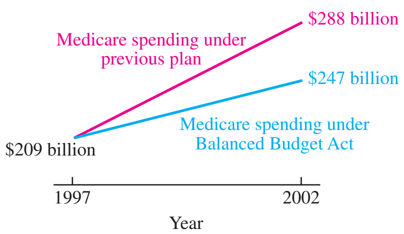
68. AIR PRESSURE Air pressure, measured in units called **Pascals** (Pa), decreases with altitude. Find the rate of change in Pascals for the fastest and the slowest decreasing steps of the graph below.
$-10{,}000$ Pa/km; $-\frac{5{,}000}{8}$ Pa/km $= -625$ Pa/km



Based on data from *The Blue Planet* (Wiley, 1995)

WRITING

69. POLITICS The graph shows how federal Medicare spending would have continued if the Republican-sponsored Balanced Budget Act hadn't become law in 1997. Write a brief statement explaining why Democrats could argue that the budget act "cut spending." Then write a brief statement explaining why Republicans could respond by saying, "There was no cut in spending—only a reduction in the rate of growth of spending."



Based on information supplied by Congressman David Drier's office

70. NUCLEAR ENERGY Since 1998, the number of nuclear reactors licensed for operation in the United States has remained about the same. Knowing this, what can be said about the rate of change in the number of reactors since 1998? Explain your answer.

71. Explain why a vertical line has no defined slope.

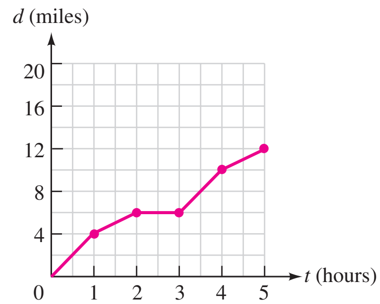
72. Explain how to determine from their slopes whether two lines are parallel, perpendicular, or neither.

REVIEW

73. HALLOWEEN CANDY A candy maker wants to make a 60-pound mixture of two candies to sell for \$2 per pound. If black licorice bits sell for \$1.90 per pound and orange gumdrops sell for \$2.20 per pound, how many pounds of each should be used? 40 lb licorice, 20 lb gumdrops

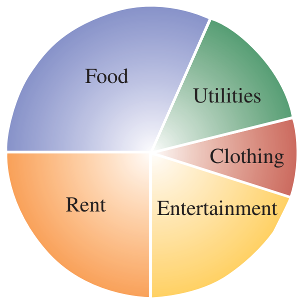
74. MEDICATIONS A doctor prescribes an ointment that is 2% hydrocortisone. A pharmacist has 1% and 5% concentrations in stock. How many ounces of each should the pharmacist use to make a 1-ounce tube? 0.75 oz of the 1%, 0.25 oz of the 5%

75. READING GRAPHS In the graph below, d is the distance a person has walked after t hours. How many hours did it take for the person to walk 10 miles? 4 hr



76. CIRCLE GRAPHS In the illustration, each part of the circle represents the amount of money spent in each of five categories. Approximately what percent was spent on rent? 25%

Monthly Expenses of Joe Sigueri



Objectives

1. Use point–slope form to write the equation of a line.
2. Use slope–intercept form to write the equation of a line.
3. Use slope as an aid when graphing.
4. Recognize parallel and perpendicular lines.
5. Write a linear equation model for straight-line depreciation.
6. Write a linear equation model to fit a collection of data.

SECTION 2.4
Writing Equations of Lines

We have seen that linear relationships are often presented in graphs. In this section, we begin a discussion of how to write an equation to model a linear relationship.

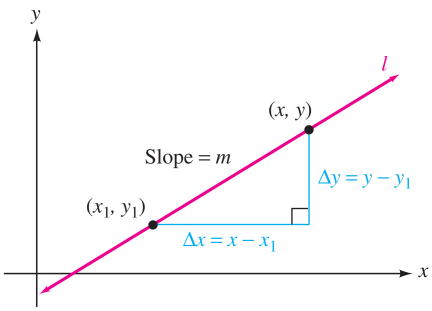
1 Use point–slope form to write the equation of a line.

Suppose that line l in the figure below has a slope of m and passes through (x_1, y_1). If (x, y) is a second point on line l, we have

$$m = \frac{y - y_1}{x - x_1}$$

or if we multiply both sides by $x - x_1$, we have

(1) $y - y_1 = m(x - x_1)$



Because Equation 1 displays the coordinates of the point (x_1, y_1) on the line and the slope m of the line, it is called the **point–slope form** of the equation of a line.

Point–Slope Form

The equation of the line that passes through (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

Self Check 1

Write an equation of the line that has a slope of $\frac{5}{4}$ and passes through $(4, 10)$.

Now Try **Problem 20**

Self Check 1 Answer

$y = \frac{5}{4}x + 5$

Teaching Example 1 Write an equation of the line that has a slope of $-\frac{1}{2}$ and passes through $(6, -5)$.

Answer:

$y = -\frac{1}{2}x - 2$

EXAMPLE 1 Write an equation of the line that has a slope of $-\frac{2}{3}$ and passes through $(-4, 5)$.

Strategy We will use the point–slope form, $y - y_1 = m(x - x_1)$, to write the equation of the line.

WHY We are given the slope of the line and the coordinates of a point that it passes through.

Solution

We substitute $-\frac{2}{3}$ for m, -4 for x_1, and 5 for y_1 into the point–slope form and simplify.

$y - y_1 = m(x - x_1)$ This is the point–slope form.

$y - 5 = -\frac{2}{3}[x - (-4)]$ Substitute $-\frac{2}{3}$ for m, -4 for x_1, and 5 for y_1.

$y - 5 = -\frac{2}{3}(x + 4)$ Simplify the expression within the brackets.

$y - 5 = -\frac{2}{3}x - \frac{8}{3}$ Distribute the multiplication by $-\frac{2}{3}$.

$y = -\frac{2}{3}x + \frac{7}{3}$ To solve for y, add 5 in the form of $\frac{15}{3}$ to both sides and simplify.

The equation of the line is $y = -\frac{2}{3}x + \frac{7}{3}$.

EXAMPLE 2 Write an equation of the line passing through $(-5, 4)$ and $(8, -6)$.

Strategy We will use the point–slope form, $y - y_1 = m(x - x_1)$, to write the equation of the line.

WHY Since we know two points on the line, we can calculate the slope of the line and solve the problem using the method of Example 1.

Solution

First we find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{This is the slope formula.}$$

$$= \frac{-6 - 4}{8 - (-5)} \quad \text{Substitute } -6 \text{ for } y_2, 4 \text{ for } y_1, 8 \text{ for } x_2, \text{ and } -5 \text{ for } x_1.$$

$$= -\frac{10}{13}$$

Since the line passes through $(-5, 4)$ and $(8, -6)$, we can choose either point and substitute its coordinates into the point–slope form. If we choose $(-5, 4)$, we substitute -5 for x_1, 4 for y_1, and $-\frac{10}{13}$ for m and proceed as follows.

$$y - y_1 = m(x - x_1) \quad \text{This is the point–slope form.}$$

$$y - 4 = -\frac{10}{13}[x - (-5)] \quad \text{Substitute } -\tfrac{10}{13} \text{ for } m, -5 \text{ for } x_1, \text{ and 4 for } y_1.$$

$$y - 4 = -\frac{10}{13}(x + 5) \quad \text{Simplify the expression within the brackets.}$$

$$y - 4 = -\frac{10}{13}x - \frac{50}{13} \quad \text{Distribute the multiplication by } -\tfrac{10}{13}.$$

$$y = -\frac{10}{13}x + \frac{2}{13} \quad \text{To solve for } y \text{, add 4 in the form of } \tfrac{52}{13} \text{ to both sides and simplify.}$$

The equation of the line is $y = -\frac{10}{13}x + \frac{2}{13}$.

Self Check 2

Write an equation of the line passing through $(-2, 5)$ and $(4, -3)$.

Now Try **Problem 28**

Self Check 2 Answer

$y = -\frac{4}{3}x + \frac{7}{3}$

Teaching Example 2 Write an equation of the line passing through $(-7, 4)$ and $(-4, 10)$.

Answer:

$y = 2x + 18$

2 Use slope–intercept form to write the equation of a line.

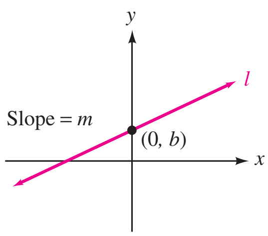
Since the y-intercept of the line l shown in the figure is the point $(0, b)$, we can write its equation by substituting 0 for x_1 and b for y_1 in the point–slope form and simplifying.

$$y - y_1 = m(x - x_1)$$

$$y - b = m(x - 0)$$

$$y - b = mx$$

$$(2) \qquad y = mx + b \quad \text{To solve for } y \text{, add } b \text{ to both sides.}$$



Because Equation 2 displays the slope m and the y-coordinate b of the y-intercept, it is called the **slope–intercept form** of the equation of a line.

Slope–Intercept Form

The equation of the line having slope m and y-intercept $(0, b)$ is

$$y = mx + b$$

Self Check 3

Use the slope–intercept form to write an equation of the line that has a slope of -2 and passes through $(-2, 8)$.

Now Try **Problem 34**

Self Check 3 Answer
$y = -2x + 4$

Teaching Example 3 Use slope–intercept form to write an equation of the line that has a slope of $\frac{2}{3}$ and passes through $(-6, 5)$.
Answer:
$y = \frac{2}{3}x + 9$

EXAMPLE 3 Use the slope–intercept form to write an equation of the line that has a slope of 4 and passes through (5, 9).

Strategy We will substitute 4 for m, 5 for x, and 9 for y in the slope–intercept form $y = mx + b$ and solve for b.

WHY The directions indicate to use the slope–intercept form to find the equation of the line.

Solution

$y = mx + b$	This is the slope–intercept form.
$9 = 4(5) + b$	Substitute 9 for y, 4 for m, and 5 for x.
$9 = 20 + b$	Perform the multiplication.
$-11 = b$	To solve for b, subtract 20 from both sides.

Because $m = 4$ and $b = -11$, the equation is $y = 4x - 11$.

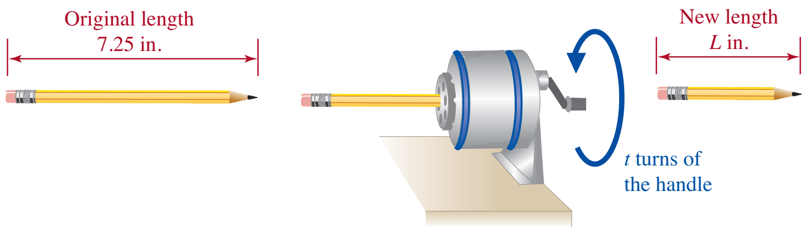
When an equation of a line is written in slope–intercept form, the coefficient of the x-term is the line's slope and the constant term gives the y-coordinate of the y-intercept.

$$y = \underbrace{m}_{\text{slope}}x + \underbrace{b}_{y\text{-intercept: }(0,\, b)}$$

Linear equation	Slope	y-intercept
$y = 4x - 3$	4	$(0, -3)$
$y = -\frac{5}{6}x$	$-\frac{5}{6}$	$(0, 0)$

Caution! For equations in $y = mx + b$ form, the slope of the line is the coefficient of x, not the x-term. For example, the graph of $y = 4x - 3$ has slope 4, not $4x$.

When an equation describing a linear relationship between two quantities is written in slope–intercept form, two pieces of information about the relationship are easily seen. As an example, let's consider the equation $L = -0.05t + 7.25$. If we begin with a pencil 7.25 inches long, this linear model gives the new length L in inches of the pencil after it has been inserted into a sharpener and the handle turned t times.



The value of m (in this case, -0.05) gives the change in the length of the pencil for one turn of the handle. Because the slope is negative, we know that the length of the pencil *decreases* by 0.05 inch for each turn of the handle. The value of b (in this case, 7.25) tells us that before any turns were made (when $t = 0$), the length of the pencil was 7.25 inches.

$$L = -0.05t + 7.25$$

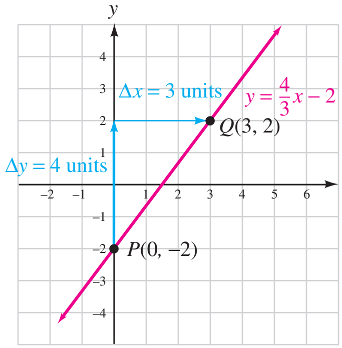
The slope is the rate of change of the length of the pencil.

The intercept is the original length of the pencil.

3 Use slope as an aid when graphing.

It is easy to graph a linear equation when it is written in slope–intercept form. For example, to graph $y = \frac{4}{3}x - 2$, we note that $b = -2$ and that the y-intercept is $(0, b) = (0, -2)$.

Because the slope of the line is $\frac{\Delta y}{\Delta x} = \frac{4}{3}$, we can locate another point Q on the line by starting at point $P(0, -2)$ and counting 3 units to the right (run) and 4 units up (rise). The change in x from point P to point Q is $\Delta x = 3$, and the corresponding change in y is $\Delta y = 4$. The line through points P and Q is the graph of the equation.



EXAMPLE 4 Find the slope and the y-intercept of the line with the equation $2x + 3y = -9$. Then graph the line.

Strategy We will solve the equation for y to write the equation in slope–intercept form $(y = mx + b)$. Then we will plot the y-intercept, and use the slope to determine a second point on the line.

WHY Once we locate two points on the line, we can draw the graph of the line.

Solution

We write an equation in the form $y = mx + b$ to find the slope m and the y-intercept $(0, b)$.

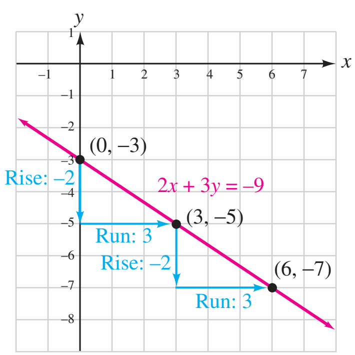
$2x + 3y = -9$ This is the given equation in standard (general) form.

$3y = -2x - 9$ Subtract $2x$ from both sides.

$\frac{3y}{3} = \frac{-2x}{3} - \frac{9}{3}$ To solve for y, divide both sides by 3.

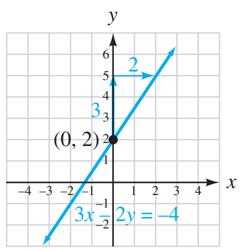
$y = -\frac{2}{3}x - 3$ Simplify both sides. We see that $m = -\frac{2}{3}$ and $b = -3$.

The slope of the line is $-\frac{2}{3}$, which can be expressed as $\frac{-2}{3}$. After plotting the y-intercept, $(0, -3)$, we move 2 units downward (rise) and then 3 units to the right (run). This locates a second point on the line, $(3, -5)$. From this point, we move another 2 units downward and 3 units to the right to locate a *third point* on the line, $(6, -7)$. Then we draw a line through the points to obtain the graph shown in the figure.



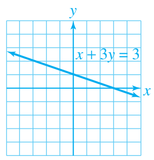
Self Check 4

Find the slope and the y-intercept of the line with the equation $3x - 2y = -4$. Then graph the line. $m = \frac{3}{2}, (0, 2)$



Now Try **Problems 40 and 45**

Teaching Example 4 Find the slope and the y-intercept of the line with the equation $x + 3y = 3$. Then graph the line.
Answer:
$m = -\frac{1}{3}, (0, 1)$



4 Recognize parallel and perpendicular lines.

Recall these two facts from Section 2.3:

- Different lines having the same slope are parallel.
- If slopes of two lines are negative reciprocals, the lines are perpendicular.

Self Check 5

Are the lines represented by $3x - 2y = 4$ and $2x = 5(y + 1)$ parallel? no

Now Try **Problem 50**

Teaching Example 5 Are the lines represented by $5x + 2y = 10$ and $4y = -10x + 8$ parallel?
Answer:
yes

EXAMPLE 5 Show that the lines represented by $4x + 8y = 10$ and $2x = 12 - 4y$ are parallel.

Strategy We will solve each equation for y to write the equation in slope–intercept form ($y = mx + b$). Then we will identify the slope of each line.

WHY If the slopes are equal, the lines are parallel.

Solution

We solve each equation for y to see that the lines are distinct and that their slopes are equal.

$$\begin{aligned} 4x + 8y &= 10 \\ 8y &= -4x + 10 \\ y &= -\frac{1}{2}x + \frac{5}{4} \end{aligned} \qquad \begin{aligned} 2x &= 12 - 4y \\ 4y &= -2x + 12 \\ y &= -\frac{1}{2}x + 3 \end{aligned}$$

Since the values of b in these equations are different $\left(\frac{5}{4} \text{ and } 3\right)$, the lines are distinct. Since the slope of each line is $-\frac{1}{2}$, they are parallel.

Self Check 6

Are the lines represented by $3x + 2y = 6$ and $2x - 3y = 6$ perpendicular? yes

Now Try **Problem 52**

Teaching Example 6 Are the lines represented by $4x + 3y = 9$ and $6x + 8y = 32$ perpendicular?
Answer:
no

EXAMPLE 6 Show that the lines represented by $4x + 8y = 10$ and $4x - 2y = 21$ are perpendicular.

Strategy We will solve each equation for y to write the equation in slope–intercept form ($y = mx + b$). Then we will identify the slope of each line.

WHY If the slopes are negative reciprocals, the lines are perpendicular.

Solution

We solve each equation for y to see that the slopes of their straight-line graphs are negative reciprocals.

$$\begin{aligned} 4x + 8y &= 10 \\ 8y &= -4x + 10 \\ y &= -\frac{1}{2}x + \frac{5}{4} \end{aligned} \qquad \begin{aligned} 4x - 2y &= 21 \\ -2y &= -4x + 21 \\ y &= 2x - \frac{21}{2} \end{aligned}$$

Since the slopes are negative reciprocals $\left(-\frac{1}{2} \text{ and } 2\right)$, the lines are perpendicular.

Self Check 7

Write an equation of the line that is parallel to the line $y = 8x - 3$ and passes through the origin. $y = 8x$

Now Try **Problems 58 and 60**

EXAMPLE 7 Write an equation of the line that passes through $(-2, 5)$ and is parallel to the line $y = 8x - 3$.

Strategy We will use the point–slope form, $y - y_1 = m(x - x_1)$, to write the equation of the line.

WHY We know that the line passes through $(-2, 5)$. We can use the fact that the lines are parallel to determine the unknown slope of the desired line.

Solution

Since the slope of the line given by $y = 8x - 3$ is the coefficient of x, the slope is 8. The desired equation is to have a graph that is parallel to the graph of $y = 8x - 3$. Its slope must also be 8.

We substitute -2 for x_1, 5 for y_1, and 8 for m in the point–slope form and simplify.

$$y - y_1 = m(x - x_1)$$

$$y - 5 = 8[x - (-2)] \quad \text{Substitute 5 for } y_1\text{, 8 for } m\text{, and } -2 \text{ for } x_1.$$

$$y - 5 = 8(x + 2) \quad \text{Simplify the expression within the brackets.}$$

$$y - 5 = 8x + 16 \quad \text{Distribute the multiplication by 8 and simplify.}$$

$$y = 8x + 21 \quad \text{Add 5 to both sides.}$$

The equation is $y = 8x + 21$.

Teaching Example 7 Write an equation of the line that is parallel to the line $y = \frac{2}{5}x + 3$ and passes through $(10, -4)$.
Answer:
$y = \frac{2}{5}x - 8$

EXAMPLE 8 Write an equation of the line that passes through $(6, -4)$ and is perpendicular to the line $3x + y = 2$.

Strategy We will use the point–slope form, $y - y_1 = m(x - x_1)$, to write the equation of the line.

WHY We know that the line passes through $(6, -4)$. We can use the fact that the lines are perpendicular to determine the unknown slope of the desired line.

Solution

To find the slope of the given line, we must first solve for y to write the equation in slope–intercept form.

$$3x + y = 2$$

$$y = -3x + 2$$

The slope of the given line is -3. The desired line is to have a graph that is perpendicular to $y = -3x + 2$. Therefore, their slopes must be negative reciprocals. Its slope must be $\frac{1}{3}$. We substitute 6 for x_1, -4 for y_1, and $\frac{1}{3}$ for m in the point–slope form and simplify.

$$y - y_1 = m(x - x_1)$$

$$y - (-4) = \frac{1}{3}(x - 6)$$

$$y + 4 = \frac{1}{3}x - 2$$

$$y = \frac{1}{3}x - 6$$

The equation is $y = \frac{1}{3}x - 6$.

Self Check 8

Write an equation of the line that passes through $(8, -2)$ that is perpendicular to the line $2x + 3y = 9$.

Now Try **Problems 63, 65, and 69**

Self Check 8 Answer
$y = \frac{3}{2}x - 14$

Teaching Example 8 Write an equation of the line that passes through $(-12, 3)$ that is perpendicular to $3x - 4y = 8$.
Answer:
$y = -\frac{4}{3}x - 13$

When asked to *write the equation of a line,* determine what you know about the graph of the line; its slope, its y-intercept, points it passes through, and so on. Then substitute the appropriate numbers into one of the following forms of a linear equation.

Forms for the Equation of a Line

Standard (general) form of a linear equation	$Ax + By = C$ A and B cannot both be 0.
Slope–intercept form of a linear equation	$y = mx + b$ The slope is m, and the y-intercept is $(0, b)$.
Point–slope form of a linear equation	$y - y_1 = m(x - x_1)$ The slope is m, and the line passes through (x_1, y_1).
A **horizontal line**	$y = b$ The slope is 0, and the y-intercept is $(0, b)$.
A **vertical line**	$x = a$ There is no defined slope, and the x-intercept is $(a, 0)$.

5 Write a linear equation model for straight-line depreciation.

For tax purposes, many businesses use *straight-line depreciation* to find the declining value of aging equipment.

Self Check 9

DEPRECIATION Find the value of the drill press in Example 9 after 4 years. $1,290

Now Try Problem 85

Teaching Example 9
DEPRECIATION A copy machine originally cost $1,050. After its useful life of 8 years, it has a salvage value of $90. Find the straight-line depreciation equation for the copy machine.
Answer:
$y = -120x + 1050$

EXAMPLE 9 *Depreciation*

After purchasing a new drill press, a machine shop owner had his accountant prepare a depreciation worksheet for tax purposes. See the illustration.

Depreciation Worksheet

Drill press $1,970
(new)

Salvage value $270
(in 10 years)

a. Assuming straight-line depreciation, write an equation that gives the value v of the drill press after x years of use.

b. Find the value of the drill press after $2\frac{1}{2}$ years of use.

c. What is the economic meaning of the v-intercept of the line?

d. What is the economic meaning of the slope of the line?

Strategy We will use the point–slope form, $y - y_1 = m(x - x_1)$, to write the equation of the line. Then we will answer the questions using the information from the equation.

WHY We know that the line passes through (0, 1,970) and (10, 270) from the depreciation worksheet. Since we know two points on the line, we can calculate the slope of the line. Then we can use the method from Example 2 to write the equation of the line.

Solution

a. The facts presented in the worksheet can be expressed as ordered pairs of the form

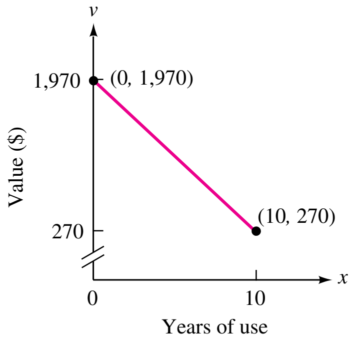
$$(x, v)$$

number of years of use — value of the drill press

- When purchased, the new \$1,970 drill press had been used 0 years: (0, 1,970).
- After 10 years of use, the value of the drill press will be \$270: (10, 270).

A simple sketch showing these ordered pairs and the line of depreciation is helpful in visualizing the situation.

Since we know two points that lie on the line, we can write its equation using the point–slope form. First, we find the slope of the line.



$$m = \frac{v_2 - v_1}{x_2 - x_1}$$ This is the slope formula written in terms of x and v.

$$= \frac{270 - 1{,}970}{10 - 0}$$ $(x_1, v_1) = (0, 1{,}970)$ and $(x_2, v_2) = (10, 270)$.

$$= \frac{-1{,}700}{10}$$

$$= -170$$

To find the equation of the line, we substitute -170 for m, 0 for x_1, and 1,970 for v_1 in the point–slope form and simplify.

$$v - v_1 = m(x - x_1)$$ This is the point–slope form written in terms x and v.

$$v - 1{,}970 = -170(x - 0)$$

$$v = -170x + 1{,}970$$ This is the straight-line depreciation equation.

The value v of the drill press after x years of use is given by the linear model $v = -170x + 1{,}970$.

b. To find the value of the drill press after $2\frac{1}{2}$ years of use, we substitute 2.5 for x in the depreciation equation and find v.

$$\begin{aligned} v &= -170x + 1{,}970 \\ &= -170(2.5) + 1{,}970 \\ &= -425 + 1{,}970 \\ &= 1{,}545 \end{aligned}$$

In $2\frac{1}{2}$ years, the drill press will be worth \$1,545.

c. From the sketch, we see that the v-intercept of the graph of the depreciation line is (0, 1,970). This gives the original cost of the drill press, \$1,970.

d. Each year, the value of the drill press decreases by \$170, because the slope of the line is -170. The slope of the line is the *annual depreciation rate*.

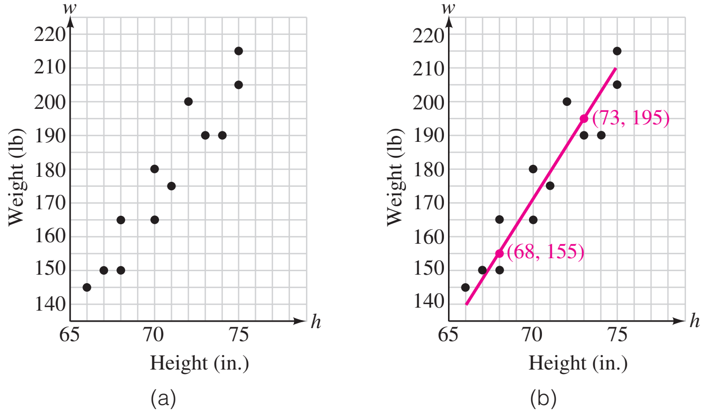
6 Write a linear equation model to fit a collection of data.

In statistics, the process of using one variable to predict another is called **regression.** For example, if we know a man's height, we can usually make a good prediction about his weight, because taller men tend to weigh more than shorter men.

The table shows the results of sampling twelve men at random and recording the height h and weight w of each. In figure (a) on the next page, the ordered

pairs (h, w) from the table are plotted to form a **scatter diagram.** Notice that the data points fall more or less along an imaginary straight line, indicating a linear relationship between h and w.

Height h in.	66	67	68	68	70	70	71	72	73	74	75	75
Weight w lb	145	150	150	165	180	165	175	200	190	190	205	215



To write a *prediction equation* (sometimes called a *regression equation*) that relates height and weight, we must find the equation of the line that comes closer to all of the data points in the scatter diagram than any other possible line. In statistics, there are exact methods to find this equation; however, they are beyond the scope of this book. In this course, we will draw "by eye" a line that we feel best fits the data points.

In figure (b) above, a straight edge was placed on the scatter diagram and a line was drawn that seemed to best fit all of the data points. Note that it passes through (68, 155) and (73, 195). To write the equation of that line, we first need to find its slope.

$$m = \frac{w_2 - w_1}{h_2 - h_1}$$ This is the slope formula written in terms of h and w.

$$= \frac{195 - 155}{73 - 68}$$ Choose $(h_1, w_1) = (68, 155)$ and $(h_2, w_2) = (73, 195)$.

$$= \frac{40}{5}$$

$$= 8$$

We then use the point–slope form to find the equation of the line. Since the line passes through (68, 155) and (73, 195), we can use either one to write its equation.

$w - w_1 = m(h - h_1)$ This is the point–slope form written in terms of h and w.

$w - 155 = 8(h - 68)$ Choose (68, 155) for (h_1, w_1).

$w - 155 = 8h - 544$ Distribute the multiplication by 8.

$w = 8h - 389$ To solve for w, add 155 to both sides.

The equation of the line that was drawn through the data points in the scatter diagram is $w = 8h - 389$. We can use this equation to predict the weight of a man who is 72 inches tall.

$w = 8h - 389$

$w = 8(72) - 389$ Substitute 72 for h.

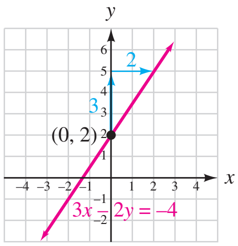
$w = 576 - 389$

$w = 187$

We predict that a 72-inch-tall man chosen at random will weigh about 187 pounds.

ANSWERS TO SELF CHECKS

1. $y = \frac{5}{4}x + 5$ **2.** $y = -\frac{4}{3}x + \frac{7}{3}$ **3.** $y = -2x + 4$ **4.** $m = \frac{3}{2}$, $(0, 2)$



5. no **6.** yes **7.** $y = 8x$ **8.** $y = \frac{3}{2}x - 14$ **9.** $1,290

SECTION 2.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. The point–slope form of the equation of a line is $\underline{y - y_1 = m(x - x_1)}$.

2. The __slope–intercept__ form of the equation of a line is $y = mx + b$.

3. Two lines are __perpendicular__ when their slopes are negative reciprocals.

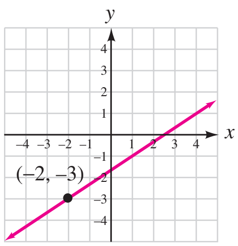
▶ **4.** Two distinct lines are __parallel__ when they have the same slope.

CONCEPTS

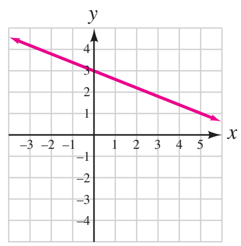
5. If you know the slope of a line, can you write its equation? no

6. If you know the coordinates of a point on a line, can you write its equation? no

7. The line graphed on the right passes through the point $(-2, -3)$. Find its slope. Then write its equation. Express the answer in point–slope form. $m = \frac{2}{3}$, $y + 3 = \frac{2}{3}(x + 2)$



8. For the line graphed below, find the slope and the y-intercept. Then write the equation of the line. Express your answer in slope–intercept form. $m = -\frac{2}{5}$, $(0, 3)$; $y = -\frac{2}{5}x + 3$



9. Find the slope and the y-intercept of the line graph of $y = -\frac{2}{3}x + 1$. $m = -\frac{2}{3}$, $(0, 1)$

▶ **10.** Find the slope of the line graph of $y - 3 = -\frac{2}{3}(x + 1)$. What point does the equation indicate the graph will pass through? $m = -\frac{2}{3}$, $(-1, 3)$

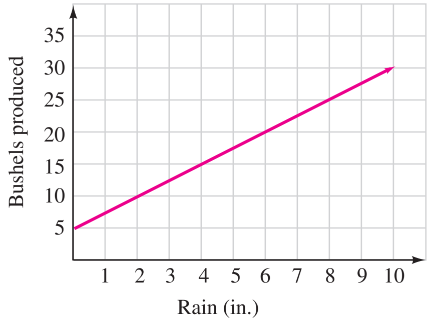
11. Do the equations $y - 2 = 3(x - 2)$, $y = 3x - 4$, and $3x - y = 4$ describe the same line? yes

▶ Selected exercises available online at
www.webassign.net/brookscole

12. See the linear model below.

a. What information does the y-intercept give?
If there is no rain, 5 bushels will be produced.

b. What information does the slope give?
The number of bushels produced increases by 5 for every 2 in. of rain.



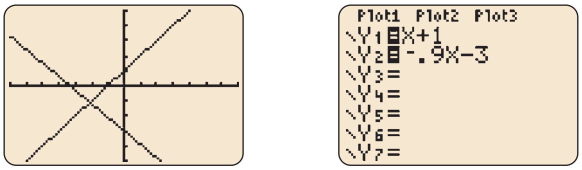
13. Find the y-intercept of the graph of each equation.

a. $y = 2x$ (0, 0) **b.** $x = 23$ none

▶ **14.** Find the slope of the graph of each line, if possible.

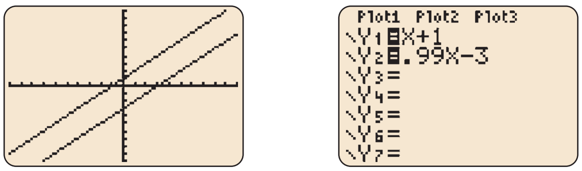
a. $y = -x$ -1 **b.** $x = -3$ undefined

15. The two lines shown in illustration (a) below appear to be perpendicular. Their equations are shown in illustration (b). Are the lines perpendicular? Explain.
No; the slopes are not negative reciprocals. Their product is not -1: $1(-0.9) = -0.9$.



(a) (b)

16. The two lines shown in illustration (a) below appear to be parallel. Their equations are shown in illustration (b). Are the lines parallel? Explain.
No; the slopes are not the same: $1 \neq 0.99$.



(a) (b)

NOTATION

Complete each solution.

17. Write $y + 2 = \frac{1}{3}(x + 3)$ in slope–intercept form.

$$y + 2 = \frac{1}{3}(x + 3)$$
$$y + 2 = \boxed{\tfrac{1}{3}x} + 1$$
$$y + 2 - \boxed{2} = \frac{1}{3}x + 1 - \boxed{2}$$
$$y = \frac{1}{3}x - \boxed{1}$$
$$m = \boxed{\tfrac{1}{3}}, b = \boxed{-1}$$

18. Write an equation of the line that has slope -2 and passes through the point (3, 1).

$$y - y_1 = m(x - x_1)$$
$$y - \boxed{1} = -2(x - \boxed{3})$$
$$y - 1 = \boxed{-2x} + 6$$
$$y = -2x + 7$$

GUIDED PRACTICE

Use point–slope form to write an equation of the line with the given properties. Then write each equation in slope–intercept form. See Example 1.

19. $m = 5$, passing through (0, 7) $y = 5x + 7$

▶ **20.** $m = -8$, passing through (0, −2) $y = -8x - 2$

21. $m = -3$, passing through (2, 0) $y = -3x + 6$

22. $m = 4$, passing through (−5, 0) $y = 4x + 20$

Use point–slope form to write an equation of the line passing through the two given points. Then write each equation in slope–intercept form. See Example 2.

23. (0, 0), (4, 4)
$y = x$

▶ **24.** (−5, 5), (0, 0)
$y = -x$

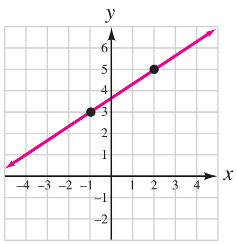
25. (3, 4), (0, −3)
$y = \frac{7}{3}x - 3$

26. (4, 0), (6, −8)
$y = -4x + 16$

27. (4, 0), (−4, −6)
$y = \frac{3}{4}x - 3$

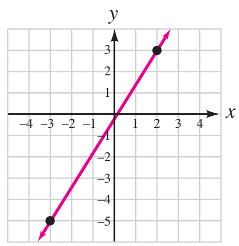
28. (5, −1),(−10, 5)
$y = -\frac{2}{5}x + 1$

▶ **29.**



$y = \frac{2}{3}x + \frac{11}{3}$

30.



$y = \frac{8}{5}x - \frac{1}{5}$

Use the slope–intercept form to write an equation of the line with the given properties. **See Example 3.**

31. $m = 3, b = 17$ $y = 3x + 17$

32. $m = -2, b = 11$ $y = -2x + 11$

33. $m = -7$, passing through $(7, 5)$ $y = -7x + 54$

▶ **34.** $m = 3$, passing through $(-2, -5)$ $y = 3x + 1$

35. $m = 0$, passing through $(2, -4)$ $y = -4$

36. $m = -7$, passing through the origin $y = -7x$

37. passing through $(6, 8)$ and $(2, 10)$ $y = -\frac{1}{2}x + 11$

38. passing through $(-4, 5)$ and $(2, -6)$ $y = -\frac{11}{6}x - \frac{7}{3}$

Write each equation in slope–intercept form. Then find the slope and the y-intercept of the line determined by the equation. **See Example 4.**

39. $3x - 2y = 8$ $\frac{3}{2}, (0, -4)$

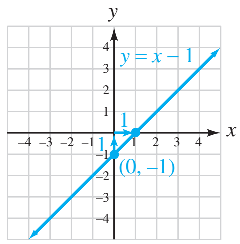
▶ **40.** $-2x + 4y = 12$ $\frac{1}{2}, (0, 3)$

41. $-2(x + 3y) = 5$ $-\frac{1}{3}, \left(0, -\frac{5}{6}\right)$

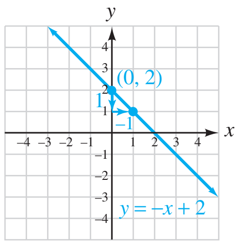
42. $5(2x - 3y) = 4$ $\frac{2}{3}, \left(0, -\frac{4}{15}\right)$

Find the slope and y-intercept and use them to graph the line. **See Example 4.**

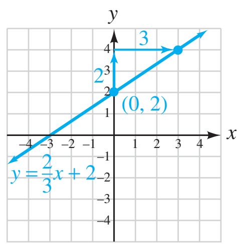
43. $y = x - 1$ $1, (0, -1)$



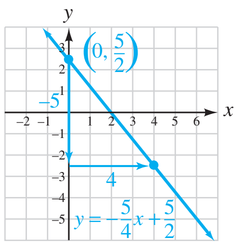
44. $y = -x + 2$ $-1, (0, 2)$



45. $y = \frac{2}{3}x + 2$ $\frac{2}{3}, (0, 2)$



▶ **46.** $y = -\frac{5}{4}x + \frac{5}{2}$ $-\frac{5}{4}, \left(0, \frac{5}{2}\right)$



Determine whether the graphs of each pair of equations are parallel, perpendicular, or neither. **See Examples 5–6.**

47. $y = 3x + 4, y = 3x - 7$ parallel

48. $y = 4x - 13, y = \frac{1}{4}x + 13$ neither

49. $x + y = 2, y = x + 5$ perpendicular

▶ **50.** $x = y + 2, y = x + 3$ parallel

51. $3x + 6y = 1, y = \frac{1}{2}x$ neither

▶ **52.** $2x + 3y = 9, 3x - 2y = 5$ perpendicular

53. $y = 3, x = 4$ perpendicular

54. $y = -3, y = -7$ parallel

Write an equation of the line that passes through the given point and is parallel to the graph of the given equation. Write the answer in slope–intercept form. **See Example 7.**

55. $(0, 0), y = 4x - 7$ $y = 4x$

▶ **56.** $(0, 0), x = -3y - 12$ $y = -\frac{1}{3}x$

57. $(2, 5), 4x - y = 7$ $y = 4x - 3$

▶ **58.** $(-6, 3), y + 3x = -12$ $y = -3x - 15$

59. $(4, -2), x = \frac{5}{4}y - 2$ $y = \frac{4}{5}x - \frac{26}{5}$

60. $(1, -5), x = -\frac{3}{4}y + 5$ $y = -\frac{4}{3}x - \frac{11}{3}$

61. $(-2, 3), 2x - 3y = 12$ $y = \frac{2}{3}x + \frac{13}{3}$

62. $(5, -2), 3x + 2y = 9$ $y = -\frac{3}{2}x + \frac{11}{2}$

Write an equation of the line that passes through the given point and is perpendicular to the graph of the given equation. Write the answer in slope–intercept form. **See Example 8.**

63. $(0, 0), y = 4x - 7$ $y = -\frac{1}{4}x$

64. $(0, 0), x = -3y - 12$ $y = 3x$

65. $(2, 5), 4x + y = 7$ $y = \frac{1}{4}x + \frac{9}{2}$

66. $(-6, 3), y + 3x = -12$ $y = \frac{1}{3}x + 5$

67. $(4, -2), x = \frac{5}{4}y - 2$ $y = -\frac{5}{4}x + 3$

▶ **68.** $(1, -5), x = -\frac{3}{4}y + 5$ $y = \frac{3}{4}x - \frac{23}{4}$

69. $(-1, 4), x - 3y = 5$ $y = -3x + 1$

70. $(4, -1), 2x + y = 7$ $y = \frac{1}{2}x - 3$

TRY IT YOURSELF

Write an equation in slope–intercept form of the line with the given properties or given graph.

71. Passes through

x	y
3	4
−3	−10

$y = \frac{7}{3}x - 3$

72. Passes through

x	y
2	8
6	−8

$y = -4x + 16$

73. Slope $\frac{4}{3}$, passes through $(5, 9)$ $y = \frac{4}{3}x + \frac{7}{3}$

74. Slope $-\frac{7}{5}$, passes through $(-6, 0)$ $y = -\frac{7}{5}x - \frac{42}{5}$

75. Passes through $(2, 5)$, perpendicular to $4x - y = 7$ $y = -\frac{1}{4}x + \frac{11}{2}$

76. Passes through $(-6, 3)$, perpendicular to $y + 3x = -12$ $y = \frac{1}{3}x + 5$

77. Passes through $(-1, -1)$ and $(4, 4)$ $y = x$

78. Passes through $(-1, 1)$ and $(9, -9)$ $y = -x$

79. Passes through (0, 0), parallel to

$$y = 4x - 7 \quad y = 4x$$

80. Passes through (0, 0), parallel to

$$y = -\frac{1}{3}x + 4 \quad y = -\tfrac{1}{3}x$$

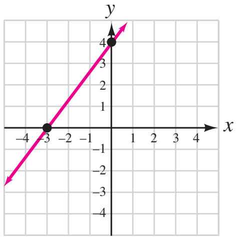
81. Passes through $\left(\frac{2}{3}, \frac{1}{4}\right)$, perpendicular to

$$y = 3x - 2 \quad y = -\tfrac{1}{3}x + \tfrac{17}{36}$$

82. Passes through $\left(\frac{4}{5}, -\frac{2}{3}\right)$, perpendicular to

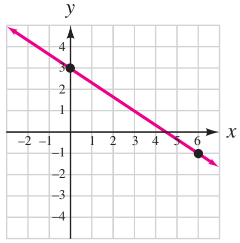
$$y = -5x - \frac{1}{2} \quad y = \tfrac{1}{5}x - \tfrac{62}{75}$$

83.



$y = \frac{4}{3}x + 4$

84.



$y = -\frac{2}{3}x + 3$

APPLICATIONS

In Exercises 85–88, assume straight-line depreciation or straight-line appreciation. See Example 9.

85. BIG-SCREEN TVs Find a linear depreciation equation for the TV in the ad shown. $y = -\frac{950}{3}x + 1{,}750$

For Sale: 3-year-old 45-inch TV, with matrix surround sound & picture within picture, remote. $1,750 new. Asking $800. Call 875-5555. Ask for Mike.

86. SALVAGE VALUE A truck was purchased for $19,984. Its salvage value at the end of 8 years is expected to be $1,600. Find the depreciation equation. $y = -2{,}298x + 19{,}984$

87. ART In 1987, the painting *Rising Sunflowers* by Vincent van Gogh sold for $36,225,000. Suppose that an art appraiser expected the painting to double in value in 20 years. Let x represent the time in years after 1987. Find an appreciation equation. $y = 1{,}811{,}250x + 36{,}225{,}000$

88. REAL ESTATE LISTINGS Use the information given in the description of the property below to write an appreciation equation for the house. $y = 4{,}000x + 114{,}000$

Vacation Home
$122,000
Only 2 years old

- Great investment property!
- Expected to appreciate $4,000/yr

Sq ft: 1,635	**Fam rm:** yes	**Den:** no
Bdrm: 3	**Ba:** 1.5	**Gar:** enclosed
A/C: yes	**Firepl:** yes	**Kit:** built-ins

89. CRIMINOLOGY City growth and the number of burglaries are related by a linear equation. Records show that 575 burglaries were reported in a year when the local population was 77,000 and that the rate of increase in the number of burglaries was 1 for every 100 new residents.

a. Using the variables p for population and B for burglaries, write an equation (in slope–intercept form) that police can use to predict future burglary statistics. $B = \frac{1}{100}p - 195$

b. How many burglaries can be expected when the population reaches 110,000? 905

90. CABLE TV Since 1990, when the average monthly basic cable TV rate in the United States was $15.81, the cost has risen by about $1.52 a year.

a. Write an equation in slope–intercept form to predict cable TV costs in the future. Use t to represent time in years after 1990 and C to represent the average basic monthly cost. $C = 1.52t + 15.81$

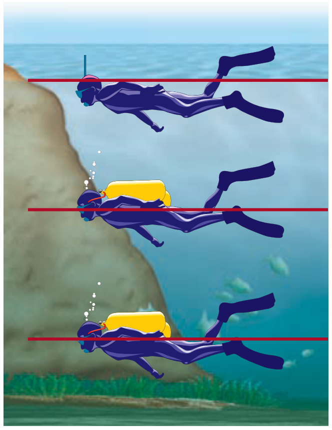
b. If the equation in part a were graphed, what would be the meaning of the C-intercept and the slope of the line? C-intercept: cost in 1990 was $15.81; slope: the yearly increase in cost was $1.52

91. COLLEGE COSTS According to the *College Board*, in 1980, the average tuition and fees at a private college were $8,850 a year. Since then, the annual cost has increased by about $514 per year.

a. Write a linear model in slope–intercept form that gives the annual cost c to attend the college t years after 1980. $c = 514t + 8{,}850$

b. Use the model to predict the tuition and fees to attend the college in the year 2050. $44,830

92. UNDERSEA DIVING The illustration below shows that the pressure p that divers experience is related to the depth d of the dive. A linear model can be used to describe this relationship.



a. Write a linear model in slope–intercept form. $p = \frac{14.7}{33}d + 14.7$ or $p = \frac{147}{330}d + 14.7$

b. Pearl and sponge divers often reach depths of 100 feet. What pressure do they experience? Round to the nearest tenth. 59.2 lb/in.^{2}

c. Scuba divers can safely dive to depths of 250 feet. What pressure do they experience? Round to the nearest tenth. 126.1 lb/in.^{2}

93. PHYSICAL FITNESS *from Campus to Careers*

A fitness instructor wants to determine the number of calories a client burns during a workout. The instructor knows that during the aerobic part of the workout, the client will burn 220 calories. He also knows that during the swimming part of the workout, the client will burn 7.8 calories per minute.


© iStockphoto.com/iofoto

a. Write a linear model in slope–intercept form that gives the total number of calories c the client will burn if she concludes a workout with m minutes of swimming. $c = 7.8m + 220$

b. If the client needs to burn 300 calories per exercise session to lose weight, how many minutes should she swim satisfy this requirement? About $10\frac{1}{4}$ min

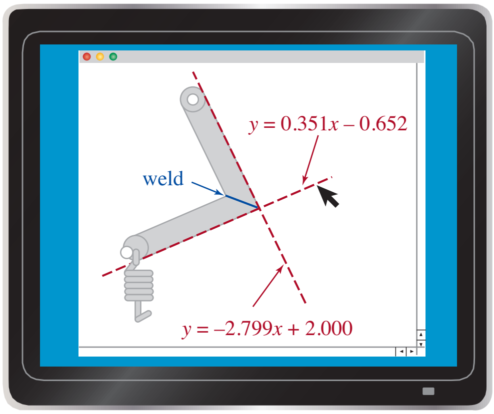
94. WINDCHILL A combination of cold and wind makes a person feel colder than the actual temperature. The table shows what temperatures of 35°F and 15°F feel like when a 15-mph wind is blowing. The relationship between the actual temperature and the windchill temperature can be modeled with a linear equation.

a. Write an equation that models this relationship. Answer in slope–intercept form. $y = 1.35x - 31.25$

b. What information is given by the y-intercept of the graph of the equation found in part a? When the actual temperature is 0°F, the windchill temperature is about −31°F.

Actual temperature	Windchill temperature
35°F	16°F
15°F	−11°F

95. DRAFTING The illustration shows a computer-generated drawing of an airplane part. When the designer clicks the mouse on a line on the drawing, the computer finds the equation of the line. Use a calculator to determine whether the angle where the weld is to be made is a right angle. not quite

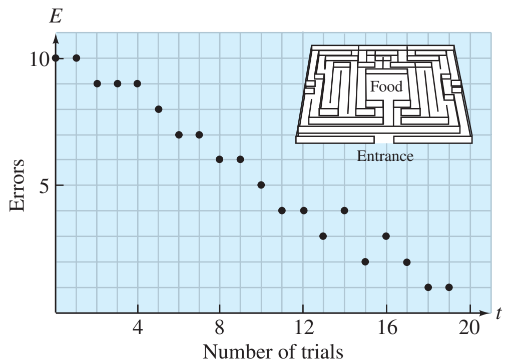


96. PSYCHOLOGY EXPERIMENTS The scattergram below shows the performance of a rat in a maze.

a. Draw a line through (1, 10) and (19, 1). Write its equation using the variables t and E. In psychology, this equation is called the *learning curve* for the rat. $E = -\frac{1}{2}t + \frac{21}{2}$

b. What does the slope of the line tell us? The number of errors is reduced by 1 for every 2 trials.

c. What information does the t-intercept of the graph give? On the 21st trial, the rat should make no errors.



WRITING

97. Explain how to find the equation of a line passing through two given points.

98. Explain what m, x_1, and y_1 represent in the point–slope form of the equation of a line.

99. Explain what m and b represent in the slope–intercept form of the equation of a line.

100. Linear relationships between two quantities can be described by an equation or a graph. Which do you think is the more informative? Why?

REVIEW

101. INVESTMENTS Equal amounts are invested at 6%, 7%, and 8% annual interest. The three investments yield a total of \$2,037 annual interest. Find the total amount of money invested. \$29,100

102. MIXING COFFEES To make a mixture of 80 pounds of coffee worth \$272, a grocer mixes coffee worth \$3.25 a pound with coffee worth \$3.85 a pound. How many pounds of cheaper coffee should the grocer use? 60 lb

103. Solve: $3x = 2x$ 0

104. Find the area of a rectangle that has a perimeter of 32 centimeters and a length of 12 centimeters. 48 cm^2

Objectives

1. Define relation, domain, and range.
2. Identify functions.
3. Use function notation.
4. Graph linear functions.
5. Find the domain and range of a function.
6. Use the vertical line test.
7. Solve applications involving functions.

SECTION 2.5
An Introduction to Functions

The concept of a *function* is one of the most important ideas in all of mathematics. To introduce this topic, we will begin with a table that might be seen on television or printed in a newspaper.

1 Define relation, domain, and range.

The following table shows the number of women serving in the U.S. House of Representatives for several recent sessions of Congress.

Women in the U.S. House of Representatives

Session of Congress	104th	105th	106th	107th	108th	109th	110th
Number of women Representatives	48	54	56	59	59	68	71

We can display the data in the table as a set of ordered pairs, where the **first component** represents the session of Congress and the **second component** represents the number of women representatives serving during that session:

$$\{(104, 48), \quad (105, 54), \quad (106, 56), \quad (107, 59), \quad (108, 59), \quad (109, 68), \quad (110, 71)\}$$

Sets of ordered pairs like this are called **relations.** The set of all first components is called the **domain of the relation,** and the set of all second components is called the **range of the relation.** A relation may consist of a finite number of ordered pairs or an infinite number of ordered pairs.

EXAMPLE 1 Find the domain and range of the relation: $\{(3, 2), (5, -7), (-8, 2), (9, 0)\}$

Strategy We will identify the first components and the second components of the ordered pairs.

WHY The set of all first components is the domain of the relation, and the set of all second components is the range.

Solution

The first components of the ordered pairs are highlighted in red, and the second components are highlighted in blue:

$\{(3, 2), (5, -7), (-8, 2), (9, 0)\}$

The domain of the relation is $\{-8, 3, 5, 9\}$.

The range of the relation is $\{-7, 0, 2\}$.

The elements of the domain and range are usually listed in increasing order, and if a value is repeated, it is only listed once.

Self Check 1

Find the domain and range of the relation: $\{(5, 6), (-12, 4), (8, 6), (-6, -6), (5, 4)\}$

Now Try Problem 22

Self Check 1 Answer

D: $\{-12, -6, 5, 8\}$, R: $\{-6, 4, 6\}$

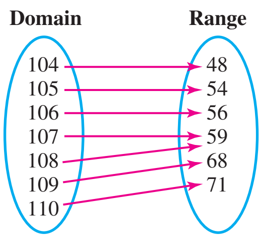
Teaching Example 1 Find the domain and range of the relation: $\{(4, -9), (0, 5), (-2, 6), (0, 7)\}$

Answer:

D: $\{-2, 0, 4\}$, R: $\{-9, 5, 6, 7\}$

2 Identify functions.

The relation in Example 1 was defined by a set of ordered pairs. Relations can also be defined using an **arrow** or **mapping diagram.** The data from the U.S. House of Representatives example is presented on the right in that form.



Notice that to each session of Congress, there corresponds exactly one number of women representatives. That is, to each member of the domain there corresponds exactly one member of the range. Relations that have this characteristic are called *functions.*

Function

A **function** is a set of ordered pairs (a relation) in which to each first component there corresponds exactly one second component. The set of first components is called the **domain of the function,** and the set of second components is called the **range of the function.**

Since we will often work with sets of ordered pairs of the form (x, y), it is helpful to define a function using the variables x and y.

y Is a Function of x

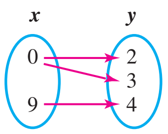
Given a relation in x and y, if to each value of x in the domain there corresponds exactly one value of y in the range, then y is said to be a function of x.

In the previous definition, since y depends on x, we call x the **independent variable** and y the **dependent variable.** The set of all possible values that can be used for the independent variable is the **domain** of the function, and the set of all values of the dependent variable is the **range** of the function.

Self Check 2

In each case, determine whether the relation defines y to be a function of x.

a.



b.

x	y
−1	−60
0	55
3	0

c. $\{(4, -1), (9, 2), (16, 15), (4, 4)\}$

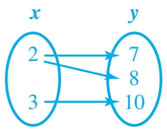
Now Try **Problems 25, 29, and 33**

Self Check 2 Answers

a. no; (0, 2), (0, 3)
b. yes
c. no; (4, −1), (4, 4)

Teaching Example 2 In each case, determine whether the relation defines y to be a function of x.

a.



b.

x	y
5	3
2	9
7	9

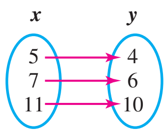
c. $\{(-3, 4), (5, 12), (-3, 6)\}$

Answers:
a. no; (2, 7), (2, 8)
b. yes **c.** no; (−3, 4), (−3, 6)

EXAMPLE 2

In each case, determine whether the relation defines y to be a function of x.

a.



b.

x	y
8	2
1	4
8	3
9	9

c. $\{(-2, 3), (-1, 3), (0, 3), (1, 3)\}$

Strategy In each case, we will determine whether there is more than one value of y that corresponds to a single value of x.

WHY If to any x-value there corresponds more than one y-value, then y is not a function of x.

Solution

a. The arrow diagram defines a function because to each value of x there corresponds exactly one value of y.

- $5 \to 4$ To the x-value 5, there corresponds exactly one y-value, 4.
- $7 \to 6$ To the x-value 7, there corresponds exactly one y-value, 6.
- $11 \to 10$ To the x-value 11, there corresponds exactly one y-value, 10.

b. The table does not define a function, because to the x-value 8 there corresponds to more than one y-value.

- In the first row, to the x-value 8, there corresponds the y-value 2.
- In the third row, to the same x-value 8, there corresponds a different y-value, 3.

When the correspondence in the table is written as a set of ordered pairs, it is apparent that the relation does not define a function

The same x-value

$\{(8, 2), (1, 4), (8, 3), (9, 9)\}$ This is not a function.

Different y-values

c. Since to each value of x, there corresponds exactly one value of y, the set of ordered pairs defines y to be a function of x.

- $(-2, 3)$ To the x-value -2, there corresponds exactly one y-value, 3.
- $(-1, 3)$ To the x-value -1, there corresponds exactly one y-value, 3.

- (0, 3) To the x-value 0, there corresponds exactly one y-value, 3.
- (1, 3) To the x-value 1, there corresponds exactly one y-value, 3.

In this case, the same y-value, 3, corresponds to each x-value.

The results from parts (b) and (c) illustrate an important fact: *Two different ordered pairs of a function can have the same y-value, but they cannot have the same x-value.*

Success Tip Every function is, by definition, a relation. However, not every relation is a function, as we see in part (b) of Example 2.

A function can also be defined by an equation. For example, $y = \frac{1}{2}x + 3$ sets up a rule in which to each value of x there corresponds exactly one value of y. To find the y-value (called an **output**) that corresponds to the x-value 4 (called an **input**), we substitute 4 for x and evaluate the right side of the equation.

$$y = \frac{1}{2}x + 3$$

$$= \frac{1}{2}(4) + 3 \quad \text{Substitute 4 for x. The input is 4.}$$

$$= 2 + 3$$

$$= 5 \quad \text{This is the output.}$$

In the function $y = \frac{1}{2}x + 3$, a y-value of 5 corresponds to an x-value of 4.

Not all equations define functions, as we will see in the next example.

EXAMPLE 3

Does $y = 2x - 3$ define y to be a function of x? If so, illustrate the function with a table and a graph.

Strategy We will determine whether there is more than one value of y that corresponds to a single value of x.

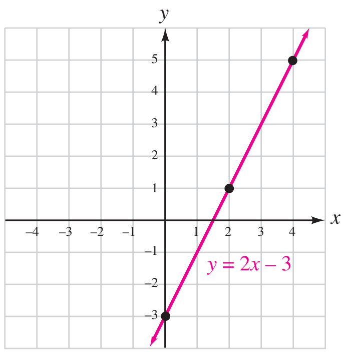
WHY If to any x-value there corresponds more than one y-value, then y is not a function of x.

Solution

For $y = 2x - 3$ to define a function, every input number x must determine one output value of y. To find y in the equation $y = 2x - 3$, we multiply x by 2 and then subtract 3. Since this arithmetic gives one result, each choice of x determines one value of y. Thus, the equation defines y to be a function of x.

A table of values and the graph appear below.

$y = 2x - 3$	
x	y
−4	−11
−2	−7
0	−3
2	1
4	5
6	9



Self Check 3

Does $y = -2x + 3$ define y to be a function of x? yes

Now Try Problem 37

Teaching Example 3 Does $y = 5x - 2$ define y to be a function of x?
Answer:
yes

As you will see in the next example, not every equation in two variables defines a function.

Self Check 4

Does $|y| = x$ define y to be a function of x? Explain your answer.

Now Try **Problem 40**

Self Check 4 Answer

No; if $x = 2$, y could be either 2 or -2.

Teaching Example 4 Does $y^2 = 12x$ define y to be a function of x?

Answer:

No; if $x = 3$, y could be either 6 or -6.

EXAMPLE 4 Does $y^2 = x$ define y to be a function of x?

Strategy We will determine whether there is more than one value of y that corresponds to a single value of x.

WHY If to any x-value there corresponds more than one y-value, then y is not a function of x.

Solution

For a function to exist, each input x must determine one output y. If we let $x = 16$, for example, y could be either 4 or -4, because $4^2 = 16$ and $(-4)^2 = 16$. Since more than one value of y is determined when $x = 16$, the equation does not represent a function.

3 Use function notation.

There is a special notation that we will use to denote functions.

Function Notation

The notation $y = f(x)$ denotes that the variable y is a function of x.

The notation $y = f(x)$ is read as "y equals f of x." Note that y and $f(x)$ are two notations for the same quantity. Thus, the equations $y = 4x + 3$ and $f(x) = 4x + 3$ are equivalent. We read $f(x) = 4x + 3$ as "f of x is equal to $4x$ plus 3."

This is the variable used to represent input values.

$$f(x) = \underbrace{4x + 3}$$

This is the name of the function.

This expression shows how to obtain an output value from a given input value.

Caution! The notation $f(x)$ does not mean "f times x."

The notation $y = f(x)$ provides a way of denoting the value of y (the dependent variable) that corresponds to some number x (the independent variable). For example, if $y = f(x)$, the value of y that is determined by $x = 3$ is denoted by $f(3)$.

Self Check 5

If $f(x) = -2x - 1$, find:

a. $f(2)$ -5

b. $f(-3)$ 5

c. $f(-t)$ $2t - 1$

Now Try **Problems 52 and 54**

EXAMPLE 5 Let $f(x) = 4x + 3$. Find:

a. $f(3)$ **b.** $f(-1)$ **c.** $f(0)$ **d.** $f(r + 1)$

Strategy We will substitute 3, -1, 0, and $r + 1$ for each x in $f(x) = 4x + 3$ and evaluate the right side.

WHY Whatever expression appears within the parentheses in $f(\ )$ is to be substituted for each x in $f(x) = 4x + 3$.

Solution

a. To find $f(3)$, we replace x with 3:

$$\begin{aligned} f(x) &= 4x + 3 \\ f(3) &= 4(3) + 3 \\ &= 12 + 3 \\ &= 15 \end{aligned}$$

b. To find $f(-1)$, we replace x with -1:

$$\begin{aligned} f(x) &= 4x + 3 \\ f(-1) &= 4(-1) + 3 \\ &= -4 + 3 \\ &= -1 \end{aligned}$$

c. To find $f(0)$, we replace x with 0:

$$\begin{aligned} f(x) &= 4x + 3 \\ f(0) &= 4(0) + 3 \\ &= 3 \end{aligned}$$

d. To find $f(r + 1)$, we replace x with $r + 1$:

$$\begin{aligned} f(x) &= 4x + 3 \\ f(r + 1) &= 4(r + 1) + 3 \\ &= 4r + 4 + 3 \\ &= 4r + 7 \end{aligned}$$

Teaching Example 5 If $f(x) = -3x - 4$, find:
a. $f(0)$ **b.** $f(-3)$ **c.** $f(t - 2)$
Answers:
a. -4 **b.** 5 **c.** $-3t + 2$

To see why function notation is helpful, we consider the following equivalent sentences:

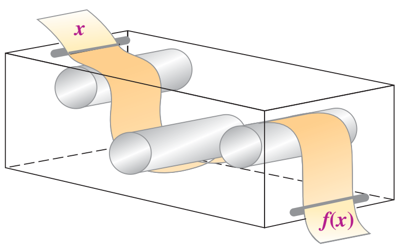
1. For the equation $y = 4x + 3$, find the value of y when x is 3.
2. For the function $f(x) = 4x + 3$, find $f(3)$.

Statement 2, which uses $f(x)$ notation, is much more concise.

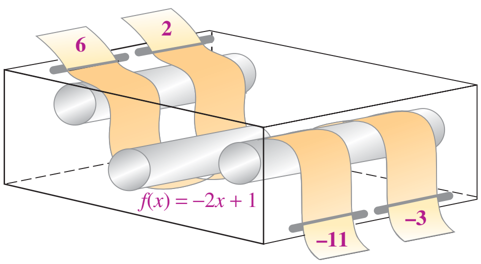
We can think of a function as a machine that takes some input x and turns it into some output $f(x)$, as shown in figure (a). The machine shown in figure (b) turns the input number 6 into the output value -11 and turns the input number 2 into the output value -3. The set of numbers that we can put into the machine is the domain of the function, and the set of numbers that comes out is the range.

The letter f used in the notation $y = f(x)$ represents the word *function*. However, other letters can be used to represent functions. For example, the notations $y = g(x)$ and $y = h(x)$ are often used to denote functions involving the independent variable x.

In Example 6, the equation $g(x) = x^2 - 2x$ defines a function, because each value of x determines a single value of $g(x)$.



(a)



(b)

EXAMPLE 6 Let $g(x) = x^2 - 2x$. Find: **a.** $g\left(\frac{2}{5}\right)$ **b.** $g(-2.4)$

Strategy We will substitute $\frac{2}{5}$ and -2.4 for each x in $g(x) = x^2 - 2x$ and evaluate the right side.

WHY Whatever expression appears within the parentheses in $g(\ \ )$ is to be substituted for each x in $g(x) = x^2 - 2x$.

Self Check 6

Let $h(x) = -\dfrac{x^2 + 2}{2}$. Find:

a. $h(4)$ -9
b. $h(-0.6)$ -1.18

Now Try **Problem 58**

Teaching Example 6 Let $h(x) = -x^2 + x - 2$. Find:
a. $h(-3)$ **b.** $h(1.5)$
Answers:
a. -14 **b.** -2.75

Solution

a. To find $g\left(\frac{2}{5}\right)$, we replace x with $\frac{2}{5}$:

$$\begin{aligned} g(x) &= x^2 - 2x \\ g\left(\frac{2}{5}\right) &= \left(\frac{2}{5}\right)^2 - 2\left(\frac{2}{5}\right) \\ &= \frac{4}{25} - \frac{4}{5} \\ &= -\frac{16}{25} \end{aligned}$$

b. To find $g(-2.4)$, we replace x with -2.4:

$$\begin{aligned} g(x) &= x^2 - 2x \\ g(-2.4) &= (-2.4)^2 - 2(-2.4) \\ &= 5.76 + 4.8 \\ &= 10.56 \end{aligned}$$

In the next example, the letter A is chosen to name a function that finds the area of a circle. The letter d is chosen as the independent variable, to help stress the fact that the area of a circle is a function of its diameter.

Self Check 7

ARCHERY In Example 7, find $A(9.6)$ to the nearest tenth. What information does it give about the archery target in the figure?

Now Try **Problem 105**

Self Check 7 Answer
72.4; the area of the bull's eye is 72.4 in.^{2}.

Teaching Example 7 ARCHERY In Example 7, find $A(28.8)$ to the nearest tenth.
Answer:

EXAMPLE 7 *Archery*

The area of a circle with a diameter of length d is given by the function $A(d) = \pi\left(\frac{d}{2}\right)^2$. Find $A(48)$ to the nearest tenth. What information does it give about the archery target shown in the figure below?

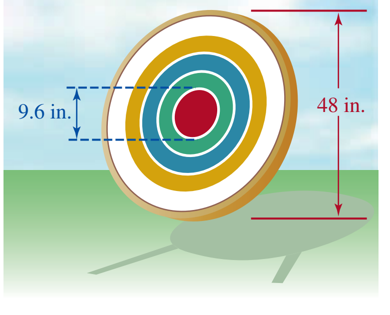
Strategy We will substitute 48 for each d in $A(d) = \pi\left(\frac{d}{2}\right)^2$ and evaluate the right side.

WHY Whatever expression appears within the parentheses in $A(\quad)$ is to be substituted for each d in $A(d) = \pi\left(\frac{d}{2}\right)^2$.

Solution

Since the diameter of the circular target is 48 inches, $A(48)$ gives the area of the target. To find $A(48)$, we replace d with 48.

$$\begin{aligned} A(d) &= \pi\left(\frac{d}{2}\right)^2 \\ A(48) &= \pi\left(\frac{48}{2}\right)^2 && \text{Substitute 48 for } d. \\ &= \pi(24)^2 \\ &= 576\pi \\ &\approx 1{,}809.557368 && \text{Use a calculator to perform the multiplication.} \end{aligned}$$



To the nearest tenth, the area of the target is 1,809.6 in.^{2}.

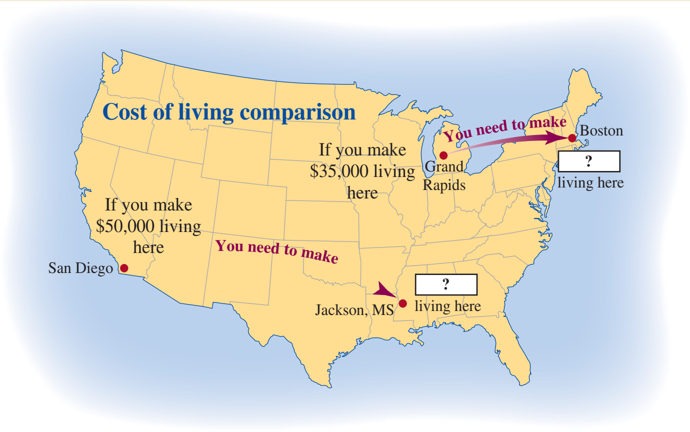
THINK IT THROUGH *Cost of Hiring*

"Whether you work for a company that sponsors relocation, or you make an independent move, you should examine living costs in the area to truly understand the full impact."

Doug Roy, consultant, Runzheimer International

Job offers often involve relocating. In such cases, it is important to consider the living costs of the new destination. The annual cost of living in an area can make or break a career move.

Cost-of-living comparisons between two cities can be expressed as functions. For example, suppose you live in San Diego, California, and have an annual income of $\$x$. The cost-of-living function $f(x) = 0.56x$ gives the salary required to maintain the same lifestyle if you move to Jackson, Mississippi. Similarly, if you earn $\$x$ annually in Grand Rapids, Michigan, the function $g(x) = 1.75x$ gives the salary needed to maintain the same lifestyle in Boston, Massachusetts. Use these functions to find the missing salaries in the illustration. $28,000, $61,250



Source: The Salary Calculator, www.homefair.com

4 Graph linear functions.

We have seen that in a function, a single value of $f(x)$ corresponds to each value of x in the domain. The "input-output" pairs that a function generates can be plotted on a rectangular coordinate system to get the graph of the function.

EXAMPLE 8

Graph the function: $f(x) = \dfrac{1}{2}x + 3$

Strategy To graph the function, we can think of $f(x)$ as y and use the same methods that we used to graph linear equations in Section 2.2.

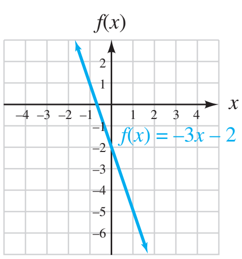
WHY The notation $f(x) = \frac{1}{2}x + 3$ is another way to write $y = \frac{1}{2}x + 3$.

Solution

We begin by constructing a table of function values. To make a table, we select several values for x and find the corresponding values of $f(x)$. If $x = -2$, we have

Self Check 8

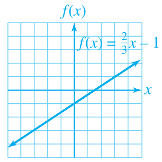
Graph: $f(x) = -3x - 2$



Now Try Problem 67

Teaching Example 8 Graph the function: $f(x) = \frac{2}{3}x - 1$

Answer:



$$f(x) = \frac{1}{2}x + 3 \quad \text{This is the function to graph.}$$

$$f(-2) = \frac{1}{2}(-2) + 3 \quad \text{Substitute } -2 \text{ for each } x.$$

$$= -1 + 3 \quad \text{Evaluate the right side.}$$

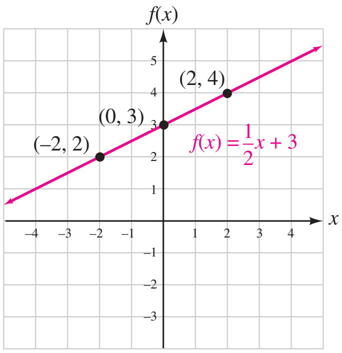
$$= 2$$

Thus, $f(-2) = 2$ and the ordered pair $(-2, 2)$ lies on the graph of f.

In a similar way, we find the corresponding values of $f(x)$ for x-values of 0 and 2 and record them in the table. Then we plot the ordered pairs and draw a straight line through the points to get the graph of $f(x) = \frac{1}{2}x + 3$.

$f(x) = \frac{1}{2}x + 3$		
x	$f(x)$	
−2	2	→ (−2, 2)
0	3	→ (0, 3)
2	4	→ (2, 4)
↑ Select x.	↑ Find $f(x)$.	↑ Plot the point.

This axis can be labeled y or $f(x)$.



Notation A table of function values is similar to a table of solutions, except that the second column is usually labeled $f(x)$ instead of y.

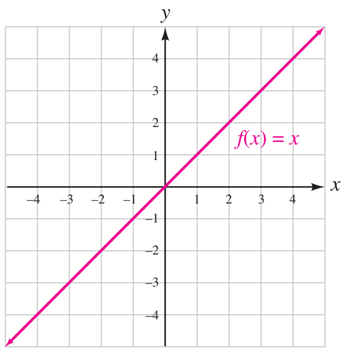
x	$f(x)$

x	y

The function $f(x) = \frac{1}{2}x + 3$ that was graphed in Example 8 is called a *linear function.* In general, a **linear function** is a function that can be written in the form $f(x) = mx + b$. Its graph is a straight line with slope m and y-intercept $(0, b)$.

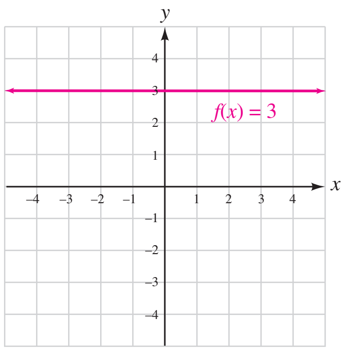
The most basic linear function is $f(x) = x$. It is called the **identity function** because it assigns each real number to itself. The graph of the identity function is a line with slope 1 and y-intercept $(0, 0)$, as shown below in figure (a).

A linear function defined by $f(x) = b$ is called a **constant function,** because for any input x, the output is the constant b. The graph of a constant function is a horizontal line. The graph of $f(x) = 3$ is shown below in figure (b).



The identity function

(a)



A constant function

(b)

5 Find the domain and range of a function.

EXAMPLE 9 Find the domain and range of each function:

a. $f(x) = 3x + 1$ **b.** $f(x) = \frac{1}{x-2}$

Strategy For the domain, we will ask, "What values of x are acceptable replacements for x in $3x + 1$ and $\frac{1}{x-2}$." For the range, we will determine the set of output values.

WHY These values of x form the domain of the function and the set of output values is the range.

Solution

a. We will be able to evaluate $3x + 1$ for any real-number input x. So the domain of the function is the set of real numbers. Since the output y can be any real number, the range is the set of real numbers.

b. To find the domain of $f(x) = \frac{1}{x-2}$, we exclude any real-number x inputs for which we would be unable to compute $\frac{1}{x-2}$. The number 2 cannot be substituted for x, because that would make the denominator equal to zero. Since any real number except 2 can be substituted for x in the equation $f(x) = \frac{1}{x-2}$, the domain is the set of all real numbers except 2. Since a fraction with a numerator of 1 cannot be 0, the range is the set of all real numbers except 0.

Self Check 9

Find the domain and range of each function:

a. $f(x) = -2x + 3$

b. $f(x) = \dfrac{2}{x+3}$

Now Try **Problems 70 and 72**

Self Check 9 Answers

a. D: the set of all real numbers, R: the set of all real numbers

b. D: the set of all real numbers except -3, R: the set of all real numbers except 0

Teaching Example 9 Find the domain and range of each function.

a. $f(x) = -5x + 3$

b. $f(x) = \frac{-7}{x+9}$

Answers:

a. D: the set of all real numbers, R: the set of all real numbers

b. D: the set of all real numbers except -9, R: the set of all real numbers except 0.

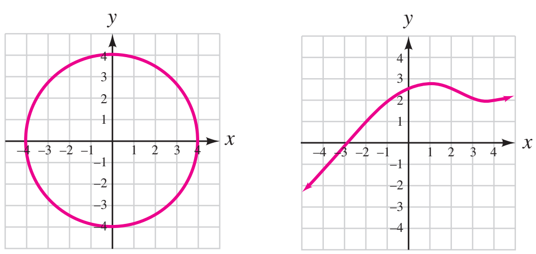
6 Use the vertical line test.

Some graphs define functions and some do not. If a vertical line intersects a graph more than once, the graph does not represent a function, because to one value of x there would correspond more than one value of y.

The Vertical Line Test

If a vertical line intersects a graph in more than one point, the graph is not the graph of a function.

EXAMPLE 10 Determine whether each of the following is the graph of a function.



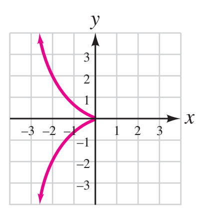
(a) (b)

Strategy We will check to see whether any vertical lines intersect the graph more than once.

WHY If any vertical line intersects the graph more than once, it is not the graph of a function.

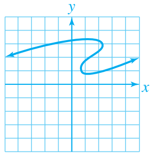
Self Check 10

Determine whether the following graph is the graph of a function. not a function



Now Try **Problems 74 and 76**

Teaching Example 10 Determine whether the following graph is the graph of a function.

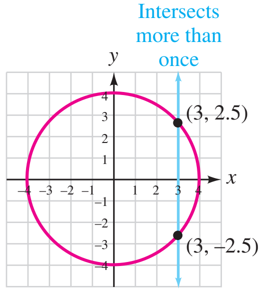


Answer:
not a function

Solution

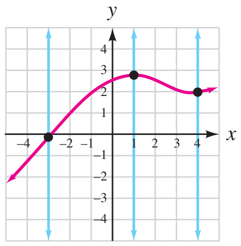
a. Refer to figure (a) below. The graph shown in red is not the graph of a function because a vertical line intersects the graph more than once. The points of intersection of the graph and the vertical line indicate that two values of y (2.5 and -2.5) correspond to the x-value 3.

b. Refer to figure (b) below. The graph shown in red is the graph of a function, because no vertical line intersects the graph more than once.



(a)

x	y
3	2.5
3	-2.5



(b)

7 Solve applications involving functions.

We can use functions to describe many relationships where one quantity depends upon another.

Self Check 11

ADVERTISING A business owner purchases engraved pens to promote his company. If the set-up charge is \$13, and the price per pen is \$1, write a linear function describing the cost of the order if he orders x pens.

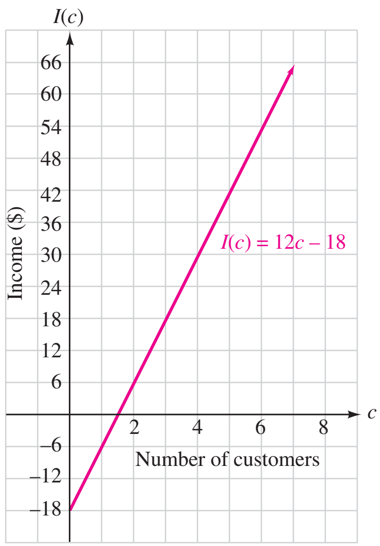
$f(x) = x + 13$

Now Try **Problem 107**

Teaching Example 11 SCHOOL SUPPLIES A school district ordered lanyards with the school's name for every employee. If the set-up charge is \$30 and each lanyard cost \$2, write a linear function describing the cost of the lanyards if they order x of them.
Answer:
$f(x) = 2x + 30$

EXAMPLE 11 *Manicurists*

A recent graduate of a cosmetology school rents a station from the owner of a beauty salon for \$18 a day. She expects to make \$12 profit from each customer she serves. Write a linear function describing her daily income if she serves c customers per day. Then graph the function.



Strategy To write a function that describes her daily income, we must write an expression that represents the total daily profit that she makes from serving c customers.

WHY Her daily income will be the profit that she makes from serving c customers less the \$18 daily station rental fee.

Solution

The manicurist makes a profit of \$12 per customer, so if she serves c customers a day, she will make \$12c. To find her income, we must *subtract* the \$18 rental fee she pays from the profit. Therefore, the income function is $I(c) = 12c - 18$.

The graph of this linear function, shown in the figure, is a line with slope 12 and intercept $(0, -18)$. Since the manicurist cannot have a negative number of customers, we do not extend the line into quadrant III.

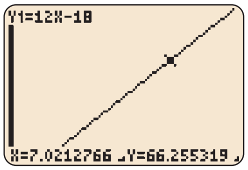
Using Your CALCULATOR Evaluating Functions

We can use a graphing calculator to find the value of a function for different input values. For example, to find the income earned by the manicurist in Example 11 for different numbers of customers, we first graph the income function $I(c) = 12c - 18$ as $y = 12x - 18$, using window settings of [0, 10] for x and [0, 100] for y. To find her income when she serves seven customers, we trace and move the cursor until the x-coordinate on the screen is nearly 7, as in figure (a). From the screen, we can read that her income is about \$66.25.

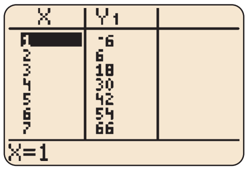
We can also use the *table* feature to evaluate a function. For example, after entering $Y_1 = 12x - 18$ to represent $I(c) = 12c - 18$, and after adjusting the *table set* (TBLSET), we press $\boxed{\text{2nd}}$ $\boxed{\text{TABLE}}$ to get the display shown in figure (b). The next-to-last line of the table indicates that $I(6) = 54$. That is, if she serves 6 customers, her income will be \$54.

With some graphing calculator models, we can evaluate a function by entering function notation. For example, to find the income earned by the manicurist if she serves 15 customers, we can use the following steps on a TI-84 Plus graphing calculator.

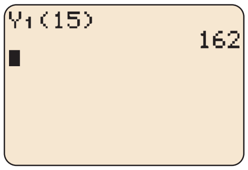
With $I(c) = 12c - 18$ entered as $Y_1 = 12x - 18$, we call up the home screen by pressing $\boxed{\text{2nd}}$ $\boxed{\text{QUIT}}$. Then we enter $\boxed{\text{VARS}}$ $\boxed{\blacktriangleright}$ $\boxed{1}$ $\boxed{\text{ENTER}}$. The symbolism Y_1 will be displayed. Next, we enter the input value 15 within parentheses and press $\boxed{\text{ENTER}}$. In figure (c), we see that $Y_1(15) = 162$. That is, $I(15) = 162$. The manicurist will earn \$162 if she serves 15 customers in one day.



(a)



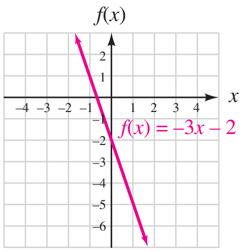
(b)



(c)

ANSWERS TO SELF CHECKS

1. D: {−12, −6, 5, 8}, R: {−6, 4, 6} **2. a.** no; (0, 2), (0, 3) **b.** yes **c.** no; (4, −1), (4, 4) **3.** yes **4.** No; if $x = 2$, y could be either 2 or −2. **5. a.** −5 **b.** 5 **c.** $2t - 1$ **6. a.** −9 **b.** −1.18 **7.** 72.4; the area of the bull's eye is 72.4 in.^{2}. **8.**



9. a. D: the set of all real numbers, R: the set of all real numbers **b.** D: the set of all real numbers except −3, R: the set of all real numbers except 0 **10.** not a function **11.** $f(x) = x + 13$

SECTION 2.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. Sets of ordered pairs are called relations.

2. In a relation, the set of all first components is called the domain of the relation.

3. In a relation, the set of all second components is called the range of the relation.

4. A function is a set of ordered pairs (a relation) in which to each first component there corresponds exactly one second component.

5. In a function, the set of first components (the input values) is called the domain of the function. The set of second components (the output values) is called the range.

6. If y is a function of x, x is called the independent variable and y is called the dependent variable.

7. $y = f(x)$ is read as y is a function of x or y equals f of x.

8. A linear function is a function that can be written in the form $f(x) = mx + b$.

9. The function $f(x) = x$ is called the identity function, because it assigns each real number to itself.

10. The function $f(x) = b$ is called the constant function, because for any input x the output is the constant b.

11. When graphing a function, input values x are associated with the horizontal axis and output values y with the vertical axis.

12. The vertical line test can be used to determine whether the graph of an equation determines a function.

CONCEPTS

13. Fill in the blank so that the following statements ask for the same thing.

a. For the equation $y = -5x + 1$, find the value of y when $x = -1$.

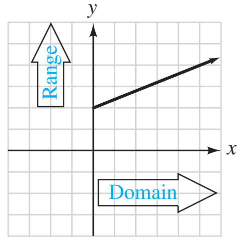
b. For the function $f(x) = -5x + 1$, find $f(-1)$.

14. For the function $f(x) = \frac{1}{x+4}$, why isn't -4 in the domain of f?
If $x = -4$, the output is a fraction whose denominator is 0.

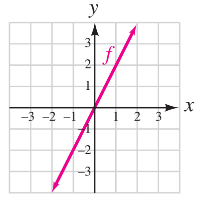
15. Consider the graph of the function in the next column.

a. Label each arrow in the illustration with the appropriate term: *domain* or *range*.

b. Give the domain and range.
D: all real numbers greater than or equal to 0, R: all real numbers greater than or equal to 2



16. Use the graph of function f on the right to find each of the following.



a. $f(-2)$ -4

b. $f(0)$ 0

c. $f(1)$ 2

NOTATION

Complete each solution.

17. If $f(x) = x^2 - 3x$, find $f(-5)$.

$$f(x) = x^2 - 3x$$
$$f(-5) = (-5)^2 - 3(\boxed{-5})$$
$$= \boxed{25} + 15$$
$$= 40$$

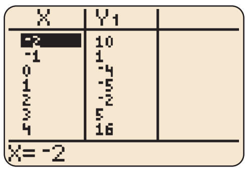
18. If $g(x) = \frac{2-x}{6}$, find $g(8)$.

$$g(x) = \frac{2-x}{6}$$
$$g(8) = \frac{2-\boxed{8}}{6}$$
$$= \frac{\boxed{-6}}{6}$$
$$= -1$$

19. Complete the sentence: $f(5) = 6$ is read "f of 5 is 6."

20. The illustration below shows a table of values generated by a graphing calculator for a function f. Find the following.

a. $f(-1)$ 1 **b.** $f(3)$ 5



▶ Selected exercises available online at **www.webassign.net/brookscole**

GUIDED PRACTICE

Find the domain and range of each relation. See Example 1.

21. $\{(-2, 1), (0, 4), (2, 5)\}$
D: $\{-2, 0, 2\}$, R: $\{1, 4, 5\}$

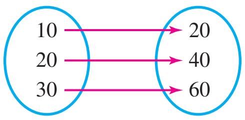
▶ **22.** $\{(15, -3), (0, 0), (4, 6), (-3, -8)\}$
D: $\{-3, 0, 4, 15\}$, R: $\{-8, -3, 0, 6\}$

23. $\{(0, 1), (-23, 35), (7, 1)\}$
D: $\{-23, 0, 7\}$, R: $\{1, 35\}$

24. $\{(1, -12), (-6, 8), (5, 8), (1, 4)\}$
D: $\{-6, 1, 5\}$, R: $\{-12, 4, 8\}$

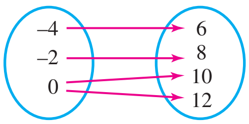
In each case, determine whether the relation defines y to be a function of x. If it does not, find two ordered pairs where more than one value of y corresponds to a single value of x. See Example 2.

▶ **25.**



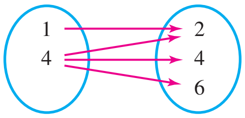
yes

26.



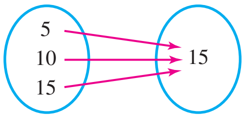
no; (0, 10), (0, 12)

27.



no; (4, 2), (4, 4), (4, 6)

28.



yes

29. $\{(3, 4), (3, -4), (4, 3), (4, -3)\}$
no, $(3, 4), (3, -4)$ or $(4, 3), (4, -3)$

▶ **30.** $\{(-1, 1), (-3, 1), (-5, 1), (-7, 1), (-9, 1)\}$
yes

31. $\{(-2, 7), (-1, 10), (0, 13), (1, 16)\}$
yes

32. $\{(-2, 4), (-3, 8), (-3, 12), (-4, 16)\}$
no, $(-3, 8), (-3, 12)$

33.

x	y
1	7
2	15
3	23
4	16
5	8

yes

34.

x	y
30	2
30	4
30	6
30	8
30	10

no; (30, 2), (30, 4) (answers may vary)

35.

x	y
−4	6
−1	0
0	−3
2	4
−1	2

no; $(-1, 0), (-1, 2)$

▶ **36.**

x	y
1	1
2	2
3	3
4	4

yes

Determine whether each equation defines y to be a function of x. If it does not, find two ordered pairs where more than one value of y corresponds to a single value of x. See Examples 3–4.

37. $y = 2x + 3$ yes

▶ **38.** $y = 4x - 1$ yes

39. $y = 4x^2$ yes

40. $y^2 = 3x$ no; $(12, 6), (12, -6)$

41. $y^4 = x$ no; $(1, 1), (1, -1)$

42. $y = \dfrac{1}{x}$ yes

43. $x = |y|$ no; $(1, 1), (1, -1)$

44. $x + 1 = |y|$ no; $(1, 2), (1, -2)$ yes

Find $f(3)$ and $f(-1)$. See Example 5a–c.

45. $f(x) = 3x$ $9, -3$

▶ **46.** $f(x) = -4x$ $-12, 4$

47. $f(x) = 2x - 3$ $3, -5$

48. $f(x) = 3x - 5$ $4, -8$

49. $f(x) = 7 + 5x$ $22, 2$

50. $f(x) = 3 + 3x$ $12, 0$

51. $f(x) = 9 - 2x$ $3, 11$

52. $f(x) = 12 + 3x$ $21, 9$

Find $g(w)$ and $g(w + 1)$. See Example 5d.

53. $g(x) = 2x$ $g(w) = 2w, g(w + 1) = 2w + 2$

▶ **54.** $g(x) = -3x$ $g(w) = -3w, g(w + 1) = -3w - 3$

55. $g(x) = 3x - 5$ $g(w) = 3w - 5, g(w + 1) = 3w - 2$

56. $g(x) = 2x - 7$ $g(w) = 2w - 7, g(w + 1) = 2w - 5$

Find $g\left(\frac{1}{2}\right)$ and $g(0.3)$. See Example 6.

57. $g(x) = x^2$ $\frac{1}{4}, 0.09$

▶ **58.** $g(x) = x^2 - 2$ $-\frac{7}{4}, -1.91$

59. $g(x) = x^3 - 1$ $-\frac{7}{8}, -0.973$

60. $g(x) = x^3$ $\frac{1}{8}, 0.027$

61. $g(x) = (x + 1)^2$ $\frac{9}{4}, 1.69$

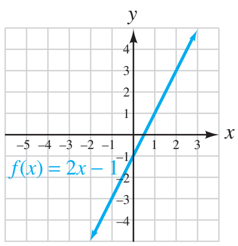
62. $g(x) = (x - 3)^2$ $\frac{25}{4}, 7.29$

63. $g(x) = 2x^2 - x$ $0, -0.12$

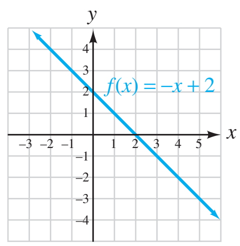
64. $g(x) = 5x^2 + 2x$ $\frac{9}{4}, 1.05$

Graph each function. See Example 8.

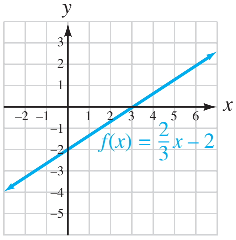
65. $f(x) = 2x - 1$



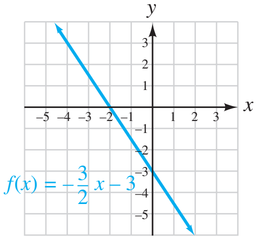
66. $f(x) = -x + 2$



▶ **67.** $f(x) = \frac{2}{3}x - 2$



68. $f(x) = -\frac{3}{2}x - 3$



Find the domain and range of each function. See Example 9.

69. $\{(-2, 3), (4, 5), (6, 7)\}$ D: $\{-2, 4, 6\}$, R: $\{3, 5, 7\}$

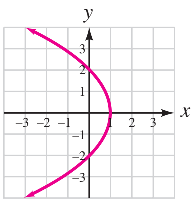
▶ **70.** $\{(0, 2), (1, 2), (3, 4)\}$ D: $\{0, 1, 3\}$, R: $\{2, 4\}$

71. $f(x) = \dfrac{1}{x - 4}$ D: the set of all real numbers except 4, R: the set of all real numbers except 0

72. $f(x) = \dfrac{5}{x + 1}$ D: the set of all real numbers except -1, R: the set of all real numbers except 0

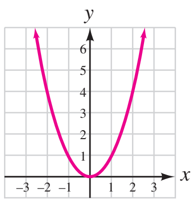
Determine whether each of the following is the graph of a function. See Example 10.

▶ **73.**



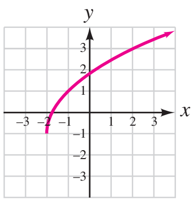
not a function

74.



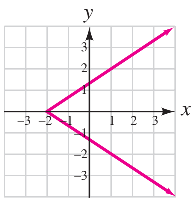
a function

▶ **75.**



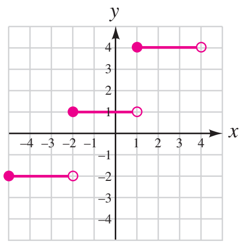
a function

76.



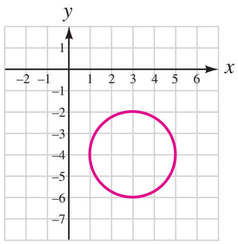
not a function

77.



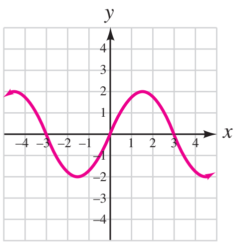
a function

78.



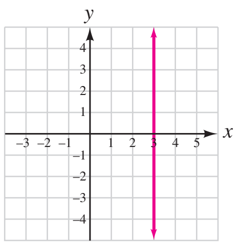
not a function

79.



a function

80.



not a function

TRY IT YOURSELF

Complete each table.

81. $f(t) = |t - 2|$

t	$f(t)$
-1.7	3.7
0.9	1.1
5.4	3.4

▶ **82.** $f(r) = -2r^2 + 1$

Input	Output
-1.7	-4.78
0.9	-0.62
5.4	-57.32

83. $g(x) = x^3$

Input	Output
$-\frac{3}{4}$	$-\frac{27}{64}$
$\frac{1}{6}$	$\frac{1}{216}$
$\frac{5}{2}$	$\frac{125}{8}$

84. $g(x) = 2\left(-x - \frac{1}{4}\right)$

x	$g(x)$
$-\frac{3}{4}$	1
$\frac{1}{8}$	$-\frac{3}{4}$
$\frac{5}{2}$	$-\frac{11}{2}$

Find the domain of each function.

85. $g(x) = \dfrac{x}{6 - x}$ D: the set of all real numbers except 6

86. $H(x) = \dfrac{4}{3 - 2x}$ D: the set of all real numbers except $\frac{3}{2}$

87. $s(x) = |x - 7|$ D: the set of all real numbers

88. $t(x) = \left|\dfrac{2x}{3} + 1\right|$ D: the set of all real numbers

▶ **89.** $f(x) = x^2$ D: the set of all real numbers

90. $g(x) = x^3$ D: the set of all real numbers

91. $s(x) = 3x + 6$ D: the set of all real numbers

92. $h(x) = \frac{4}{5}x - 8$ D: the set of all real numbers

Determine whether each equation defines a linear function.

93. $y = 3x^2 + 2$ no

94. $y = \dfrac{x - 3}{2}$ yes

95. $y = x$ yes

96. $y = 3x^3 - 4$ no

Find $h(2)$ and $h(-2)$.

97. $h(x) = |x| + 2$ 4, 4

98. $h(x) = |x| - 5$ $-3, -3$

99. $h(x) = x^2 - 2$ 2, 2

100. $h(x) = x^2 + 3$ 7, 7

101. $h(x) = \dfrac{1}{x + 3}$ $\frac{1}{5}, 1$

▶ **102.** $h(x) = \dfrac{3}{x - 4}$ $-\frac{3}{2}, -\frac{1}{2}$

103. $h(x) = \dfrac{x}{x - 3}$ $-2, \frac{2}{5}$

▶ **104.** $h(x) = \dfrac{x}{x^2 + 2}$ $\frac{1}{3}, -\frac{1}{3}$

APPLICATIONS

105. DECONGESTANTS The temperature in degrees Celsius that is equivalent to a temperature in degrees Fahrenheit is given by the linear function $C(F) = \frac{5}{9}(F - 32)$. Use this function to find the temperature range, in degrees Celsius, at which a bottle of Dimetapp should be stored. The label directions are shown below. between 20°C and 25°C

> **DIRECTIONS:** Adults and children 12 years of age and over: Two teaspoons every 4 hours. DO NOT EXCEED 6 DOSES IN A 24-HOUR PERIOD. Store at a controlled room temperature between 68°F and 77°F.

106. BODY TEMPERATURES The temperature in degrees Fahrenheit that is equivalent to a temperature in degrees Celsius is given by the linear function $F(C) = \frac{9}{5}C + 32$. Convert each of the temperatures in the following excerpt from *The Good Housekeeping Family Health and Medical Guide* to degrees Fahrenheit. (Round to the nearest degree.) 90°F, 110°F, 95°F, 106°F

In disease, the temperature of the human body may vary from about 32.2°C *to* 43.3°C *for a time, but there is grave danger to life should it drop and remain below* 35°C *or rise and remain at or above* 41°C.

107. CONCESSIONAIRES A baseball club pays a peanut vendor \$50 per game for selling bags of peanuts for \$1.75 each.

a. Write a linear function that describes the income the vendor makes for the baseball club during a game if she sells b bags of peanuts.
$I(b) = 1.75b - 50$

b. Find the income the baseball club will make if the vendor sells 110 bags of peanuts during a game. \$142.50

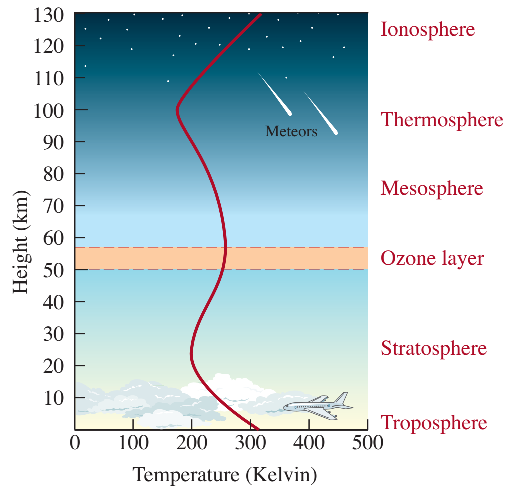
108. NEW HOME CONSTRUCTION In a proposal to some prospective clients, a housing contractor listed the following costs.

- Fees, permits, miscellaneous 12,000
- Construction, per square foot \$75

a. Write a linear function that the clients could use to determine the cost of building a home having f square feet. $C(f) = 75f + 12{,}000$

b. Find the cost to build a home having 1,950 square feet. \$158,250

109. EARTH'S ATMOSPHERE The illustration shows a graph of the temperatures of the atmosphere at various altitudes above Earth's surface. The temperature is expressed using the Kelvin scale, which is used in scientific work.



a. Estimate the coordinates of three points on the graph that have an x-coordinate of 200.
(200, 25), (200, 90), (200, 105)

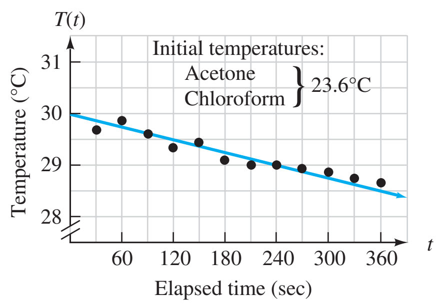
b. Explain why this is not the graph of a function.
It doesn't pass the vertical line test.

110. CHEMICAL REACTIONS When students in a chemistry laboratory mixed solutions of acetone and chloroform, they found that heat was immediately generated. As time went by, the mixture cooled down. The illustration below shows a graph of data points of the form (time, temperature) taken by the students during the experiment.

a. The linear function $T(t) = -\frac{t}{240} + 30$ models the relationship between the elapsed time t since the solutions were combined and the temperature $T(t)$ of the mixture. Graph the function in the illustration.

b. Predict the temperature of the mixture immediately after the two solutions are combined. 30°C

c. Is $T(180)$ more or less than the temperature recorded by the students for $t = 300$ on the graph? more



111. INCOME TAXES The function

$$T(a) = 700 + 0.15(a - 7{,}000)$$

(where a is adjusted gross income) is a model of the instructions given on the first line of the tax rate Schedule X shown.

a. Find $T(25{,}000)$ and interpret the result 3,400; the tax on an income of $25,000 is $3,400.

Schedule X–Use if your filing status is Single 2010

If your adjusted gross income is: Over —	But not over —	Your tax is	of the amount over —
$ 7,000	$28,400	**$ 700 + 15%**	**$ 7,000**
$28,400	$68,800	**$3,910 + 25%**	**$28,400**

b. Write a function that models the second line on Schedule X. $T(a) = 3{,}910 + 0.25(a - 28{,}400)$

▶ **112.** COST FUNCTIONS An electronics firm manufactures tape recorders, receiving $120 for each recorder it makes. If x represents the number of recorders produced, the income received is determined by the *revenue function* $R(x) = 120x$. The manufacturer has fixed costs of $12,000 per month and variable costs of $57.50 for each recorder manufactured. Thus, the *cost function* is $C(x) = 57.50x + 12{,}000$. How many recorders must the company sell for revenue to equal cost? (*Hint:* Set $R(x) = C(x)$.) 192

113. BALLISTICS The height of a toy rocket shot from the ground straight upward is given by the function $f(t) = -16t^2 + 256t$.

a. Find the height of the rocket 3 seconds after it is shot. 624 ft

b. Find $f(16)$. Interpret the result. 0; the rocket strikes the ground 16 seconds after being shot

114. PLATFORM DIVING The number of feet a diver is above the surface of the water is given by the function $h(t) = -16t^2 + 16t + 32$, where t is the elapsed time in seconds after the diver jumped. Find the height of the diver for the times shown in the table.

t	$h(t)$
0	32
0.5	36
1.5	20
2.0	0

WRITING

115. Give four ways in which a function can be described. Which do you think is the most informative? Why?

▶ **116.** Explain why we can think of a function as a machine.

REVIEW

Show that each number is a rational number by expressing it as a ratio of two integers.

117. $-3\frac{3}{4}$ $\frac{-15}{4}$

▶ **118.** 4.7 $\frac{47}{10}$

119. $0.333\ldots$ $\frac{1}{3}$

120. $-0.\overline{6}$ $\frac{-2}{3}$

Objectives

1. Find function values graphically.
2. Find the domain and range of a function graphically.
3. Graph nonlinear functions.
4. Translate graphs of functions.
5. Reflect graphs of functions.
6. Solve equations graphically.

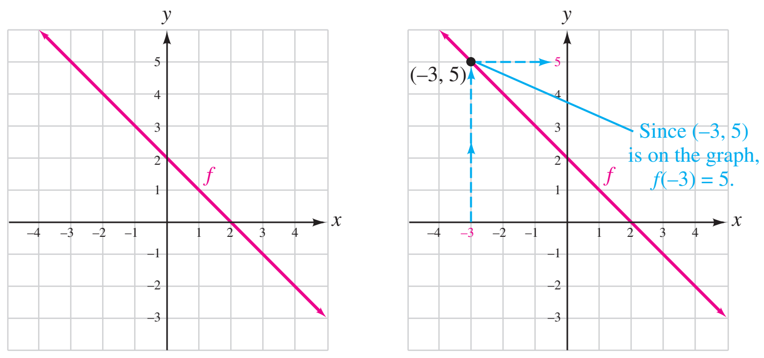
SECTION 2.6
Graphs of Functions

Since a graph is often the best way to describe a function, we need to know how to construct and interpret their graphs.

1 Find function values graphically.

From the graph of a function, we can determine function values. In general, the value of $f(a)$ is given by the y-coordinate of a point on the graph of f with x-coordinate a.

Refer to the graph of function f in figure (a). To find $f(-3)$ from the graph we need to find the y-coordinate of the point on the graph of f whose x-coordinate is -3. If we draw a vertical line through -3 on the x-axis, as shown in figure (b), the line intersects the graph of f at $(-3, 5)$. Therefore, 5 corresponds to -3, and it follows that $f(-3) = 5$.

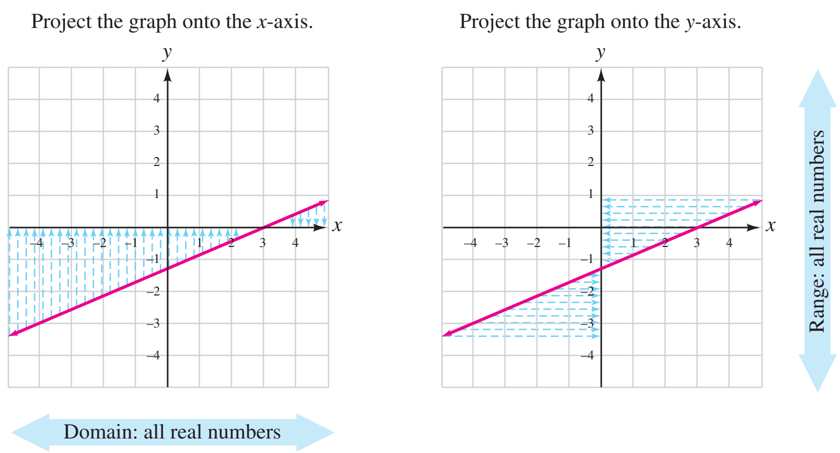


(a) (b)

2 Find the domain and range of a function graphically.

We can find the domain and range of a function from its graph. For example, to find the domain of the linear function graphed in figure (a), we *project* the graph onto the x-axis. Because the graph of the function extends indefinitely to the left and to the right, the projection includes all the real numbers. Therefore, the domain of the function is the set of real numbers.

To find the range of the same linear function, we project the graph onto the y-axis, as shown in figure (b). Because the graph of the function extends indefinitely upward and downward, the projection includes all the real numbers. Therefore, the range of the function is the set of real numbers.



(a) (b)

The Language of Algebra Think of the *projection* of a graph on an axis as the “shadow” that the graph makes on the axis.

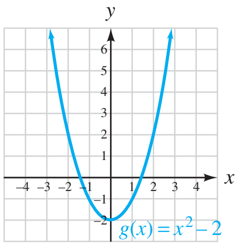
Many real-world situations can be modeled by linear functions. When these functions are graphed, we can learn information by examining the resulting line. In this section, we will discuss three more types of functions. Like linear functions, they can be used as mathematical models. But unlike linear functions, their graphs are not straight lines. Therefore, they are called **nonlinear functions.**

3 Graph nonlinear functions.

The first nonlinear function we will discuss is $f(x) = x^2$, called the **squaring function.**

Self Check 1

Graph $g(x) = x^2 - 2$. Find the domain and the range.



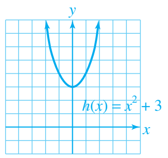
Now Try **Problem 20**

Self Check 1 Answer

D: the set of real numbers, R: the set of all real numbers greater than or equal to -2

Teaching Example 1 Graph $h(x) = x^2 + 3$. Find the domain and range.

Answer:



D: the set of all real numbers, R: the set of real numbers greater than or equal to 3

EXAMPLE 1 ***The Squaring Function*** Graph the function $f(x) = x^2$ and find the domain and range.

Strategy We will graph the function by creating a table of at least 7 function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

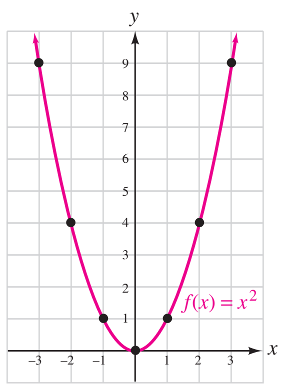
We substitute values for x in the equation and compute the corresponding values of $f(x)$. For example, if $x = -3$, we have

$$f(x) = x^2$$
$$f(-3) = (-3)^2 \quad \text{Substitute } -3 \text{ for } x.$$
$$= 9$$

Since $f(-3) = 9$, the ordered pair $(-3, 9)$ lies on the graph of f. In a similar manner, we find the corresponding values of $f(x)$ for other x-values and list the ordered pairs in the table of values. Then we plot the points and draw a smooth curve through them to get the graph, called a **parabola.**

$f(x) = x^2$		
x	$f(x)$	$(x, f(x))$
-3	9	$(-3, 9)$
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$

Choose values for x. Compute each $f(x)$.



From the graph, we can see that x can be any real number. This means that the domain of the squaring function is the set of real numbers. We can also see that y is always positive or zero. This means that the range is the set of nonnegative real numbers.

EXAMPLE 2 ***The Cubing Function*** Graph the function $f(x) = x^3$ and find the domain and range.

Strategy We will graph the function by creating a table of at least 7 function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

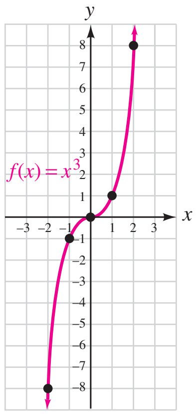
Solution

We substitute values for x in the equation and compute the corresponding values of $f(x)$. For example, if $x = -2$, we have

$$f(x) = x^3$$
$$f(-2) = (-2)^3 \quad \text{Substitute } -2 \text{ for } x.$$
$$= -8$$

Since $f(-2) = -8$, the ordered pair $(-2, -8)$ lies on the graph of f. In a similar manner, we find the corresponding values of $f(x)$ for other x-values and list the ordered pairs in the table. Then we plot the points and draw a smooth curve through them to get the graph.

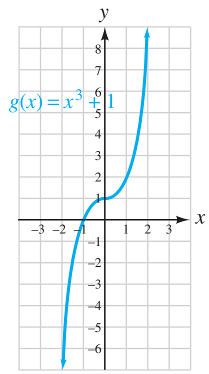
$f(x) = x^3$		
x	$f(x)$	$(x, f(x))$
-2	-8	$(-2, -8)$
-1	-1	$(-1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	8	$(2, 8)$



From the graph, we can see that x can be any real number. This means that the domain of the cubing function is the set of real numbers. We can also see that y can be any real number. This means that the range is the set of real numbers.

Self Check 2

Graph $g(x) = x^3 + 1$. Find the domain and the range.



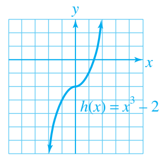
Now Try Problem 22

Self Check 2 Answer

D: the set of real numbers, R: the set of real numbers

Teaching Example 2 Graph $h(x) = x^3 - 2$. Find the domain and range.

Answer:



D: the set of all real numbers, R: the set of all real numbers

A third nonlinear function is $f(x) = |x|$, called the **absolute value function.**

EXAMPLE 3 *The Absolute Value Function*

Graph the function $f(x) = |x|$ and find the domain and range.

Strategy We will graph the function by creating a table of at least 7 function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

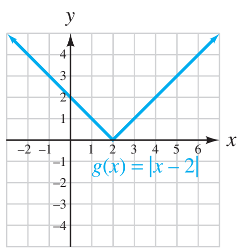
We substitute values for x in the equation and compute the corresponding values of $f(x)$. For example, if $x = -3$, we have

$$f(x) = |x|$$
$$f(-3) = |-3| \quad \text{Substitute } -3 \text{ for } x.$$
$$= 3$$

Since $f(-3) = 3$, the ordered pair $(-3, 3)$ lies on the graph of f. In a similar manner, we find the corresponding values of $f(x)$ for other x-values and list the ordered pairs in the table on the next page. Then we plot the points and connect them to get the graph.

Self Check 3

Graph $g(x) = |x - 2|$. Find the domain and the range.

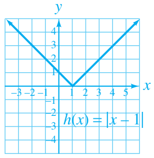


Now Try Problem 24

Self Check 3 Answer

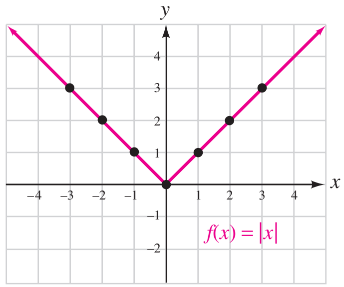
D: the set of real numbers, R: the set of nonnegative real numbers

Teaching Example 3 Graph $h(x) = |x - 1|$. Find the domain and range.
Answer:



D: the set of all real numbers, R: the set of real numbers greater than or equal to 0

$f(x) = \|x\|$		
x	$f(x)$	$(x, f(x))$
−3	3	(−3, 3)
−2	2	(−2, 2)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

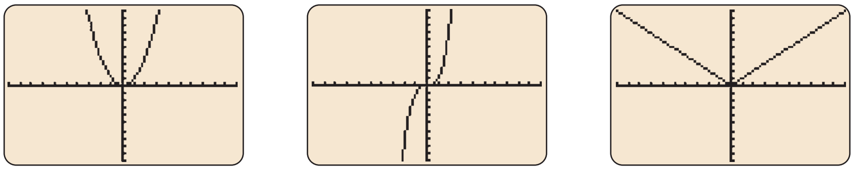


From the graph, we can see that x can be any real number. So the domain of the absolute value function is the set of real numbers. We can also see that y is always positive or zero. This means that the range is the set of nonnegative real numbers.

Using Your CALCULATOR **Graphing Functions**

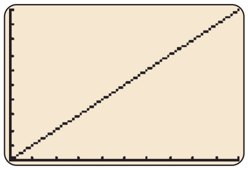
We can graph nonlinear functions with a graphing calculator. For example, to graph $f(x) = x^2$ in a standard window of $[-10, 10]$ for x and $[-10, 10]$ for y, we press $\boxed{\text{Y=}}$, and then enter the function by typing x and then $\boxed{x^2}$. Finally, we press the $\boxed{\text{GRAPH}}$ key to obtain the graph shown in figure (a).

To graph $f(x) = x^3$, we enter the function by typing x^3 and then press the $\boxed{\text{GRAPH}}$ key to obtain the graph shown in figure (b). To graph $f(x) = |x|$, we enter the function by selecting abs from the NUM option within the MATH menu, typing x, and pressing the $\boxed{\text{GRAPH}}$ key to obtain the graph shown in figure (c).



(a) (b) (c)

When using a graphing calculator, we must be sure that the viewing window does not show a misleading graph. For example, if we graph $f(x) = |x|$ in the window $[0, 10]$ for x and $[0, 10]$ for y, we will obtain a misleading graph that looks like a line shown in figure (d). This is not correct. The proper graph is the V-shaped graph shown in figure (c). One of the challenges of using graphing calculators is finding an appropriate viewing window.

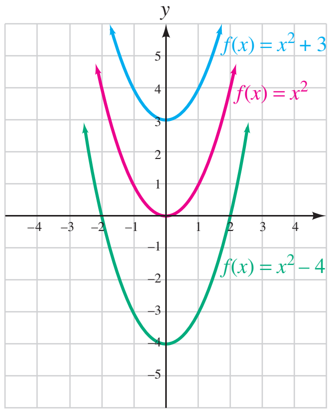


(d)

4 Translate graphs of functions.

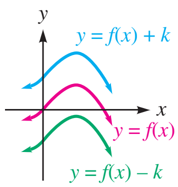
Examples 1, 2, and 3 and their Self Checks suggest that the graphs of different functions may be identical except for their positions in the xy-plane. For example, the figure on the next page shows the graph of $f(x) = x^2 + k$ for three different values of k. If

$k = 0$, we get the graph of $f(x) = x^2$. If $k = 3$, we get the graph of $f(x) = x^2 + 3$, which is identical to the graph of $f(x) = x^2$ except that it is shifted 3 units upward. If $k = -4$, we get the graph of $f(x) = x^2 - 4$, which is identical to the graph of $f(x) = x^2$ except that it is shifted 4 units downward. These shifts are called **vertical translations.**



In general, we can make these observations.

Vertical Translations



If f is a function and k represents a positive number, then

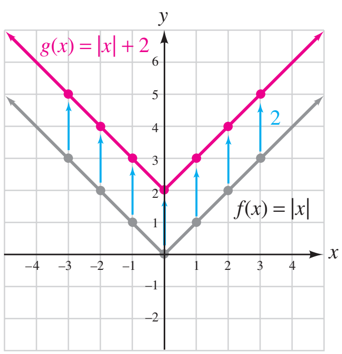
- The graph of $y = f(x) + k$ is identical to the graph of $y = f(x)$ except that it is translated k units upward.
- The graph of $y = f(x) - k$ is identical to the graph of $y = f(x)$ except that it is translated k units downward.

EXAMPLE 4 Graph: $g(x) = |x| + 2$

Strategy We will graph $g(x) = |x| + 2$ by translating (shifting) the graph of $f(x) = |x|$ upward 2 units.

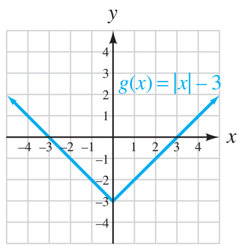
WHY The addition of 2 in $g(x) = |x| + 2$ causes a vertical shift of the graph of the absolute value function 2 units upward.

Solution
The graph of $g(x) = |x| + 2$ will be the same V-shaped graph as $f(x) = |x|$, except that it is shifted 2 units up. The graph appears below.



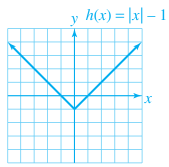
To graph $g(x) = |x| + 2$, translate each point on the graph of $f(x) = |x|$ up 2 units.

Self Check 4
Graph: $g(x) = |x| - 3$

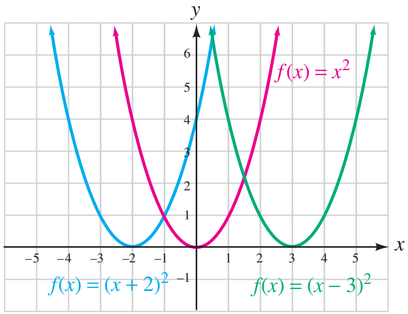


Now Try **Problem 28**

Teaching Example 4 Graph: $h(x) = |x| - 1$
Answer:

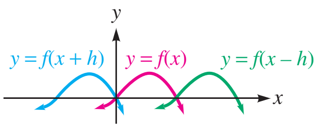


The figure below shows the graph of $f(x) = (x + h)^2$ for three different values of h. If $h = 0$, we get the graph of $f(x) = x^2$. The graph of $f(x) = (x - 3)^2$ is identical to the graph of $f(x) = x^2$ except that it is shifted 3 units to the right. The graph of $f(x) = (x + 2)^2$ is identical to the graph of $f(x) = x^2$ except that it is shifted 2 units to the left. These shifts are called **horizontal translations.**



In general, we can make these observations.

Horizontal Translations

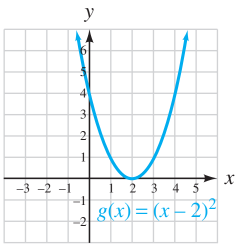


If f is a function and h is a positive number, then

- The graph of $y = f(x - h)$ is identical to the graph of $y = f(x)$ except that it is translated h units to the right.
- The graph of $y = f(x + h)$ is identical to the graph of $y = f(x)$ except that it is translated h units to the left.

Self Check 5

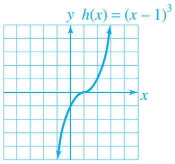
Graph: $g(x) = (x - 2)^2$



Now Try **Problem 30**

Teaching Example 5 Graph: $h(x) = (x - 1)^3$

Answer:



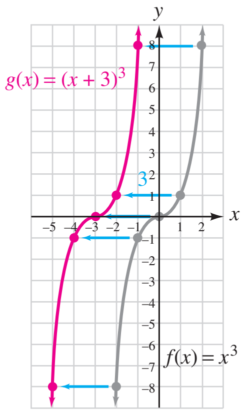
EXAMPLE 5 Graph: $g(x) = (x + 3)^3$

Strategy We will graph $g(x) = (x + 3)^3$ by translating (shifting) the graph of $f(x) = x^3$ left 3 units.

WHY The addition of 3 to x in $g(x) = (x + 3)^3$ causes a horizontal shift of the graph of the cubic function 3 units left.

Solution

The graph appears below.



To graph $g(x) = (x + 3)^3$, translate each point on the graph of $f(x) = x^3$ to the left 3 units.

EXAMPLE 6

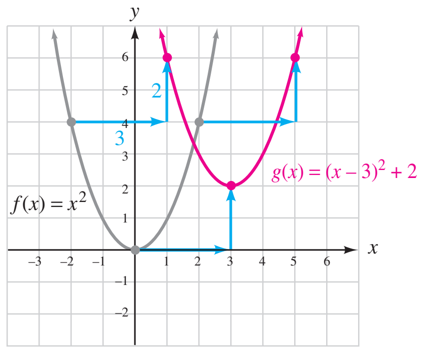
Graph: $g(x) = (x - 3)^2 + 2$

Strategy We will graph $g(x) = (x - 3)^2 + 2$ by translating (shifting) the graph of $f(x) = x^2$ right 3 units and then 2 units upward.

WHY The subtraction of 3 from x in $g(x) = (x - 3)^2 + 2$ causes a horizontal shift of the graph of the squaring function 3 units right. The addition of 2 in $g(x) = (x - 3)^2 + 2$ causes a vertical shift of 2 units upward.

Solution

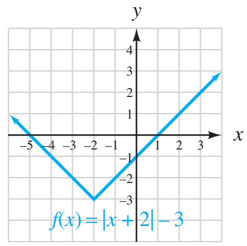
In this example, two translations are made to a basic graph. We can graph this equation by translating the graph of $f(x) = x^2$ to the right 3 units and then 2 units up, as shown in the figure.



To graph $g(x) = (x - 3)^2 + 2$, translate each point on the graph of $f(x) = x^2$ to the right 3 units and then 2 units up.

Self Check 6

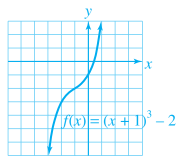
Graph: $f(x) = |x + 2| - 3$



Now Try **Problem 32**

Teaching Example 6 Graph $f(x) = (x + 1)^3 - 2$

Answer:



5 Reflect graphs of functions.

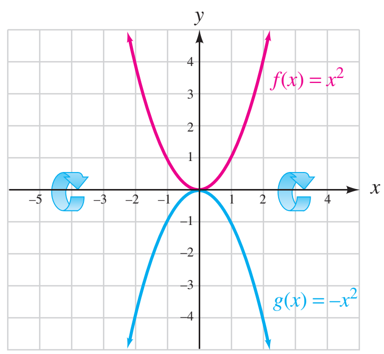
The figure below shows a table of values for $f(x) = x^2$ and for $g(x) = -x^2$. We note that for a given value of x, the corresponding y-values in the tables are opposites. When graphed, we see that the − in $g(x) = -x^2$ has the effect of flipping the graph of $f(x) = x^2$ over the x-axis so that the parabola opens downward. We say that the graph of $g(x) = -x^2$ is a **reflection** of the graph of $f(x) = x^2$ about the x-axis.

$f(x) = x^2$

x	$f(x)$	$(x, f(x))$
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)

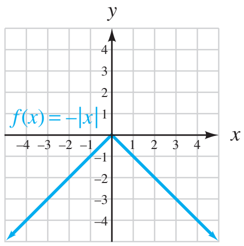
$g(x) = -x^2$

x	$g(x)$	$(x, g(x))$
−2	−4	(−2, −4)
−1	−1	(−1, −1)
0	0	(0, 0)
1	−1	(1, −1)
2	−4	(2, −4)



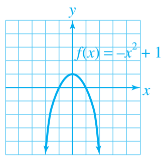
Self Check 7

Graph: $f(x) = -|x|$



Now Try **Problem 36**

Teaching Example 7 Graph:
$f(x) = -x^2 + 1$
Answer:



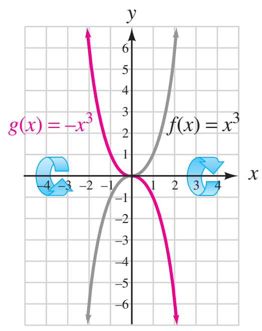
EXAMPLE 7 Graph: $g(x) = -x^3$

Strategy We will graph $g(x) = -x^3$ by reflecting the graph of $f(x) = x^3$ about the x-axis.

WHY The $-$ in $g(x) = -x^3$ causes a reflection of the graph of the cubing function about the x-axis.

Solution

To graph $g(x) = -x^3$, we use the graph of $f(x) = x^3$ from Example 2. First, we reflect the portion of the graph of $f(x) = x^3$ in quadrant I to quadrant IV, as shown in the figure on the right. Then we reflect the portion of the graph of $f(x) = x^3$ in quadrant III to quadrant II.

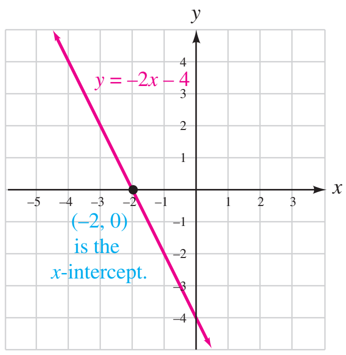


Reflection of a Graph

The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis.

6 Solve equations graphically.

Some of the graphing concepts discussed in this chapter can be used to solve equations. For example, the solution of $-2x - 4 = 0$ is the number x that will make y equal to 0 in the equation $y = -2x - 4$. To find this number, we inspect the graph of $y = -2x - 4$ and locate the point on the graph that has a y-coordinate of 0. In the figure to the right, we see that the point is $(-2, 0)$, which is the x-intercept of the graph. We can conclude that the x-coordinate of the x-intercept, $x = -2$, is the solution of $-2x - 4 = 0$.



$$\begin{aligned} \textbf{Check:} \quad -2x - 4 &= 0 \\ -2(-2) - 4 &\stackrel{?}{=} 0 \\ 4 - 4 &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

Another approach that can be used to solve equations involves a graphical determination of the value of the variable that will make the sides of the equation equal.

Self Check 8

Solve $2x + 4 = 2$ graphically. -1

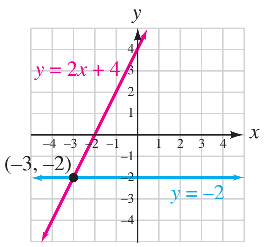
EXAMPLE 8 Solve $2x + 4 = -2$ graphically.

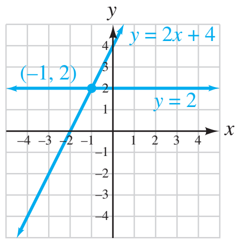
Strategy We will graph $y =$ the left side of the equation and $y =$ the right side of the equation.

WHY To solve $2x + 4 = -2$ graphically, we need to find the value of the x-coordinate at the point where the two graphs intersect.

Solution

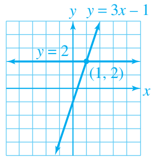
The graphs of $y = 2x + 4$ and $y = -2$ are shown in the figure. To solve the equation $2x + 4 = -2$, we need to find the value of x that makes $2x + 4$ equal -2. The point of intersection of the graphs is $(-3, -2)$. This tells us that if $x = -3$, the expression $2x + 4$ equals -2. So the solution of $2x + 4 = -2$ is $x = -3$.





Now Try **Problem 39**

Teaching Example 8 Solve $3x - 1 = 2$ graphically.
Answer:



$x = 1$

To solve more complex equations by graphing, we can use a graphing calculator.

Using Your CALCULATOR Solving Equations Graphically

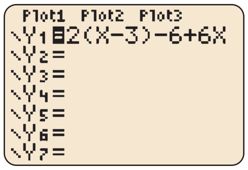
To solve $2(x - 3) - 6 = -6x$ with a graphing calculator, we add $6x$ to both sides so that the right-hand side of the equation is 0.

$$2(x - 3) - 6 = -6x$$

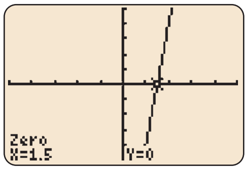
$$2(x - 3) - 6 + 6x = 0 \quad \text{Add } 6x \text{ to both sides.}$$

Then we enter the left-hand side of the equation into a graphing calculator in the form $y = 2(x - 3) - 6 + 6x$, as shown in figure (a). Note that it is not necessary to simplify the expression $2(x - 3) - 6 + 6x$.

Next, we enter window settings of $[-5, 5]$ for x and $[-5, 5]$ for y and select the ZERO feature found under the CALC menu. After we guess left and right bounds and press $\boxed{\text{ENTER}}$, the cursor automatically locates the x-intercept of the graph and displays its coordinates. (See figure (b).) The x-coordinate of the x-intercept of the graph is called a **zero** of $y = 2(x - 3) - 6 + 6x$. The zero (in this case, 1.5) is the solution of $2(x - 3) - 6 = -6x$.



(a)



(b)

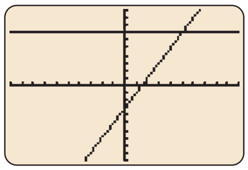
To solve the equation $2(x - 3) + 3 = 7$ by the method of Example 8 using a graphing calculator, we graph the left-hand side and the right-hand side of the equation in the same window by entering

$$Y_1 = 2(x - 3) + 3$$
$$Y_2 = 7$$

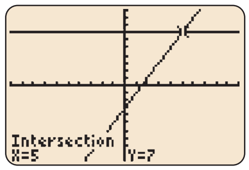
Figure (c) shows the graphs, generated using settings of $[-10, 10]$ for x and $[-10, 10]$ for y.

The coordinates of the point of intersection of the graphs can be determined using the INTERSECT feature found on most graphing calculators. With this feature, the cursor automatically highlights the intersection point, and the x- and y-coordinates are displayed. Consult your owner's manual for specific instructions concerning this feature.

In figure (d), we see that the point of intersection is (5, 7), which indicates that 5 is a solution of $2(x - 3) + 3 = 7$.



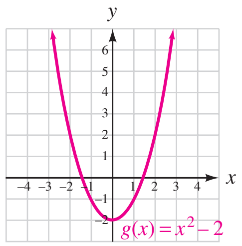
(c)



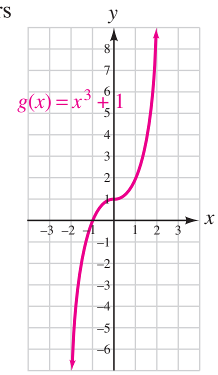
(d)

ANSWERS TO SELF CHECKS

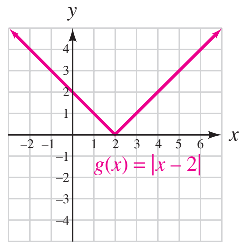
1. D: the set of real numbers, R: the set of all real numbers greater than -2



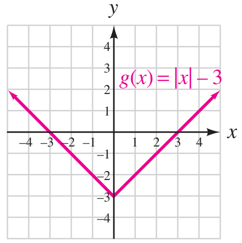
2. D: the set of real numbers, R: the set of real numbers



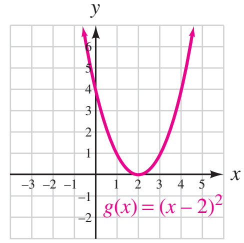
3. D: the set of real numbers, R: the set of nonnegative real numbers



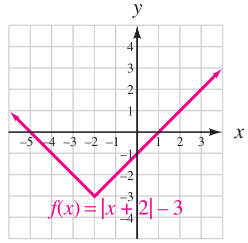
4.



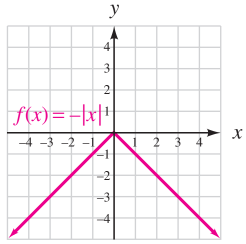
5.



6.



7.



8. -1

SECTION 2.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. Functions whose graphs are not straight lines are called nonlinear functions.

2. The function $f(x) = x^2$ is called the squaring function.

3. The function $f(x) = x^3$ is called the cubing function.

4. The function $f(x) = |x|$ is called the absolute value function.

5. Shifting the graph of an equation up or down is called a vertical translation. Shifting the graph of an equation to the left or to the right is called a horizontal translation.

6. The graph of $f(x) = -x^2$ is a reflection of the graph of $f(x) = x^2$ about the x-axis.

▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

Fill in the blanks.

7. The graph of $f(x) = (x + 4)^3$ is the same as the graph of $f(x) = x^3$ except that it is shifted 4 units to the left.

8. The graph of $f(x) = x^3 - 2$ is the same as the graph of $f(x) = x^3$ except that it is shifted 2 units down.

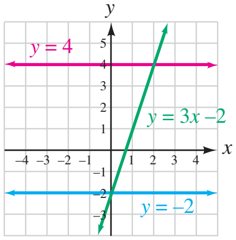
9. The graph of $f(x) = x^2 + 5$ is the same as the graph of $f(x) = x^2$ except that it is shifted 5 units up.

10. The graph of $f(x) = |x - 5|$ is the same as the graph of $f(x) = |x|$ except that it is shifted 5 units to the right.

11. Use the graphs on the right to solve each equation.

a. $3x - 2 = 4$ 2

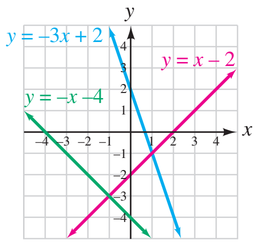
b. $3x - 2 = -2$ 0



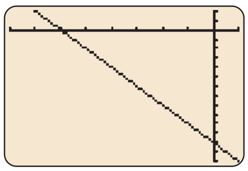
12. Use the graphs on the right to solve each equation.

a. $-3x + 2 = x - 2$ 1

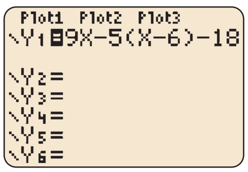
b. $-x - 4 = x - 2$ −1

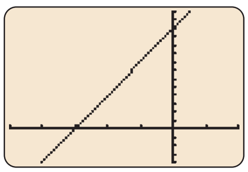


13. Use the graph of $y = 2(x - 6) - 4x$, shown on the right, to solve $2(x - 6) - 4x = 0$. −6

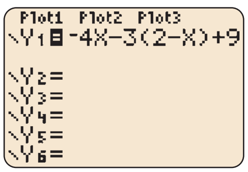


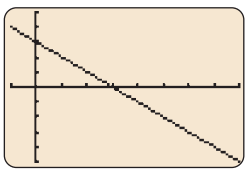
14. a. Use the information in the given calculator displays to solve $9x - 5(x - 6) - 18 = 0$. −3



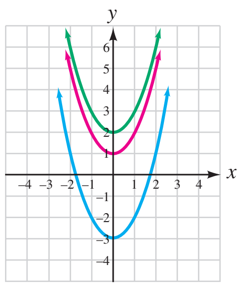


b. Use the information in the given calculator displays to solve $-4x - 3(2 - x) = -9$. 3





15. The illustration on the right shows the graph of $f(x) = x^2 + k$, for three values of k. Find the three values. −3, 1, 2

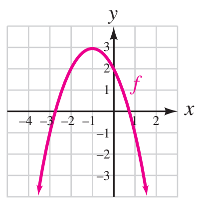


16. Use the graph on the right to find each function value.

a. $f(-2)$ 2

b. $f(0)$ 2

c. $f(1)$ −1

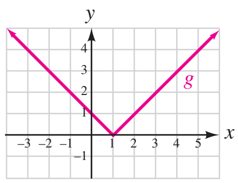


17. Use the graph on the right to find each function value.

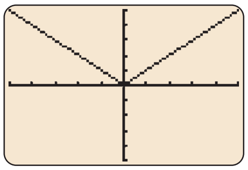
a. $g(-2)$ 3

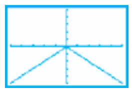
b. $g(1)$ 0

c. $g(2.5)$ 1.5



18. Use a graphing calculator to sketch the reflection of the graph shown below.

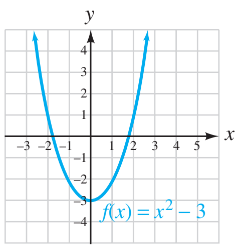




GUIDED PRACTICE

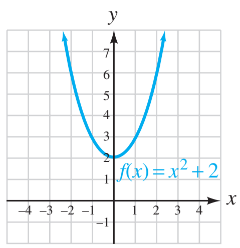
Use plotting points to graph each function. Use the graph to find the domain and range of each function. See Objectives 1–2 and Examples 1–3.

19. $f(x) = x^2 - 3$



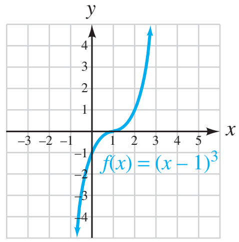
D: the set of real numbers, R: all real numbers greater than or equal to −3

20. $f(x) = x^2 + 2$



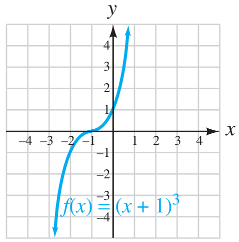
D: the set of real numbers, R: the set of all real numbers greater than or equal to 2

▶ **21.** $f(x) = (x - 1)^3$



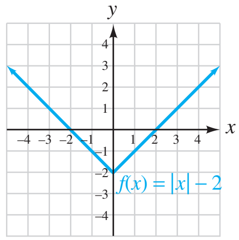
D: the set of real numbers,
R: the set of real numbers

▶ **22.** $f(x) = (x + 1)^3$



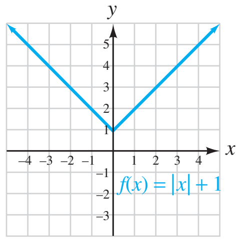
D: the set of real numbers,
R: the set of real numbers

▶ **23.** $f(x) = |x| - 2$



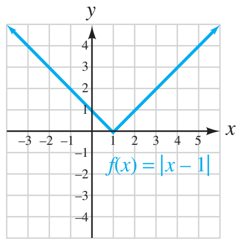
D: the set of real numbers,
R: the set of all real numbers greater than or equal to -2

▶ **24.** $f(x) = |x| + 1$



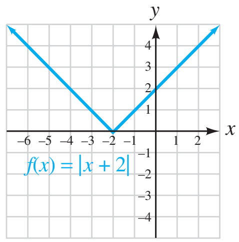
D: the set of real numbers,
R: the set of all real numbers greater than or equal to 1

▶ **25.** $f(x) = |x - 1|$



D: the set of real numbers,
R: the set of real numbers greater than or equal to 0

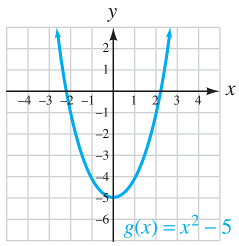
▶ **26.** $f(x) = |x + 2|$



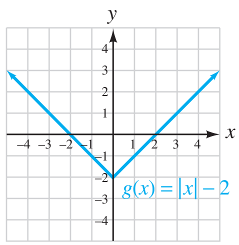
D: the set of real numbers,
R: the set of real numbers greater than or equal to 0

For each function, first sketch the graph of its associated function, $f(x) = x^2$, $f(x) = x^3$, or $f(x) = |x|$. Then draw each graph using a translation. See Examples 4–6.

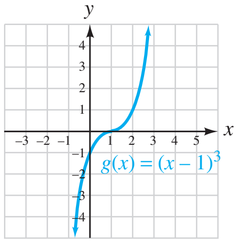
27. $g(x) = x^2 - 5$



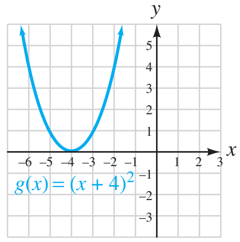
▶ **28.** $g(x) = |x| - 2$



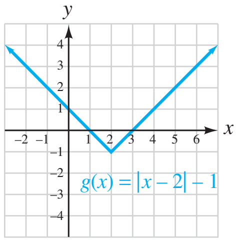
29. $g(x) = (x - 1)^3$



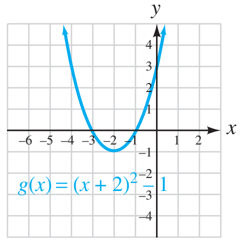
30. $g(x) = (x + 4)^2$



31. $g(x) = |x - 2| - 1$

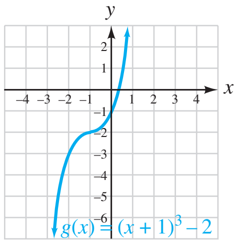


32. $g(x) = (x + 2)^2 - 1$

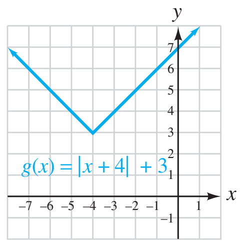


For each function, first sketch the graph of its associated function, $f(x) = x^2$, $f(x) = x^3$, or $f(x) = |x|$. Then draw each graph using translations and/or reflections.

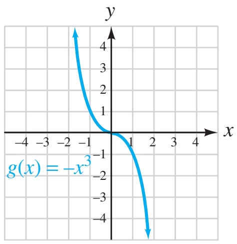
33. $g(x) = (x + 1)^3 - 2$



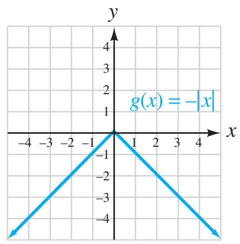
34. $g(x) = |x + 4| + 3$



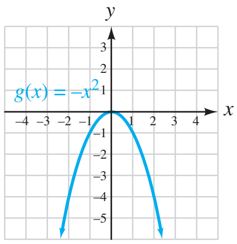
35. $g(x) = -x^3$



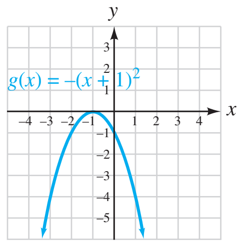
36. $g(x) = -|x|$



37. $g(x) = -x^2$



▶ **38.** $g(x) = -(x + 1)^2$



Use graphing to solve each equation. See Example 8.

39. $4(x - 1) = 3x$ 4

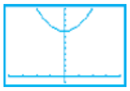
40. $4(x - 3) - x = x - 6$ 3

41. $11x + 6(3 - x) = 3$ -3

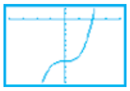
42. $2(x + 2) = 2(1 - x) + 10$ 2

Graph each function using window settings of $[-4, 4]$ for x and $[-4, 4]$ for y. The graph is not what it appears to be. Pick a better viewing window and find a better representation of the true graph.

43. $f(x) = x^2 + 8$



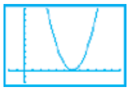
44. $f(x) = x^3 - 8$



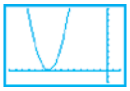
45. $f(x) = |x + 5|$

46. $f(x) = |x - 5|$

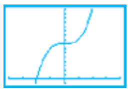
47. $f(x) = (x - 6)^2$



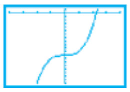
48. $f(x) = (x + 9)^2$



49. $f(x) = x^3 + 8$

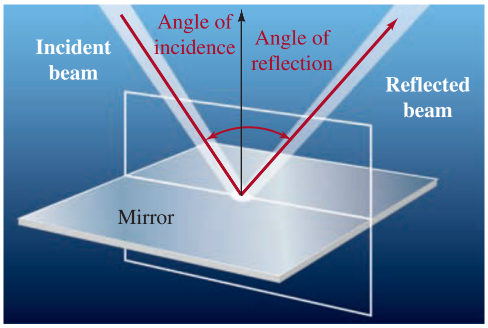


50. $f(x) = x^3 - 12$

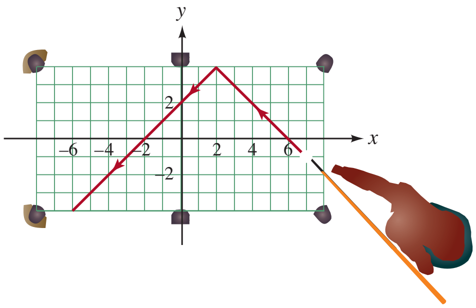


APPLICATIONS

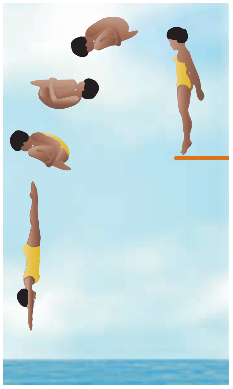
51. OPTICS The law of reflection states that the angle of reflection is equal to the angle of incidence. What function studied in this section mathematically models the path of the reflected light beam with an angle of incidence measuring 45°? $f(x) = |x|$



52. BILLIARDS In the illustration below, a rectangular coordinate system has been superimposed over a billiard table. Write a function that mathematically models the path of the ball that is shown banking off of the far cushion. $f(x) = -|x - 2| + 4$



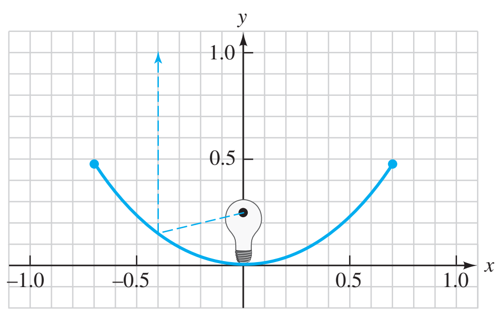
53. CENTER OF GRAVITY As a diver performs a $1\frac{1}{2}$ somersault in the tuck position, her center of gravity follows a path that can be described by a graph shape studied in this section. What graph shape is that? a parabola



54. LIGHT Light beams coming from a bulb are reflected outward by a parabolic mirror as parallel rays.

a. The cross-sectional view of a parabolic mirror is given by the function $f(x) = x^2$ for the following values of x: $-0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$. Sketch the parabolic mirror using the graph below.

b. From the light bulb filament at $(0, 0.25)$, draw a line segment representing a beam of light that strikes the mirror at $(-0.4, 0.16)$ and then reflects outward, parallel to the y-axis.



WRITING

55. Explain how to graph a function by plotting points.

56. Explain why the correct choice of window settings is important when using a graphing calculator.

57. Explain how to solve the equation $2x + 6 = 0$ graphically.

58. What does it mean when we say that the domain of a function is the set of all real numbers?

REVIEW

Solve each formula for the indicated variable.

59. $T - W = ma$ for W $W = T - ma$

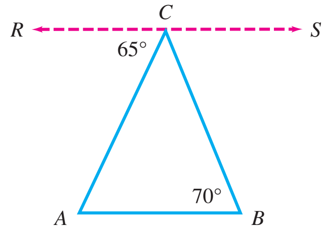
60. $a + (n - 1)d = l$ for n $n = \frac{l - a + d}{d}$

61. $s = \frac{1}{2}gt^2 + vt$ for g $g = \frac{2(s - vt)}{t^2}$

62. $e = mc^2$ for m $m = \frac{e}{c^2}$

63. BUDGETING Last year, Rock Valley College had an operating budget of $4.5 million. Due to salary increases and a new robotics program, the budget was increased by 20%. Find the operating budget for this year. $5.4 million

64. In the illustration, the line passing through points R, C, and S is parallel to line segment AB. Find the measure of $\angle ACB$. (Read $\angle ACB$ as "angle ACB." *Hint:* Recall from geometry that alternate interior angles have the same measure.) 45°



STUDY SKILLS CHECKLIST

Preparing for the Chapter 2 Test

There are common difficulties that students have when working with the topics of Chapter 2. To make sure you are prepared for the test and to help you overcome these difficulties, study the checklist below.

☐ When finding the slope of a line, if you are given two ordered pairs, use the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Find the slope of the line that passes through $(-2, 5), (6, -1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - 5}{6 - (-2)}$$
$$= \frac{-6}{8}$$
$$= -\frac{3}{4}$$

The slope of the line is $-\frac{3}{4}$.

☐ When finding the slope of a line, if you are given an equation, isolate y, and the slope is the *coefficient* of x.

$$2x + 3y = 6$$
$$3y = -2x + 6 \quad \text{Subtract 2x from both sides.}$$
$$y = -\frac{2}{3}x + 2 \quad \text{Divide both sides by 3.}$$

The slope of the line is $-\frac{2}{3}$.

☐ Remember, the slope of a line is a ratio, and the midpoint of a line segment is a point. To find the midpoint of the line segment connecting $(-5, 4)$ and $(-2, -3)$ use

$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$M\left(\frac{-5 + -2}{2}, \frac{4 + (-3)}{2}\right)$$
$$M\left(-\frac{7}{2}, \frac{1}{2}\right)$$

☐ To graph a linear equation in two variables:

1. Find three ordered pairs that are solutions of the equation by selecting three values for x and calculating the corresponding values of y.
2. Plot these points on a rectangular coordinate system.
3. Draw a straight line passing through them.

(continued)

☐ To graph a nonlinear equation in two variables:

1. Find at least *seven* ordered pairs that are solutions of the equation by selecting *seven* values for x and calculating the corresponding values of y.
2. Plot these points on a rectangular coordinate system.
3. Draw a curve passing through them.

☐ To write an equation of a line, you:

1. need the slope of the line, m.
2. need a point that line passes through (x_1, y_1).
3. input these values in the equation, $y - y_1 = m(x - x_1)$.

☐ To find the value of a function at an x-value, replace each x in the function with the x-value and evaluate the right side.

$$\begin{aligned} f(x) &= -6x + 7 \\ f(-2) &= -6(-2) + 7 \\ &= 12 + 7 \\ &= 19 \end{aligned}$$

Thus, $f(-2) = 19$.

CHAPTER 2 SUMMARY AND REVIEW

SECTION 2.1 The Rectangular Coordinate System

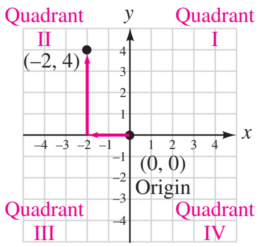
DEFINITIONS AND CONCEPTS

A **rectangular coordinate system** is formed by two intersecting perpendicular number lines called the ***x*-axis** and the ***y*-axis.** The x- and y-axes divide the plane into four **quadrants.**

The process of locating a point in the coordinate plane is called **plotting** or **graphing** that point.

EXAMPLES

To plot the point with coordinates $(-2, 4)$, begin at the origin, move 2 units to the left along the x-axis and then 4 units up, and draw a dot.



Midpoint formula: The midpoint of a line segment with endpoints at (x_1, y_1) and (x_2, y_2) is the point M with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

To find the midpoint of the segment joining $(-3, 7)$ and $(5, -8)$, let $(x_1, y_1) = (-3, 7)$ and $(x_2, y_2) = (5, -8)$ and substitute the coordinates into the midpoint formula.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-3 + 5}{2}, \frac{7 + (-8)}{2}\right)$$

$$= \left(\frac{2}{2}, \frac{-1}{2}\right)$$

$$= \left(1, -\frac{1}{2}\right) \quad \text{This is the midpoint.}$$

REVIEW EXERCISES

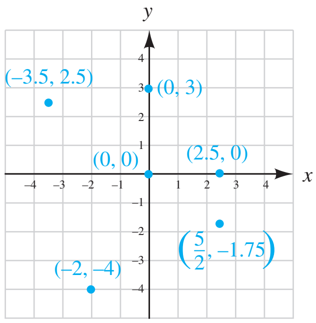
Plot each point on the rectangular coordinate system shown below.

1. $(0, 3)$ **2.** $(-2, -4)$

3. $\left(\frac{5}{2}, -1.75\right)$ **4.** the origin

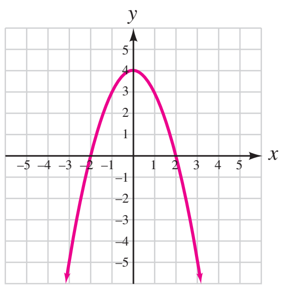
5. $(2.5, 0)$

6. $(-3.5, 2.5)$

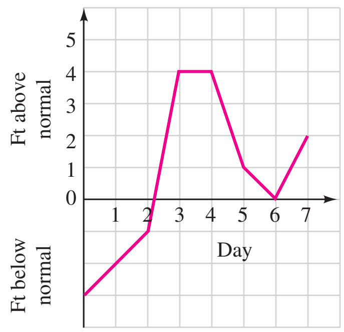


7. Use the information in the graph to complete the table.

x	y	(x, y)
-3	-5	$(-3, -5)$
-2	0	$(-2, 0)$
-1	3	$(-1, 3)$
0	4	$(0, 4)$
1	3	$(1, 3)$
2	0	$(2, 0)$
3	-5	$(3, -5)$



The graph below shows how the height of the water in a flood control channel changed over a 7-day period.

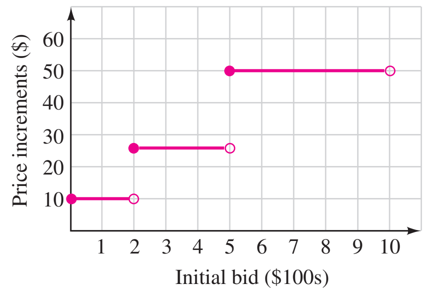


8. Describe the height of the water at the beginning of day 2. 1 ft below its normal level

9. By how much did the water level increase or decrease from day 4 to day 5? decreased by 3 ft

10. During what time period did the water level stay the same? from day 3 to the beginning of day 4

The dollar increments used by an auctioneer during the bidding process depend on what initial price the auctioneer began with for the item. See the step graph.



11. What increments are used by the auctioneer if the bidding on an item began at \$150? \$10 increments

12. If the first bid on an item being auctioned is \$750, what will be the next price asked for by the auctioneer? \$800

13. Find the midpoint of the line segment joining $(8, -2)$ and $(-4, 6)$. $(2, 2)$

14. The midpoint of a line segment has coordinates of $\left(-\frac{1}{2}, \frac{1}{2}\right)$. If one endpoint has coordinates of $(-3, 7)$, what are the coordinates of the other endpoint? $(2, -6)$

SECTION 2.2 Graphing Linear Equations

DEFINITIONS AND CONCEPTS | **EXAMPLES**

A **solution of an equation in two variables** is an ordered pair of numbers that makes the equation a true statement when the pair is substituted for the variables in the equation.

To determine whether $(1, -5)$ is a solution of $3x - y = 8$, substitute the coordinates into the equation.

$$3x - y = 8$$

$$3(1) - (-5) \stackrel{?}{=} 8 \quad \text{Substitute 1 for } x \text{ and } -5 \text{ for } y.$$

$$3 + 5 \stackrel{?}{=} 8 \quad \text{Evaluate the left side.}$$

$$8 = 8 \quad \text{True}$$

Since the result is true, $(1, -5)$ is a solution.

The **standard** or **general form** of a **linear equation** in two variables is $Ax + By = C$, where A, B, and C are real numbers and A and B are not both 0.

Linear equations:

$$3x + 4y = -8, \qquad y = \frac{2}{3}x - 5, \qquad y = -\frac{5}{2}, \qquad \text{and} \qquad x = 3$$

Graphing Linear Equations Solved for y by plotting points:

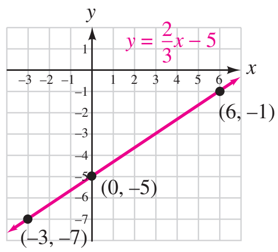
1. Find three ordered pairs that are solutions of the equation by selecting three values for x and calculating the corresponding values of y.
2. Plot the solutions on a rectangular coordinate system.
3. Draw a line passing through the points. If the points do not lie on a line, check your computations.

To graph $y = \frac{2}{3}x - 5$, construct a table of solutions, plot the points, and draw the line.

$$y = \frac{2}{3}x - 5$$

x	y	(x, y)
-3	-7	$(-3, -7)$
0	-5	$(0, -5)$
6	-1	$(6, -1)$

Select x | Find y | Plot (x, y)



To find the ***y*-intercept** of a line, substitute 0 for x in the equation and solve for y. To find the ***x*-intercept** of a line, substitute 0 for y in the equation and solve for x.

Plotting the x- and y-intercepts of a graph and drawing a line through them is called the **intercept method for graphing a line.**

Use the y- and x-intercepts to graph $3x + 4y = -8$.

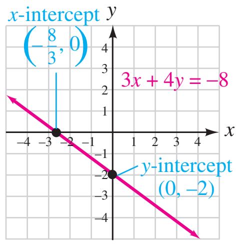
***y*-intercept: let $x = 0$**

$$3x + 4y = -8$$
$$3(0) + 4y = -8$$
$$4y = -8$$
$$y = -2$$

***x*-intercept: let $y = 0$**

$$3x + 4y = -8$$
$$3x + 4(0) = -8$$
$$3x = -8$$
$$x = -\frac{8}{3}$$

The y-intercept is $(0, -2)$ and the x-intercept is $\left(-\frac{8}{3}, 0\right)$.

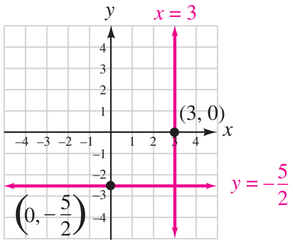


The graph of the equation $x = a$ is a **vertical line** with x-intercept at $(a, 0)$.

The graph of the equation $y = b$ is a **horizontal line** with y-intercept at $(0, b)$.

Graph: $x = 3$

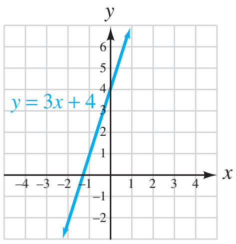
Graph: $y = -\dfrac{5}{2}$



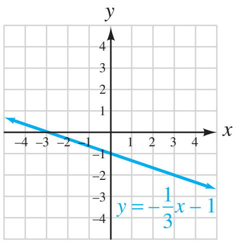
REVIEW EXERCISES

Plot points to graph each equation.

15. $y = 3x + 4$

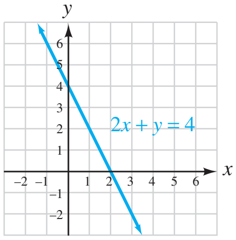


16. $y = -\dfrac{1}{3}x - 1$

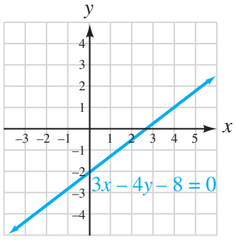


Use intercepts to graph each equation.

17. $2x + y = 4$

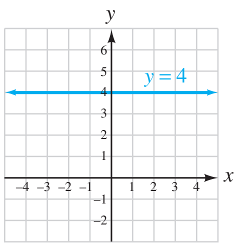


18. $3x - 4y - 8 = 0$

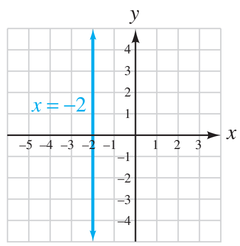


Graph each equation.

19. $y = 4$



20. $x = -2$



Complete the table of solutions for each equation.

21. $y = -3x$

x	y
-3	9
0	0
3	-9

22. $y = \dfrac{1}{2}x - \dfrac{5}{2}$

x	y
-3	-4
0	$-\frac{5}{2}$
3	-1

SECTION 2.3 Rate of Change and the Slope of a Line

DEFINITIONS AND CONCEPTS	EXAMPLES

DEFINITIONS AND CONCEPTS

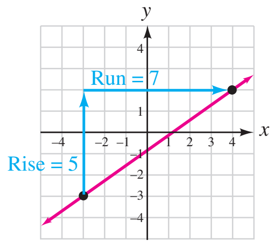
The **slope m** of a line is a ratio that compares the vertical and horizontal change as we move along the line from one point to another.

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$$

EXAMPLES

Find the slope of the line.

$$m = \frac{\text{rise}}{\text{run}} = \frac{5}{7}$$



DEFINITIONS AND CONCEPTS

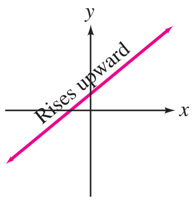
Lines that rise from left to right have a **positive slope.**

Lines that fall from left to right have a **negative slope.**

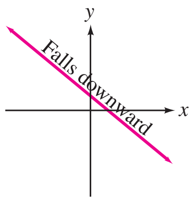
Horizontal lines have **zero slope.**

Vertical lines have **undefined slope.**

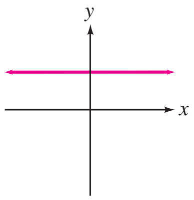
EXAMPLES



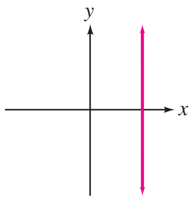
$m > 0$



$m < 0$



$m = 0$



Slope undefined

DEFINITIONS AND CONCEPTS

We can also find the slope of a line using the **slope formula:**

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{if } x_1 \neq x_2$$

EXAMPLES

To find the slope of the line passing through $(-5, -2)$ and $(7, -14)$, substitute into the slope formula.

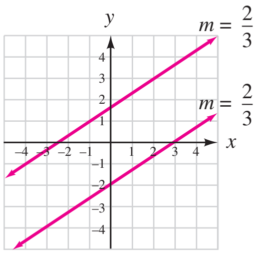
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-14 - (-2)}{7 - (-5)} \\ &= \frac{-12}{12} \\ &= -1 \end{aligned}$$

DEFINITIONS AND CONCEPTS

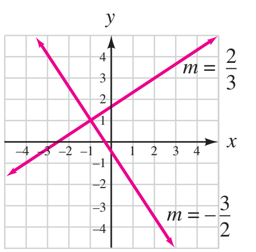
Parallel lines have the same slope.

The slopes of two nonvertical **perpendicular lines** are negative reciprocals. The product of their slopes is -1.

EXAMPLES



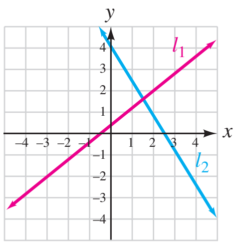
Parallel lines



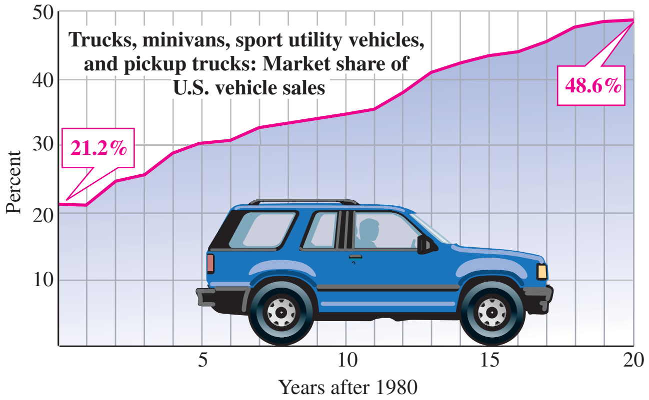
Perpendicular lines

REVIEW EXERCISES

23. Find the slope of lines l_1 and l_2 in the illustration.
Slope of $l_1 = \frac{4}{5}$, slope of $l_2 = -\frac{8}{5}$



24. U.S. VEHICLE SALES On the graph below, draw a line through the points (0, 21.2) and (20, 48.6). Use this linear model to estimate the rate of increase in the market share of minivans, sport utility vehicles, and light trucks over the years 1980–2000.
1.37% per yr



Source: American Automotive Association and U.S. Bureau of Economic Analysis

Find the slope of the line passing through the given points.

25. (2, 5) and (5, 8)
1

26. (3, −2) and (−6, 12)
$-\frac{14}{9}$

27. (−2, 4) and (8, 4)
0

28. (−5, −4) and (−5, 8)
undefined

Find the slope of the graph of each equation, if one exists.

29. $y = \frac{2}{3}x = 18$
$\frac{2}{3}$

30. $4x + 2y = 8$
−2

31. $x = 10$
undefined

32. $y = 7$
0

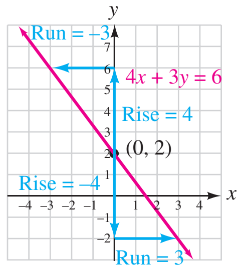
Determine whether the lines with the given slopes are parallel, perpendicular, or neither.

33. $m_1 = 4, m_2 = -\frac{1}{4}$
perpendicular

34. $m_1 = 0.5, m_2 = \frac{1}{2}$
parallel

SECTION 2.4 Writing Equations of Lines

DEFINITIONS AND CONCEPTS	EXAMPLES
If a linear equation is written in **slope–intercept** form $y = mx + b$ the graph of the equation is a line with slope m and y-intercept $(0, b)$.	The equation of the line with slope $\frac{2}{3}$ and y-intercept (0, 7) is: $y = \frac{2}{3}x + 7$
Slope can be used as an aid in graphing.	To find the slope and y-intercept of the line whose equation is $4x + 3y = 6$, we solve the equation for y. $4x + 3y = 6$; $3y = -4x + 6$ Subtract 4x from both sides.; $y = -\frac{4}{3}x + 2$ To isolate y, divide both sides by 3. The slope of the line is $-\frac{4}{3}$ and the y-intercept is (0, 2).

	To graph the line, plot the y-intercept and use the rise and run components of the slope to locate other points on the graph. 
If a linear equation is written in **point–slope form** $y - y_1 = m(x - x_1)$ the graph of the equation is a line with slope m and passing through the point with coordinates of (x_1, y_1).	To find the equation of the line with slope -5 that passes through $(-1, 3)$, write the equation in point–slope form and substitute the coordinates of the point. $y - y_1 = m(x - x_1)$ This is point–slope form. $y - 3 = -5[x - (-1)]$ Substitute. $y - 3 = -5(x + 1)$ Simplify within the brackets. $y - 3 = -5x - 5$ Distribute. $y = -5x - 2$ To isolate y, add 3 to both sides. This is slope–intercept form.
Slopes can be used to identify **parallel** and **perpendicular** lines.	The graph of the lines $y = 7x + 5$ and $y = 7x - 3$ are parallel because each line has a slope of 7. The graph of the lines $y = 6x + 1$ and $y = -\frac{1}{6}x$ are perpendicular because their slopes are 6 and $-\frac{1}{6}$, which are negative reciprocals.

REVIEW EXERCISES

Write an equation of the line with the given properties. Express the result in slope–intercept form.

35. Slope of 3, passing through $(-8, 5)$ $y = 3x + 29$

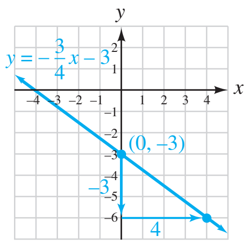
36. Passing through $(-2, 4)$ and $(6, -9)$ $y = -\frac{13}{8}x + \frac{3}{4}$

Write an equation of the line with the given properties. Write the answer in standard (general) form.

37. Passing through $(-3, -5)$, parallel to the graph of $3x - 2y = 7$ $3x - 2y = 1$

38. Passing through $(-3, -5)$, perpendicular to the graph of $3x - 2y = 7$ $2x + 3y = -21$

39. Write $3x + 4y = -12$ in slope–intercept form. Give the slope and y-intercept of the graph of the equation. Then use this information to graph the line. $y = -\frac{3}{4}x - 3$; $m = -\frac{3}{4}$, $(0, -3)$



40. DEPRECIATION A business purchased a copy machine for \$8,700 and will depreciate it on a straight-line basis over the next 5 years. At the end of its useful life, it will be sold as scrap for \$100. Find its depreciation equation. $y = -1{,}720x + 8{,}700$

SECTION 2.5 An Introduction to Functions

<table>
<tr><th>DEFINITIONS AND CONCEPTS</th><th>EXAMPLES</th></tr>
<tr><td>

A **relation** is a set of ordered pairs. The set of first components is called the **domain** of the relation and the set of second components is called the **range.**

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The relation $\{(2, 5), (7, -3), (4, 6)\}$ has domain of $\{2, 4, 7\}$ and range $\{-3, 5, 6\}$.

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A **function** is a set of ordered pairs (a relation) in which to each first component there corresponds exactly one second component.

***y* is a function of *x*:** Given a relation in x and y, if to each value of x in the domain there corresponds exactly one value of y in the range, then y is said to be a function of x.

Since y depends on x, we call x the **independent variable** and y the **dependent variable.**

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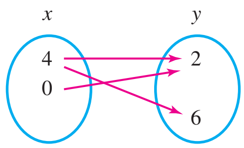
The relation $\{(2, 5), (7, -3), (4, 6)\}$ defines a function because to each first component there corresponds exactly one second component.

The relation $\{(-9, 1), (\mathbf{3}, 8), (0, 0), (\mathbf{3}, 24)\}$ does not define a function because to the first component 3 there corresponds two second components: 8 and 24.

The arrow diagram does not define y as a function of x because to the x-value 4 there corresponds more than one y-value: 2 and 6.



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A function can be defined by an equation.

Not all equations in two variables define functions.

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The equation $y = 2x - 7$ defines y as a function of x because to each value of x there corresponds exactly one value of y.

In the previous function, x is the independent variable and y is the dependent variable.

The equation $x = |y|$ does not define y as a function of x because more than one value of y corresponds to a single value of x. If $x = 2$, for example, the equation becomes $2 = |y|$ and y can be either 2 or -2.

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The **function notation** $y = f(x)$ indicates that y is a function of x. It is read as "f of x."

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If $f(x) = 2x + 1$, find $f(-2)$ and $f(n + 1)$.

$$\begin{aligned} f(\mathbf{x}) &= 2\mathbf{x} + 1 \\ f(\mathbf{-2}) &= 2(\mathbf{-2}) + 1 \\ &= -4 + 1 \\ &= -3 \end{aligned} \qquad \begin{aligned} f(\mathbf{x}) &= 2\mathbf{x} + 1 \\ f(\mathbf{n + 1}) &= 2(\mathbf{n + 1}) + 1 \\ &= 2n + 2 + 1 \\ &= 2n + 3 \end{aligned}$$

Thus $f(-2) = -3$. Thus $f(n + 1) = 2n + 3$.

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The input-output pairs that a function generates can be written as ordered pairs and plotted on a rectangular coordinate system to give the **graph of the function.**

A **linear function** is a function that can be written in the form $f(x) = mx + b$.

The graph of a linear function is a straight line.

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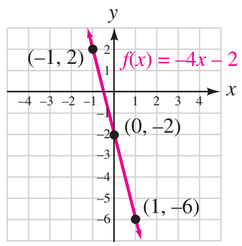
To graph the linear function $f(x) = -4x - 2$, we can make a table of values, plot the points, and draw the line.

x	$f(x)$	
-1	2	→ $(-1, 2)$
0	-2	→ $(0, -2)$
1	-6	→ $(1, -6)$

↑ Select x. ↑ Find $f(x)$. ↑ Plot the point.



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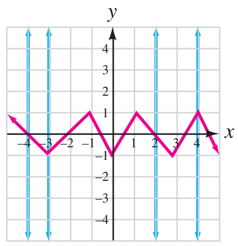
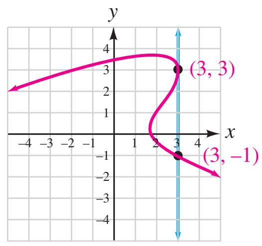
The **domain** of a function is the set of input values. The **range** is the set of output values.	To find the domain of $f(x) = \frac{10}{x+3}$, note that -3 cannot be substituted for x because that would make the denominator 0. Since any real number except -3 can be substituted for x, the domain is the set of all real numbers except -3.
The **vertical line test:** If a vertical line intersects a graph in more than one point, the graph is not the graph of a function.	 function  not a function

REVIEW EXERCISES

Determine whether each equation determines y to be a function of x.

41. $y = 6x - 4$ yes

42. $y = 4 - x^2$ yes

43. $y^2 = x$ no

44. $|y| = x$ no

Assume that $f(x) = 3x + 2$ and $g(x) = \frac{x^2 - 4x + 4}{2}$ and find each value.

45. $f(-3)$ -7

46. $g(8)$ 18

47. $g(-2)$ 8

48. $f(t)$ $3t + 2$

Find the domain and range of each function.

49. $f(x) = 4x - 1$
D: the set of real numbers,
R: the set of real numbers

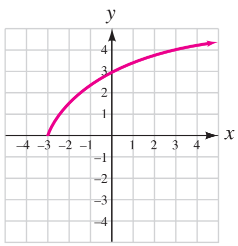
50. $f(x) = x^2 + 1$
D: the set of real numbers,
R: the set of all real numbers greater than or equal to 1

51. $f(x) = \dfrac{4}{2 - x}$
D: the set of all real numbers except 2,
R: the set of all real numbers except 0

52. $y = -|4x|$
D: the set of real numbers,
R: the set of nonpositive real numbers

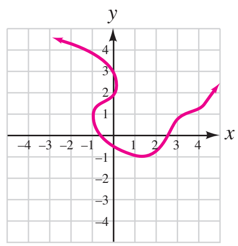
Use the vertical line test to determine whether each graph represents a function.

53.



function

54.



not a function

***VIDEOS** A video production company charges a $175 setup fee and $105 per hour to tape a wedding and reception.*

55. Write a function that gives the cost to tape a wedding and reception that lasts x hours. $c(x) = 105x + 175$

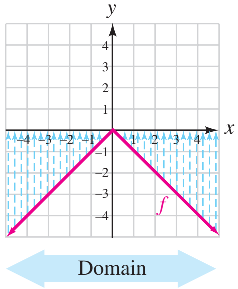
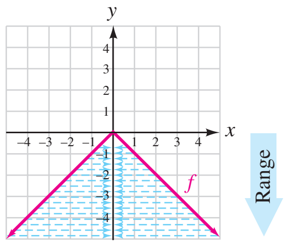
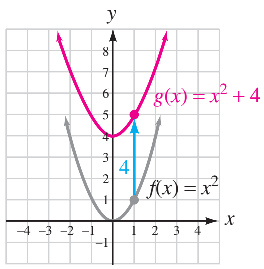
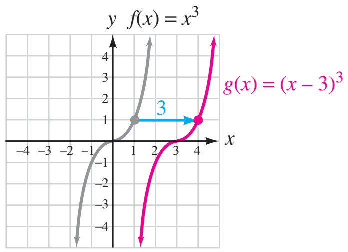
56. Use your answer to Exercise 55 to find the cost to tape a wedding and reception lasting 6 hours. $805

Determine which are linear functions.

57. $f(x) = 3x + 2$ yes

58. $f(x) = x^2 - 25$ no

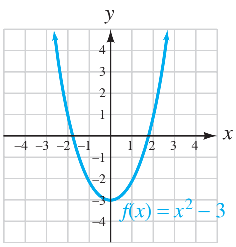
SECTION 2.6 Graphs of Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
We can find the domain and range of a function from its graph. The **domain of a function** is the projection of its graph onto the x-axis. **The range of a function** is the projection of its graph onto the y-axis. **Squaring function:** $f(x) = x^2$ **Cubing function:** $f(x) = x^3$ **Absolute value function:** $f(x) = \|x\|$	Find the domain and range of function f.  D: The set of real numbers  R: The set of nonpositive real numbers
A **vertical translation** shifts a graph upward or downward. A **horizontal translation** shifts a graph left or right. A **reflection** "flips" a graph about the x-axis.	 To graph $g(x) = x^2 + 4$, translate each point on the graph of $f(x) = x^2$ up 4 units.  To graph $g(x) = (x - 3)^3$, translate each point on the graph of $f(x) = x^3$ to the right 3 units.

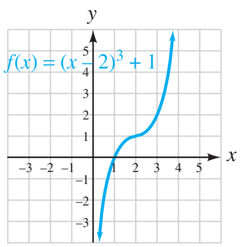
REVIEW EXERCISES

Graph each function.

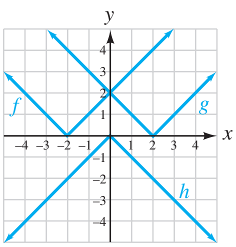
59. $f(x) = x^2 - 3$



60. $f(x) = (x - 2)^3 + 1$

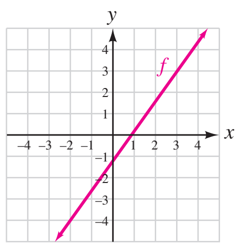


61. Graph $f(x) = |x + 2|$, $g(x) = |x - 2|$, and $h(x) = -|x|$ on the same coordinate system.



62. Use the graph to find each function value.

a. $f(-2)$ -4 **b.** $f(3)$ 3

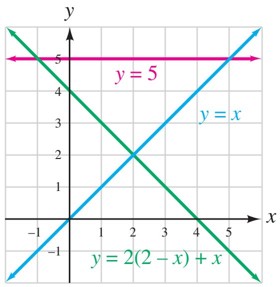


Use the graphs below to solve each equation.

63. $2(2 - x) + x = 0$ 4

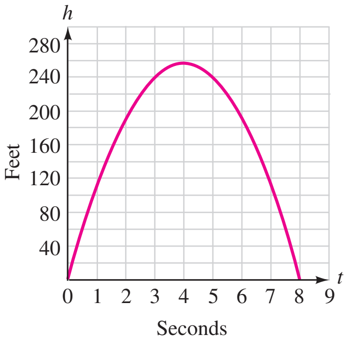
64. $2(2 - x) + x = 5$ -1

65. $2(2 - x) + x = x$ 2



CHAPTER 2 TEST

Refer to the graph below, which shows the height of an object at different times after it was shot straight up into the air.



1. How high was the object 3 seconds into the flight? 240 ft

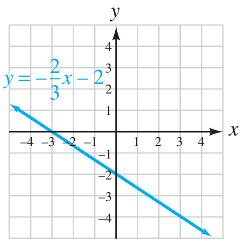
2. At what times was the object about 110 feet above the ground? 1 sec and 7 sec

3. What was the maximum height reached by the object? about 260 ft

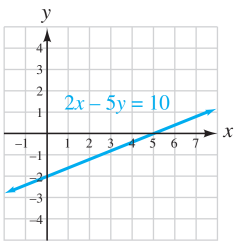
4. How long did the flight take? 8 sec

5. Find the coordinates of the midpoint of the line segment joining $(-2, -5)$ and $(7, 8)$. $\left(\frac{5}{2}, \frac{3}{2}\right)$

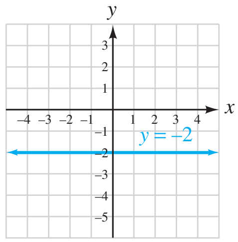
6. Graph: $y = -\dfrac{2}{3}x - 2$



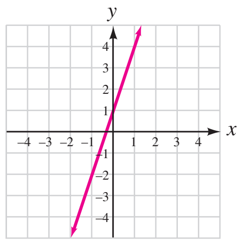
7. Find the x- and y-intercepts of the graph of $2x - 5y = 10$. Then graph the equation. $(5, 0)$, $(0, -2)$



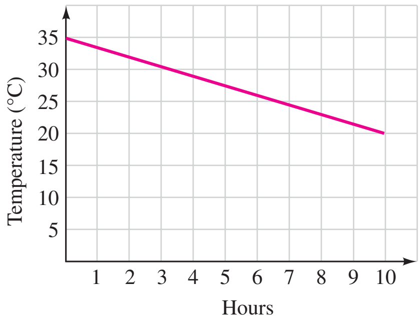
8. Graph: $y = -2$



9. Find the slope of the line shown below. 3



10. Find the rate of change of the temperature for the period of time shown in the graph below. -1.5 degree/hr



Find the slope of each line, if possible.

11. The line that passes through $(-2, 4)$ and $(6, 8)$ $\frac{1}{2}$

12. The graph of $2x - 3y = 8$ $\frac{2}{3}$

13. The graph of $x = 12$ undefined

14. The graph of $y = 12$ 0

15. Write an equation of the line that has slope of $\frac{2}{3}$ and passes through $(4, -5)$. Give the answer in slope–intercept form. $y = \frac{2}{3}x - \frac{23}{3}$

16. Write an equation of the line that passes through $(-2, 6)$ and $(-4, -10)$. Give the answer in general form. $8x - y = -22$

17. Find the slope and the y-intercept of the graph of $-2x + 6 = 6y + 15$. $m = -\frac{1}{3}, \left(0, -\frac{3}{2}\right)$

18. Determine whether the graphs of $4x - y = 12$ and $y = \frac{1}{4}x + 3$ are parallel, perpendicular, or neither. neither

19. Write an equation of the line that passes through the origin and is parallel to the graph of $y = \frac{3}{2}x - 7$. $y = \frac{3}{2}x$

20. ACCOUNTING After purchasing a new color copier, a business owner had his accountant prepare a depreciation worksheet for tax purposes. (See the illustration below.)

a. Assuming straight-line depreciation, write an equation that gives the value v of the copier after x years of use. $v = -600x + 4{,}000$

b. If the depreciation equation is graphed, explain the significance of its v-intercept.
(0, 4,000): it gives the value of the copier when new: $4,000

Depreciation Worksheet

Color copier $4,000
(new)

Salvage value $400
(in 6 years)

21. Does $|y| = x$ define y to be a function of x? no

22. Does the table define y as a function of x? yes

x	y
−3	4
4	−3
1	4
2	5

23. Find the domain and range of the function $f(x) = |x|$.
D: the set of real numbers, R: the set of nonnegative real numbers

24. Find the domain and range of the function $f(x) = x^3$.
D: the set of real numbers, R: the set of real numbers

Let $f(x) = 3x + 1$ and $g(x) = x^2 - 2x - 1$. Find each value.

25. $f(3)$ 10

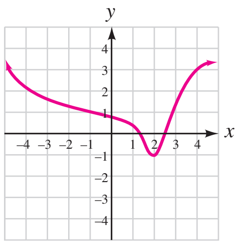
26. $g(0)$ −1

27. $f\left(\frac{2}{3}\right)$ 3

28. $g(r)$ $r^2 - 2r - 1$

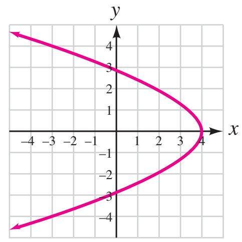
Determine whether each graph represents a function.

29.



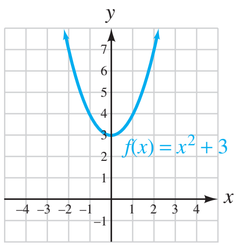
function

30.

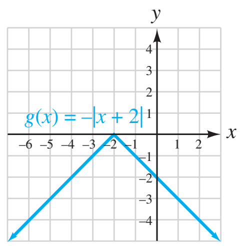


not a function

31. Graph: $f(x) = x^2 + 3$

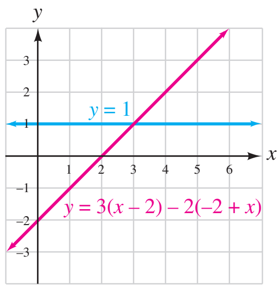


32. Graph: $g(x) = -|x + 2|$



33. a. See the illustration below. Explain how to solve $3(x - 2) - 2(-2 + x) = 0$ graphically. What is the solution?
Find the x-coordinate of the x-intercept of the graph of $y = 3(x - 2) - 2(-2 + x)$; 2

b. See the illustration below. Explain how to solve $3(x - 2) - 2(-2 + x) = 1$ graphically. What is the solution?
Find the x-coordinate of the point of intersection of the graph of $y = 3(x - 2) - 2(-2 + x)$ and $y = 1$; 3



34. Describe four different ways to represent a function.
answers vary

35. Explain why the graph of a circle does not represent a function.
answers vary

36. Explain why the graph of a nonvertical line is a function.
answers vary

CHAPTERS 1–2 CUMULATIVE REVIEW

List the elements of $\left\{-2, 0, 1, 2, \frac{13}{12}, 6, 7, \sqrt{5}, \pi\right\}$ *that belong to the following set.* [Section 1.2]

1. Natural numbers
1, 2, 6, 7

2. Whole numbers
0, 1, 2, 6, 7

3. Rational numbers
$-2, 0, 1, 2, \frac{13}{12}, 6, 7$

4. Irrational numbers
$\sqrt{5}, \pi$

5. Negative numbers
-2

6. Real numbers
$-2, 0, 1, 2, \frac{13}{12}, 6, 7, \sqrt{5}, \pi$

7. Prime numbers
2, 7

8. Composite numbers
6

9. Even numbers
$-2, 0, 2, 6$

10. Odd numbers
1, 7

Evaluate each expression. [Section 1.3]

11. $-|5| + |-3|$ -2

12. $\dfrac{|-5| + |-3|}{-|4|}$ -2

13. $2 + 4 \cdot 5$ 22

14. $\dfrac{8-4}{2-4}$ -2

15. $-\dfrac{16}{5} \div \left(-\dfrac{10}{3}\right)$ $\dfrac{24}{25}$

16. $\dfrac{(9-8)^4 + 21}{3^3 - \left(\sqrt{16}\right)^2}$ 2

Evaluate each expression for $x = 2$ *and* $y = -3$. [Section 1.3]

17. $-x - 2y$ 4

18. $\dfrac{x^2 - y^2}{2x + y}$ -5

Determine which property of real numbers justifies each statement. [Section 1.4]

19. $(a + b) + c = a + (b + c)$ assoc. prop. of add.

20. $3(x + y) = 3x + 3y$ distrib. prop.

21. $(a + b) + c = c + (a + b)$ commut. prop. of add.

22. $(ab)c = a(bc)$ assoc. prop. of mult.

Simplify each expression. [Section 1.4]

23. $12y - 17y$ $-5y$

24. $-7s(-4t)(-1)$ $-28st$

25. $3x^2 + 2x^2 - 5x^2$ 0

26. $-(4 + z) + 2z$ $z - 4$

Solve each equation. [Section 1.5]

27. $2x - 5 = 11$ 8

28. $\dfrac{2x - 6}{3} = x + 7$ -27

29. $4(y - 3) + 4 = -3(y + 5)$ -1

30. $2x - \dfrac{3(x-2)}{2} = 7 - \dfrac{x-3}{3}$ 6

31. $-3 = -\dfrac{9}{8}s$ $\frac{8}{3}$

32. $0.04(24) + 0.02x = 0.04(12 + x)$ 24

Solve each formula. [Section 1.6]

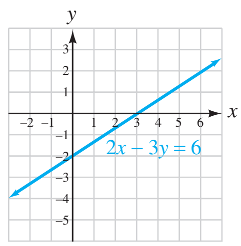
33. $s = \dfrac{n(a + l)}{2}$ for a $a = \frac{2S}{n} - l$ or $a = \frac{2S - ln}{n}$

34. $A = \dfrac{1}{2}h(b_1 + b_2)$ for h $h = \frac{2A}{b_1 + b_2}$

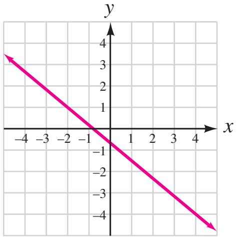
35. INVESTMENTS A woman invested part of \$20,000 at 6% and the rest at 7%. If her annual interest is \$1,260, how much did she invest at 6%? [Section 1.8] \$14,000

36. DRIVING RATES John drove to a distant city in 5 hours. When he returned, there was less traffic, and the trip took only 3 hours. If he drove 26 mph faster on the return trip, how fast did he drive each way? [Section 1.8] 39 mph going, 65 mph returning

37. Graph: $2x - 3y = 6$ [Section 2.2]



38. Find the slope of the line shown below. [Section 2.3] $-\frac{5}{6}$



39. Write an equation of the line passing through $(-2, 5)$ and $(8, -9)$. Give the answer in slope–intercept form. [Section 2.4] $y = -\frac{7}{5}x + \frac{11}{5}$

40. Write an equation of the line passing through $(-2, 3)$ and parallel to the graph of $3x + y = 8$. Give the answer in slope–intercept form. [Section 2.4]
$y = -3x - 3$

Let $f(x) = 3x^2 + 2$ and $g(x) = -2x - 1$. Find each value. [Section 2.5]

41. $f(-1)$ 5

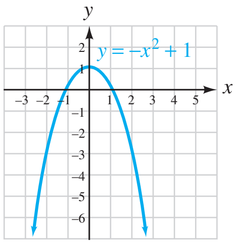
42. $g(0)$ -1

43. $g(-2)$ 3

44. $f(-r)$ $3r^2 + 2$

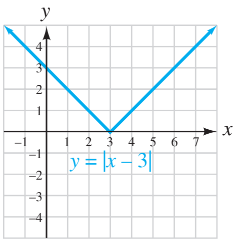
Graph each equation and determine whether it is a function. If it is a function, give the domain and range. [Section 2.6]

45. $y = -x^2 + 1$



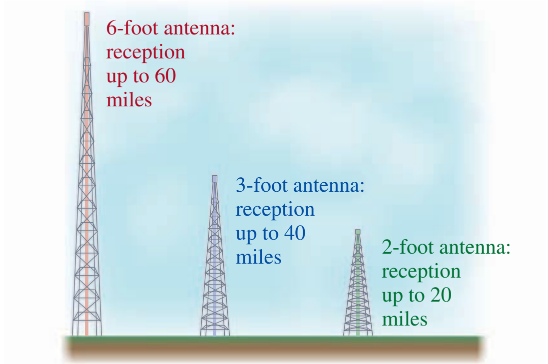
a function; D: the set of real numbers, R: the set of all real numbers less than or equal to 1

46. $y = |x - 3|$



a function; D: the set of real numbers, R: the set of nonnegative real numbers

47. See the illustration below. Explain why there is not a linear relationship between the height of the antenna and its maximum range of reception. [Section 2.5]

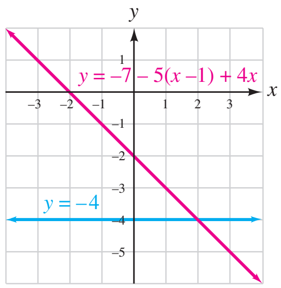


The points (2, 20), (3, 40) and (6, 60) do not lie on a straight line.

48. Explain how to use the following illustration to solve each equation. Give each solution. [Section 2.6]

a. $-7 - 5(x - 1) + 4x = 0$
Find the x-coordinate of the x-intercept of the graph of $y = -7 - 5(x - 1) + 4x$; -2

b. $-7 - 5(x - 1) + 4x = -4$
Find the x-coordinate of the point of intersection of the graph of $y = -7 - 5(x - 1) + 4x$ and $y = -4$; 2



3

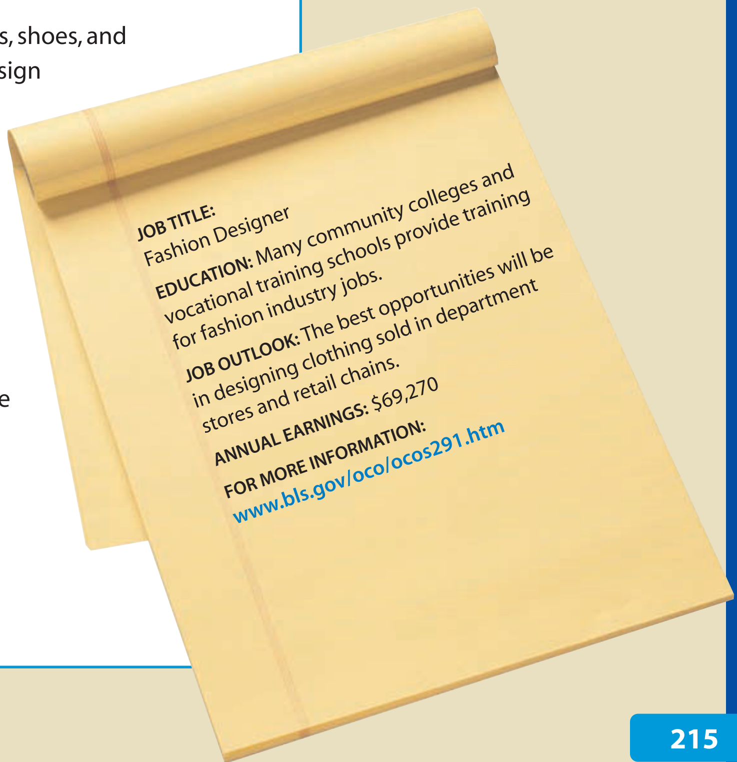
Systems of Equations

from Campus to Careers

Fashion Designer

Fashion designers help create the billions of clothing articles, shoes, and accessories purchased every year by consumers. Fashion design relies heavily on mathematical skills, including knowledge of lines, angles, curves, and measurement. Designers also use mathematics in the manufacturing and marketing parts of the industry as they calculate labor costs and determine the markups and markdowns involved in retail pricing.

In **Problem 11** of **Study Set 3.5,** we will examine the production side of fashion design as we determine the number of coats, shirts, and slacks that can be made with the available labor.



Objectives

1 Determine whether an ordered pair is a solution of a system.

2 Solve systems of linear equations by graphing.

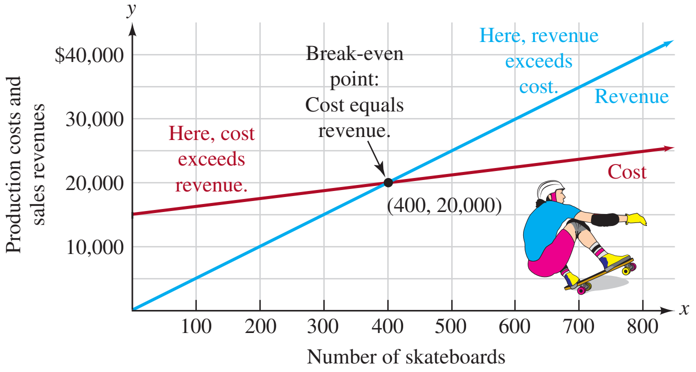
3 Use graphing to identify inconsistent systems and dependent equations.

SECTION 3.1

Solving Systems of Equations by Graphing

The red line in the graph shows the cost for a company to produce a given number of skateboards. The blue line shows the revenue the company will earn for selling a given number of those skateboards. The graph offers the company important financial information.

- The production costs exceed the revenue earned if fewer than 400 skateboards are sold. In this case, the company loses money.
- The revenue earned exceeds the production costs if more than 400 skateboards are sold. In this case, the company makes a profit.
- Production costs equal revenue earned if exactly 400 skateboards are sold. This fact is indicated by the point of intersection of the two lines, (400, 20,000), which is called the **break-even point.**



From Chapter 2, we know that the lines in the graph that show the cost and the revenue can be modeled by linear equations in two variables. Together, such a set of equations is called a *system of equations.*

In general, when two equations with the same variables are considered simultaneously (at the same time), we say that they form a **system of equations.** We will use a left brace { when writing a system of equations. An example is

$$\begin{cases} 2x + 5y = -1 \\ x - y = -4 \end{cases}$$

Read as "the system of equations $2x + 5y = -1$ and $x - y = -4$."

1 Determine whether an ordered pair is a solution of a system.

A **solution of a system** of equations in two variables is an ordered pair that satisfies both equations of the system.

Self Check 1

Determine whether $(6, -2)$ is a solution of the system:

$$\begin{cases} x - 2y = 10 \\ y = 3x - 20 \end{cases}$$

yes

Now Try **Problem 16**

EXAMPLE 1 Determine whether $(-3, 1)$ is a solution of each system of equations.

a. $\begin{cases} 2x + 5y = -1 \\ x - y = -4 \end{cases}$ **b.** $\begin{cases} 5y = 2 - x \\ y = 3x \end{cases}$

Strategy We will substitute the x- and y-coordinates of $(-3, 1)$ for the corresponding variables in both equations of the system.

WHY If both equations are satisfied (made true) by the x- and y-coordinates, the ordered pair is a solution of the system.

Solution

a. To determine whether $(-3, 1)$ is a solution, we substitute -3 for x and 1 for y in each equation.

Check:

$$\begin{aligned} 2x + 5y &= -1 && \text{First equation.} \\ 2(-3) + 5(1) &\stackrel{?}{=} -1 \\ -6 + 5 &\stackrel{?}{=} -1 \\ -1 &= -1 && \text{True} \end{aligned}$$

$$\begin{aligned} x - y &= -4 && \text{Second equation.} \\ -3 - 1 &\stackrel{?}{=} -4 \\ -4 &= -4 && \text{True} \end{aligned}$$

Since $(-3, 1)$ satisfies both equations, it is a solution of the system.

b. Again, we substitute -3 for x and 1 for y in each equation.

Check:

$$\begin{aligned} 5y &= 2 - x && \text{First equation.} \\ 5(1) &\stackrel{?}{=} 2 - (-3) \\ 5 &\stackrel{?}{=} 2 + 3 \\ 5 &= 5 && \text{True} \end{aligned}$$

$$\begin{aligned} y &= 3x && \text{Second equation.} \\ 1 &\stackrel{?}{=} 3(-3) \\ 1 &= -9 && \text{False} \end{aligned}$$

Although $(-3, 1)$ satisfies the first equation, it does not satisfy the second. Because it does not satisfy both equations, $(-3, 1)$ is not a solution of the system.

Teaching Example 1 Determine whether $(4, -3)$ is a solution of the system: $\begin{cases} 3x - 2y = 18 \\ 4y = 3x \end{cases}$
Answer:
no

2 Solve systems of linear equations by graphing.

To **solve a system** of equations means to find all the solutions of the system. One way to solve a system of linear equations in two variables is to graph each equation and find where the graphs intersect.

The Graphing Method

1. Graph each equation on the same rectangular coordinate system.
2. Find the coordinates of the point (or points) where the graphs intersect. These coordinates give the solution of the system.
3. If the graphs have no point in common, the system has no solution.
4. Check the proposed solution in both of the original equations.

When a system of equations (as in Example 2) has at least one solution, the system is called a **consistent system.**

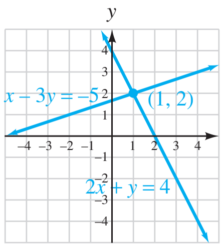
EXAMPLE 2 Solve the system by graphing: $\begin{cases} x + 2y = 4 \\ 2x - y = 3 \end{cases}$

Strategy We will graph both equations on the same coordinate system.

WHY The graph of a linear equation is a picture of its solutions. If both equations are graphed on the same coordinate system, we can see whether they have any common solutions.

Self Check 2

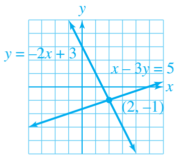
Solve the system by graphing:
$\begin{cases} x - 3y = -5 \\ 2x + y = 4 \end{cases}$ (1, 2)



Now Try **Problem 21**

Teaching Example 2 Solve the system by graphing: $\begin{cases} x - 3y = 5 \\ y = -2x + 3 \end{cases}$

Answer:
(2, −1)



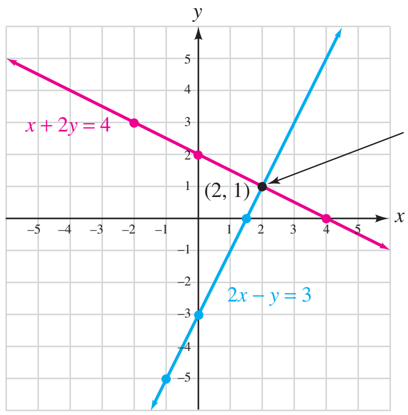
Solution

Although infinitely many ordered pairs (x, y) satisfy $x + 2y = 4$, and infinitely many ordered pairs (x, y) satisfy $2x - y = 3$, only the coordinates of the point where the graphs intersect satisfy both equations. From the graph, it appears that the intersection point has coordinates (2, 1). To verify that it is the solution, we substitute 2 for x and 1 for y in both equations and verify that (2, 1) satisfies each one, as shown below.

$x + 2y = 4$		
x	y	(x, y)
4	0	(4, 0)
0	2	(0, 2)
−2	3	(−2, 3)

$2x - y = 3$		
x	y	(x, y)
$\frac{3}{2}$	0	$\left(\frac{3}{2}, 0\right)$
0	−3	(0, −3)
−1	−5	(−1, −5)

Use the intercept method to graph each line.



The point of intersection gives the solution of the system.

Check: **The first equation**

$$x + 2y = 4$$
$$2 + 2(1) \stackrel{?}{=} 4$$
$$2 + 2 \stackrel{?}{=} 4$$
$$4 = 4$$

The second equation

$$2x - y = 3$$
$$2(2) - 1 \stackrel{?}{=} 3$$
$$4 - 1 \stackrel{?}{=} 3$$
$$3 = 3$$

Since (2, 1) makes both equations true, it is the solution of the system.

Success Tip Since accuracy is crucial when using the graphing method to solve a system:

- Use graph paper.
- Use a sharp pencil.
- Use a straight edge.

3 Use graphing to identify inconsistent systems and dependent equations.

When a system has no solution (as in Example 3), it is called an **inconsistent system.**

EXAMPLE 3

Solve the system $\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 24 \end{cases}$ by graphing, if possible.

Strategy We will graph both equations on the same coordinate system.

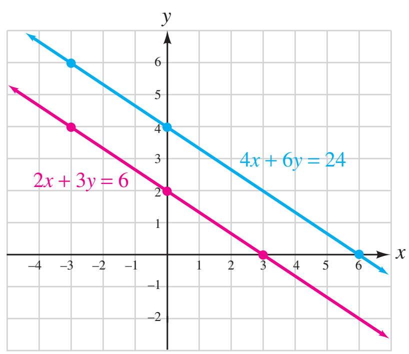
WHY The graph of a linear equation is a picture of it solutions. If both equations are graphed on the same coordinate system, we can see whether they have any common solutions.

Solution

Using the intercept method, we graph both equations on one set of coordinate axes, as shown below.

$2x + 3y = 6$		
x	y	(x, y)
3	0	(3, 0)
0	2	(0, 2)
−3	4	(−3, 4)

$4x + 6y = 24$		
x	y	(x, y)
6	0	(6, 0)
0	4	(0, 4)
−3	6	(−3, 6)



In this example, the graphs are parallel, because the slopes of the two lines are equal and they have different y-intercepts. We can see that the slope of each line is $-\frac{2}{3}$ by writing each equation in slope-intercept form.

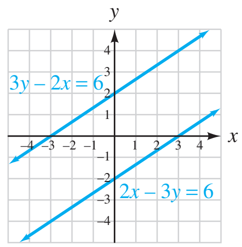
$$\begin{aligned} 2x + 3y &= 6 \\ 3y &= -2x + 6 \\ y &= -\frac{2}{3}x + 2 \end{aligned} \qquad \begin{aligned} 4x + 6y &= 24 \\ 6y &= -4x + 24 \\ y &= -\frac{2}{3}x + 4 \end{aligned}$$

Since the graphs are parallel lines, the lines do not intersect, and the system does not have a solution. It is an **inconsistent system.** The solution set is the empty set, which is written ∅.

When the equations of a system of two equations in two variables have different graphs (as in Examples 2 and 3), the equations are called **independent equations.** Two equations with the same graph are called **dependent equations.**

Self Check 3

Solve the system $\begin{cases} 3y - 2x = 6 \\ 2x - 3y = 6 \end{cases}$ by graphing, if possible.



Now Try **Problem 23**

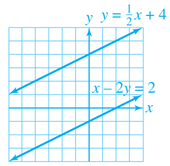
Self Check 3 Answer

no solutions

Teaching Example 3 Solve the system $\begin{cases} y = \frac{1}{2}x + 4 \\ x - 2y = 2 \end{cases}$ by graphing, if possible.

Answer:

no solutions

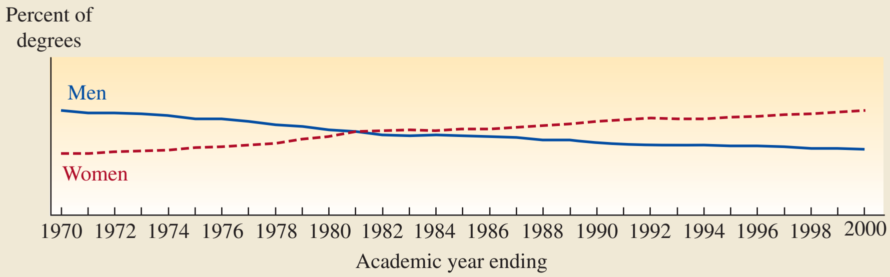


THINK IT THROUGH *Bachelor's Degrees*

"Women now earn more associate's, bachelor's and master's degrees than their male counterparts. Women also earned nearly half of the Ph.D.s as well as professional degrees, which include medical, law and dental degrees."

CNN/Money, April 27, 2004

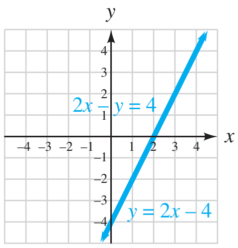
Examine the graph below. Determine the point of intersection of the graph and explain its importance. (*Note:* The y-axis does not need to be scaled for you to answer this question.)
(1981, 50); in 1981, 50% of the bachelor's degrees that were awarded went to men and 50% went to women.



Source: U.S. Department of Education, National Center for Education Statistics

Self Check 4

Solve the system by graphing:
$$\begin{cases} 2x - y = 4 \\ y = 2x - 4 \end{cases}$$

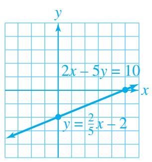


Now Try **Problem 25**

Self Check 4 Answer
There are infinitely many solutions; three of them are $(0, -4)$, $(2, 0)$, and $(4, 4)$.

Teaching Example 4 Solve the system by graphing: $\begin{cases} 2x - 5y = 10 \\ y = \frac{2}{5}x - 2 \end{cases}$

Answer:
There are infinitely many solutions; three of them are $(5, 0)$, $(0, -2)$, and $(-5, -4)$.



EXAMPLE 4

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x + 2 \\ 2x + 8 = 4y \end{cases}$

Strategy We will graph both equations on the same coordinate system.

WHY If both equations are graphed on the same coordinate system, we can see whether they have any common solutions.

Solution

We graph each equation on one set of coordinate axes, as shown below. Since the graphs coincide (are the same), the system has infinitely many solutions. Any ordered pair (x, y) that satisfies one equation also satisfies the other. From the graph we see that $(-4, 0)$, $(0, 2)$, and $(2, 3)$ are three of the infinitely many solutions. Because the two equations have the same graph, they are dependent equations.

Graph by using the slope and y-intercept.

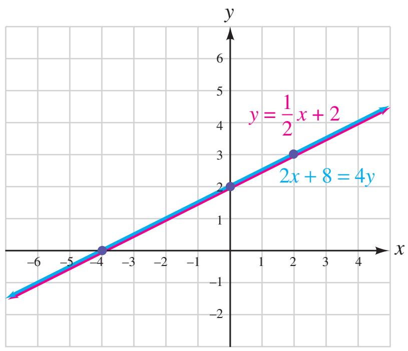
$$y = \frac{1}{2}x + 2$$

$m = \frac{1}{2}$ $\quad b = 2$

Slope $= \frac{1}{2}$ $\quad$ y-intercept: $(0, 2)$

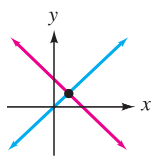
Graph by using the intercept method.

$2x + 8 = 4y$		
x	y	(x, y)
-4	0	$(-4, 0)$
0	2	$(0, 2)$
2	3	$(2, 3)$

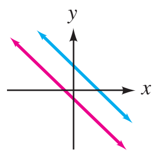


We now summarize the possibilities that can occur when two linear equations, each in two variables, are graphed.

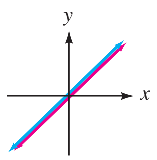
Solving a System of Equations by the Graphing Method



If the lines are different and intersect, the equations are independent, and the system is consistent. **One solution exists.** It is the point of intersection.



If the lines are different and parallel, the equations are independent, and the system is inconsistent. **No solution exists.**



If the lines coincide, the equations are dependent, and the system is consistent. **Infinitely many solutions exist.** Any point on the line is a solution.

If each equation in one system is equivalent to a corresponding equation in another system, the systems are called **equivalent.**

EXAMPLE 5

Solve the system by graphing: $\begin{cases} \frac{3}{2}x - y = \frac{5}{2} \\ x + \frac{1}{2}y = 4 \end{cases}$

Strategy We will use the multiplication property of equality to clear both equations of fractions and solve the resulting equivalent system by graphing.

WHY It is usually easier to solve systems that do not contain fractions.

Solution

We multiply both sides of $\frac{3}{2}x - y = \frac{5}{2}$ by 2 to eliminate the fractions and obtain the equation $3x - 2y = 5$. We multiply both sides of $x + \frac{1}{2}y = 4$ by 2 to eliminate the fractions and obtain the equation $2x + y = 8$.

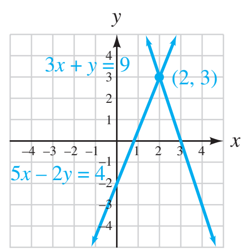
The original system / The new system

$$\begin{cases} \dfrac{3}{2}x - y = \dfrac{5}{2} \xrightarrow{\text{Multiply by 2}} 2\left(\dfrac{3}{2}x - y\right) = 2\left(\dfrac{5}{2}\right) \xrightarrow{\text{Simplify}} 3x - 2y = 5 \\ x + \dfrac{1}{2}y = 4 \xrightarrow[\text{Multiply by 2}]{} 2\left(x + \dfrac{1}{2}y\right) = 2(4) \xrightarrow[\text{Simplify}]{} 2x + y = 8 \end{cases}$$

is equivalent to the original system and is easier to solve, since it has no fractions. If we graph each equation in the new system, it appears that the lines intersect at (3, 2). Verify that (3, 2) is the solution by substituting 3 for x and 2 for y in each equation of the original system.

Self Check 5

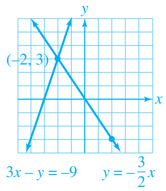
Solve the system by graphing: $\begin{cases} \frac{5}{2}x - y = 2 \\ x + \frac{1}{3}y = 3 \end{cases}$ (2, 3)



Now Try Problem 27

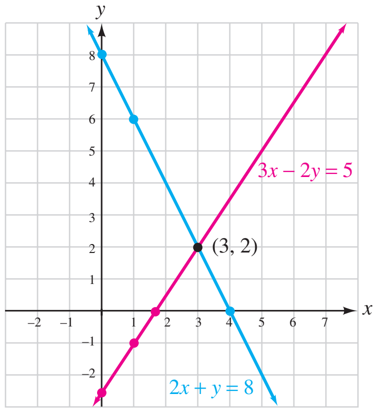
Teaching Example 5 Solve the system by graphing: $\begin{cases} x = -\frac{2}{3}y \\ \frac{x}{2} - \frac{y}{6} = \frac{-3}{2} \end{cases}$

Answer:
$(-2, 3)$



$3x - 2y = 5$		
x	y	(x, y)
$\frac{5}{3}$	0	$\left(\frac{5}{3}, 0\right)$
0	$-\frac{5}{2}$	$\left(0, -\frac{5}{2}\right)$
1	−1	$(1, -1)$

$2x + y = 8$		
x	y	(x, y)
4	0	$(4, 0)$
0	8	$(0, 8)$
1	6	$(1, 6)$



Caution! When checking a solution of a system of equations, always substitute the values of the variables into the original equations.

Using Your CALCULATOR Solving Systems by Graphing

The graphing method has limitations. First, the method is limited to equations with two variables. Systems with three or more variables cannot be solved graphically. Second, it is often difficult to find exact solutions graphically. However, the TRACE and ZOOM capabilities of graphing calculators enable us to get very good approximations of such solutions.

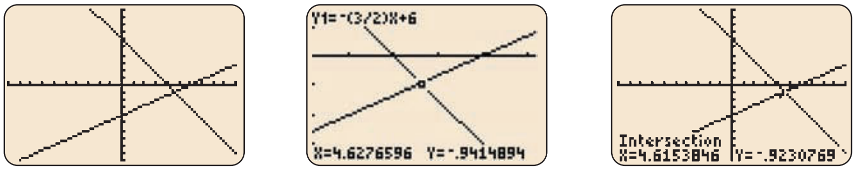
For example, to solve the system $\begin{cases} 3x + 2y = 12 \\ 2x - 3y = 12 \end{cases}$

with a graphing calculator, we first solve each equation for y so that we can enter the equations into the calculator. After solving for y, we obtain the equivalent system:

$$\begin{cases} y = -\frac{3}{2}x + 6 \\ y = \frac{2}{3}x - 4 \end{cases}$$

If we use window settings of $[-10, 10]$ for x and $[-10, 10]$ for y, the graphs of the equations will look like those in figure (a) on the next page. If we zoom in on the intersection point of the two lines and trace, we will get an approximate solution like the one shown in figure (b). To get better results, we can do more zooms. We would then find that, to the nearest hundredth, the solution is $(4.62, -0.92)$.

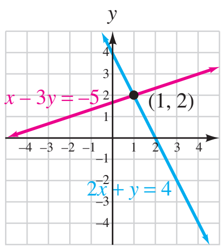
We can also find the intersection of the two lines by using the INTERSECT feature found under the CALC menu. After graphing the lines and using INTERSECT, we obtain the display shown in figure (c), which shows the approximate coordinates of the point of intersection to be $(4.62, -0.92)$.



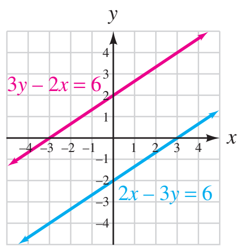
(a) (b) (c)

ANSWERS TO SELF CHECKS

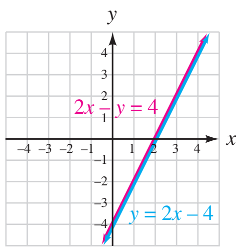
1. yes **2.** $(1, 2)$



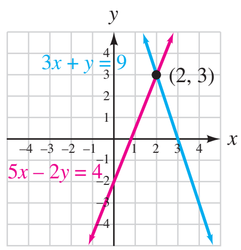
3. no solutions



4. There are infinitely many solutions; three of them are $(0, -4)$, $(2, 0)$, and $(4, 4)$.



5. $(2, 3)$



SECTION 3.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\begin{cases} x + 2y = 4 \\ 2x - y = 3 \end{cases}$ is called a __system__ of linear equations.

2. When a system of equations has one or more solutions, it is called a __consistent__ system.

3. If a system has no solutions, it is called an __inconsistent__ system.

▶ **4.** If two equations have different graphs, they are called __independent__ equations.

▶ **5.** Two equations with the same graph are called __dependent__ equations.

▶ **6.** When solving a system of two linear equations by the graphing method, we look for the point of __intersection__ of the two lines.

CONCEPTS

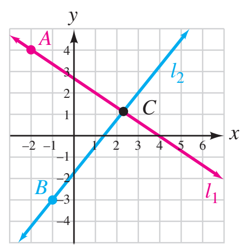
7. Refer to the illustration. Determine whether a true or a false statement would be obtained when the coordinates of

a. point A are substituted into the equation for line l_1. true

b. point B are substituted into the equation for line l_1. false

c. point C are substituted into the equation for line l_1. true

d. point C are substituted into the equation for line l_2. true



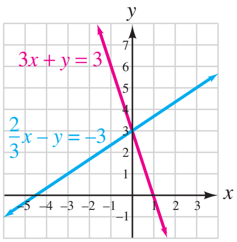
▶ Selected exercises available online at **www.webassign.net/brookscole**

8. Refer to the illustration.

a. How many ordered pairs satisfy the equation $3x + y = 3$? Name three.
infinitely many; $(-1, 6), (0, 3), (1, 0)$

b. How many ordered pairs satisfy the equation $\frac{2}{3}x - y = -3$? Name three.
infinitely many; $(-3, 1), (0, 3), (3, 5)$

c. How many ordered pairs satisfy both equations? Name it or them. one; $(0, 3)$



9. a. The intercept method can be used to graph the equation $2x - 4y = -8$. Complete the following table.

x	y	(x, y)
-4	0	$(-4, 0)$
0	2	$(0, 2)$
2	3	$(2, 3)$

b. What is the x-intercept of the graph of $2x - 4y = -8$? What is the y-intercept? $(-4, 0)$, $(0, 2)$

▶ **10. a.** To graph $y = 3x + 1$, we can pick three numbers for x and find the corresponding values of y. Complete the following table.

x	y	(x, y)
-1	-2	$(-1, -2)$
0	1	$(0, 1)$
2	7	$(2, 7)$

b. We can also graph $y = 3x + 1$ if we know the slope and the y-intercept of the line. What are they? $m = 3$, $(0, 1)$

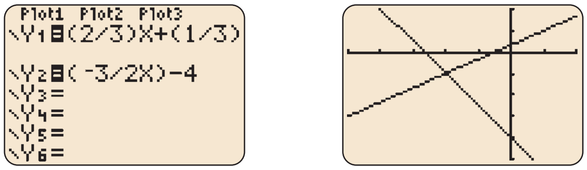
11. Write a system of two linear equations that has

a. only one solution, $(2, 3)$.
$\begin{cases} x + y = 5 \\ x - y = -1 \end{cases}$ (answers may vary)

b. an infinite number of solutions.
$\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$ (answers may vary)

c. no solution.
$\begin{cases} x + y = 5 \\ x + y = 4 \end{cases}$ (answers may vary)

12. Estimate the solution of the system of linear equations shown in the display below. Then check your answer. $(-2, -1)$



NOTATION

Fill in the blanks.

13. A left brace is often used when writing a system of equations.

▶ **14.** The solution of a system of two linear equations in two variables is written as an ordered pair.

GUIDED PRACTICE

Determine whether the ordered pair is a solution of the system of equations. See Example 1.

15. $(1, 2)$; $\begin{cases} 2x - y = 0 \\ y = \frac{1}{2}x + \frac{3}{2} \end{cases}$
yes

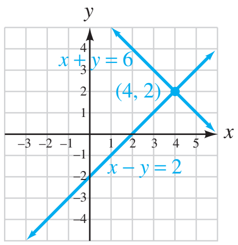
▶ **16.** $(-1, 2)$; $\begin{cases} y = 3x + 5 \\ y = x + 4 \end{cases}$
no

17. $(2, -3)$; $\begin{cases} y + 2 = \frac{1}{2}x \\ 3x + 2y = 0 \end{cases}$
no

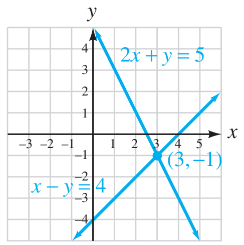
▶ **18.** $(-4, 3)$; $\begin{cases} 4x - y = -19 \\ 3x + 2y = -6 \end{cases}$
yes

Solve each system by graphing. See Example 2.

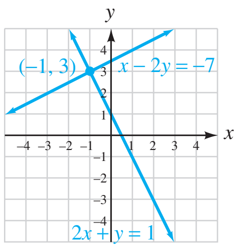
19. $\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$



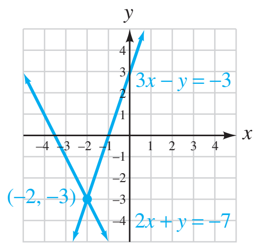
20. $\begin{cases} x - y = 4 \\ 2x + y = 5 \end{cases}$



21. $\begin{cases} 2x + y = 1 \\ x - 2y = -7 \end{cases}$



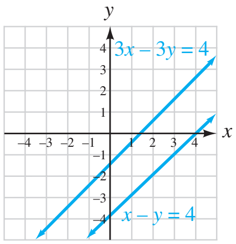
▶ **22.** $\begin{cases} 3x - y = -3 \\ 2x + y = -7 \end{cases}$



Solve each system by graphing. If a system is inconsistent or if the equations are dependent, so indicate. See Examples 3–4.

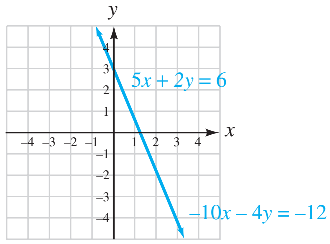
23. $\begin{cases} 3x - 3y = 4 \\ x - y = 4 \end{cases}$

no solution, inconsistent system

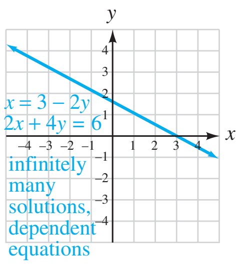


24. $\begin{cases} 5x + 2y = 6 \\ -10x - 4y = -12 \end{cases}$

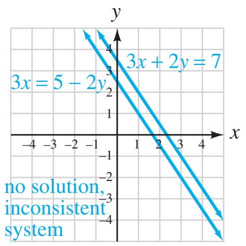
infinitely many solutions, dependent equations



▶ **25.** $\begin{cases} x = 3 - 2y \\ 2x + 4y = 6 \end{cases}$

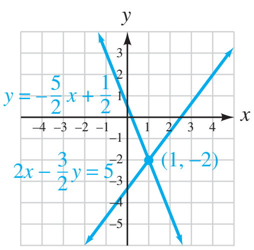


▶ **26.** $\begin{cases} 3x = 5 - 2y \\ 3x + 2y = 7 \end{cases}$

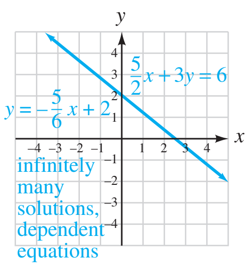


Solve each system by graphing. See Example 5.

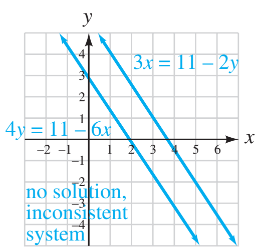
27. $\begin{cases} y = -\frac{5}{2}x + \frac{1}{2} \\ 2x - \frac{3}{2}y = 5 \end{cases}$



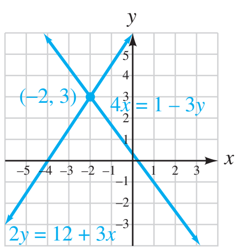
28. $\begin{cases} \frac{5}{2}x + 3y = 6 \\ y = -\frac{5}{6}x + 2 \end{cases}$



29. $\begin{cases} x = \dfrac{11 - 2y}{3} \\ y = \dfrac{11 - 6x}{4} \end{cases}$



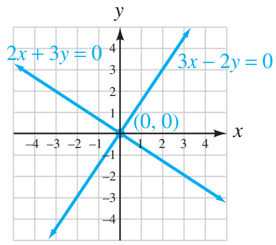
▶ **30.** $\begin{cases} x = \dfrac{1 - 3y}{4} \\ y = \dfrac{12 + 3x}{2} \end{cases}$



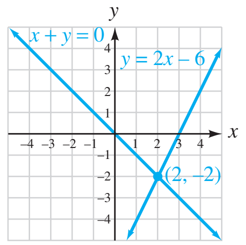
TRY IT YOURSELF

Solve each system by graphing. If a system is inconsistent or if the equations are dependent, so indicate.

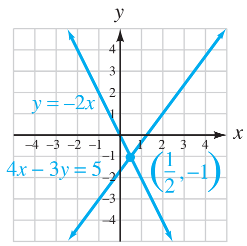
31. $\begin{cases} 3x - 2y = 0 \\ 2x + 3y = 0 \end{cases}$



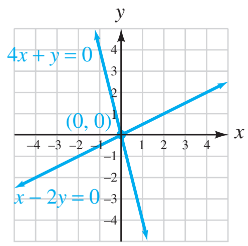
32. $\begin{cases} x + y = 0 \\ 2x - y = 6 \end{cases}$



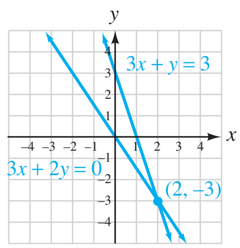
▶ **33.** $\begin{cases} 4x - 3y = 5 \\ 2x + y = 0 \end{cases}$



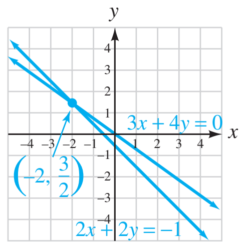
34. $\begin{cases} 4x + y = 0 \\ x - 2y = 0 \end{cases}$



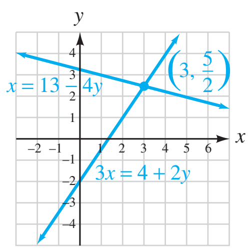
▶ **35.** $\begin{cases} 3x + y = 3 \\ 3x + 2y = 0 \end{cases}$



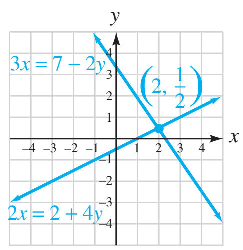
36. $\begin{cases} 2x + 2y = -1 \\ 3x + 4y = 0 \end{cases}$



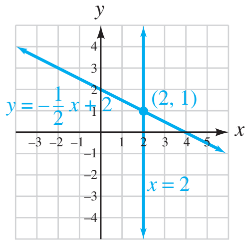
37. $\begin{cases} x = 13 - 4y \\ 3x = 4 + 2y \end{cases}$



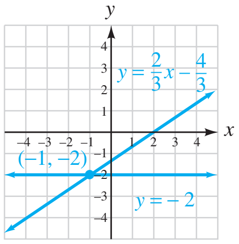
▶ **38.** $\begin{cases} 3x = 7 - 2y \\ 2x = 2 + 4y \end{cases}$



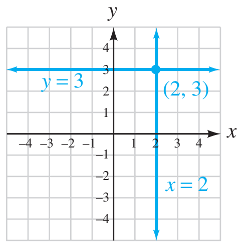
39. $\begin{cases} x = 2 \\ y = -\frac{1}{2}x + 2 \end{cases}$



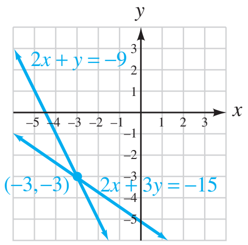
40. $\begin{cases} y = -2 \\ y = \frac{2}{3}x - \frac{4}{3} \end{cases}$



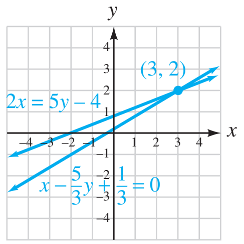
41. $\begin{cases} y = 3 \\ x = 2 \end{cases}$



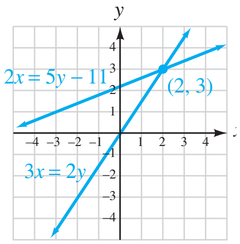
▶ **42.** $\begin{cases} 2x + 3y = -15 \\ 2x + y = -9 \end{cases}$



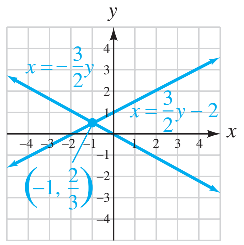
43. $\begin{cases} x = \dfrac{5y - 4}{2} \\ x - \dfrac{5}{3}y + \dfrac{1}{3} = 0 \end{cases}$



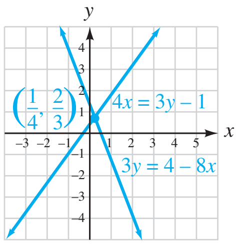
44. $\begin{cases} 2x = 5y - 11 \\ 3x = 2y \end{cases}$



45. $\begin{cases} x = -\frac{3}{2}y \\ x = \frac{3}{2}y - 2 \end{cases}$



46. $\begin{cases} 4x = 3y - 1 \\ 3y = 4 - 8x \end{cases}$



▶ **47.** $\begin{cases} x + 3y = 6 \\ y = -\frac{1}{3}x + 2 \end{cases}$ infinitely many solutions, dependent equations

▶ **48.** $\begin{cases} 2x - y = -4 \\ 2y = 4x - 6 \end{cases}$ no solution, inconsistent system

Use a graphing calculator to solve each system. Give all answers to the nearest hundredth.

49. $\begin{cases} y = 3.2x - 1.5 \\ y = -2.7x - 3.7 \end{cases}$ (−0.37, −2.69)

▶ **50.** $\begin{cases} y = -0.45x + 5 \\ y = 5.55x - 13.7 \end{cases}$ (3.12, 3.60)

51. $\begin{cases} 1.7x + 2.3y = 3.2 \\ y = 0.25x + 8.95 \end{cases}$ (−7.64, 7.04)

▶ **52.** $\begin{cases} 2.75x = 12.9y - 3.79 \\ 7.1x - y = 35.76 \end{cases}$ (5.24, 1.41)

APPLICATIONS

53. MAPS See the following illustration. Name the cities that lie along Interstate 40. Name the cities that lie along Interstate 25. What city lies on both interstate highways?
Gallup, Grants, Albuquerque, Tucumcari; Las Vegas, Santa Fe, Albuquerque, Socorro, Las Cruces; Albuquerque

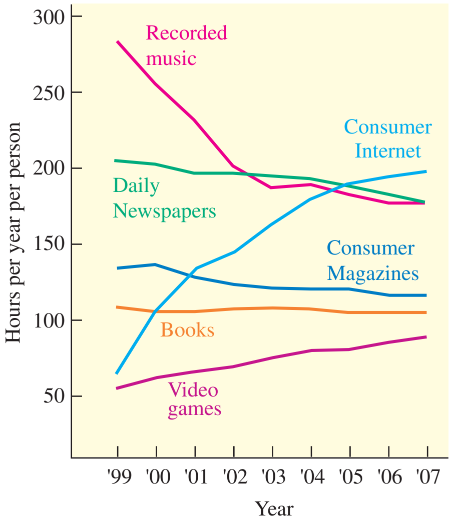
54. THE INTERNET The graph on the next page shows the growing importance of the Internet in the daily lives of Americans. Determine when the time spent on the following activities was the same. Approximately how many hours per year were spent on each?

a. Internet and reading magazines 2001, 130 hr

b. Internet and reading newspapers 2005, 185 hr

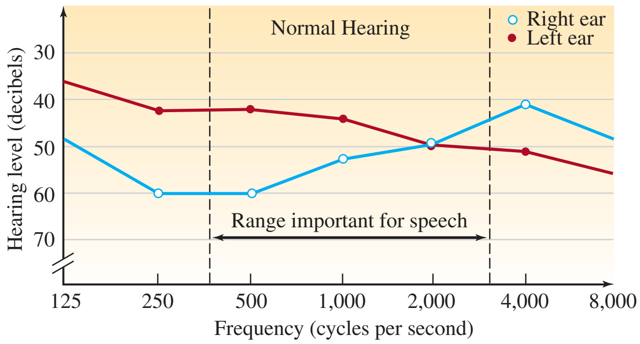
c. Internet and reading books 2000, 105 hr

d. Reading newspapers and listening to recorded music 2002, 200 hr and 2007, 175 hr

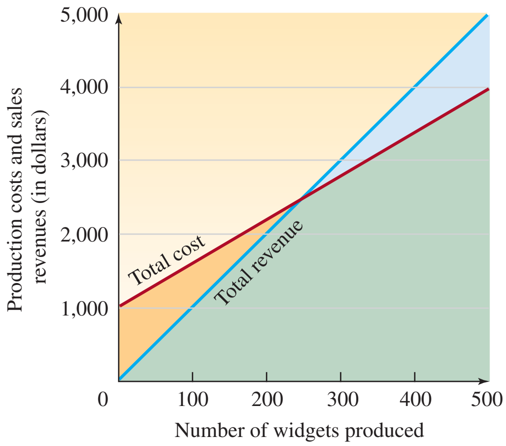


Source: Veronis Suhler Stevenson

55. HEARING TESTS See the illustration below. At what frequency and decibel level were the hearing test results the same for the left and right ear? Write your answer as an ordered pair. (2,000, 50)



56. BUSINESS Estimate the break-even point (where cost = revenue) on the graph below. Then explain why is it called the break-even point. (250, 2,500)

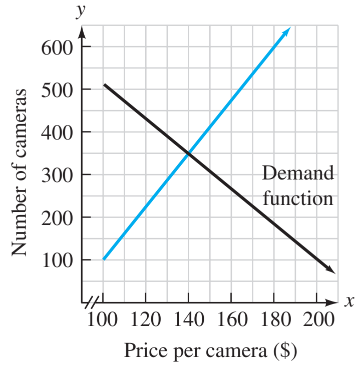


57. SUPPLY AND DEMAND The demand function in the illustration describes the relationship between the price x of a certain camera and the demand for the camera.

a. The supply function, $S(x) = \frac{25}{4}x - 525$, describes the relationship between the price x of the camera and the number of cameras the manufacturer is willing to supply. Graph this function.

b. For what price will the supply of cameras equal the demand? $140

c. As the price of the camera is increased, what happens to supply and what happens to demand? Supply increases and demand decreases.



58. COST AND REVENUE The function $C(x) = 200x + 400$ gives the cost for a college to offer x sections of an introductory class in CPR (cardiopulmonary resuscitation). The function $R(x) = 280x$ gives the amount of revenue the college brings in when offering x sections of CPR.

a. Find the break-even point (where cost = revenue) by graphing each function on the same coordinate system. (5, 1,400)

b. How many sections does the college need to offer to make a profit on the CPR training course? more than 5

59. NAVIGATION The paths of two ships are tracked on the same coordinate system. One ship is following a path described by the equation $2x + 3y = 6$, and the other is following a path described by the equation $y = \frac{2}{3}x - 3$.

a. Is there a possibility of a collision? yes

b. What are the coordinates of the danger point? (3.75, −0.5)

c. Is a collision a certainty? no

60. AIR TRAFFIC CONTROL Two airplanes flying at the same altitude are tracked using the same coordinate system on a radar screen. One plane is following a path described by the equation $y = \frac{2}{5}x - 2$, and the other is following a path described by the equation $2x = 5y + 7$. Is there a possibility of a collision? no

WRITING

61. Suppose the solution of a system of two linear equations is $\left(\frac{14}{5}, -\frac{8}{3}\right)$. Knowing this, explain any drawbacks with solving the system by the graphing method.

62. Can a system of two linear equations have exactly two solutions? Why or why not?

REVIEW

Let $f(x) = -x^3 + 2x - 2$ and $g(x) = \dfrac{2 - x}{9 + x}$. Find each value.

63. $f(-1)$ -3

64. $f(10)$ -982

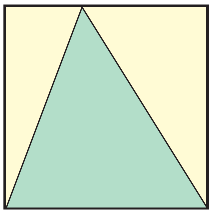
65. $g(2)$ 0

66. $g(-20)$ -2

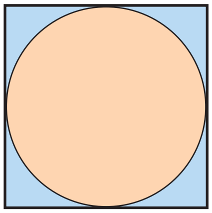
67. Determine the domain and range of $f(x) = x^2 - 2$. D: the set of real numbers, R: the set of all real numbers greater than or equal to -2

68. Find the slope of the line passing through the points $(-4, 8)$ and $(3, 8)$. 0

69. The area of the square on the right is 81 square centimeters. Find the area of the shaded triangle. 40.5 cm^2



70. If the area of the circle on the right is 49π square centimeters, find the area of the square. 196 cm^2



Objectives

1. Solve systems of linear equations by substitution.
2. Solve systems of linear equations by the addition (elimination) method.
3. Use substitution and addition (elimination) to identify inconsistent systems and dependent equations.
4. Determine the most efficient method to solve a linear system.

SECTION 3.2 Solving Systems of Equations Algebraically

The graphing method provides a way to visualize the process of solving systems of equations. However, it can sometimes be difficult to determine the exact coordinates of the point of intersection. We now discuss two other methods, called the *substitution* and the *addition* methods, that can be used to find the exact solutions of systems of equations.

1 Solve systems of linear equations by substitution.

To solve a system of two equations in two variables by the **substitution method,** we follow these steps.

The Substitution Method

1. Solve one equation for a variable—preferably one with a coefficient of 1 or -1. If this is already done, go to step 2. We call the equation found in step 1 the **substitution equation.**
2. Substitute the expression for x or for y obtained in step 1 into the other equation and solve it.
3. Substitute the value of the variable found in step 2 into the substitution equation to find the value of the remaining variable.
4. State the solution.
5. Check the proposed solution in both of the original equations. Write the solution as an ordered pair.

EXAMPLE 1 Solve the system by substitution: $\begin{cases} 4x + y = 13 \\ -2x + 3y = -17 \end{cases}$

Strategy We will use the substitution method. Since the system does not contain an equation solved for x or for y, we must choose an equation and solve it for x or y. It is easiest to solve for y in the first equation, because y has a coefficient of 1 in that equation.

WHY Solving $4x + y = 13$ for x or solving $-2x + 3y = -17$ for x or for y would involve working with cumbersome fractions.

Solution

Step 1 We solve the first equation for y, because y has a coefficient of 1.

$$4x + y = 13$$

$$y = -4x + 13$$ To isolate y, subtract $4x$ from both sides. This is the substitution equation.

Step 2 We then substitute $-4x + 13$ for y in the second equation of the system. This step will eliminate the variable y from that equation. The result will be an equation containing only one variable, x.

$$-2x + 3y = -17$$ This is the second equation of the system.

$$-2x + 3(-4x + 13) = -17$$ Substitute $-4x + 13$ for y. The variable y is eliminated from the equation.

Write parentheses so that the multiplication by 3 is distributed over both terms of $-4x + 13$.

$$-2x - 12x + 39 = -17$$ Distribute the multiplication by 3.

$$-14x = -56$$ To solve for x, first combine like terms and then subtract 39 from both sides.

$$x = 4$$ Divide both sides by -14. This is the x-value of the solution.

Step 3 To find y, we substitute 4 for x in the substitution equation and simplify:

$$y = -4x + 13$$ This is the substitution equation.

$$= -4(4) + 13$$ Substitute 4 for x.

$$= -3$$ This is the y-value of the solution.

Step 4 The solution is $(4, -3)$. The graphs of the equations of the given system would intersect at the point $(4, -3)$.

Step 5 To verify that this result satisfies both equations, we substitute 4 for x and -3 for y into the original equations of the system and simplify.

Check: **The first equation**

$$4x + y = 13$$
$$4(4) + (-3) \stackrel{?}{=} 13$$
$$16 - 3 \stackrel{?}{=} 13$$
$$13 = 13 \quad \text{True}$$

The second equation

$$-2x + 3y = -17$$
$$-2(4) + 3(-3) \stackrel{?}{=} -17$$
$$-8 - 9 \stackrel{?}{=} -17$$
$$-17 = -17 \quad \text{True}$$

Since $(4, -3)$ satisfies both equations of the system, it checks.

Success Tip With this method, the objective is to use an appropriate substitution to obtain *one* equation in *one* variable.

Self Check 1

Solve the system by substitution: $\begin{cases} x + 3y = 9 \\ 2x - y = -10 \end{cases}$ $(-3, 4)$

Now Try Problems 14 and 16

Teaching Example 1 Solve the system by substitution: $\begin{cases} 2x + 5y = -3 \\ 3x - y = 4 \end{cases}$

Answer:
$(1, -1)$

Self Check 2

Solve the system by substitution:
$$\begin{cases} \frac{x}{8} + \frac{y}{4} = \frac{1}{2} \\ 0.01y = -0.02x + 0.04 \end{cases}$$

Now Try **Problem 20**

Self Check 2 Answer
$\left(\frac{4}{3}, \frac{4}{3}\right)$

Teaching Example 2 Solve the system by substitution: $\begin{cases} \frac{3}{8}x - \frac{3}{4}y = \frac{1}{2} \\ 0.1x = 0.1y + 0.1 \end{cases}$

Answer:
$\left(\frac{2}{3}, -\frac{1}{3}\right)$

EXAMPLE 2 Solve the system by substitution: $\begin{cases} \frac{2}{9}x - \frac{2}{9}y = \frac{2}{3} \\ 0.1x = 0.2 - 0.1y \end{cases}$

Strategy We will use the multiplication property of equality to clear both equations of fractions and solve the resulting equivalent system by substitution.

WHY It is usually easier to solve a system of equations that involves only integers.

Solution

To clear the first equation of fractions, we multiply both sides by 9, which is the least common denominator of the fractions in the equation. To clear the second equation of the decimals, we multiply both sides by 10.

The original system ***An equivalent system***

$$\begin{cases} \frac{2}{9}x - \frac{2}{9}y = \frac{2}{3} \xrightarrow{\text{Multiply by 9}} 9\left(\frac{2}{9}x - \frac{2}{9}y\right) = 9\left(\frac{2}{3}\right) \xrightarrow{\text{Simplify}} 2x - 2y = 6 \\ 0.1x = 0.2 - 0.1y \xrightarrow[\text{Multiply by 10}]{} 10(0.1x) = 10(0.2 - 0.1y) \xrightarrow[\text{Simplify}]{} x = 2 - y \end{cases}$$

These results form the following equivalent system, which has the same solution as the original one.

$$\begin{aligned} &(1) \quad 2x - 2y = 6 \\ &(2) \quad x = 2 - y \end{aligned}$$

This is the substitution equation.

Since the variable x is isolated in equation 2, we will substitute $2 - y$ for x in equation 1. This step will eliminate x from equation 1, leaving an equation containing only one variable, y. We then solve for y.

$$\begin{aligned} 2x - 2y &= 6 && \text{This is equation 1 of the equivalent system.} \\ 2(2 - y) - 2y &= 6 && \text{Substitute } 2 - y \text{ for } x. \\ 4 - 2y - 2y &= 6 && \text{Distribute the multiplication by 2.} \\ -4y &= 2 && \text{To solve for } y\text{, combine like terms and then subtract 4 from both sides.} \\ y &= -\frac{1}{2} && \text{Divide both sides by } -4 \text{ and then simplify the fraction.} \end{aligned}$$

We can find x by substituting $-\frac{1}{2}$ for y in the substitution equation and simplifying:

$$\begin{aligned} x &= 2 - y \\ x &= 2 - \left(-\frac{1}{2}\right) && \text{Substitute } -\tfrac{1}{2} \text{ for } y. \\ &= 2 + \frac{1}{2} \\ &= \frac{5}{2} && 2 + \tfrac{1}{2} = \tfrac{4}{2} + \tfrac{1}{2} = \tfrac{5}{2}. \end{aligned}$$

The solution is $\left(\frac{5}{2}, -\frac{1}{2}\right)$. Verify that this solution satisfies both equations in the original system.

2 Solve systems of linear equations by the addition (elimination) method.

Another method for solving a system of linear equations is the **addition method.** In this method, we combine the equations in a way that will eliminate the terms involving one of the variables.

The Addition Method

1. Write both equations of the system in standard (general) form: $Ax + By = C$.
2. Multiply the terms of one or both of the equations by nonzero numbers chosen to make the coefficients of x (or y) opposites.
3. Add the equations to eliminate the terms involving x (or y).
4. Solve the equation resulting from step 3.
5. Substitute the value obtained in step 4 into either of the original equations and solve for the remaining variable.
6. Check the proposed solution in both of the original equations. Write the solution as an ordered pair.

EXAMPLE 3

Solve the system by addition: $\begin{cases} 4x + y = 13 \\ -2x + 3y = -17 \end{cases}$

Strategy To use the addition method to solve this system, we will multiply both sides of the second equation by 2 and add the equations. This will eliminate the terms involving the variable x.

WHY This will give one equation involving only the variable y.

Solution

Step 1 This is the system discussed in Example 1. In this example, we will solve it by the addition method. Since both equations are already written in general form, step 1 is unnecessary.

Step 2 We note that the coefficient of x in the first equation is 4. If we multiply both sides of the second equation by 2, the coefficient of x in that equation will be -4. Then the coefficients of x will be opposites.

$$\begin{cases} 4x + y = 13 \\ -4x + 6y = -34 \end{cases}$$

The original system ***An equivalent system***

$$\begin{aligned} 4x + y &= 13 &\xrightarrow{\text{Unchanged}}\quad 4x + y &= 13 &\xrightarrow{\text{Unchanged}}\quad 4x + y &= 13 \\ -2x + 3y &= -17 &\xrightarrow[\text{Multiply by 2}]{}\quad 2(-2x + 3y) &= 2(-17) &\xrightarrow[\text{Simplify}]{}\quad -4x + 6y &= -34 \end{aligned}$$

Step 3 When these equations are added, the terms involving x drop out (or are eliminated), and we get an equation that contains only the variable y.

$$\begin{aligned} 4x + y &= 13 \\ + \; -4x + 6y &= -34 \\ \hline 7y &= -21 \end{aligned}$$

Add the like terms, column by column: $4x + (-4x) = 0$, $y + 6y = 7y$, and $13 + (-34) = -21$.

Step 4 Solve the resulting equation for y.

$7y = -21$ To solve for y, divide both sides by 7.

$y = -3$ This is the y-value of the equation.

Step 5 To find x, we substitute -3 for y in either of the original equations and solve for x. If we use the first equation, we have

Self Check 3

Solve the system by addition: $\begin{cases} 3x + 2y = 0 \\ 2x - y = -7 \end{cases}$ $(-2, 3)$

Now Try **Problems 24 and 28**

Teaching Example 3 Solve the system by addition: $\begin{cases} 3x - y = 7 \\ 5x + 2y = 8 \end{cases}$

Answer: $(2, -1)$

$$4x + y = 13$$
$$4x + (-3) = 13 \quad \text{Substitute } -3 \text{ for } y.$$
$$4x = 16 \quad \text{To solve for } x \text{, add 3 to both sides.}$$
$$x = 4 \quad \text{Divide both sides by 4. This is the } x\text{-value of the equation.}$$

Step 6 The solution is $(4, -3)$. The check was completed in Example 1.

Self Check 4

Solve the system by addition:
$\begin{cases} 4(2x - y) = 18 \\ 3(x - 3) = 2y - 1 \end{cases}$ $\left(1, -\frac{5}{2}\right)$

Now Try **Problem 32**

Teaching Example 4 Solve the system by addition:
$\begin{cases} 7(x - 1) = 3(y - 4) + 7 \\ -2(x + 4) = 2y \end{cases}$
Answer:
$(-1, -3)$

EXAMPLE 4

Solve the system by addition: $\begin{cases} 4x = 3(2 + y) \\ 3(x - 10) = -2y \end{cases}$

Strategy We will write each equation in standard (general) form and use the addition (elimination) method to solve the resulting equivalent system.

WHY In their current form, the equations do not contain terms with coefficients that are opposites.

Solution

To write each equation in standard (general) form, we use the distributive property to remove the parentheses and then rearrange the terms.

The first equation	**The second equation**
$4x = 3(2 + y)$	$3(x - 10) = -2y$
$4x = 6 + 3y$	$3x - 30 = -2y$
$4x - 3y = 6$	$3x + 2y = 30$

We now solve the equivalent system

$$\begin{array}{ll} \textbf{(1)} & \begin{cases} 4x - 3y = 6 \\ 3x + 2y = 30 \end{cases} \\ \textbf{(2)} & \end{array}$$

Since the coefficients of y already have opposite signs, we choose to eliminate y. To make the y-terms drop out when we add the equations, we multiply both sides of equation 1 by 2 and both sides of equation 2 by 3 to get

$$4x - 3y = 6 \xrightarrow{\text{Multiply by 2}} 2(4x - 3y) = 2(6) \xrightarrow{\text{Simplify}} 8x - 6y = 12$$
$$3x + 2y = 30 \xrightarrow[\text{Multiply by 3}]{} 3(3x + 2y) = 3(30) \xrightarrow[\text{Simplify}]{} 9x + 6y = 90$$

When these equations are added, the y-terms drop out, and we can solve for x.

$$\begin{array}{rll} 8x - 6y = & 12 & \\ 9x + 6y = & 90 & \text{Add like terms, column by column:} \\ \hline 17x \quad = & 102 & 8x + 9x = 17x, \ -6y + 6y = 0, \text{ and } 12 + 90 = 102. \\ x = & 6 & \text{To solve for } x\text{, divide both sides by 17.} \end{array}$$

To find y, we can substitute 6 for x in any equation that contains both variables. It appears the computations will be simplest if we use equation 2.

$$3x + 2y = 30$$
$$3(6) + 2y = 30 \quad \text{Substitute 6 for } x.$$
$$18 + 2y = 30 \quad \text{Perform the multiplication.}$$
$$2y = 12 \quad \text{Subtract 18 from both sides.}$$
$$y = 6 \quad \text{Divide both sides by 2.}$$

The solution is $(6, 6)$. Check this result.

Success Tip We create the term $-6y$ from $-3y$ and the term $6y$ from $2y$. Note that the *least common multiple* of 2 and 3 is 6.

2, 4, **6,** 8, 10, 12, 14, …

3, **6,** 9, 12, …

3 Use substitution and elimination (addition) to identify inconsistent systems and dependent equations.

EXAMPLE 5

Solve the system, if possible: $\begin{cases} y = 2x + 4 \\ 8x - 4y = 7 \end{cases}$

Strategy We will use the substitution method to solve this system.

WHY The substitution method works well when one of the equations of the system (in this case $y = 2x + 4$) is solved for a variable.

Solution

Because the first equation is already solved for y, we use the substitution method.

$$8x - 4y = 7 \quad \text{This is the second equation.}$$
$$8x - 4(2x + 4) = 7 \quad \text{Substitute } 2x + 4 \text{ for } y.$$

We then solve this equation for x:

$$8x - 8x - 16 = 7 \quad \text{Use the distributive property to remove parentheses.}$$
$$-16 \neq 7 \quad \text{Combine like terms.}$$

The x-terms drop out. The false result, $-16 = 7$, indicates that the system has *no solution* and is, therefore, inconsistent. The solution set is $\varnothing$. Since the system has no solution, the graphs of the equations in the system will be parallel.

Self Check 5

Solve the system, if possible: $\begin{cases} x = -2.5y + 8 \\ y = -0.4x + 2 \end{cases}$

Now Try Problem 35

Self Check 5 Answer
no solution, $\varnothing$; inconsistent system

Teaching Example 5 Solve the system, if possible: $\begin{cases} 4x - 12y = 5 \\ x = 3y - 4 \end{cases}$

Answer:
no solution, $\varnothing$; inconsistent system

EXAMPLE 6

Solve the system: $\begin{cases} 4x + 6y = 12 \\ -2x - 3y = -6 \end{cases}$

Strategy We will use the addition (elimination) method to solve this system.

WHY The addition method works well when each equation is written in general form.

Solution

We copy the first equation and multiply both sides of the second equation by 2 to get

$$\begin{aligned} 4x + 6y &= 12 \\ -4x - 6y &= -12 \\ \hline 0 &= 0 \end{aligned}$$

When we add like terms, column by column, the result is $0x + 0y = 0$, which simplifies to $0 = 0$.

Here, both the x- and y-terms drop out. The true statement $0 = 0$ indicates that the equations are dependent and that the system is consistent with infinitely many solutions.

Note that the equations of the system are equivalent, because when the second equation is multiplied by -2, it becomes the first equation. The graphs of these equations would coincide. Any ordered pair that satisfies one of the equations also satisfies the other. To find some solutions, we can substitute 0, 3, and -3 for x in either original equation to obtain $(0, 2)$, $(3, 0)$, and $(-3, 4)$.

Self Check 6

Solve the system: $\begin{cases} x - \frac{5}{2}y = \frac{19}{2} \\ -\frac{2}{5}x + y = -\frac{19}{5} \end{cases}$

Now Try Problem 37

Self Check 6 Answer
There are infinitely many solutions; three of them are $(2, -3)$, $(12, 1)$, and $\left(\frac{19}{2}, 0\right)$.

Teaching Example 6 Solve the system: $\begin{cases} -7x + y = -2 \\ 14x - 2y = 4 \end{cases}$

Answer:
There are infinitely many solutions; three of them are $(1, 5)$, $(-1, -9)$, and $(0, -2)$.

Examples 5 and 6 illustrate the following facts.

Inconsistent Systems and Dependent Equations

When solving a system of two linear equations in two variables using substitution or addition (elimination):

1. If the variables drop out and a true statement (identity) is obtained, the system has an infinite number of solutions. The equations are dependent and the system is consistent.
2. If the variables drop out and a false statement (contradiction) is obtained, the system has no solution and is inconsistent.

4 Determine the most efficient method to use to solve a linear system.

If no method is specified for solving a particular linear system, the following guidelines can be helpful in determining whether to use graphing, substitution, or addition.

1. If you want to show trends and see the point that two graphs have in common, use the **graphing method.** However, this method can be lengthy and is not exact.
2. If one of the equations is solved for one of the variables, or easily solved for one of the variables, use the **substitution method.**
3. If both equations are in standard (general) $Ax + By = C$ form, and no variable has a coefficient of 1 or -1, use the **addition (elimination) method.**
4. If the coefficient of one of the variables is 1 or -1, you have a choice. You can write each equation in standard form ($Ax + By = C$) and use elimination, or you can solve for the variable with coefficient 1 or -1 and use substitution.

Here are some examples of suggested approaches:

$\begin{cases} 5x + 6y = 1 \\ y = x - 4 \end{cases}$ Substitution

$\begin{cases} a - 16 = 5b \\ 4a - 8b = -11 \end{cases}$ Substitution

$\begin{cases} 3x + 2y = 17 \\ 7x + 5y = -33 \end{cases}$ Addition

$\begin{cases} 3x - y = -1 \\ 9x + 8y = 2 \end{cases}$ Addition or substitution

ANSWERS TO SELF CHECKS

1. $(-3, 4)$ **2.** $\left(\frac{4}{3}, \frac{4}{3}\right)$ **3.** $(-2, 3)$ **4.** $\left(1, -\frac{5}{2}\right)$ **5.** no solution, $\varnothing$; inconsistent system
6. There are infinitely many solutions; three of them are $(2, -3)$, $(12, 1)$, and $\left(\frac{19}{2}, 0\right)$

SECTION 3.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. $Ax + By = C$ is the standard (general) form of a linear equation.
2. In the equation $x + 3y = -1$, the x-term has an understood coefficient of 1.
3. When we add the two equations of the system $\begin{cases} x + y = 5 \\ x - y = -3 \end{cases}$, the y-terms are eliminated.
4. To solve $\begin{cases} y = 3x \\ x + y = 4 \end{cases}$ we can substitute $3x$ for y in the second equation.

CONCEPTS

5. If the system $\begin{cases} 4x - 3y = 7 \\ 3x - y = 6 \end{cases}$ is to be solved using the substitution method, what variable in what equation would it be easier to solve for? y; second equation

▶ Selected exercises available online at www.webassign.net/brookscole

6. If the system $\begin{cases} 4x - 3y = 7 \\ 3x - 2y = 6 \end{cases}$ is to be solved using the elimination (addition) method, by what constants should each equation be multiplied if

   a. the x-terms are to drop out? $3; -4$ (answers may vary)

   b. the y-terms are to drop out? $2; -3$ (answers may vary)

7. Can the system $\begin{cases} 2x + 5y = 7 \\ 4x - 3y = 16 \end{cases}$ be solved more easily by the substitution or the addition method? addition method

▶ 8. Given the equation $3x + y = -4$.

   a. Solve for x. $x = \frac{-y-4}{3}$

   b. Solve for y. $y = -3x - 4$

   c. Which variable was easier to solve for? Explain why. y; it involved one less step

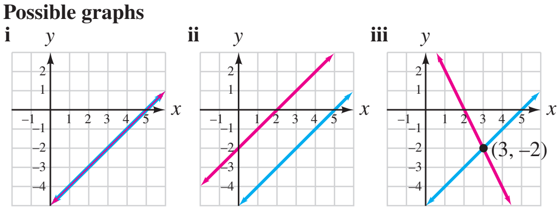
9. The substitution method was used to solve three systems of linear equations. The results after y was eliminated and the remaining equation was solved for x are listed below. Match each result with a possible graph of the system from the illustration.

   a. $-2 = 3$ ii

   b. $x = 3$ iii

   c. $3 = 3$ i

Possible graphs



10. Consider the system: $\begin{cases} \frac{2}{3}x - \frac{y}{6} = \frac{16}{9} \\ 0.03x + 0.02y = 0.03 \end{cases}$

   a. What algebraic step should be performed to clear the first equation of fractions? Multiply both sides by 18.

   b. What algebraic step should be performed to clear the second equation of decimals? Multiply both sides by 100.

GUIDED PRACTICE

Solve each system by substitution. See Example 1.

11. $\begin{cases} y = x \\ x + y = 4 \end{cases}$ $(2, 2)$

▶ 12. $\begin{cases} y = x + 2 \\ x + 2y = 16 \end{cases}$ $(4, 6)$

13. $\begin{cases} x = 2 + y \\ 2x + y = -5 \end{cases}$ $(-1, -3)$

14. $\begin{cases} x = -5 + y \\ 3x - 2y = -7 \end{cases}$ $(3, 8)$

15. $\begin{cases} x - y = 2 \\ 2x + y = 13 \end{cases}$ $(5, 3)$

16. $\begin{cases} x - y = -4 \\ 3x - 2y = -5 \end{cases}$ $(3, 7)$

17. $\begin{cases} x + 2y = 6 \\ 3x - y = -10 \end{cases}$ $(-2, 4)$

▶ 18. $\begin{cases} 2x - y = -21 \\ 4x + 5y = 7 \end{cases}$ $(-7, 7)$

Solve each system by substitution. See Example 2.

19. $\begin{cases} \frac{1}{3}x - \frac{2}{9}y = 1 \\ 0.1x = 0.3 - 0.1y \end{cases}$ $(3, 0)$

20. $\begin{cases} \frac{1}{4}x + \frac{1}{4}y = 1 \\ 0.4y = 1.6 - 0.4x \end{cases}$ $(0, 4)$

▶ 21. $\begin{cases} 0.3a + 0.1b = 0.5 \\ \frac{4}{3}a + \frac{1}{3}b = 3 \end{cases}$ $(4, -7)$

22. $\begin{cases} 0.9p + 0.2q = 1.2 \\ \frac{2}{3}p + \frac{1}{9}q = 1 \end{cases}$ $(2, -3)$

Solve each system by addition (elimination). See Example 3.

23. $\begin{cases} x - y = 3 \\ x + y = 7 \end{cases}$ $(5, 2)$

▶ 24. $\begin{cases} x + y = 1 \\ x - y = 7 \end{cases}$ $(4, -3)$

25. $\begin{cases} 2x + y = -10 \\ 2x - y = -6 \end{cases}$ $(-4, -2)$

26. $\begin{cases} x + 2y = -9 \\ x - 2y = -1 \end{cases}$ $(-5, -2)$

27. $\begin{cases} 3x - 4y = 9 \\ x + 2y = 8 \end{cases}$ $\left(5, \frac{3}{2}\right)$

28. $\begin{cases} 2x + 3y = -15 \\ 2x + y = -9 \end{cases}$ $(-3, -3)$

29. $\begin{cases} 5x + 2y = 11 \\ 7x + 6y = 9 \end{cases}$ $(3, -2)$

30. $\begin{cases} 3x + 4y = -24 \\ 5x + 12y = -72 \end{cases}$ $(0, -6)$

Solve each system by addition (elimination). See Example 4.

31. $\begin{cases} 4(x - 2) = -9y \\ 2(x - 3y) = -3 \end{cases}$ $\left(\frac{1}{2}, \frac{2}{3}\right)$

▶ 32. $\begin{cases} 2(2x + 3y) = 5 \\ 8x = 3(1 + 3y) \end{cases}$ $\left(\frac{3}{4}, \frac{1}{3}\right)$

▶ 33. $\begin{cases} 2(x + y) + 1 = 0 \\ 3x + 4y = 0 \end{cases}$ $\left(-2, \frac{3}{2}\right)$

34. $\begin{cases} 5x + 3y = -7 \\ 3(x - y) - 7 = 0 \end{cases}$ $\left(0, -\frac{7}{3}\right)$

Solve each system, if possible. If a system is inconsistent or if the equations are dependent, so indicate. See Examples 5–6.

35. $\begin{cases} \frac{3}{2}x + 2 = y \\ 6x - 4y = -4 \end{cases}$ no solution, inconsistent system

▶ 36. $\begin{cases} 2x - \frac{5}{2} = y \\ 4x - 2y = 5 \end{cases}$ infinitely many solutions, dependent equations

37. $\begin{cases} 16x - 8y = 32 \\ 2x - 4 = y \end{cases}$ infinitely many solutions, dependent equations

38. $\begin{cases} x = \frac{3}{2}y + 5 \\ 2x - 3y = 8 \end{cases}$ no solution, inconsistent system

TRY IT YOURSELF

Solve each system of equations. If a system is inconsistent or if the equations are dependent, so indicate.

39. $\begin{cases} 2x + 3y = 8 \\ 3x - 2y = -1 \end{cases}$ $(1, 2)$

40. $\begin{cases} 5x - 2y = 19 \\ 3x + 4y = 1 \end{cases}$ $(3, -2)$

41. $\begin{cases} 3x - 2y = -10 \\ 6x + 5y = 25 \end{cases}$
$(0, 5)$

42. $\begin{cases} 2x + 1 = -2y \\ 3x + 4y = 0 \end{cases}$
$\left(-2, \frac{3}{2}\right)$

43. $\begin{cases} 2x + 3(x + y) + 7 = 0 \\ 3(x - y - 1) - 4 = 0 \end{cases}$
$\left(0, -\frac{7}{3}\right)$

▶ **44.** $\begin{cases} 0.6y - 0.9x = -3.9 \\ 3x - 17 = 4y \end{cases}$
$(3, -2)$

45. $\begin{cases} x = \frac{2}{3}y \\ y = 4x + 5 \end{cases}$
$(-2, -3)$

46. $\begin{cases} 0.5x + 0.5y = 6 \\ \frac{x}{2} - \frac{y}{2} = -2 \end{cases}$
$(4, 8)$

47. $\begin{cases} \frac{x}{2} - \frac{y}{3} = -4 \\ \frac{x}{2} + \frac{y}{9} = 0 \end{cases}$
$(-2, 9)$

48. $\begin{cases} \frac{3}{4}x + \frac{2}{3}y = 7 \\ \frac{3}{5}x - \frac{1}{2}y = 18 \end{cases}$
$(20, -12)$

49. $\begin{cases} \frac{2}{3}x - \frac{1}{4}y = -8 \\ 0.5x - 0.375y = -9 \end{cases}$
$(-6, 16)$

50. $\begin{cases} \frac{3x}{2} - \frac{2y}{3} = 0 \\ \frac{3x}{4} + \frac{4y}{3} = \frac{5}{2} \end{cases}$
$\left(\frac{2}{3}, \frac{3}{2}\right)$

51. $\begin{cases} 3c = 2(6 - d) \\ 2d = 3(4 - c) \end{cases}$
infinitely many solutions, dependent equations

52. $\begin{cases} 2(a + b) = a + 12 \\ a = 14 - 2b \end{cases}$
no solution, inconsistent system

▶ **53.** $\begin{cases} \frac{3x}{5} + \frac{5y}{3} = 2 \\ \frac{6x}{5} - \frac{5y}{3} = 1 \end{cases}$
$\left(\frac{5}{3}, \frac{3}{5}\right)$

54. $\begin{cases} 12x - 5y - 21 = 0 \\ \frac{3}{4}x - \frac{2}{3}y = \frac{19}{8} \end{cases}$
$\left(\frac{1}{2}, -3\right)$

55. $\begin{cases} 3a + b = 5 \\ \frac{4}{3}a + \frac{1}{3}b = 3 \end{cases}$
$(4, -7)$

56. $\begin{cases} 9p + 2q = 12 \\ \frac{2}{3}p + \frac{1}{9}q = 1 \end{cases}$
$(2, -3)$

57. $\begin{cases} \frac{x}{4} = 1 + \frac{y}{5} \\ x = \frac{4}{5}(y + 10) \end{cases}$
no solution, inconsistent system

58. $\begin{cases} \frac{5}{3}x + 2 = 2(y + 6) \\ 3y + 5 = \frac{5}{2}(x - 4) \end{cases}$
infinitely many solutions, dependent equations

59. $\begin{cases} 4y + 5x - 7 = 0 \\ \frac{10}{7}x - \frac{4}{9}y = \frac{17}{21} \end{cases}$
$\left(\frac{4}{5}, \frac{3}{4}\right)$

60. $\begin{cases} \frac{x}{2} - \frac{y}{3} = -4 \\ 0.009x + 0.002y = 0 \end{cases}$
$(-2, 9)$

The following systems involve variables other than x and y. When writing the solution as an ordered pair, write the values for the variables in alphabetical order.

61. $\begin{cases} \frac{3}{2}p + \frac{1}{3}q = 2 \\ \frac{2}{3}p + \frac{1}{9}q = 1 \end{cases}$
$(2, -3)$

62. $\begin{cases} a + \frac{b}{3} = \frac{5}{3} \\ \frac{a + b}{3} = 3 - a \end{cases}$
$(4, -7)$

▶ **63.** $\begin{cases} \frac{m - n}{5} + \frac{m + n}{2} = 6 \\ \frac{m - n}{2} - \frac{m + n}{4} = 3 \end{cases}$
$(9, -1)$

▶ **64.** $\begin{cases} \frac{r - 2}{5} + \frac{s + 3}{2} = 5 \\ \frac{r + 3}{2} + \frac{s - 2}{3} = 6 \end{cases}$
$(7, 5)$

Solve each system. To do this, substitute a for $\frac{1}{x}$ and b for $\frac{1}{y}$ and solve for a and b. Then find x and y using the fact that and $a = \frac{1}{x}$ and $b = \frac{1}{y}$.

▶ **65.** $\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{6} \end{cases}$
$(2, 3)$

▶ **66.** $\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{9}{20} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{20} \end{cases}$
$(4, 5)$

67. $\begin{cases} \frac{1}{x} + \frac{2}{y} = -1 \\ \frac{2}{x} - \frac{1}{y} = -7 \end{cases}$
$\left(-\frac{1}{3}, 1\right)$

68. $\begin{cases} \frac{3}{x} - \frac{2}{y} = -30 \\ \frac{2}{x} - \frac{3}{y} = -30 \end{cases}$
$\left(-\frac{1}{6}, \frac{1}{6}\right)$

WRITING

69. Which method would you use to solve the system $\begin{cases} 4x + 6y = 5 \\ 8x - 3y = 3 \end{cases}$? Explain why.

▶ **70.** Which method would you use to solve the system $\begin{cases} x - 2y = 5 \\ 2x + 3y = 11 \end{cases}$? Explain why.

71. When solving a problem using two variables, why must we write two equations?

72. Write a problem that can be solved by solving the system $\begin{cases} x + y = 36 \\ \$1.29x + \$2.29y = \$72.44 \end{cases}$

73. Write a problem to fit the information given in the table.

Solution	Ounces	% insecticide	Amount of insecticide
Weak	x	0.02	$0.02x$
Strong	y	0.10	$0.10y$
Mixture	80	0.07	0.07(80)

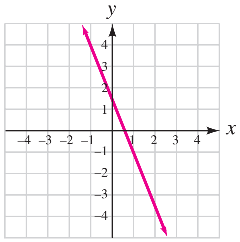
74. Make a table like the one below, listing one advantage and one disadvantage for each of the methods that can be used to solve a system of two linear equations.

Method	Advantage	Disadvantage
Graphing		
Substitution		
Addition		

REVIEW

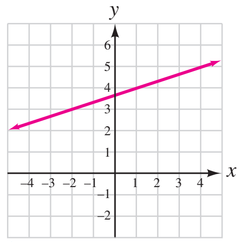
Find the slope of each line.

75.



$-\frac{5}{2}$

76.



$\frac{1}{3}$

77. The line passing through $(0, -8)$ and $(-5, 0)$ $-\frac{8}{5}$

▶ **78.** The line with equation $y = -3x + 4$ -3

79. The line with equation $4x - 3y = -3$ $\frac{4}{3}$

▶ **80.** The line with equation $y = 3$ 0

SECTION 3.3

Problem Solving Using Systems of Two Equations

Objectives

1. Assign variables to two unknowns.
2. Use systems to solve geometry problems.
3. Use systems to solve number-value problems.
4. Use systems to find the break point.
5. Use systems to solve interest, uniform motion, and mixture problems.

In Chapter 1, we solved applied problems involving two unknown quantities by modeling the situation with an equation in one variable. It's often easier to solve such problems using a two-variable approach. We write two equations in two variables to model the situation, and then we use the methods of this chapter to solve the system formed by the pair of equations.

1 Assign variables to two unknowns.

The following steps are helpful when solving problems involving two unknown quantities.

Problem-Solving Strategy

1. **Analyze the problem** by reading it carefully to understand the given facts. Often a diagram or table will help you visualize the facts of the problem.
2. Pick different variables to represent two unknown quantities. Translate the words of the problem to **form two equations** involving each of the two variables.
3. **Solve the system** of equations using graphing, substitution, or elimination.
4. **State the conclusion.**
5. **Check the results** in the words of the problem.

EXAMPLE 1 ***Pets*** In 2007, there were 163 million dogs and cats owned in the United States. If the number of dogs was 13 million less than the number of cats, how many dogs and how many cats were owned in the United States that year? (Source: Humane Society)

Analyze

- In 2007, the total number of dogs and cats was 163 million.
- The number of dogs was 13 million less than the number of cats.
- Find the number of dogs and the number of cats that year.

Form We will let x = the number of dogs (in millions) and y = the number of cats (in millions). We can translate the words of the problem into two equations, each involving x and y.

The number of dogs	plus	the number of cats	was	163 million.
x	$+$	y	$=$	163

The number of dogs	was	13 million	less than	the number of cats.
x	$=$	y	$-$	13

The resulting system is $\begin{cases} x + y = 163 \\ x = y - 13 \end{cases}$

Solve Since the second equation is solved for x, we will use substitution to solve the system.

Self Check 1

An iPod classic and iHome together cost \$348. If the iPod is \$150 more than the cost of the iHome, how much does each cost? iPod: \$249, iHome: \$99

Now Try Problem 13

Teaching Example 1 In 2008, there were 305.3 million people in the United States. If the number of females was 4.1 million more than the number of males, how many males and how many females were in the United States in 2008? (*Source:* U.S. Census Bureau, www.census.gov/popest/national/asrh/2007-nat-res.html)
Answer:
150.6 million males, 154.7 million females

$$
\begin{aligned}
x + y &= 163 && \text{This is the first equation of the system.}\\
y - 13 + y &= 163 && \text{Substitute } y - 13 \text{ for } x.\\
2y - 13 &= 163 && \text{Combine like terms.}\\
2y &= 176 && \text{Add 13 to both sides.}\\
y &= 88 && \text{Divide both sides by 2. This is the number of cats (in millions).}
\end{aligned}
$$

To find x, we substitute 88 for y in the second equation of the system.

$$
\begin{aligned}
x &= y - 13\\
&= 88 - 13 && \text{Substitute 88 for } y.\\
&= 75 && \text{This is the number of dogs (in millions).}
\end{aligned}
$$

State In 2007, the number of dogs owned in the United States was 75 million and the number of cats was 88 million.

Check Since 75 million + 88 million = 163 million and 75 million is 13 million less than 88 million, the results check.

Caution! If two variables are used to represent two unknown quantities, we must form a system of two equations to find the unknowns.

2 Use systems to solve geometry problems.

Sometimes we can use geometric facts or formulas to solve application problems.

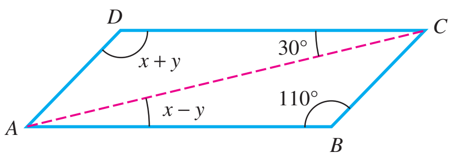
Self Check 2

In a right triangle, the sum of the two acute angles is 90°. If one of the acute angles is 36° more than the other, find the measure of the two angles. 27°, 63°

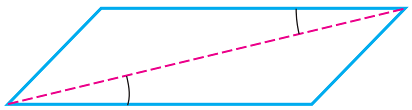
Now Try **Problem 19**

Teaching Example 2 In a parallelogram, consecutive angles are supplementary, meaning their sum is 180°. If one of the angles is twice the other, find the measure of the consecutive angles.
Answer:
60°, 120°

EXAMPLE 2 ***Parallelograms*** Refer to parallelogram $ABCD$. Find the unknown degree measures represented by x and y.



Analyze To solve this problem, we will use two facts about parallelograms.



Alternate interior angles

When a diagonal intersects two parallel sides of a parallelogram, pairs of alternate interior angles have the same measure.

Opposite angles

Opposite angles of a parallelogram have the same measure.

Form In the figure, $\angle BAC$ and $\angle DCA$ are alternate interior angles and therefore have the same measure. Thus, $x - y = 30$. Since $\angle B$ and $\angle D$ in the figure are opposite angles of the parallelogram, $x + y = 110$.

Solve To find x and y, we solve the following system:

$$
\begin{cases}
x - y = 30\\
x + y = 110
\end{cases}
$$

Since the coefficients of the terms $-y$ and y are opposites, we will use addition to solve the system.

$$\begin{aligned} x - y &= 30 \\ x + y &= 110 \\ \hline 2x \quad &= 140 && \text{Add the equations. The } y\text{-terms drop out.} \\ x &= 70 && \text{To solve for } x\text{, divide both sides by 2.} \end{aligned}$$

We can substitute 70 for x in the second equation and solve for y.

$$\begin{aligned} x + y &= 110 \\ 70 + y &= 110 && \text{Substitute 70 for } x. \\ y &= 40 && \text{To solve for } y\text{, subtract 70 from both sides.} \end{aligned}$$

State Thus, $x = 70°$ and $y = 40°$.

Check Since $70° - 40° = 30°$ and $70° + 40° = 110°$, the results check.

3 Use systems to solve number-value problems.

EXAMPLE 3 ***Wedding Pictures*** A professional photographer offers two different packages for wedding pictures. Use the information in the figure to determine the cost of one 8×10-inch photograph and the cost of one 5×7-inch photograph.

Analyze

- Eight 8×10 and twelve 5×7 pictures cost \$133.
- Six 8×10 and twenty-two 5×7 pictures cost \$168.
- Find the cost of one 8×10 photograph and the cost of one 5×7 photograph.

Form Let x = the cost of one 8×10 photograph (in dollars), and let y = the cost of one 5×7 photograph (in dollars). We can use the fact that **Number · value = total value** to write equations that model the cost of each package. For the first package, the cost of eight 8×10 photos is $8 \cdot \$x = \$8x$, and the cost of twelve 5×7 photos is $12 \cdot \$y = \$12y$. For the second package, the cost of six 8×10 photos is $\$6x$, and the cost of twenty-two 5×7 photos is $\$22y$. To find x and y, we must write and solve two equations.

The cost of eight 8×10 photographs	plus	the cost of twelve 5×7 photographs	is	the cost of the first package.
$8x$	$+$	$12y$	$=$	133

The cost of six 8×10 photographs	plus	the cost of twenty-two 5×7 photographs	is	the cost of the second package.
$6x$	$+$	$22y$	$=$	168

Solve To find the cost of one 8×10 and one 5×7 photograph, we must solve the following system:

$$\begin{aligned} &\textbf{(1)} \\ &\textbf{(2)} \end{aligned} \begin{cases} 8x + 12y = 133 \\ 6x + 22y = 168 \end{cases}$$

Self Check 3

At a movie theater, 1 large popcorn and 2 medium drinks cost \$10.50. Another package offers 1 large popcorn and 3 medium drinks for \$13.00. Find the cost of one large popcorn and 1 medium drink.

Now Try **Problem 25**

Self Check 3 Answer
popcorn: \$5.50, medium drink: \$2.50

Teaching Example 3 A soccer picture package that includes one team photo and 8 trading cards cost \$15.00. Another package that includes two team photos and 12 trading cards cost \$25.00. Find the cost of one team photo and one trading card.
Answer:
team photo; \$5.00, trading card: \$1.25

We will use elimination to solve this system. To make the x-terms drop out, we multiply both sides of equation 1 by 3. Then we multiply both sides of equation 2 by -4, add the resulting equations, and solve for y:

$$\begin{aligned} 24x + 36y &= 399 && \text{This is } 3(8x + 12y) = 3(133). \\ -24x - 88y &= -672 && \text{This is } -4(6x + 22y) = -4(168). \\ \hline -52y &= -273 && \text{Add the terms, column by column. The } x\text{-terms drop out.} \\ y &= 5.25 && \text{Divide both sides by } -52. \text{ This is the cost of one } 5 \times 7 \text{ photograph.} \end{aligned}$$

To find x, we substitute 5.25 for y in equation 1 and solve for x:

$$\begin{aligned} 8x + 12\mathbf{y} &= 133 \\ 8x + 12(\mathbf{5.25}) &= 133 && \text{Substitute 5.25 for } y. \\ 8x + 63 &= 133 && \text{Do the multiplication.} \\ 8x &= 70 && \text{Subtract 63 from both sides.} \\ x &= 8.75 && \text{Divide both sides by 8. This is the cost of one } 8 \times 10 \text{ photograph.} \end{aligned}$$

State The cost of one 8×10 photo is \$8.75, and the cost of one 5×7 photo is \$5.25.

Check If the first package contains eight 8×10 and twelve 5×7 photographs, the value of the package is $8(\$8.75) + 12(\$5.25) = \$70 + \$63 = \$133$. If the second package contains six 8×10 and twenty-two 5×7 photographs, the value of the package is $6(\$8.75) + 22(\$5.25) = \$52.50 + \$115.50 = \$168$. The results check.

Caution! In this problem we are to find two unknowns, the cost of an 8×10 photo and the cost of a 5×7 photo. Remember to give both in the *State* step of the solution.

4 Use systems to find the break point.

Running a machine involves both *setup costs* and *unit costs*. Setup costs include the cost of preparing a machine to do a certain job. The costs to make one item are unit costs. They depend on the number of items to be manufactured, including costs of raw materials and labor.

Self Check 4

If in Example 4, the setup cost of the smaller machine is \$600 with a \$.30 cost to mill each plate, and the larger machine's setup is \$1,000 with \$0.22 to mill each plate, what would the break point be? 5,000

EXAMPLE 4 ***Manufacturing*** The setup cost of a machine that mills brass plates is \$750. After setup, it costs \$0.25 to mill each plate. Management is considering the purchase of a larger machine that can produce a plate at a cost of \$0.20 per plate. If the setup cost of the larger machine is \$1,200, how many plates would the company have to produce to make the purchase worthwhile?

Analyze We need to find the number of plates (called the **break point**) that will cost equal amounts to produce on either machine.

Form We can let c = the cost (in dollars) of milling p plates. If we call the machine currently in use machine 1 and the new, larger one machine 2, we can form two equations.

The cost of making p plates using machine 1	equals	the setup cost of machine 1	plus	the cost per plate of machine 1	times	the number of plates p to be made.
c	$=$	750	$+$	0.25	$\cdot$	p

The cost of making p plates using machine 2	equals	the setup cost of machine 2	plus	the cost per plate of machine 2	times	the number of plates p to be made.
c	$=$	1,200	$+$	0.20	$\cdot$	p

To find the break point, we must solve the system $\begin{cases} c = 750 + 0.25p \\ c = 1{,}200 + 0.20p \end{cases}$

Solve Since the costs are equal for the break point, we can use substitution to solve the system.

$$\begin{cases} c = 750 + 0.25p \\ c = 1{,}200 + 0.20p \end{cases}$$

$$750 + 0.25p = 1{,}200 + 0.20p \quad \text{Substitute } 750 + 0.25p \text{ for } c \text{ in the second equation.}$$

$$0.25p = 450 + 0.20p \quad \text{Subtract 750 from both sides.}$$

$$0.05p = 450 \quad \text{Subtract } 0.20p \text{ from both sides.}$$

$$p = 9{,}000 \quad \text{Divide both sides by 0.05.}$$

State Since the cost will be the same on either machine when 9,000 plates are milled, 9,000 is the break point.

Check We can check the result by substituting 9,000 for p in each equation of the system and verifying that 3,000 is the value of c in both cases.

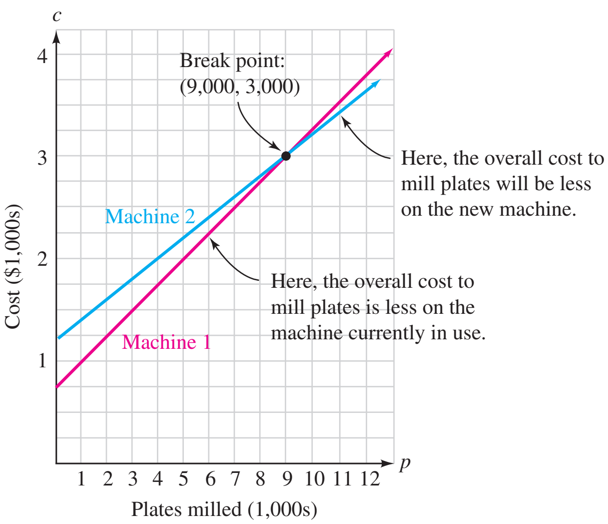
If we graph the two equations, we can illustrate the break point.

Machine 1
$c = 750 + 0.25p$

p	c
0	750
1,000	1,000
3,000	1,500

Machine 2
$c = 1{,}200 + 0.20p$

p	c
0	1,200
4,000	2,000
12,000	3,600



Now Try **Problem 31**

Teaching Example 4 Find the break point for a smaller machine that has a setup of $550 with a per-unit cost of $0.45 compared to a larger machine that has a setup cost of $1,250 and a per-unit cost of $0.38.
Answer:
10,000

5 Use systems to solve interest, uniform motion, and mixture problems.

To compare one-variable and two-variable approaches, we will solve the investment problem of Example 6 in Chapter 1 using two variables to find the unknown investment amounts.

Self Check 5

Parents saving for their child's college education decide to invest \$12,000. They put some of the money in a 3.75% certificate of deposit and the rest in a money market paying 4.25% annually. If the annual interest from the two accounts is \$475, how much did they invest in each account?

Now Try **Problem 39**

Self Check 5 Answer
\$7,000 at 3.75%, \$5,000 at 4.25%

Teaching Example 5 A student has \$15,000 to invest, some at 8% and the rest at 3.5%. If the annual interest earned from the two accounts is \$840, how much was invested in each account?
Answer:
\$7,000 at 8%, \$8,000 at 3.5%

EXAMPLE 5 ***Interest Income*** To protect against a major loss, a financial analyst suggested the following plan for a client who has \$50,000 to invest for 1 year.

1. Alco Development, Inc. Builds mini-malls. High yield: 12% per year. Risky!
2. Certificate of deposit (CD). Insured, safe. Low yield: 4.5% annual interest.

If the client puts some money in each investment and wants to earn \$3,600 in interest, how much should be invested at each rate?

Analyze A total of \$50,000 is invested at two different rates for 1 year. The total interest earned is \$3,600.

Form Let x = the number of dollars invested at 12% and y = the number of dollars invested at 4.5%. We will use the formula $I = Prt$ to determine that \$$x$ invested at 12% for 1 year will earn \$$0.12x$ and \$$y$ invested at 4.5% for 1 year will earn \$$0.045y$. This information is shown in the table.

	P ·	r ·	t =	I
Alco	x	0.12	1	$0.12x$
CD	y	0.045	1	$0.045y$
Total	50,000			3,600

One equation comes from this column. The other equation comes from this column.

Success Tip With a one-variable approach, we let x = the number of dollars invested at 12% and $50{,}000 - x$ = the number of dollars invested at 4.5%. With a two-variable approach, again we let x = the number of dollars invested at 12%, but then we let y = the number of dollars invested at 4.5%.

The facts of the problem give the following two equations.

The amount invested in Alco	plus	the amount invested in the CD	is	\$50,000.
x	$+$	y	$=$	50,000

The interest earned at 12%	plus	the interest earned at 4.5%	equals	the total interest earned.
$0.12x$	$+$	$0.045y$	$=$	3,600

Solve To find out how much was invested at each rate, we solve the following system:

$$\begin{array}{ll} \textbf{(1)} & \left\{\begin{array}{l} x + y = 50{,}000 \\ 0.12x + 0.045y = 3{,}600 \end{array}\right. \\ \textbf{(2)} & \end{array}$$

To solve this system by substitution, we can solve equation 1 for y:

$$\begin{aligned} x + y &= 50{,}000 \\ y &= 50{,}000 - x \qquad \text{This is the substitution equation.} \end{aligned}$$

Then we substitute $50{,}000 - x$ for y in equation 2 and solve for x.

$$\begin{aligned} 0.12x + 0.045\mathbf{y} &= 3{,}600 \\ 0.12x + 0.045(\mathbf{50{,}000 - x}) &= 3{,}600 && \text{Substitute } 50{,}000 - x \text{ for } y. \\ 120x + 45(50{,}000 - x) &= 3{,}600{,}000 && \text{To clear the equation of the decimals, multiply both sides by 1,000.} \\ 120x + 2{,}250{,}000 - 45x &= 3{,}600{,}000 && \text{Distribute the multiplication by 45.} \\ 75x &= 1{,}350{,}000 && \text{Combine like terms and subtract 2,250,000 from both sides.} \\ x &= 18{,}000 && \text{Divide both sides by 75. This is the amount that should be invested at 12\%.} \end{aligned}$$

To find y, we can substitute 18,000 for x in the substitution equation:

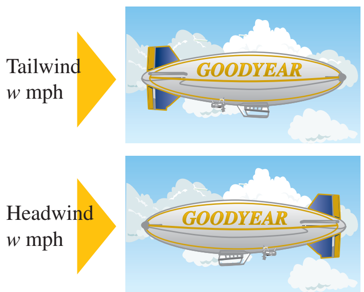
$$\begin{aligned} y &= 50{,}000 - \mathbf{x} \\ &= 50{,}000 - \mathbf{18{,}000} && \text{Substitute 18,000 for } x. \\ &= 32{,}000 && \text{This is the amount that should be invested at 4.5\%.} \end{aligned}$$

State \$18,000 should be invested at 12% and \$32,000 should be invested at 4.5%.

Check We note that \$18,000 plus \$32,000 equals the required amount of money invested. The annual interest on \$18,000 is 0.12(\$18,000) = \$2,160. The interest earned on \$32,000 is 0.045(\$32,000) = \$1,440. The total interest is \$2,160 + \$1,440 = \$3,600. The results check.

EXAMPLE 6 Blimps

The Spirit of America, one of the Goodyear blimps, flew 175 miles in 5 hours with the wind. The return trip took 7 hours flying against the wind. Find the speed of the blimp in still air and the speed of the wind.



Analyze Traveling with the wind, the speed of the blimp will be faster than it would be in still air. Traveling against the wind, the speed of the blimp will be slower than it would be in still air.

Form Let s = the speed of the blimp (in mph) in still air and w = the speed of the wind (in mph). Then the speed of the blimp flying with the wind is $s + w$ and the speed of the blimp flying against the wind is $s - w$. Using the formula $d = rt$, we find that $5(s + w)$ represents the distance traveled with the wind and $7(s - w)$ represents the distance traveled against the wind. This information is shown in the table.

	r	$\cdot\ t =$	d
With the wind	$s + w$	5	$5(s + w)$
Against the wind	$s - w$	7	$7(s - w)$

Enter this information first.

Each of these expressions for distance traveled is equal to 175.

Self Check 6

A boat traveled 63 miles downstream in 3 hours. The return trip took 7 hours going upstream. Find the speed of the boat in still water and the speed of the current.

Now Try **Problem 45**

Self Check 6 Answer
boat: 15 mph, current: 6 mph

Teaching Example 6 A boat traveled 40 miles downstream in 2 hours. The return trip upstream took 5 hours. Find the speed of the boat in still water and the speed of the current.
Answer:
boat: 14 mph, current: 6 mph

Since each trip is 175 miles long, the information in the Distance column of the table can be used to form two equations in two variables. To write each equation in standard form, we use the distributive property.

$$\begin{cases} 5(s + w) = 175 \\ 7(s - w) = 175 \end{cases} \xrightarrow{\text{Distribute}} \begin{cases} 5s + 5w = 175 & \textbf{(1)} \\ 7s - 7w = 175 & \textbf{(2)} \end{cases}$$

Solve Since the coefficients of w have opposite signs, we will eliminate w. To do this, we will create terms of $35w$ and $-35w$ by multiplying both sides of the equation 1 by 7 and both sides of the equation 2 by 5.

$$\begin{aligned} 35s + 35w &= 1{,}225 && \text{This is } 7(5s + 5w) = 7(175). \\ \underline{35s - 35w} &\underline{= 875} && \text{This is } 5(7s - 7w) = 5(175). \\ 70s &= 2{,}100 \\ s &= 30 && \text{Divide both sides by 70. This is the speed of the blimp in still air.} \end{aligned}$$

To find w, we will substitute 30 for s in equation 1.

$$\begin{aligned} 5s + 5w &= 175 \\ 5(30) + 5w &= 175 && \text{Substitute 30 for } s. \\ 150 + 5w &= 175 && \text{Do the multiplication.} \\ 5w &= 25 && \text{Subtract 150 from both sides.} \\ w &= 5 && \text{Divide both sides by 5. This is the speed of the wind.} \end{aligned}$$

State The speed of the blimp in still air is 30 mph and the speed of the wind is 5 mph.

Check The blimp's speed traveling with a 5-mph wind will be $30 + 5 = 35$ mph. In 5 hours, it will travel $35 \cdot 5 = 175$ miles. The blimp's speed traveling against a 5-mph wind will be $30 - 5 = 25$ mph. In 7 hours, it will travel $25 \cdot 7 = 175$ miles. The results check.

Self Check 7

How many pounds of peanuts (selling for \$4 per pound) and how many pounds of M&M's (selling for \$3.52 per pound) must be combined to get a 12-pound mixture worth \$3.75 per pound?

Now Try **Problem 51**

Self Check 7 Answer

peanuts: $5\frac{3}{4}$ lb, M&M's: $6\frac{1}{4}$ lb

Teaching Example 7 How many pounds of cashews (selling for \$5.25 per pound) and how many pounds of peanuts (selling for \$4.00 per pound) must be combined to get a 15-pound mixture worth \$4.50 per pound?
Answer:
cashews: 6 lb, peanuts: 9 lb

EXAMPLE 7 ***Popcorn*** A tin of jalapeño-flavored popcorn sells for \$36, while the same size tin of cheddar cheese–flavored popcorn sells for \$24. How many tins of each type of popcorn should be used to create 10 tins of a jalapeño–cheddar mix that can be sold for \$27 per tin?

Analyze We will use a two-variable approach to solve this dry mixture problem.

Form Let x = the number of tins of jalapeño popcorn and y = the number of tins of cheddar cheese popcorn that should be mixed. The value of the mixture and the value of each of its components are given by

Amount · price = total value

Thus, the value of x tins of jalapeño popcorn is $\$36x$ and the value of y tins of cheddar cheese popcorn is $\$24y$. The sum of these values is also equal to the total value of the final mixture that is $10 \cdot \$27$ or \$270. This information is shown in the table.

	Amount ·	Price =	Total value
Jalapeño popcorn	x	36	$36x$
Cheddar cheese popcorn	y	24	$24y$
Mixture	10	27	10(27)

The facts in the table give the following equations:

The number of tins of jalapeño popcorn	plus	the number of tins of cheddar cheese popcorn	equals	10.
x	$+$	y	$=$	10

The value of the jalapeño popcorn	plus	the value of the cheddar cheese popcorn	equals	the value of the mixture.
$36x$	$+$	$24y$	$=$	$10(27)$

Solve To find how many tins of each popcorn are needed, we solve the following system:

$$\begin{cases} x + y = 10 \\ 36x + 24y = 270 \end{cases} \quad \text{Multiply: } 10(27) = 270.$$

We will use elimination to solve this system. To make the y-terms drop out, we multiply both sides of the first equation of the system by -24 and add the resulting equations to solve for x:

$$\begin{aligned} -24x - 24y &= -240 && \text{This is } -24(x + y) = -24(10). \\ 36x + 24y &= 270 \\ \hline 12x &= 30 && \text{Add like terms, column by column.} \\ x = \frac{30}{12} &= 2.5 && \text{Divide both sides by 12 and simplify. This is the number of tins of jalapeño popcorn needed.} \end{aligned}$$

To find y, we substitute 2.5 for x in the first equation of the system and solve for y:

$$\begin{aligned} x + y &= 10 \\ 2.5 + y &= 10 && \text{Substitute 2.5 for } x. \\ y &= 7.5 && \text{Subtract 2.5 from both sides. This is the number of tins of cheddar cheese popcorn needed.} \end{aligned}$$

State To obtain 10 tins of jalapeño–cheddar popcorn, 2.5 tins of japaleño and 7.5 tins of cheddar cheese popcorn should be combined.

Check When 2.5 tins and 7.5 tins are combined, the result is 10 tins. The 2.5 tins of jalapeño popcorn are valued at 2.5(\$36) = \$90 and the 7.5 tins of cheddar cheese popcorn are valued at 7.5(\$24) = \$180. The sum of those values, \$90 + \$180 = \$270, is the same as the value of the mixture, 10(\$27) = \$270. The results check.

EXAMPLE 8 ***Water Treatment*** A technician determines that 100 fluid ounces of a 15% muriatic acid solution needs to be added to the water in a swimming pool to kill a growth of algae. If the technician has 5% and 20% muriatic solutions on hand, how many ounces of each must be combined to create the 15% solution?

Analyze We will use a two-variable approach to solve this liquid mixture problem. We need to find the number of ounces of a 5% solution and the number of ounces of a 20% solution that must be combined to obtain 100 ounces of a 15% solution.

Form Let x = the number of ounces of the 5% solution and let y = the number of ounces of the 20% solution that are to be mixed. The amount of pure muriatic acid in each solution is given by

Amount of solution · strength = amount pure muriatic acid

Self Check 8

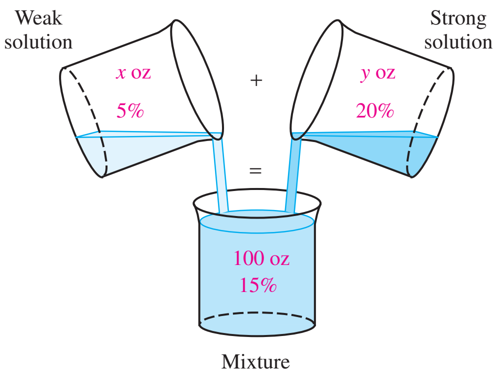
A nutritionist suggested that 20 fluid ounces of 4% butterfat milk should be included in an underweight toddler's diet. If the mother has cream that is approximately 22% butterfat and 2% milk, how many ounces of each must be combined to create the 4% solution?

Now Try **Problem 57**

Self Check 8 Answer
cream: 2 oz, 2% milk: 18 oz

Teaching Example 8 A container that is partially filled with 4% butterfat milk is to be mixed with 1% milk to obtain 36 liters of 2% butterfat milk. How many liters of the 4% and the 1% milk must be combined to create the 2% milk?
Answer:
4% milk: 12 L, 1% milk: 24 L

Thus, the amount of muriatic acid in the 5% solution is $0.05x$ ounces, and the amount of muriatic acid in the 20% solution is $0.20y$ ounces. The sum of these amounts is also the amount of muriatic acid in the final mixture, which is 15% of 100 ounces or $0.15(100)$. This information is shown in the table.



	Amount $\cdot$	**Strength** $=$	**Amount of muriatic acid**
Weak	x	0.05	$0.05x$
Strong	y	0.20	$0.20y$
Mixture	100	0.15	$0.15(100)$

One equation comes from the information in this column. (Amount column)

Another equation comes from the information in this column. (Amount of muriatic acid column)

The facts in the table give the following equations:

The number of ounces of 5% solution	plus	the number of ounces of 20% solution	equals	the total number of ounces in the 15% mixture.
x	$+$	y	$=$	100
The number of ounces of acid in the 5% solution	plus	the number of ounces of acid in the 20% solution	is	the number of ounces of acid in the 15% mixture.
$0.05x$	$+$	$0.20y$	$=$	$0.15(100)$

Solve To find out how many ounces of each are needed, we solve the following system:

$$\begin{array}{ll} \textbf{(1)} & \begin{cases} x + y = 100 \\ 0.05x + 0.20y = 15 \end{cases} \\ \textbf{(2)} & \end{array}$$

Multiply: $0.15(100) = 15$.

To solve this system by substitution, we can solve the first equation for y:

$$x + y = 100$$
$$y = 100 - x \quad \text{This is the substitution equation.}$$

Then we substitute $100 - x$ for y in equation 2 and solve for x.

$0.05x + 0.20y = 15$	
$0.05x + 0.20(100 - x) = 15$	Substitute $100 - x$ for y.
$5x + 20(100 - x) = 1{,}500$	To clear the equation of decimals, multiply both sides by 100.
$5x + 2{,}000 - 20x = 1{,}500$	Distribute the multiplication by 20.
$-15x = -500$	Combine like terms and subtract 2,000 from both sides.
$x = \frac{-500}{-15}$	To solve for x, divide both sides by -15.
$x = \frac{100}{3}$	Simplify. This is the number of ounces of the 5% acid solution that is needed.

To find y, we can substitute $\frac{100}{3}$ for x in the substitution equation:

$y = 100 - x$	
$= 100 - \frac{100}{3}$	Substitute $\frac{100}{3}$ for x.
$= \frac{200}{3}$	To subtract, think: $\frac{300}{3} - \frac{100}{3}$. This is the number of ounces of the 20% acid solution that is needed.

State To obtain 100 ounces of a 15% solution, the technician must mix $\frac{100}{3}$ or $33\frac{1}{3}$ ounces of the 5% solution with $\frac{200}{3}$ or $66\frac{2}{3}$ ounces of the 20% solution.

Check We note that $33\frac{1}{3}$ ounces of solution plus $66\frac{2}{3}$ ounces of solution equals the required 100 ounces of solution. The $33\frac{1}{3} = \frac{100}{3}$ ounces of 5% solution contains $0.05\left(\frac{100}{3}\right) = \frac{5}{3}$ ounces of muriatic acid, and the $66\frac{2}{3} = \frac{200}{3}$ ounces of 20% solution contains $0.20\left(\frac{200}{3}\right) = \frac{40}{3}$ ounces of muriatic acid—a total of $\frac{5}{3} + \frac{40}{3} = \frac{45}{3}$ or 15 ounces of muriatic acid. The 100 ounces of the 15% mixture contains $0.15(100) = 15$ ounces of acid. The results check.

ANSWERS TO SELF CHECKS

1. iPod: \$249, iHome: \$99 **2.** 27°, 63° **3.** popcorn: \$5.50, medium drink: \$2.50 **4.** 5,000 **5.** \$7,000 at 3.75%, \$5000 at 4.25% **6.** boat: 15 mph, current: 6 mph **7.** peanuts: $5\frac{3}{4}$ lb, M&M's: $6\frac{1}{4}$ lb **8.** cream: 2 oz, 2% milk: 18 oz

SECTION 3.3 STUDY SET

VOCABULARY

Fill in the blanks.

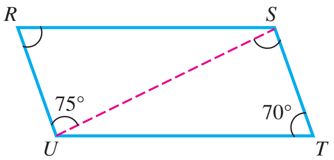
1. A parallelogram is a four-sided figure with two pairs of __parallel__ sides.

2. Suppose a hammer can be manufactured in two different ways. The number of hammers that will cost equal amounts to produce either way is called the __break__ point.

CONCEPTS

3. a. Refer to the parallelogram below. Find the measure of $\angle SRU$ and $\angle TSU$. 70°, 75°

b. Fill in the blank: $\angle SUR$ and $\angle TSU$ are called __alternate__ __interior__ angles.



Selected exercises available online at
www.webassign.net/brookscole

4. A company charges a $75 setup fee plus $5.25 per shirt to silkscreen a design on specialty t-shirts. Write an equation which gives the cost of purchasing x shirts. $c = 5.25x + 75$

5. In still water, a person swims at the rate of x mph. Find the speed of the swimmer for each of the following situations.

a. With the current $x + c$

b. Against the current $x - c$

6. a. Write an expression that represents the total value of x ounces of ginseng tea that costs $32 per pound. $\$32x$

b. Write an expression that represents the amount of hydrochloric acid in x gallons of a 3% hydrochloric acid solution. $0.03x$ gal

NOTATION

7. Write each percent as a decimal.

a. 6% 0.06 **b.** 4.8% 0.048 **c.** $13\frac{1}{2}\%$ 0.135

8. What is the formula that finds

a. Simple interest $I = Prt$

b. Distance traveled $d = rt$

GUIDED PRACTICE

Complete each table and write a system of two equations that can be used to find x and y. DO NOT SOLVE THE SYSTEM.
See Examples 5–8.

▶ **9.** INVESTMENTS A total of $25,000 was invested in two accounts for 1 year and earned a total of $1,050 in interest.

	Principal ·	Rate ·	Time =	Interest
Township Bank	x	0.05	1	$0.05x$
Ameritech Savings	y	0.04	1	$0.04y$
Total	25,000			1,050

$$\begin{cases} x + y = 25{,}000 \\ 0.05x + 0.04y = 1{,}050 \end{cases}$$

▶ **10.** PHYSICAL FITNESS A jogger and cyclist started at the same point and traveled for 2 hours in opposite directions until they were 42 miles apart. The cyclist traveled 10 mph faster than the jogger.

	Rate ·	Time =	Distance
Jogger	x	2	$2x$
Cyclist	y	2	$2y$
Total			42

$$\begin{cases} 2x + 2y = 42 \\ y = x + 10 \end{cases}$$

▶ **11.** CANDY A company combines dark chocolate (selling for $13.90 per pound) with white chocolate (selling for $5.10 per pound) to get 6 pounds of a mixture that will sell for $10.25 per pound.

	Amount ·	Value =	Total value
Dark chocolate	x	13.90	$13.90x$
White chocolate	y	5.10	$5.10y$
Mixture	6	10.25	61.50

$$\begin{cases} x + y = 6 \\ 13.90x + 5.10y = 61.50 \end{cases}$$

▶ **12.** DISINFECTANTS A 1% bleach solution is to be mixed with a 5% bleach solution to obtain 15 ounces of a 3% bleach solution.

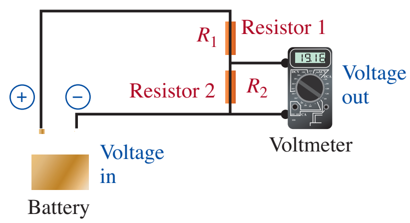
	Amount ·	Strength =	Amount of bleach
Weak	x	0.01	$0.01x$
Strong	y	0.05	$0.05y$
Mix	15	0.03	0.45

$$\begin{cases} x + y = 15 \\ 0.01x + 0.05y = 0.45 \end{cases}$$

APPLICATIONS

Write a system of two equations in two variables to solve each problem.

▶ **13.** ELECTRONICS In the illustration, two resistors in the voltage divider circuit have a total resistance of 1,375 ohms. To provide the required voltage, R_1 must be 125 ohms greater than R_2. Find both resistances. 750 ohms, 625 ohms.



14. DESSERTS A slice of Mrs. Smith's apple pie and one scoop of Häagen-Dazs vanilla bean ice cream totals 600 calories. The pie has 20 more calories than the ice cream. Find the number of calories in each. pie: 310, ice cream: 290

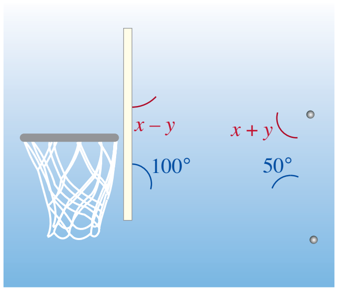
15. AREA CODES The entire state of Montana has just one telephone area code. The same is true for Idaho. The sum of their area codes is 614 and the difference is 198, and Montana has the numerically larger one. Find the area code of each of these states. Montana: 406, Idaho: 208

▶ **16.** HIKING The Pacific Crest Trail runs from the U.S. border with Mexico to its border with Canada. The Appalachian Trail extends between Springer Mountain in Georgia and Mount Katahdin in Maine. The sum of the lengths of the trails is 4,824 miles, the difference is 476 miles, and the Pacific Crest Trail is the longer. Find the length of each trail. Pacific Crest Trail: 2,650 mi, Appalachian Trail: 2,174 mi

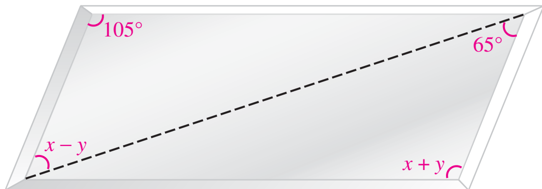
17. AVALANCHES For the 2005–2006 snow season, the total number of avalanche fatalities in the United States and Canada was 32. If the number in the United States was three times greater than the number in Canada, how many avalanche fatalities were there in each country? (Source: Avalanche.org) Canada: 8, United States: 24

▶ **18.** CLOTHING STORES During the years 2005 and 2006, the Abercrombie and Fitch Company opened a total of 167 new stores. The number of new stores opened in 2006 was 22 less than twice the number opened in 2005. Find the number of new stores that the company opened each of those years. (Source: International Council of Shopping Centers) 2005: 63, 2006: 104

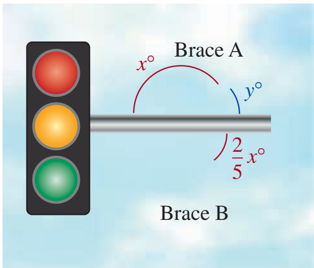
19. BRACING The bracing of a basketball backboard forms a parallelogram. Find the unknown degree measures represented by x and y. 75°, 25°



20. DECORATIVE MIRRORS An interior designer is ordering a mirror that is the shape of a parallelogram as shown below. Find the unknown degree measures represented by x and y. 85°, 20°

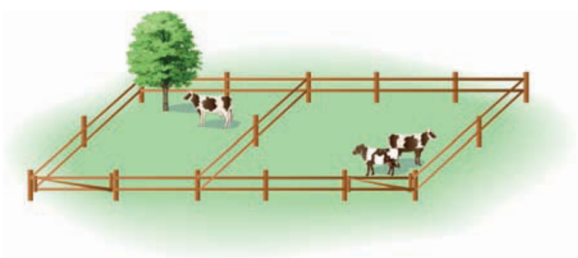


▶ **21.** TRAFFIC SIGNALS In the illustration, braces A and B are perpendicular. Find the values of x and y. 150°, 30°



22. GEOMETRY An acute angle is an angle with measure less than 90°. In a right triangle, the measure of one acute angle is 15° greater than two times the measure of the other acute angle. Find the measure of each acute angle. 65°, 25°

▶ **23.** FENCING A FIELD The perimeter of a rectangular field is surrounded by 72 meters of fencing. If the field is partitioned into two parts as shown, a total of 88 meters of fencing is required. Find the dimensions of the field. 16 m by 20 m



24. NEW YORK CITY The triangular-shaped Flatiron Building in Manhattan has a perimeter of 499 feet at its base. It is bordered on each side by a street. The 5th Avenue front of the building is 198 feet long. The Broadway front is 43 feet more than twice as long as the East 22nd Street front. Find the length of the Broadway front and East 22nd Street front. (Source: New York Public Library)
Broadway: 215 ft, East 22nd Street: 86 ft

© Bill Ross/Corbis

25. ADVERTISING Use the information in the ad to find the cost of a 15-second and a 30-second radio commercial on radio station KLIZ.
15 sec: $475, 30 sec: $800

ADVERTISE YOUR COMPANY ON THE RADIO
KLIZ
1250 AM

Plan 1:
Four 30-second spots, six 15-second spots
Cost: $6,050

Plan 2:
Three 30-second spots, five 15-second spots
Cost: $4,775

26. TEMPORARY HELP A law firm hired several workers to help finish a large project. From the following billing records, determine the daily fee charged by the employment agency for a clerk-typist and for a computer programmer.
Clerk-typist: $105, computer programmer: $185

TEMPORARY EMPLOYMENT, INC.
We meet your employment needs!
Billed to: Archer Law Offices Attn: B. Kinsell

Day	Position/Employee Name	Total cost
Mon. 3/22	*Clerk-typists:* K. Amad, B. Tran, S. Smith *Programmers:* T. Lee, C. Knox	$685
Tues. 3/23	*Clerk-typists:* K. Amad, B. Tran, S. Smith, W. Morada *Programmers:* T. Lee, C. Knox, B. Morales	$975

27. PRODUCTION PLANNING A manufacturer builds racing bikes and mountain bikes, with the per unit manufacturing costs shown in the table. The company has budgeted $26,150 for materials and $31,800 for labor. How many bicycles of each type can be built? 85 racing bikes, 120 mountain bikes

Model	Cost of materials	Cost of labor
Racing	$110	$120
Mountain	$140	$180

28. FARMING A farmer keeps some animals on a strict diet. Each animal is to receive 15 grams of protein and 7.5 grams of carbohydrates. The farmer uses two food mixes, with nutrients as shown in the table. How many grams of each mix should be used to provide the correct nutrients for each animal? 50 g of A, 60 g of B

Mix	Protein	Carbohydrates
Mix A	12%	9%
Mix B	15%	5%

29. CONCERTS According to *StubHub.com,* in 2006, two tickets to a Rolling Stones concert and two tickets to a Jimmy Buffet concert cost, on average, a total of $792. At those prices, four tickets to see the Stones and two tickets to see Jimmy Buffet cost $1,320. What was the average cost of a Rolling Stones ticket and a Jimmy Buffet ticket in 2006? Rolling Stones: $264, Jimmy Buffet: $132

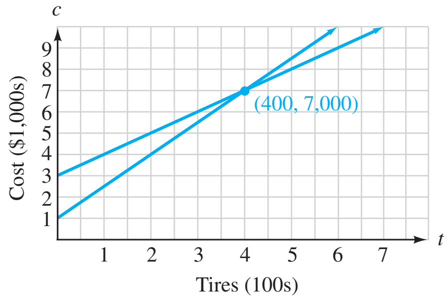
30. MEASUREMENT A *furlong* is a measure of distance and is used to express the length of certain horse races. A *fathom* is a measure of distance and is used to express depths of water. Four furlongs and five fathoms is a total of 2,670 feet. Five furlongs and four fathoms is a total of 3,324 feet. Find the length of a furlong and a fathom. furlong: 660 ft, fathom: 6 ft

31. MAKING TIRES A company has two molds to form tires. One mold has a setup cost of $1,000 and the other has a setup cost of $3,000. The cost to make each tire with the first mold is $15, and the cost to make each tire with the second mold is $10.

a. Find the break point. 400 tires

b. Check your result by graphing both equations on the coordinate system on the next page.

c. If a production run of 500 tires is planned, determine which mold should be used.
the second mold



32. CHOOSING A FURNACE A high-efficiency 90+ furnace can be purchased for $2,250 and costs an average of $824 per year to operate in Chicago, Illinois. An 80+ furnace can be purchased for only $1,710, but it costs $932 per year to operate.
   a. Find the break point. 5 yr
   b. If you intended to live in Chicago for 4 years, which furnace would you choose? 80+

▶ 33. PUBLISHING A printer has two presses. The older press has a setup cost of $210 and can print the pages of a certain book for $5.98. The newer press has a setup cost of $350 and can print the pages of the same book for $5.95.
   a. Find the break point. $4{,}666\frac{2}{3}$ books
   b. If the publisher has advanced orders for 5,100 copies of the book, which press should be used? the newer press

34. COSMETOLOGY A beauty shop specializing in permanents has fixed costs of $2,101.20 per month. The owner estimates that the cost for each permanent is $23.60, which covers labor, chemicals, and electricity. If her shop can give as many permanents as she wants at a price of $44 each, how many must be given each month to break even? 103

35. RECORDING COMPANIES Three people invest a total of $105,000 to start a recording company that will produce reissues of classic jazz. Each release will be a set of 3 CDs that will retail for $45 per set. If each set can be produced for $18.95, how many sets must be sold for the investors to make a profit? 4,031

▶ 36. PRODUCTION PLANNING A paint manufacturer can choose between two processes for manufacturing house paint, with monthly costs as shown in the table. Assume that the paint sells for $18 per gallon.

Process	Fixed costs	Unit cost (per gallon)
A	$32,500	$13
B	$80,600	$5

   a. Find the break even point where production costs equal revenue earned for process A.
   6,500 gal per month
   b. Find the break even point where production costs equal revenue earned for process B.
   6,200 gal per month
   c. If expected sales are 7,000 gallons per month, which process should the company use?
   B

37. MANUFACTURING A manufacturer of automobile water pumps is considering retooling for one of two manufacturing processes, with monthly fixed costs and unit costs as indicated in the table. Each water pump can be sold for $50.

Process	Fixed costs	Unit cost
A	$12,390	$29
B	$20,460	$17

   a. Find the break even point where production costs equal revenue earned for process A.
   590 units per month
   b. Find the break even point where production costs equal revenue earned for process B.
   620 units per month
   c. If expected sales are 550 per month, which process should be used?
   A (smaller loss)

▶ 38. SALARY OPTIONS A sales clerk can choose from two salary plans:
   1. a straight 7% commission
   2. $150 + 2% commission

   How much would the clerk have to sell for each plan to produce the same monthly paycheck?
   $3,000

39. INVESTMENT CLUBS Part of $8,000 was invested by an investment club at 10% interest and the rest at 12%. If the annual income from these investments is $900, how much was invested at each rate?
$3,000 at 10%, $5,000 at 12%

▶ 40. RETIREMENT INCOME A retired couple invested part of $12,000 at 6% interest and the rest at 7.5%. If their annual income from these investments is $810, how much was invested at each rate?
$6,000 at 6%, $6,000 at 7.5%

41. INVESTING A woman invested some money in a credit union paying 5% annual simple interest and three times as much in a money market account paying 4.25% annual simple interest. If she earned $1,420 interest in one year, how much did she invest in each account?
credit union: $8,000, money market: $24,000

42. IRA ACCOUNTS A teacher started an Individual Retirement Account (IRA) by investing a total of \$30,500 in two municipal bond funds, one paying 5% annual interest and the other paying $5\frac{1}{2}$% annual interest. At the end of the year, the funds earned a total of \$1,615 in interest. How much did the teacher invest in each fund?
\$12,500 at 5%, \$18,000 at $5\frac{1}{2}$%

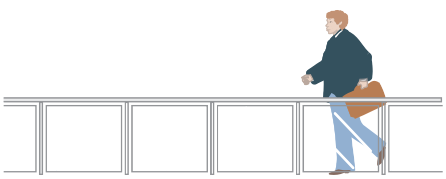
43. CREDIT CARDS A couple stopped using their VISA credit card charging 1.5% per month interest and their Robinsons-May credit card charging 1.75% per month interest because they had built up a combined debt of \$16,500 on the two cards. For 1 month, they made no purchases or payments on the accounts. If the total amount of interest the credit cards accumulated during the month was \$259.25, what amount did they owe on each card when they stopped using them?
VISA: \$11,800, Robinsons-May: \$4,700

44. LOANS A student had a car loan charging 0.75% interest per month and a tuition loan charging 0.5% interest per month. How much did he owe on each account if he paid a total of \$95.50 monthly interest on a total debt of \$14,200?
car loan: \$9,800, tuition loan: \$4,400

45. AVIATION The jet stream is a wind current that flows across the United States from west to east. Flying with the jet stream, an airplane flew 2,700 miles in 4.5 hours. Against the same wind, the return trip took 6 hours. Find the speed of the plane in still air and the speed of the jet stream.
525 mph, 75 mph

46. SALMON It takes a salmon 40 minutes to swim 10,000 feet upstream and 8 minutes to swim that same portion of a river downstream. Find the speed of the salmon in still water and the speed of the current.
750 ft/min, 500 ft/min

47. AIRPORT WALKWAYS A man walks at a steady pace as he steps onto a moving walkway. It takes him 40 seconds to reach the end, 320 feet away. If he walks at the same rate against the flow of the walkway, it would take him 80 seconds to reach the end. Find his rate of walking and the rate of the moving walkway.
walking: 6 ft per sec, moving walkway: 2 ft per sec

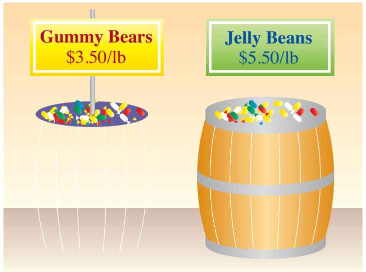


48. SNOWMOBILING A man rode a snowmobile at the rate of 20 mph and then skied cross country at the rate of 4 mph. During the 6-hour trip, he traveled 48 miles. How long did he snowmobile, and how long did he ski?
snowmobile: 1.5 hr, ski: 4.5 hr

49. JET SKIS A Jet Ski rider can travel 10 miles against the current of the lower Mississippi River in $\frac{1}{2}$ hour and make the return trip with the current in $\frac{1}{3}$ hour. Find the speed of the Jet Ski in still water and the speed of the current.
25 mph, 5 mph

50. ROLLERBLADING An in-line skater headed west at the rate of 6 mph. One hour later, a moped rider left the same spot and headed west on the same road at 30 mph. How long will it take the moped rider to catch the skater?
$\frac{1}{4}$ hr

51. MIXING CANDY How many pounds of each candy shown must be mixed to obtain 60 pounds of candy that would be worth \$4 per pound?
gummy bears: 45 lb, jelly beans: 15 lb



52. GASOLINE A truck owner drove his pickup to a service station to fill the nearly empty 24-gallon gas tank. If the truck runs on 89-octane gasoline, but the station only sells 87-octane and 93-octane gas, how many gallons of each should be pumped to fill the tank with an 89-octane blend?
87-octane: 16 gal, 93-octane: 8 gal

53. MIXING COFFEE How many pounds of regular coffee (selling for \$4 per pound) and how many pounds of Kona coffee (selling for \$11.50 per pound) must be combined to get 20 pounds of a mixture worth \$6 per pound?
regular: $14\frac{2}{3}$ lb, Kona: $5\frac{1}{3}$ lb

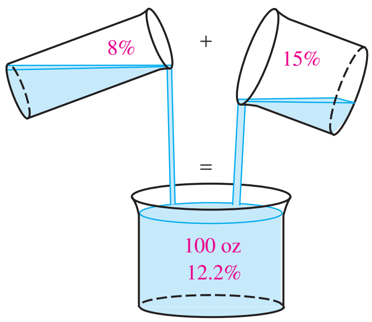
54. ROOM FRESHENER A florist sells mixtures of dried, fragrant plant material that provides a gentle natural scent for houses. She wants to mix lilac (that sells for \$18.25 a pound) with lavender (that sells for \$12.25 a pound) to create 30 pounds of a blend that sells for \$15 a pound. How many pounds of each should the florist use?
lilac: $13\frac{3}{4}$ lb, lavender: $16\frac{1}{4}$ lb

55. CONFETTI How many pounds of \$14.50-per-pound small flake confetti should be mixed with \$24.50-per-pound mylar confetti stars to obtain one ton of a confetti mix that would be worth \$20 per pound? (*Hint:* How many pounds equal one ton?)
small flake: 900 lb, mylar: 1,100 lb

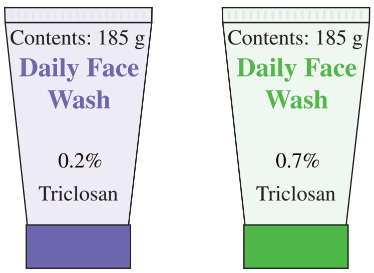
56. BATH SALTS The owner of a kitchen and bath store wants to combine \$7.92-per-pound eucalyptus bath salt with 88¢-per-pound Epsom salt to create 40 pounds of a \$2.64-per-pound bath salt mix. How many pounds of each should be used?
eucalyptus salt: 10 lb, Epsom salt: 30 lb

57. ANTIFREEZE How many pints of a 10% antifreeze solution and how many pints of a 40% antifreeze solution must be mixed to obtain 24 pints of a 30% solution?
10%: 8 pints, 40%: 16 pints

▶ **58.** MIXING SOLUTIONS How many ounces of the two alcohol solutions must be mixed to obtain 100 ounces of a 12.2% solution?
40 oz of 8% solution, 60 oz of 15% solution



59. DERMATOLOGY Tests of an antibacterial face wash cream showed that a mixture containing 0.3% Triclosan (active ingredient) gave the best results. How many grams of cream from each tube shown in the illustration should be used to make an equal-size tube of the 0.3% cream?
148 g of the 0.2%, 37 g of the 0.7%



▶ **60.** SALADS A chef wants to make 1 gallon (128 ounces) of a 50% vinegar-to-oil salad dressing. He only has pure vinegar and a mild 4% vinegar-to-oil salad dressing on hand. How many ounces of each should he mix to make the desired dressing?
4% dressing: $66\frac{2}{3}$ oz, pure vinegar: $61\frac{1}{3}$ oz

WRITING

61. Write a problem that can be solved by solving the system:

$$\begin{cases} x + y = 36 \\ \$1.29x + \$2.29y = \$72.44 \end{cases}$$

62. Write a problem to fit the information given in the table.

	Ounces ·	Strength =	Amount of insecticide
Weak	x	0.02	$0.02x$
Strong	y	0.10	$0.10y$
Mixture	80	0.07	0.07(80)

63. What is a *break point?* Give an example.

64. A woman paid \$219 for two blouses and four pairs of pants. If we let x = the cost of a blouse and y = the cost of a pair of pants, an equation modeling the purchase is $2x + 4y = 219$. Explain why there is not enough information to determine the cost of a blouse or the cost of a pair of pants.

65. To solve mixture problems, do you prefer the one-variable or two-variable solution strategy? Explain why.

66. Write a system of two equations in two variables to attempt to solve the following problem. Then explain why the problem has no solution.

How many gallons of a 20% salt solution and how many gallons of a 30% salt solution should be mixed to obtain 10 gallons of a 50% salt solution?

REVIEW

Fill in the blanks.

67. A rational number is any number that can be written as a fraction with an integer numerator and a nonzero integer denominator.

68. The coefficient of the term $-8c$ is -8.

69. An equation that is true for all values of its variable is called an identity.

70. The volume of a three-dimensional geometric solid is the amount of space it encloses.

71. If a triangle has two sides with equal measures, it is called an isosceles triangle.

72. The reciprocal of $\frac{5}{9}$ is $\frac{9}{5}$.

Objectives

1. Determine whether an ordered triple is a solution of a system.
2. Solve systems of three linear equations in three variables.
3. Solve systems of equations with missing variable terms.
4. Identify inconsistent systems and dependent equations.

SECTION 3.4 Solving Systems of Equations in Three Variables

In previous sections, we solved systems of linear equations in two variables. We will now extend this discussion to consider systems of linear equations in *three* variables.

1 Determine whether an ordered triple is a solution of a system.

The equation $x - 5y + 7z = 10$, where each variable is raised to the first power, is an example of a linear equation in three variables. In general, we have the following definition.

Standard (General) Form

A **linear equation in three variables** is an equation that can be written in the form

$$Ax + By + Cz = D$$

where A, B, C, and D are real numbers and A, B, and C are not all 0.

A solution of a linear equation in three variables is an **ordered triple** of numbers of the form (x, y, z) whose coordinates satisfy the equation. For example, $(2, 0, 1)$ is a solution of $x + y + z = 3$ because a true statement results when we substitute 2 for x, 0 for y, and 1 for z: $2 + 0 + 1 = 3$.

A **solution of a system of three linear equations** in three variables is an ordered triple that satisfies each equation of the system.

Self Check 1

Determine whether $(6, -3, 1)$ is a solution of the system:

$$\begin{cases} x - y + z = 10 \\ x + 4y - z = -7 \\ 3x - y + 4z = 24 \end{cases}$$ no

Now Try **Problem 11**

Teaching Example 1 Determine whether $(-2, 3, -1)$ is a solution of the system: $\begin{cases} 3x - y + 2z = -11 \\ x + 2y - 3z = 7 \\ 3x + 2y + z = 1 \end{cases}$

Answer: no

EXAMPLE 1 Determine whether $(-4, 2, 5)$ is a solution of the system:

$$\begin{cases} 2x + 3y + 4z = 18 \\ 3x + 4y + z = 1 \\ x + y + 3z = 13 \end{cases}$$

Strategy We will substitute the x-, y-, and z-coordinates of $(-4, 2, 5)$ for the corresponding variables in each equation of the system.

WHY If each equation is satisfied by the x-, y-, and z-coordinates, the ordered triple is a solution of the system.

Solution

We substitute -4 for x , 2 for y, and 5 for z in each equation.

The first equation

$$\begin{aligned} 2x + 3y + 4z &= 18 \\ 2(-4) + 3(2) + 4(5) &\stackrel{?}{=} 18 \\ -8 + 6 + 20 &\stackrel{?}{=} 18 \\ 18 &= 18 \quad \text{True} \end{aligned}$$

The second equation

$$\begin{aligned} 3x + 4y + z &= 1 \\ 3(-4) + 4(2) + 5 &\stackrel{?}{=} 1 \\ -12 + 8 + 5 &\stackrel{?}{=} 1 \\ 1 &= 1 \quad \text{True} \end{aligned}$$

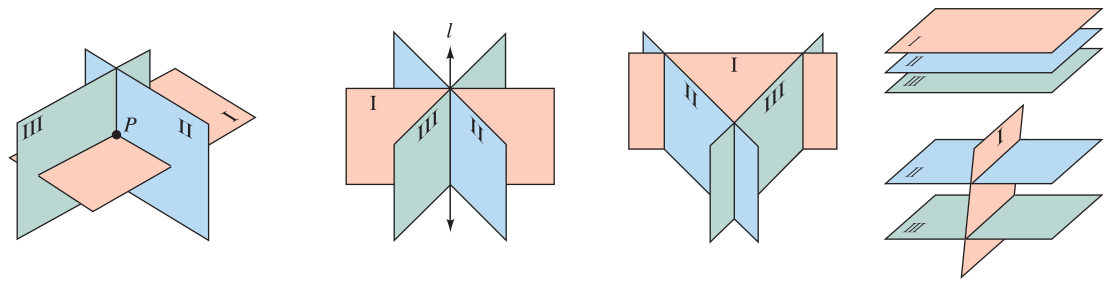
The third equation

$$\begin{aligned} x + y + 3z &= 13 \\ -4 + 2 + 3(5) &\stackrel{?}{=} 13 \\ -4 + 2 + 15 &\stackrel{?}{=} 13 \\ 13 &= 13 \quad \text{True} \end{aligned}$$

Since $(-4, 2, 5)$ satisfies each equation, it is a solution of the system.

The graph of an equation of the form $Ax + By + Cz = D$ is a flat surface called a **plane.** A system of three linear equations with three variables is consistent or inconsistent, depending on how the three planes corresponding to the three equations intersect. The following illustration shows some of the possibilities. A system of three linear equations in three variables can have exactly one solution, no solution, or infinitely many solutions.

Consistent system | Consistent system | Inconsistent systems



The three planes intersect at a single point P: one solution

(a)

The three planes have a line l in common: infinitely many solutions

(b)

The three planes have no point in common: no solutions

(c)

> ***The Language of Algebra*** Recall that when a system of equations has at least one solution, the system is called a *consistent* system, and if a system has no solution, the system is called *inconsistent.*

2 Solve systems of three linear equations in three variables.

To **solve a system of three linear equations** in three variables means to find all of the solutions of the system. Solving such a system by graphing is not practical because we would need a coordinate system with three axes.

The substitution method is useful to solve systems of three equations where one or more equations have only two variables. However, the best way to solve systems of three linear equations in three variables is usually the addition method.

Solving a System of Three Linear Equations by Addition

1. Write each equation in standard (general) form $Ax + By + Cz = D$ and clear any decimals or fractions.
2. Pick any two equations and eliminate a variable.
3. Pick a different pair of equations and eliminate the same variable as in step 1.
4. Solve the resulting pair of two equations in two variables.
5. To find the value of the third variable, substitute the values of the two variables found in step 4 into any equation containing all three variables and solve the equation.
6. Check the proposed solution in all three of the original equations. Write the solution as an ordered triple.

Self Check 2

Solve the system:

$$\begin{cases} 2x + y + 4z = 16 \\ x + 2y + 2z = 11 \\ 3x - 3y - 2z = -9 \end{cases}$$

(1, 2, 3)

Now Try **Problem 16**

Teaching Example 2 Solve the system:

$$\begin{cases} 2x - 3y + 2z = -7 \\ x + 4y - z = 10 \\ 3x + 2y + z = 4 \end{cases}$$

Answer:
(2, 1, −4)

EXAMPLE 2

Solve the system: $\begin{cases} 2x + y + 4z = 12 \\ x + 2y + 2z = 9 \\ 3x - 3y - 2z = 1 \end{cases}$

Strategy Since the coefficients of the z-terms are opposites in the second and third equations, we will add the left and right sides of those equations to eliminate z. Then we will choose another pair of equations and eliminate z again.

WHY The result will be a system of two equations in x and y that we can solve by addition.

Solution

Step 1 We can skip step 1 because each equation is written in standard form and there are no fractions or decimals to clear. We will number each equation and move to step 2.

$$\begin{array}{l} \textbf{(1)} \\ \textbf{(2)} \\ \textbf{(3)} \end{array} \begin{cases} 2x + y + 4z = 12 \\ x + 2y + 2z = 9 \\ 3x - 3y - 2z = 1 \end{cases}$$

Step 2 If we pick equations 2 and 3 and add them, the variable z is eliminated.

$$\begin{array}{ll} \textbf{(2)} & x + 2y + 2z = 9 \\ \textbf{(3)} & \underline{3x - 3y - 2z = 1} \\ \textbf{(4)} & 4x - y \quad = 10 \end{array}$$ This equation does not contain z.

Step 3 We now pick a different pair of equations (equations 1 and 3) and eliminate z again. If each side of equation 3 is multiplied by 2, and the resulting equation is added to equation 1, z is eliminated.

$$\begin{array}{ll} \textbf{(1)} & 2x + y + 4z = 12 \\ & \underline{6x - 6y - 4z = 2} \\ \textbf{(5)} & 8x - 5y \quad = 14 \end{array}$$

This is $2(3x - 3y - 2z) = 2(1)$.
This equation does not contain z.

Step 4 Equations 4 and 5 form a system of two equations in two variables, x and y.

$$\begin{array}{l} \textbf{(4)} \\ \textbf{(5)} \end{array} \begin{cases} 4x - y = 10 \\ 8x - 5y = 14 \end{cases}$$

To solve this system, we multiply equation 4 by −5 and add the resulting equation to equation 5 to eliminate y:

$$\begin{array}{ll} & -20x + 5y = -50 \\ \textbf{(5)} & \underline{\quad 8x - 5y = 14} \\ & -12x \quad = -36 \\ & \quad x = 3 \end{array}$$

This is $-5(4x - y) = -5(10)$.
Divide both sides by −12. This is the x-value of the solution.

To find y, we substitute 3 for x in any equation containing x and y only (such as equation 5) and solve for y:

$$\begin{array}{ll} \textbf{(5)} & 8x - 5y = 14 \\ & 8(3) - 5y = 14 \\ & 24 - 5y = 14 \\ & -5y = -10 \\ & y = 2 \end{array}$$

Substitute 3 for x.
Simplify.
Subtract 24 from both sides.
Divide both sides by −5. This is the y-value of the solution.

Step 5 To find z, we substitute 3 for x and 2 for y in any equation containing x, y, and z (such as equation 1) and solve for z:

$$\begin{aligned}
&\textbf{(1)} & 2x + y + 4z &= 12 \\
&& 2(3) + 2 + 4z &= 12 && \text{Substitute 3 for } x \text{ and 2 for } y. \\
&& 8 + 4z &= 12 && \text{Simplify.} \\
&& 4z &= 4 && \text{Subtract 8 from both sides.} \\
&& z &= 1 && \text{Divide both sides by 4. This is the } z\text{-value of the solution.}
\end{aligned}$$

The solution of the system is $(x, y, z) = (3, 2, 1)$. Because this system has a solution, it is a consistent system.

Step 6 Verify that these values satisfy each equation in the original system.

3 Solve systems of equations with missing variable terms.

When one or more of the equations of a system is missing a term, the elimination of a variable that is normally performed in step 2 of the solution process can be skipped.

EXAMPLE 3

Solve the system: $\begin{cases} 3x = 6 - 2y + z \\ -y - 2z = -8 - x \\ x = 1 - 2z \end{cases}$

Strategy Since the third equation does not contain the variable y, we will work with the first and second equations to obtain another equation that does not contain y.

WHY Then we can use the elimination method to solve the resulting system of two equations in x and z.

Solution

Step 1 First, we write each equation in $Ax + By + Cz = D$ form.

$$\begin{aligned}
&\textbf{(1)} & \begin{cases} 3x + 2y - z = 6 \\ x - y - 2z = -8 \\ x + 2z = 1 \end{cases} && \begin{array}{l} \text{Add } 2y \text{ and subtract } z \text{ from both sides of } 3x = 6 - 2y + z. \\ \text{Add } x \text{ to both sides of } -y - 2z = -8 - x. \\ \text{Add } 2z \text{ to both sides of } x = 1 - 2z. \end{array}
\end{aligned}$$

(rows numbered **(1)**, **(2)**, **(3)**)

Step 2 Since equation 3 does not have a y-term, we can proceed to step 3, where we will find another equation that does not contain a y-term.

Step 3 If each side of equation 2 is multiplied by 2 and the resulting equation is added to equation 1, y is eliminated.

$$\begin{aligned}
&\textbf{(1)} & 3x + 2y - z &= 6 \\
&& \underline{2x - 2y - 4z} &= \underline{-16} && \text{This is } 2(x - y - 2z) = 2(-8). \\
&\textbf{(4)} & 5x \qquad - 5z &= -10
\end{aligned}$$

Step 4 Equations 3 and 4 form a system of two equations with two variables, x and z:

$$\begin{aligned}
&\textbf{(3)} \\ &\textbf{(4)}
\end{aligned} \begin{cases} x + 2z = 1 \\ 5x - 5z = -10 \end{cases}$$

To solve this system, we multiply equation 3 by -5 and add the resulting equation to equation 4 to eliminate x:

$$\begin{aligned}
&& -5x - 10z &= -5 && \text{This is } -5(x + 2z) = -5(1) \\
&\textbf{(4)} & \underline{5x - 5z} &= \underline{-10} \\
&& -15z &= -15 \\
&& z &= 1 && \text{Divide both sides by } -15. \text{ This is the } z\text{-value of the solution.}
\end{aligned}$$

Self Check 3

Solve the system: $\begin{cases} x + 2y - z = 1 \\ 2x - y + z = 3 \\ x + z = 3 \end{cases}$ (1, 1, 2)

Now Try **Problem 24**

Teaching Example 3 Solve the system: $\begin{cases} y + 3z = 35 - x \\ x + 3y = -20 \\ 2y = -z - 35 \end{cases}$

Answer:
(40, −20, 5)

To find x, we substitute 1 for z in equation 3.

$$\begin{aligned} \textbf{(3)} \quad x + 2z &= 1 \\ x + 2(1) &= 1 && \text{Substitute 1 for } z. \\ x + 2 &= 1 && \text{Multiply.} \\ x &= -1 && \text{Subtract 2 from both sides. This is the } x\text{-value of the solution.} \end{aligned}$$

Step 5 To find y, we substitute -1 for x and 1 for z in equation 1:

$$\begin{aligned} \textbf{(1)} \quad 3x + 2y - z &= 6 \\ 3(-1) + 2y - 1 &= 6 && \text{Substitute } -1 \text{ for } x \text{ and 1 for } z. \\ -3 + 2y - 1 &= 6 && \text{Multiply.} \\ 2y &= 10 && \text{Add 4 to both sides.} \\ y &= 5 && \text{Divide both sides by 2. This is the } y\text{-value of the solution.} \end{aligned}$$

The solution of the system is $(-1, 5, 1)$.

Step 6 Check the solution in all three of the original equations.

4 Identify inconsistent systems and dependent equations.

Recall that when a system has no solution, it is called an **inconsistent system.**

Self Check 4

Solve the system, if possible:
$$\begin{cases} 2a + b - 3c = 8 \\ 3a - 2b + 4c = 10 \\ 4a + 2b - 6c = -5 \end{cases}$$

Now Try **Problem 28**

Self Check 4 Answer
no solution, $\varnothing$; inconsistent system

Teaching Example 4 Solve the system, if possible: $\begin{cases} 4a + 2b + c = 1 \\ -9a + b - c = 3 \\ 14a - 4b + c = -2 \end{cases}$

Answer:
no solution, $\varnothing$; inconsistent system

EXAMPLE 4

Solve the system, if possible: $\begin{cases} 2a + b - 3c = -3 \\ 3a - 2b + 4c = 2 \\ 4a + 2b - 6c = -7 \end{cases}$

Strategy Since the coefficient of the b-term is 1, we will eliminate the variable b.

WHY It is easier to determine the number needed to multiply to both sides of the equation to force opposites when one of the coefficients is 1. Then we will have a system of two equations in a and c.

Solution
We can multiply the first equation of the system by 2 and add the resulting equation to the second equation to eliminate b:

$$\begin{aligned} 4a + 2b - 6c &= -6 && \text{Multiply both sides of the first equation by 2.} \\ 3a - 2b + 4c &= 2 \\ \hline \textbf{(1)} \quad 7a \qquad - 2c &= -4 \end{aligned}$$

Now add the second and third equations of the system to eliminate b again:

$$\begin{aligned} 3a - 2b + 4c &= 2 \\ 4a + 2b - 6c &= -7 \\ \hline \textbf{(2)} \quad 7a \qquad - 2c &= -5 \end{aligned}$$

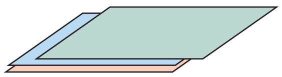
Equations 1 and 2 form the system

$$\begin{aligned} &\textbf{(1)} \\ &\textbf{(2)} \end{aligned} \begin{cases} 7a - 2c = -4 \\ 7a - 2c = -5 \end{cases}$$

Since $7a - 2c$ cannot equal both -4 and -5, the system is inconsistent and has no solution.

When the equations in a system of two equations in two variables are dependent, the system has infinitely many solutions. This is not always true for systems of three equations in three variables. In fact, a system can have dependent equations and still be inconsistent. The figure illustrates the different possibilities.

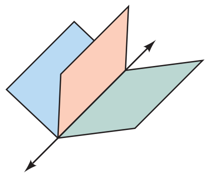
Consistent system



When three planes coincide, the equations are dependent, and there are infinitely many solutions.

(a)

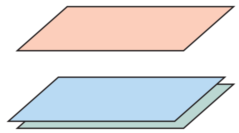
Consistent system



When three planes intersect in a common line, the equations are dependent, and there are infinitely many solutions.

(b)

Inconsistent system



When two planes coincide and are parallel to a third plane, the system is inconsistent, and there are no solutions.

(c)

EXAMPLE 5

Solve the system: $\begin{cases} 3x - 2y + z = -1 \\ 2x + y - z = 5 \\ 5x - y = 4 \end{cases}$

Strategy Since the third equation does not contain the variable z, we will work with the first and second equations to obtain another equation that does not contain z.

WHY Then we can use the addition method to solve the resulting system of two equations in x and y.

Solution

We can add the first two equations to get

$$\begin{aligned} 3x - 2y + z &= -1 \\ \underline{2x + y - z} &= \underline{5} \\ \textbf{(1)}\quad 5x - y &= 4 \end{aligned}$$

Since equation 1 is the same as the third equation of the system, the equations of the system are dependent, and there will be infinitely many solutions. From a graphical perspective, the equations represent three planes that intersect in a common line, as shown in figure (b).

To write the general solution of this system, we can solve equation 1 for y to get

$$\begin{aligned} 5x - y &= 4 \\ -y &= -5x + 4 && \text{Subtract } 5x \text{ from both sides.} \\ y &= 5x - 4 && \text{Multiply both sides by } -1. \end{aligned}$$

We can then substitute $5x - 4$ for y in the first equation of the system and solve for z to get

$$\begin{aligned} 3x - 2y + z &= -1 \\ 3x - 2(5x - 4) + z &= -1 && \text{Substitute } 5x - 4 \text{ for } y. \\ 3x - 10x + 8 + z &= -1 && \text{Use the distributive property to remove parentheses.} \\ -7x + 8 + z &= -1 && \text{Combine like terms.} \\ z &= 7x - 9 && \text{Add } 7x \text{ and } -8 \text{ to both sides.} \end{aligned}$$

Since we have found the values of y and z in terms of x, every solution of the system has the form $(x, 5x - 4, 7x - 9)$, where x can be any real number. For example,

If $x = 1$, a solution is $(1, 1, -2)$. $5(1) - 4 = 1$, and $7(1) - 9 = -2$.

If $x = 2$, a solution is $(2, 6, 5)$. $5(2) - 4 = 6$, and $7(2) - 9 = 5$.

If $x = 3$, a solution is $(3, 11, 12)$. $5(3) - 4 = 11$, and $7(3) - 9 = 12$.

Self Check 5

Solve the system:
$\begin{cases} 3x + 2y + z = -1 \\ 2x - y - z = 5 \\ 5x + y = 4 \end{cases}$

Now Try **Problem 30**

Self Check 5 Answer
Infinitely many solutions. A general solution is $(x, 4 - 5x, -9 + 7x)$. Three solutions are $(1, -1, -2)$, $(2, -6, 5)$, and $(3, -11, 12)$.

Teaching Example 5 Solve the system:
$\begin{cases} \frac{4}{5}x - y + z = \frac{53}{5} \\ x - 2y - z = 8 \\ 0.2x - 0.3y + 0.1z = 2.3 \end{cases}$

Answer:
There are infinitely many solutions. A general solution is $\left(x, \frac{3x - 31}{5}, \frac{-x + 22}{5}\right)$.

ANSWERS TO SELF CHECKS

1. no **2.** (1, 2, 3) **3.** (1, 1, 2) **4.** no solution, ∅; inconsistent system **5.** infinitely many solutions, $(x, 4 - 5x, -9 + 7x)$; three solutions are $(1, -1, -2)$, $(2, -6, 5)$, and $(3, -11, 12)$

SECTION 3.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\begin{cases} 2x + y - 3z = 0 \\ 3x - y + 4z = 5 \\ 4x + 2y - 6z = 0 \end{cases}$ is called a __system__ of three linear equations.

2. If the first two equations of the system in Exercise 1 are added, the variable y is __eliminated__.

3. The equation $2x + 3y + 4z = 5$ is a linear equation in __three__ variables.

4. The graph of the equation $2x + 3y + 4z = 5$ is a flat surface called a __plane__.

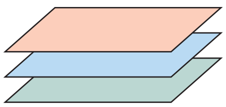
5. When three planes coincide, the equations of the system are __dependent__, and there are infinitely many solutions.

▶ **6.** When three planes intersect in a line, the system will have __infinitely__ many solutions.

CONCEPTS

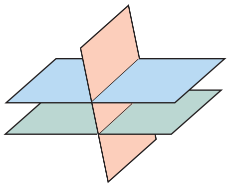
7. For each graph of a system of three equations, tell whether the solution set contains one solution, infinitely many solutions, or no solution.

a.



no solution

b.



no solution

▶ **8.** Consider the system: $\begin{cases} -2x + y + 4z = 3 & \textbf{(1)} \\ x - y + 2z = 1 & \textbf{(2)} \\ x + y - 3z = 2 & \textbf{(3)} \end{cases}$

a. What is the result if equation 1 and equation 2 are added? $-x + 6z = 4$

b. What is the result if equation 2 and equation 3 are added? $2x - z = 3$

c. What variable was eliminated in the steps performed in parts a and b? y

NOTATION

▶ **9.** Write the equation $3z - 2y = x + 6$ in $Ax + By + Cz = D$ form. $x + 2y - 3z = -6$

▶ **10.** Fill in the blank to make a true statement: Solutions of a system of three equations in three variables, x, y, and z, are written in the form (x, y, z) and are called ordered __triples__.

GUIDED PRACTICE

Determine whether the given ordered triple is a solution of given system. See Example 1.

11. $(2, 1, 1)$, $\begin{cases} x - y + z = 2 \\ 2x + y - z = 4 \\ 2x - 3y + z = 2 \end{cases}$ yes

▶ **12.** $(-3, 2, -1)$, $\begin{cases} 2x + 2y + 3z = -1 \\ 3x + y - z = -6 \\ x + y + 2z = 1 \end{cases}$ no

13. $(6, -7, -5)$, $\begin{cases} 3x - 2y - z = 37 \\ x - 3y = 27 \\ 2x + 7y + 2z = -48 \end{cases}$ no

▶ **14.** $(-4, 0, 9)$, $\begin{cases} x + 2y - 3z = -31 \\ 2x + 6y = 46 \\ 3x - y = -12 \end{cases}$ no

Solve each system. See Example 2.

15. $\begin{cases} x + y + z = 4 \\ 2x + y - z = 1 \\ 2x - 3y + z = 1 \end{cases}$ (1, 1, 2)

16. $\begin{cases} x + y + z = 4 \\ x - y + z = 2 \\ x - y - z = 0 \end{cases}$ (2, 1, 1)

17. $\begin{cases} 3x + 2y - 5z = 3 \\ 4x - 2y - 3z = -10 \\ 5x - 2y - 2z = -11 \end{cases}$ $(-1, 3, 0)$

18. $\begin{cases} 5x + 4y + 2z = -2 \\ 3x + 4y - 3z = -27 \\ 2x - 4y - 7z = -23 \end{cases}$ $(0, -3, 5)$

19. $\begin{cases} 2x + 2y + 3z = 10 \\ 3x + y - z = 0 \\ x + y + 2z = 6 \end{cases}$ (0, 2, 2)

▶ **20.** $\begin{cases} x - y + z = 4 \\ x + 2y - z = -1 \\ x + y - 3z = -2 \end{cases}$ $(2, -1, 1)$

21. $\begin{cases} 4x - 5y - 8z = -52 \\ 2x - 3y - 4z = -26 \\ 3x + 7y + 8z = 31 \end{cases}$ $(-3, 0, 5)$

22. $\begin{cases} 2x + 6y + 3z = 9 \\ 5x - 3y - 5z = 3 \\ 4x + 3y + 2z = 15 \end{cases}$ $(3, -1, 3)$

Solve each system. See Example 3.

23. $\begin{cases} 3x + 3z = 6 - 4y \\ 7x - 5z = 46 + 2y \\ 4x = 31 - z \end{cases}$
$(7, -6, 3)$

24. $\begin{cases} 5x + 6z = 4y - 21 \\ 9x + 2y = 3z - 47 \\ 3x + y = -19 \end{cases}$
$(-5, -4, -2)$

25. $\begin{cases} 2x + z = -2 + y \\ 8x - 3y = -2 \\ 6x - 2y + 3z = -4 \end{cases}$
$\left(\frac{1}{2}, 2, -1\right)$

26. $\begin{cases} 3y + z = -1 \\ -x + 2z = -9 + 6y \\ 9y + 3z = -9 + 2x \end{cases}$
$\left(3, \frac{1}{3}, -2\right)$

Solve each system. If a system is inconsistent or if the equations are dependent, so indicate. See Examples 4–5.

27. $\begin{cases} 2x + y - z = 1 \\ x + 2y + 2z = 2 \\ 4x + 5y + 3z = 3 \end{cases}$
no solution, inconsistent system

28. $\begin{cases} 2a = 2 - 3b - c \\ 4a + 6b + 2c - 5 = 0 \\ a + c = 3 + 2b \end{cases}$
no solution, inconsistent system

29. $\begin{cases} 2x + 3y + 4z = 6 \\ 2x - 3y - 4z = -4 \\ 4x + 6y + 8z = 12 \end{cases}$
infinitely many solutions, dependent equations

30. $\begin{cases} x - 3y + 4z = 2 \\ 2x + y + 2z = 3 \\ 4x - 5y + 10z = 7 \end{cases}$
infinitely many solutions, dependent equations

TRY IT YOURSELF

Solve each system, if possible. If a system is inconsistent or if the equations are dependent, so state.

31. $\begin{cases} 2a + 3b - 2c = 18 \\ 5a - 6b + c = 21 \\ 4b - 2c - 6 = 0 \end{cases}$
$(8, 4, 5)$

32. $\begin{cases} r - s + t = 0 \\ r + 2s - t = -5 \\ r + s - 3t = -6 \end{cases}$
$(-2, -1, 1)$

33. $\begin{cases} a + b + c = 180 \\ \frac{a}{4} + \frac{b}{2} + \frac{c}{3} = 60 \\ 2b + 3c - 330 = 0 \end{cases}$
$(60, 30, 90)$

34. $\begin{cases} 2a + 3b - 2c = 18 \\ 5a - 6b + c = 21 \\ 4b - 2c - 6 = 0 \end{cases}$
$(8, 4, 5)$

35. $\begin{cases} 0.5a + 0.3b = 2.2 \\ 1.2c - 8.5b = -24.4 \\ 3.3c + 1.3a = 29 \end{cases}$
$(2, 4, 8)$

36. $\begin{cases} 4a - 3b = 1 \\ 6a - 8c = 1 \\ 2b - 4c = 0 \end{cases}$
$\left(-\frac{1}{2}, -1, -\frac{1}{2}\right)$

37. $\begin{cases} x + \frac{1}{3}y + z = 13 \\ \frac{1}{2}x - y + \frac{1}{3}z = -2 \\ x + \frac{1}{2}y - \frac{1}{3}z = 2 \end{cases}$
$(2, 6, 9)$

38. $\begin{cases} x - \frac{1}{5}y - z = 9 \\ \frac{1}{4}x + \frac{1}{5}y - \frac{1}{2}z = 5 \\ 2x + y + \frac{1}{6}z = 12 \end{cases}$
$(4, 5, -6)$

39. $\begin{cases} r + s + 4t = 3 \\ 3r + 7t = 0 \\ 3s + 5t = 0 \end{cases}$
no solution, inconsistent system

40. $\begin{cases} 2x + 3y + 4z = 6 \\ 2x - 3y - 4z = -4 \\ 4x + 6y + 8z = 12 \end{cases}$
infinitely many solutions, dependent equations

41. $\begin{cases} x - y = 3 \\ 2x - y + z = 1 \\ x + z = -2 \end{cases}$
infinitely many solutions, dependent equations

42. $\begin{cases} 2x + y - z = 1 \\ x + 2y + 2z = 2 \\ 4x + 5y + 3z = 3 \end{cases}$
no solution, inconsistent system

43. $\begin{cases} b + 2c = 7 - a \\ a + c = 8 - 2b \\ 2a + b + c = 9 \end{cases}$
$(3, 2, 1)$

44. $\begin{cases} 4x + 3z = 4 \\ 2y - 6z = -1 \\ 8x + 4y + 3z = 9 \end{cases}$
$\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{3}\right)$

WRITING

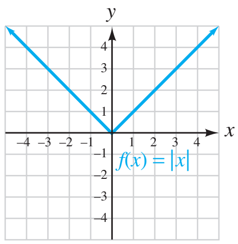
45. Explain how a system of three equations in three variables can be reduced to a system of two equations in two variables.

46. What makes a system of three equations in three variables inconsistent?

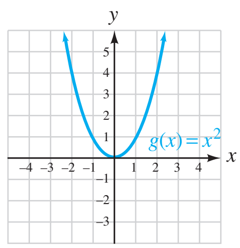
REVIEW

Graph each function.

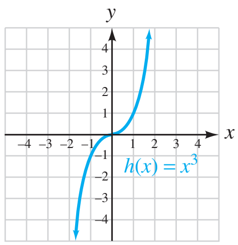
47. $f(x) = |x|$



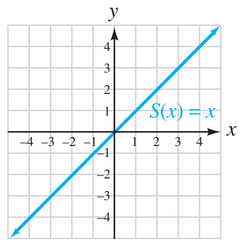
48. $g(x) = x^2$



49. $h(x) = x^3$



50. $S(x) = x$



Objectives

1. Assign variables to three unknowns.
2. Use systems to solve curve-fitting problems.

SECTION 3.5

Problem Solving Using Systems of Three Equations

Problems that involve three unknown quantities can be solved using a strategy similar to that for solving problems involving two unknowns. To solve such problems, we will write three equations in three variables to model the situation and then we will use the methods of Section 3.4 to solve the system formed by the three equations.

1 Assign variables to three unknowns.

Self Check 1

A manufacturer of memory cards makes 1-GB, 2-GB, and 4-GB storage size cards. The cost of manufacturing each is \$2, \$3, and \$5, respectively. Each day the cost of manufacturing 500 cards is \$1,500. The cards sell for \$15, \$20, and \$30, respectively, with a daily revenue of \$10,000. How many memory cards of each type are manufactured?

Now Try **Problem 7**

Self Check 1 Answer
1-GB: 200, 2-GB: 200, 4-GB: 100

Teaching Example 1 At a local bakery 3 price levels of pies are available. Cream pies sell for \$10 per pie, fruit pies sell for \$12 per pie, and specialty pies sell for \$15 per pie. The cost of producing the pies is \$4.00, \$4.50, and \$5.00, respectively. Each day, the revenue from the 50 pies sold is \$585 and the cost to make them is \$220. How many pies of each type are made?
Answer:
cream: 15, fruit: 30, specialty: 5

EXAMPLE 1 ***Tool Manufacturing*** A company makes three types of hammers, which are marketed as "good," "better," and "best." The cost of manufacturing each type of hammer is \$4, \$6, and \$7, respectively, and the hammers sell for \$6, \$9, and \$12. Each day, the cost of manufacturing 100 hammers is \$520, and the daily revenue from their sale is \$810. How many hammers of each type are manufactured?

Analyze We need to find how many of each type of hammer are manufactured daily. Since there are three unknowns, we must write three equations to find them.

Form Let x = the number of good hammers, y = the number of better hammers, and z = the number of best hammers. We know that

The cost of manufacturing

- the good hammers is \$$4x$ (\$4 times x hammers).
- the better hammers is \$$6y$ (\$6 times y hammers).
- the best hammers is \$$7z$ (\$7 times z hammers).

The revenue received by selling

- the good hammers is \$$6x$ (\$6 times x hammers).
- the better hammers is \$$9y$ (\$9 times y hammers).
- the best hammers is \$$12z$ (\$12 times z hammers).

We can use the facts of the problem to write three equations.

The number of good hammers	plus	the number of better hammers	plus	the number of best hammers	is	the total number of hammers.
x	$+$	y	$+$	z	$=$	100

The cost of good hammers	plus	the cost of better hammers	plus	the cost of best hammers	is	the total cost.
$4x$	$+$	$6y$	$+$	$7z$	$=$	520

The revenue from good hammers	plus	the revenue from better hammers	plus	the revenue from best hammers	is	the total revenue.
$6x$	$+$	$9y$	$+$	$12z$	$=$	810

Solve To find how many hammers of each type are manufactured, we must solve the following system of three equations in three variables:

$$\begin{array}{ll} \textbf{(1)} & \\ \textbf{(2)} & \\ \textbf{(3)} & \end{array} \begin{cases} x + y + z = 100 \\ 4x + 6y + 7z = 520 \\ 6x + 9y + 12z = 810 \end{cases}$$

If we multiply equation 1 by -7 and add the result to equation 2, we get

$$\begin{array}{lrl} & -7x - 7y - 7z = -700 & \text{This is } -7(x + y + z) = -7(100). \\ \textbf{(2)} & \underline{4x + 6y + 7z = 520} & \\ \textbf{(4)} & -3x - y = -180 & \end{array}$$

If we multiply equation 1 by -12 and add the result to equation 3, we get

$$\begin{array}{lrl} & -12x - 12y - 12z = -1{,}200 & \text{This is } -12(x + y + z) = -12(100). \\ \textbf{(3)} & \underline{6x + 9y + 12z = 810} & \\ \textbf{(5)} & -6x - 3y = -390 & \end{array}$$

We can multiply equation 4 by -3 and add it to equation 5 to eliminate y.

$$\begin{array}{lrl} & 9x + 3y = 540 & \text{This is } -3(-3x - y) = -3(-180). \\ \textbf{(5)} & \underline{-6x - 3y = -390} & \\ & 3x = 150 & \\ & x = 50 & \text{To solve for } x \text{, divide both sides by 3.} \\ & & \text{This is the number of good hammers manufactured.} \end{array}$$

To find y, we substitute 50 for x in equation 4:

$$\begin{aligned} -3x - y &= -180 \\ -3(50) - y &= -180 && \text{Substitute 50 for } x. \\ -150 - y &= -180 \\ -y &= -30 && \text{Add 150 to both sides.} \\ y &= 30 && \text{To solve for } y \text{, divide both sides by } -1. \\ & && \text{This is the number of better hammers manufactured.} \end{aligned}$$

To find z, we substitute 50 for x and 30 for y in equation 1:

$$\begin{aligned} x + y + z &= 100 \\ 50 + 30 + z &= 100 \\ z &= 20 && \text{To solve for } z \text{, subtract 80 from both sides.} \\ & && \text{This is the number of best hammers manufactured.} \end{aligned}$$

State Each day, the company manufactures 50 good hammers, 30 better hammers, and 20 best hammers.

Check If the company manufactures **50** good hammers, **30** better hammers, and **20** best hammers each day, that is a total of $50 + 30 + 20 = 100$ hammers. The cost of manufacturing the three types of hammers is $\$4(50) + \$6(30) + \$7(20) = \$200 + \$180 + \$140$ or $520. The revenue from the sale of the hammers is $\$6(50) + \$9(30) + \$12(20) = \$300 + \$270 + \$240$ or $810. The results check.

Success Tip If three variables are used to represent three unknowns, then three equations must be formed, and a three-part check must be performed to verify the results.

Self Check 2

The three countries that are at the top of the list for the most all-time medal wins are the United States, the Soviet Union/Unified Team, and Great Britain. Together they won 3,996.5 medals, with the U.S. medal count 1,075 more than the Soviet Union/Unified's, and the Soviet Union/Unified's medal count 444.5 more than Great Britain. Find the number of medals won by each country.

Now Try **Problem 13**

Self Check 2 Answer
U.S.: 2,197, Soviet Union/Unified: 1,122, Great Britain: 677.5

Teaching Example 2 In the 2008 summer Olympic games, the top three countries that won the most medals were the United States, China, and Russia. Together they won a total of 282 medals, with the U.S. medal count 10 more than China's, and China's medal count was 28 more than Russia's. Find the number of medals won by each country.
Answer:
United States: 110, China: 100, Russia: 72
(*Source:* nbcolympics.com/medals/alltime/index.html)

EXAMPLE 2 ***The Olympics*** The three countries that won the most medals in the 2004 summer Olympic games were the United States, Russia, and China. Together they won a total of 258 medals, with the U.S. medal count 11 more than Russia's and Russia's medal count 29 more than China's. Find the number of medals won by each country.

Analyze We need to find how many medals the United States, Russia, and China won. Since there are three unknowns, we must write three equations to find them.

Form Let x = the number of medals won by the United States, y = the number of medals won by Russia, and z = the number of medals won by China.

We can use the facts of the problem to write three equations.

The number of medals won by the United States	plus	the number of medals won by Russia	plus	the number of medals won by China	was	258.
x	$+$	y	$+$	z	$=$	258

The United States' medal count	was	11 more than	Russia's medal count.
x	$=$	y	$+$ 11

Russia's medal count	was	29 more than	China's medal count.
y	$=$	z	$+$ 29

Solve We can use substitution to solve the resulting system of three equations. If we solve equation 3 for z, then the resulting equation 4 and equation 2 can serve as substitution equations.

$$\begin{array}{l} \textbf{(1)} \\ \textbf{(2)} \\ \textbf{(3)} \end{array} \begin{cases} x + y + z = 258 \\ x = y + 11 \\ y = z + 29 \end{cases} \begin{array}{c} \xrightarrow{\text{Unchanged}} \\ \xrightarrow{\text{Unchanged}} \\ \xrightarrow[\text{Solve for } z]{} \end{array} \begin{cases} x + y + z = 258 \\ x = y + 11 \\ z = y - 29 \end{cases} \begin{array}{l} \textbf{(1)} \\ \textbf{(2)} \\ \textbf{(4)} \end{array}$$

When we substitute for x and z in equation 1, we obtain an equation in one variable, y.

$$\begin{aligned} x + y + z &= 258 && \text{This is equation 1.} \\ y + 11 + y + y - 29 &= 258 && \text{Substitute } y + 11 \text{ for } x \text{ and } y - 29 \text{ for } z. \\ 3y - 18 &= 258 && \text{On the left side, combine like terms.} \\ 3y &= 276 && \text{Add 18 to both sides.} \\ y &= 92 && \text{To solve for } y \text{, divide both sides by 3.} \\ & && \text{This is the number of medals won by Russia.} \end{aligned}$$

To find x, we substitute 92 for y in equation 2. To find z, we substitute 92 for y in equation 4.

$$\begin{aligned} x &= y + 11 && \text{This is equation 2.} \\ x &= 92 + 11 \\ x &= 103 && \text{This is the U.S. medal count.} \end{aligned} \qquad \begin{aligned} z &= y - 29 && \text{This is equation 4.} \\ z &= 92 - 29 \\ z &= 63 && \text{This is China's medal count.} \end{aligned}$$

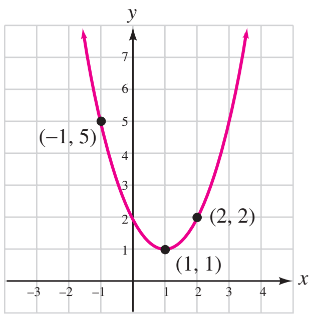
State In the 2004 summer Olympics, the United States won 103 medals, Russia won 92, and China won 63.

Check The sum of $103 + 92 + 63$ is 258. Furthermore, 103 is 11 more than 92, and 92 is 29 more than 63. The results check.

2 Use systems to solve curve-fitting problems.

The process of determining an equation whose graph contains given points is called **curve fitting.**

EXAMPLE 3 The equation of a parabola opening upward or downward is of the form $y = ax^2 + bx + c$. Find the equation of the parabola graphed on the right by determining the values of a, b, and c.



Strategy We will substitute the x- and y-coordinates of three points that lie on the graph into the equation $y = ax^2 + bx + c$. This will produce a system of three equations in three variables that we can solve to find a, b, and c.

WHY Once we know a, b, and c, we can write the equation.

Solution

Since the parabola passes through the points $(-1, 5)$, $(1, 1)$, and $(2, 2)$, each pair of coordinates must satisfy the equation $y = ax^2 + bx + c$. If we substitute each pair into $y = ax^2 + bx + c$, we will get a system of three equations in three variables.

***Substitute* (−1, 5)**	***Substitute* (1, 1)**	***Substitute* (2, 2)**
$y = ax^2 + bx + c$	$y = ax^2 + bx + c$	$y = ax^2 + bx + c$
$5 = a(-1)^2 + b(-1) + c$	$1 = a(1)^2 + b(1) + c$	$2 = a(2)^2 + b(2) + c$
$5 = a - b + c$	$1 = a + b + c$	$2 = 4a + 2b + c$
This is equation 1.	This is equation 2.	This is equation 3.

Success Tip If a point lies on the graph of an equation, it is a solution of the equation, and the coordinates of the point satisfy the equation.

The three equations above give the system, which we can solve to find a, b, and c.

$$\begin{aligned}&\textbf{(1)}\\ &\textbf{(2)}\\ &\textbf{(3)}\end{aligned}\quad \begin{cases} a - b + c = 5 \\ a + b + c = 1 \\ 4a + 2b + c = 2 \end{cases}$$

If we add equations 1 and 2, we obtain

$$\begin{array}{lrl} & a - b + c &= 5 \\ & \underline{a + b + c} &\underline{= 1} \\ \textbf{(4)} & 2a \qquad + 2c &= 6 \end{array}$$

If we multiply equation 1 by 2 and add the result to equation 3, we get

$$\begin{array}{lrl} & 2a - 2b + 2c &= 10 \\ \textbf{(3)} & \underline{4a + 2b + c} &\underline{= 2} \\ \textbf{(5)} & 6a \qquad + 3c &= 12 \end{array}$$

Self Check 3

Find the equation of the parabola, $y = ax^2 + bx + c$, that passes through $(1, 6)$, $(-4, 1)$, and $(-3, -2)$. $y = x^2 + 4x + 1$

Now Try **Problem 25**

Teaching Example 3 Find the equation of the parabola, $y = ax^2 + bx + c$, that passes through $(-2, 0)$, $(-1, 1)$, and $(1, -3)$.
Answer:
$y = -x^2 - 2x$

We can then divide both sides of equation 4 by 2 to get equation 6 and divide both sides of equation 5 by 3 to get equation 7. We now have the system

$$\begin{array}{ll} \textbf{(6)} & \begin{cases} a + c = 3 \\ 2a + c = 4 \end{cases} \\ \textbf{(7)} & \end{array}$$

To eliminate c, we multiply equation 6 by -1 and add the result to equation 7. We get

$$\begin{array}{r} -a - c = -3 \quad \text{This is } -1(a + c) = -1(3). \\ \underline{2a + c = 4} \\ a = 1 \end{array}$$

To find c, we can substitute 1 for a in equation 6 and find that $c = 2$. To find b, we can substitute 1 for a and 2 for c in equation 2 and find that $b = -2$.

After we substitute these values of a, b, and c into the equation $y = ax^2 + bx + c$, we have the equation of the parabola.

$$y = ax^2 + bx + c$$
$$y = 1x^2 - 2x + 2$$
$$y = x^2 - 2x + 2$$

ANSWERS TO SELF CHECKS

1. 1-GB: 200, 2-GB: 200, 4-GB: 100 **2.** U.S.: 2,197, Soviet Union/Unified: 1,122, Great Britain: 677.5 **3.** $y = x^2 + 4x + 1$

SECTION 3.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. If a point lies on the graph of an equation, it is a solution of the equation, and the coordinates of the point _satisfy_ the equation.

2. The process of determining an equation whose graph contains given points is called curve _fitting_.

CONCEPTS

Write a system of three equations in three variables that models the situation. Do not solve the system.

▶ **3.** DESSERTS A bakery makes three kinds of pies: chocolate cream, which sells for \$5; apple, which sells for \$6; and cherry, which sells for \$7. The cost to make the pies is \$2, \$3, and \$4, respectively. Let $x =$ the number of chocolate cream pies made daily, $y =$ the number of apple pies made daily, and $z =$ the number of cherry pies made daily.

- Each day, the bakery makes 50 pies.
- Each day, the revenue from the sale of the pies is \$295.
- Each day, the cost to make the pies is \$145.

$$\begin{cases} x + y + z = 50 \\ 5x + 6y + 7z = 295 \\ 2x + 3y + 4z = 145 \end{cases}$$

▶ **4.** FAST FOODS Let $x =$ the number of calories in a Big Mac hamburger, $y =$ the number of calories in a small order of french fries, and $z =$ the number of calories in a medium Coca-Cola.

- The total number of calories in a Big Mac hamburger, a small order of french fries, and a medium Coke is 1,000.
- The number of calories in a Big Mac is 260 more than in a small order of french fries.
- The number of calories in a small order of french fries is 40 more than in a medium Coke. (Source: McDonald's USA)

$$\begin{cases} x + y + z = 1{,}000 \\ x = y + 260 \\ y = z + 40 \end{cases}$$

5. What equation results when the coordinates of the point $(2, -3)$ are substituted into $y = ax^2 + bx + c$? $-3 = 4a + 2b + c$

6. The equation $y = 5x^2 - 6x + 1$ is written in the form $y = ax^2 + bx + c$. What are a, b, and c? 5, −6, 1

▶ Selected exercises available online at **www.webassign.net/brookscole**

APPLICATIONS

7. MAKING STATUES An artist makes three types of ceramic statues (large, medium, and small) at a monthly cost of $650 for 180 statues. The manufacturing costs for the three types are $5, $4, and $3. If the statues sell for $20, $12, and $9, respectively, how many of each type should be made to produce $2,100 in monthly revenue?
30 large, 50 medium, 100 small

8. PUPPETS A toy company makes a total of 500 puppets in three sizes during a production run. The small puppets cost $5 to make and sell for $8 each, the standard size puppets cost $10 to make and sell for $16 each, and the super-size puppets cost $15 to make and sell for $25. The total cost to make the puppets is $4,750 and the revenue from their sale is $7,700. How many small, standard, and super-size puppets are made during a production run?
small: 150, standard: 250, super-size: 100

9. NUTRITION A dietitian is to design a meal that will provide a patient with exactly 14 grams (g) of fat, 9 g of carbohydrates, and 9 g of protein. She is to use a combination of the three foods listed in the table. If one ounce of each of the foods has the nutrient content shown in the table, how many ounces of each food should be used?
Food A: 2, Food B: 3, Food C: 1

Food	Fat	Carbohydrates	Protein
A	2 g	1 g	2 g
B	3 g	2 g	1 g
C	1 g	1 g	2 g

(g stands for gram)

10. NUTRITIONAL PLANNING One ounce of each of three foods has the vitamin and mineral content shown in the table. How many ounces of each must be used to provide exactly 22 milligrams (mg) of niacin, 12 mg of zinc, and 20 mg of vitamin C?
Food A: 2, Food B: 4, Food C: 6

Food	Niacin	Zinc	Vitamin C
A	1 mg	1 mg	2 mg
B	2 mg	1 mg	1 mg
C	2 mg	1 mg	2 mg

(mg stands for milligram)

11. A clothing manufacturer makes coats, shirts, and slacks. The time required for cutting, sewing, and packaging each item is shown in the table. How many of each should be made to use all available labor hours?
120 coats, 200 shirts, 150 slacks

from Campus to Careers
Fashion Designer

© Radius Images/Alamy

	Coats	Shirts	Slacks	Time available
Cutting	20 min	15 min	10 min	115 hr
Sewing	60 min	30 min	24 min	280 hr
Packaging	5 min	12 min	6 min	65 hr

12. SCULPTING A wood sculptor carves three types of statues with a chainsaw. The number of hours required for carving, sanding, and painting a totem pole, a bear, and a deer are shown in the table. How many of each should be produced to use all available labor hours? 3 poles, 2 bears, 4 deer

	Totem pole	Bear	Deer	Time available
Carving	2 hr	2 hr	1 hr	14 hr
Sanding	1 hr	2 hr	2 hr	15 hr
Painting	3 hr	2 hr	2 hr	21 hr

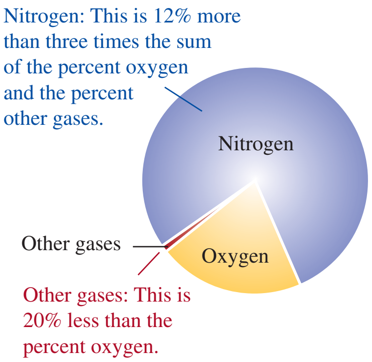
13. NFL RECORDS Jerry Rice, who played the majority of his career with the San Francisco 49ers and the Oakland Raiders, holds the all-time record for touchdown (TD) passes caught. Here are some interesting facts about this feat.

- He caught 30 more TD passes from Steve Young than he did from Joe Montana.
- He caught 39 more TD passes from Joe Montana than he did from Rich Gannon.
- He caught a total of 156 TD passes from Young, Montana, and Gannon.

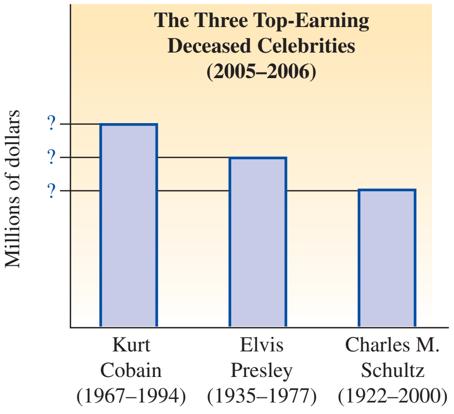
Determine the number of touchdown passes Rice has caught from Young, from Montana, and from Gannon. Young: 85, Montana: 55, Gannon: 16

14. HOT DOGS In 12 minutes, the top three finishers in the 2007 Nathan's Hot Dog Eating Contest consumed a total of 178 hot dogs. The winner, Joey Chestnut, ate 3 more hot dogs than the runner-up, Takeru Kobayashi. Pat Bertoletti finished a distant third, 14 hot dogs behind Kobayashi. How many hot dogs did each person eat?
Chestnut: 66, Kobayashi: 63, Bertoletti: 49

▶ **15.** EARTH'S ATMOSPHERE Use the information in the circle graph to determine what percent of Earth's atmosphere is nitrogen, is oxygen, and is other gases.
nitrogen: 78%, oxygen: 21%, other gases: 1%

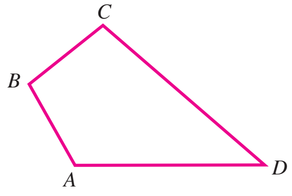


▶ **16.** DECEASED CELEBRITIES Between October 2005 and October 2006, the estates of Kurt Cobain, Elvis Presley, and Charles M. Schultz (Snoopy cartoonist) earned a total of $127 million. Together, the Presley and Schultz estates earned $27 million more than the Cobain estate. If the Schultz estate had earned $15 million more, it would equal the value of the Cobain estate. Use this information to label the vertical axis of the graph below. (Source: Forbes.com)
Cobain: $50 million, Presley: $42 million, Schultz: $35 million



17. TRIANGLES The sum of the measures of the angles of any triangle is 180°. In ΔABC, $\angle A$ measures 100° less than the sum of the measures of $\angle B$ and $\angle C$, and the measure of $\angle C$ is 40° less than twice the measure of $\angle B$. Find the measure of each angle of the triangle.
$\angle A = 40°$, $\angle B = 60°$, $\angle C = 80°$

▶ **18.** QUADRILATERALS A quadrilateral is a four-sided polygon. The sum of the measures of the angles of any quadrilateral is 360°. In the illustration below, the measures of $\angle A$ and $\angle B$ are the same. The measure of $\angle C$ is 20° greater than the measure of $\angle A$, and the measure of $\angle D$ is 60° less than $\angle B$. Find the measure of $\angle A$, $\angle B$, $\angle C$, and $\angle D$.
$\angle A = 100°$, $\angle B = 100°$, $\angle C = 120°$, $\angle D = 40°$



▶ **19.** TV HISTORY *X-Files, Will & Grace,* and *Seinfeld* are three of the most popular television shows of all time. The total number of episodes of these three shows is 575. There are 21 more episodes of *X-Files* than *Seinfeld,* and the difference between the number of episodes of *Will & Grace* and *Seinfeld* is 14. Find the number of episodes of each show.
X-Files: 201, *Will & Grace:* 194, *Seinfeld:* 180

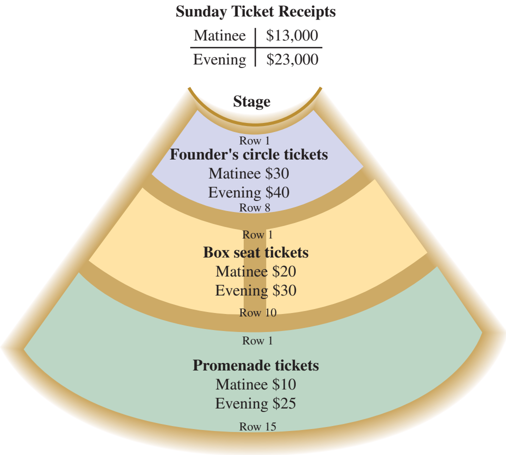
▶ **20.** TRAFFIC LIGHTS At a traffic light, one cycle through green-yellow-red lasts for 80 seconds. The green light is on eight times longer than the yellow light, and the red light is on eleven times longer than the yellow light. For how long is each colored light on during one cycle?
green: 32 sec, yellow: 4 sec, red: 44 sec

▶ **21.** POTPOURRI The owner of a home decorating shop wants to mix dried rose petals selling for $6 per pound, dried lavender selling for $5 per pound, and buckwheat hulls selling for $4 per pound to get 10 pounds of a mixture that would sell for $5.50 per pound. She wants to use twice as many pounds of rose petals as lavender. How many pounds of each should she use?
6 lb rose petals, 3 lb lavender, 1 lb buckwheat hulls

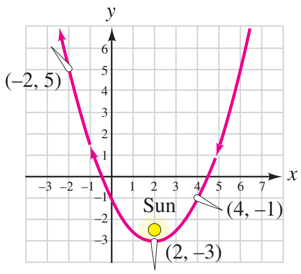
▶ **22.** MIXING NUTS The owner of a candy store wants to mix some peanuts worth $3 per pound, some cashews worth $9 per pound, and some Brazil nuts worth $9 per pound to get 50 pounds of a mixture that will sell for $6 per pound. She uses 15 fewer pounds of cashews than peanuts. How many pounds of each did she use?
25 lb peanuts, 10 lb cashews, 15 lb Brazil nuts

▶ **23.** PIGGY BANKS When a child breaks open her piggy bank, she finds a total of 64 coins, consisting of nickels, dimes, and quarters. The total value of the coins is $6. If the nickels were dimes, and the dimes were nickels, the value of the coins would be $5. How many nickels, dimes, and quarters were in the piggy bank? nickels: 20, dimes: 40, quarters: 4

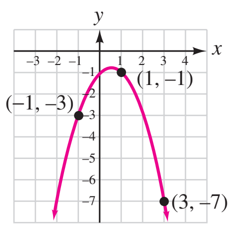
24. THEATER SEATING The illustration shows the cash receipts and the ticket prices from two sold-out Sunday performances of a play. Find the number of seats in each of the three sections of the 800-seat theater.
founder's circle: 100, box seats: 300, promenade: 400



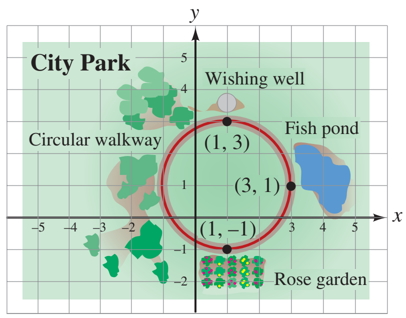
25. ASTRONOMY Comets have elliptical orbits, but the orbits of some comets are so large that they are indistinguishable from parabolas. Find an equation of the form $y = ax^2 + bx + c$ for the parabola that closely describes the orbit of the comet shown in the illustration. $y = \frac{1}{2}x^2 - 2x - 1$



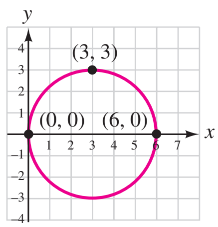
26. CURVE FITTING Find an equation of the form $y = ax^2 + bx + c$ for the parabola shown in the illustration. $y = -x^2 + x - 1$



27. WALKWAYS A circular sidewalk is to be constructed in a city park. The walk is to pass by three particular areas of the park, as shown in the illustration. If an equation of a circle is of the form $x^2 + y^2 + Cx + Dy + E = 0$, find an equation that describes the path of the sidewalk by determining C, D, and E. $x^2 + y^2 - 2x - 2y - 2 = 0$



28. CURVE FITTING The equation of a circle is of the form $x^2 + y^2 + Cx + Dy + E = 0$. Find an equation of the circle shown in the illustration by determining C, D, and E.
$x^2 - 6x + y^2 = 0$



WRITING

29. Explain why the following problem does not give enough information to answer the question: The sum of three integers is 48. If the first integer is doubled, the sum is 60. Find the integers.

30. Write an application problem that can be solved using a system of three equations in three variables.

REVIEW

Determine whether each equation defines y to be a function of x. If it does not, find two ordered pairs where more than one value of y corresponds to a single value of x.

31. $y = \dfrac{1}{x}$
yes

32. $y^4 = x$
no; (1, 1), (1, −1)

33. $xy = 9$
yes

34. $y = |x|$
yes

35. $x + 1 = |y|$
no; (1, 2), (1, −2)

36. $y = \dfrac{1}{x^2}$
yes

37. $y^2 = x$
no; (4, 2), (4, −2)

38. $x = |y|$
no; (1, 1), (1, −1)

Objectives

1. Define a matrix and determine its order.
2. Write the augmented matrix for a system.
3. Perform elementary row operations on matrices.
4. Use matrices to solve a system of two equations.
5. Use matrices to solve a system of three equations.
6. Use matrices to identify inconsistent systems and dependent equations.

SECTION 3.6

Solving Systems of Equations Using Matrices

In this section, we will discuss another way to solve systems of linear equations. This technique uses a mathematical tool called a *matrix* in a series of steps that are based on the elimination (addition) method.

1 Define a matrix and determine its order.

Another method of solving systems of equations involves rectangular arrays of numbers called **matrices** (plural for **matrix**).

Matrices

A **matrix** is any rectangular array of numbers arranged in rows and columns, written within brackets.

Some examples of matrices are

$$A = \begin{array}{c} \begin{bmatrix} 1 & -3 & 8 \\ 2 & 5 & -1 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Column} & \text{Column} & \text{Column} \\ 1 & 2 & 3 \end{array} \end{array} \qquad B = \begin{array}{c} \begin{bmatrix} 1 & 4 & -2 & -4 \\ 6 & -2 & 6 & 1 \\ 3 & 8 & -3 & 12 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \leftarrow \text{Row 3} \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{Column} & \text{Column} & \text{Column} & \text{Column} \\ 1 & 2 & 3 & 4 \end{array} \end{array}$$

The numbers in each matrix are called **elements.** A matrix with m rows and n columns has **order** $m \times n$, which is read "m by n." Because matrix A has two rows and three columns, its order is a 2×3 matrix (read "2 by 3" matrix). Matrix B is a 3×4 matrix (three rows and four columns).

2 Write the augmented matrix for a system.

To show how to use matrices to solve systems of linear equations, we consider the system

$$\begin{cases} x - y = 4 \\ 2x + y = 5 \end{cases}$$

which can be represented by the following matrix, called an **augmented matrix:**

$$\left[\begin{array}{cc:c} 1 & -1 & 4 \\ 2 & 1 & 5 \end{array}\right]$$

Each row of the augmented matrix represents one equation of the system. The first two columns of the augmented matrix are determined by the coefficients of x and y in the equations of the system. The last column is determined by the constants in the equations.

$$\begin{array}{c} \left[\begin{array}{cc:c} 1 & -1 & 4 \\ 2 & 1 & 5 \end{array}\right] \begin{array}{l} \text{This row represents the equation } x - y = 4. \\ \text{This row represents the equation } 2x + y = 5. \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{Coefficients of } x & \text{Coefficients of } y & \text{Constants} \end{array} \end{array}$$

EXAMPLE 1 Represent each system of equations using an augmented matrix:

a. $\begin{cases} 3x + y = 11 \\ x - 8y = 0 \end{cases}$ **b.** $\begin{cases} 2a + b - 3c = -3 \\ 9a + 4c = 2 \\ a - b - 6c = -7 \end{cases}$

Strategy First we note that each of the equations are in general form. Then we will write the coefficients of the variables and the constants from each equation in rows to form a matrix. The coefficients are written to the left of the vertical dashed line and constants to the right.

WHY In an augmented matrix, each row represents one equation of the system.

Solution

a. $\begin{cases} 3x + y = 11 \\ x - 8y = 0 \end{cases} \begin{matrix} \leftrightarrow \\ \leftrightarrow \end{matrix} \left[\begin{array}{rr:r} 3 & 1 & 11 \\ 1 & -8 & 0 \end{array}\right]$

b. $\begin{cases} 2a + b - 3c = -3 \\ 9a + 4c = 2 \\ a - b - 6c = -7 \end{cases} \begin{matrix} \leftrightarrow \\ \leftrightarrow \\ \leftrightarrow \end{matrix} \left[\begin{array}{rrr:r} 2 & 1 & -3 & -3 \\ 9 & 0 & 4 & 2 \\ 1 & -1 & -6 & -7 \end{array}\right]$

Self Check 1

Represent each system using an augmented matrix:

a. $\begin{cases} 2x - 4y = 9 \\ 5x - y = -2 \end{cases}$

b. $\begin{cases} a + b - c = -4 \\ -2b + 7c = 0 \\ 10a + 8b - 4c = 5 \end{cases}$

Now Try **Problems 18 and 20**

Self Check 1 Answers

a. $\left[\begin{array}{rr:r} 2 & -4 & 9 \\ 5 & -1 & -2 \end{array}\right]$

b. $\left[\begin{array}{rrr:r} 1 & 1 & -1 & -4 \\ 0 & -2 & 7 & 0 \\ 10 & 8 & -4 & 5 \end{array}\right]$

Teaching Example 1 Represent each system using an augmented matrix:

a. $\begin{cases} 5a - 7b = 11 \\ 3a + 5b = 2 \end{cases}$

a. $\begin{cases} 2a + b - 3c = 1 \\ a - 2b + c = 7 \\ 3a + b - 4c = 2 \end{cases}$

Answers:

a. $\left[\begin{array}{rr:r} 5 & -7 & 11 \\ 3 & 5 & 2 \end{array}\right]$

b. $\left[\begin{array}{rrr:r} 2 & 1 & -3 & 1 \\ 1 & -2 & 1 & 7 \\ 3 & 1 & -4 & 2 \end{array}\right]$

3 Perform elementary row operations on matrices.

To solve a 2×2 system of equations by **Gaussian elimination,** we transform the augmented matrix into the following matrix, which has 1's down its main diagonal and a 0 below the 1 in the first column.

$$\left[\begin{array}{cc:c} 1 & a & b \\ 0 & 1 & c \end{array}\right]$$

a, b, and c represent real numbers

Main diagonal

To write the augmented matrix in this form, called **row echelon form,** we use three operations called **elementary row operations.**

Elementary Row Operations

Type 1: Any two rows of a matrix can be interchanged.
Type 2: Any row of a matrix can be multiplied by a nonzero constant.
Type 3: Any row of a matrix can be changed by adding a nonzero constant multiple of another row to it.

- A type 1 row operation corresponds to interchanging two equations of the system.
- A type 2 row operation corresponds to multiplying both sides of an equation by a nonzero constant.
- A type 3 row operation corresponds to adding a nonzero multiple of one equation to another.

None of these row operations will change the solution of the given system of equations.

The Language of Algebra Two matrices are *equivalent* if they represent systems that have the same solution set.

Self Check 2

Refer to Example 2.

a. Interchange the rows of matrix B.

b. To the numbers in row 1 of matrix A, add the results of multiplying each number in row 2 by -2.

c. Interchange rows 2 and 3 of matrix C.

Now Try **Problems 22 and 24**

Self Check 2 Answers

a. $\left[\begin{array}{cc:c} 4 & -8 & 0 \\ 1 & -1 & 2 \end{array}\right]$

b. $\left[\begin{array}{cc:c} 0 & 20 & -3 \\ 1 & -8 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc:c} 2 & 1 & -8 & 4 \\ 0 & 0 & -6 & 24 \\ 0 & 1 & 4 & -2 \end{array}\right]$

Teaching Example 2 Refer to Example 2.

a. Interchange rows 1 and 2 of matrix C.

b. Multiply row 2 of matrix B by $\frac{1}{4}$.

c. To the numbers in row 1 of matrix A, add the results of multiplying each number in row 2 by -2.

Answers:

a. $\left[\begin{array}{ccc:c} 0 & 1 & 4 & -2 \\ 2 & 1 & -8 & 4 \\ 0 & 0 & -6 & 24 \end{array}\right]$

b. $\left[\begin{array}{cc:c} 1 & -1 & 2 \\ 1 & -2 & 0 \end{array}\right]$

c. $\left[\begin{array}{cc:c} 0 & 20 & -3 \\ 1 & -8 & 0 \end{array}\right]$

EXAMPLE 2 Consider the matrices:

$$A = \left[\begin{array}{cc:c} 2 & 4 & -3 \\ 1 & -8 & 0 \end{array}\right] \quad B = \left[\begin{array}{cc:c} 1 & -1 & 2 \\ 4 & -8 & 0 \end{array}\right] \quad C = \left[\begin{array}{ccc:c} 2 & 1 & -8 & 4 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & -6 & 24 \end{array}\right]$$

a. Interchange rows 1 and 2 of matrix A.

b. Multiply row 3 of matrix C by $-\frac{1}{6}$.

c. To the numbers in row 2 of matrix B, add the results of multiplying each number in row 1 by -4.

Strategy We will perform elementary row operations on each matrix as if we were performing those operations on the equations of a system.

WHY The rows of an augmented matrix correspond to the equations of a system.

Solution

a. Interchanging the rows of matrix A, we obtain: $\left[\begin{array}{cc:c} 1 & -8 & 0 \\ 2 & 4 & -3 \end{array}\right]$

b. We multiply each number in row 3 of matrix C by $-\frac{1}{6}$. Rows 1 and 2 remain unchanged.

$$\left[\begin{array}{ccc:c} 2 & 1 & -8 & 4 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

We can represent the instruction to multiply the third row by $-\frac{1}{6}$ with the notation $-\frac{1}{6}R_3$.

c. If we multiply each number in row 1 of matrix B by -4, we get

$$-4 \quad 4 \quad -8$$

We then add these numbers to row 2. (Note that row 1 remains unchanged.)

$$\left[\begin{array}{cc:c} 1 & -1 & 2 \\ 4 + \mathbf{(-4)} & -8 + \mathbf{4} & 0 + \mathbf{(-8)} \end{array}\right]$$

We can abbreviate this procedure using the notation $-4R_1 + R_2$, which means "Multiply row 1 by -4 and add the result to row 2."

After simplifying, we have the matrix

$$\left[\begin{array}{cc:c} 1 & -1 & 2 \\ 0 & -4 & -8 \end{array}\right]$$

4 Use matrices to solve a system of two equations.

We now solve a system of two linear equations using the **Gaussian elimination** process, which involves a series of elementary row operations.

Self Check 3

Solve the system using matrices:

$$\begin{cases} 3x - 2y = -5 \\ x - y = -4 \end{cases} \quad (3, 7)$$

Now Try **Problem 30**

Teaching Example 3 Solve the system using matrices: $\begin{cases} 3x + 4y = 11 \\ x - 3y = -5 \end{cases}$

Answer:
(1, 2)

EXAMPLE 3 Solve the system using matrices: $\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$

Strategy We will represent the system with an augmented matrix and use a series of elementary row operations to produce an equivalent matrix in row echelon form.

WHY When the resulting row echelon form matrix is written as a system of two equations, we will know the value of one variable, and the value of the other can be found using substitution.

Solution

We can represent the system with the following augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -1 & 4 \end{array}\right]$$

First, we want to get a 1 in the top row of the first column where the red 2 is. This can be achieved by applying a type 1 row operation: Interchange rows 1 and 2.

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 2 & 1 & 5 \end{array}\right]$$ Interchanging row 1 and row 2 can be abbreviated as $R_1 \leftrightarrow R_2$.

To get a 0 under the 1 in the first column, we use a type 3 row operation. To row 2, we add the results of multiplying each number in row 1 by -2.

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 3 & -3 \end{array}\right]$$ $-2R_1 + R_2$.

To get a 1 in the bottom row of the second column, we use a type 2 row operation: Multiply row 2 by $\frac{1}{3}$.

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & -1 \end{array}\right]$$ $\frac{1}{3}R_2$.

This augmented matrix represents the equations

$$1x - 1y = 4$$
$$0x + 1y = -1$$

Writing the equations without the coefficients of 1 and -1, we have

(1) $x - y = 4$

(2) $y = -1$

From Equation 2, we see that $y = -1$. We can back substitute -1 for y in Equation 1 to find x.

$$x - y = 4$$
$$x - (-1) = 4 \quad \text{Substitute } -1 \text{ for } y.$$
$$x + 1 = 4 \quad -(-1) = 1.$$
$$x = 3 \quad \text{Subtract 1 from both sides.}$$

The solution of the system is $(3, -1)$. Verify that this ordered pair satisfies the original system.

In general, if a system of linear equations has a single solution, we can use the following steps to solve the system using matrices.

Solving Systems of Linear Equations Using Matrices

1. Write an augmented matrix for the system.
2. Use elementary row operations to transform the augmented matrix into a matrix with 1's down its main diagonal and 0's under the 1's.
3. When step 2 is complete, write the resulting system. Then use back substitution to find the solution.
4. Check the proposed solution in each equation of the original system.

5 Use matrices to solve a system of three equations.

To show how to use matrices to solve systems of three linear equations containing three variables, we consider the system that can be represented by the augmented matrix to its right.

$$\begin{cases} x - 2y - z = 6 \\ 2x + 2y - z = 1 \\ -x - y + 2z = 1 \end{cases} \qquad \left[\begin{array}{rrr|r} 1 & -2 & -1 & 6 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 2 & 1 \end{array}\right]$$

To solve a 3×3 system by Gaussian elimination, we transform the augmented matrix into a matrix with 1's down its main diagonal and 0's below its main diagonal.

$$\left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

$a, b, c, \ldots, f$ represent real numbers

Main diagonal

Self Check 4

Use matrices to solve the system:

$$\begin{cases} 2x - y + z = 5 \\ x + y - z = -2 \\ -x + 2y + 2z = 1 \end{cases}$$

$(1, -1, 2)$

Now Try **Problem 34**

Teaching Example 4 Use matrices to solve the system:

$$\begin{cases} 2x + 3y - z = -8 \\ x - y - z = -2 \\ -4x + 3y + z = 6 \end{cases}$$

Answer:
$(-2, -1, 1)$

EXAMPLE 4 Use matrices to solve the system: $\begin{cases} 3x + y + 5z = 8 \\ 2x + 3y - z = 6 \\ x + 2y + 2z = 10 \end{cases}$

Strategy We will represent the system with an augmented matrix and use a series of elementary row operations to produce an equivalent matrix in row echelon form.

WHY When the resulting row echelon form matrix is written as a system of three equations, we will know the value of one variable, and the values of the other two can be found using substitution.

Solution

This system can be represented by the augmented matrix shown on the left below. To get a 1 in the first column in place of the red 3, we perform a type 1 row operation and interchange rows 1 and 3, as shown on the right.

$$\left[\begin{array}{rrr|r} \mathbf{3} & 1 & 5 & 8 \\ 2 & 3 & -1 & 6 \\ 1 & 2 & 2 & 10 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ \mathbf{2} & 3 & -1 & 6 \\ 3 & 1 & 5 & 8 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

To get a 0 under the 1 in the first column in place of the red 2, we perform a type 3 row operation: Multiply row 1 by -2 to get

$$-2 \quad \mathbf{-4} \quad -4 \quad -20$$

and add these numbers to the entries in row 2.

$$\begin{bmatrix} 1 & 2 & 2 & 10 \\ 2 + (-2) & 3 + (-4) & -1 + (-4) & 6 + (-20) \\ 3 & 1 & 5 & 8 \end{bmatrix}$$

After simplifying we get

$$\begin{bmatrix} 1 & 2 & 2 & 10 \\ 0 & \mathbf{-1} & \mathbf{-5} & \mathbf{-14} \\ 3 & 1 & 5 & 8 \end{bmatrix} \quad -2R_1 + R_2$$

To get a 0 under the 0 in the first column in place of the red 3, we perform another type 3 row operation: Multiply row 1 by -3 and add the results to row 3. Row 1 remains the same. (See below right.)

$$\left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ 0 & -1 & -5 & -14 \\ \mathbf{3} & 1 & 5 & 8 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 2 & 2 & 10 \\ 0 & \mathbf{-1} & -5 & -14 \\ 0 & -5 & -1 & -22 \end{array}\right] \quad -3R_1 + R_3$$

To get a 1 under the 2 in the second column in place of the red -1, we perform a type 2 row operation: Multiply row 2 by -1. (See below left.)

To get a 0 under the 1 in the second column in place of the red -5, we perform a type 3 row operation: Multiply row 2 by 5 and add the results to row 3. Row 2 remains the same. (See below right.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & 1 & 5 & 14 \\ 0 & -5 & -1 & -22 \end{array}\right] \quad -1R_2 \qquad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & 1 & 5 & 14 \\ 0 & 0 & 24 & 48 \end{array}\right] \quad 5R_2 + R_3$$

To get a 1 under the 5 in the third column in place of the red 24, we perform a type 2 row operation: Multiply row 3 by $\frac{1}{24}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 0 & 1 & 5 & 14 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \tfrac{1}{24}R_3$$

The final matrix represents the system

$$\begin{cases} 1x + 2y + 2z = 10 \\ 0x + 1y + 5z = 14 \\ 0x + 0y + 1z = 2 \end{cases} \quad \text{which can be written without the coefficients of 0 and 1 as} \quad \begin{cases} x + 2y + 2z = 10 & \textbf{(1)} \\ y + 5z = 14 & \textbf{(2)} \\ z = 2 & \textbf{(3)} \end{cases}$$

From equation 3, we can read that z is 2. To find y, we back substitute 2 for z in equation 2 and solve for y:

$$\begin{aligned} y + 5z &= 14 && \text{This is equation 2.} \\ y + 5(2) &= 14 && \text{Substitute 2 for } z. \\ y + 10 &= 14 \\ y &= 4 && \text{Subtract 10 from both sides.} \end{aligned}$$

Thus, y is 4. To find x, we back substitute 2 for z and 4 for y in equation 1 and solve for x:

$$\begin{aligned} x + 2y + 2z &= 10 && \text{This is equation 1.} \\ x + 2(4) + 2(2) &= 10 && \text{Substitute 2 for } z \text{ and 4 for } y. \\ x + 8 + 4 &= 10 \\ x + 12 &= 10 \\ x &= -2 && \text{Subtract 12 from both sides.} \end{aligned}$$

Thus, x is -2. The solution of the given system is $(-2, 4, 2)$. Verify that this ordered triple satisfies each equation of the original system.

6 Use matrices to identify inconsistent systems and dependent equations.

In the next example, we consider a system with no solution.

EXAMPLE 5 Use matrices to solve the system: $\begin{cases} x + y = -1 \\ -3x - 3y = -5 \end{cases}$

Strategy We will represent the system with an augmented matrix and use a series of elementary row operations to produce an equivalent matrix in row echelon form.

WHY When the resulting row echelon form matrix is written as a system of two equations, we can determine whether the system is consistent or inconsistent and whether the equations are dependent or independent.

Self Check 5

Use matrices to solve the system: $\begin{cases} 4x - 8y = 9 \\ x - 2y = -5 \end{cases}$

Now Try Problem 38

Self Check 5 Answer

no solution, $\varnothing$, inconsistent system

Teaching Example 5 Use matrices to solve the system: $\begin{cases} x + 3y = 4 \\ -2x - 6y = 16 \end{cases}$

Answer:
no solution, $\varnothing$; inconsistent system

Solution

This system can be represented by the augmented matrix

$$\left[\begin{array}{cc:c} 1 & 1 & -1 \\ -3 & -3 & -5 \end{array}\right]$$

Since the matrix has a 1 in the top row of the first column, we proceed to get a 0 under it by multiplying row 1 by 3 and adding the results to row 2.

$$\left[\begin{array}{cc:c} 1 & 1 & -1 \\ 0 & 0 & -8 \end{array}\right] \quad 3R_1 + R_2.$$

This matrix represents the system

$$\begin{cases} x + y = -1 \\ 0 + 0 = -8 \end{cases}$$

This system has no solution, because the second equation is never true. Therefore, the system is inconsistent. It has no solutions.

In the next example, we consider a system with infinitely many solutions.

Self Check 6

Use matrices to solve the system:
$$\begin{cases} 5x - 10y + 15z = 35 \\ -3x + 6y - 9z = -21 \\ 2x - 4y + 6z = 14 \end{cases}$$

Now Try **Problem 42**

Self Check 6 Answer
There are infinitely many solutions, dependent equations

Teaching Example 6 Use matrices to solve the system: $\begin{cases} 3x + y - z = 4 \\ x - y + z = 4 \\ 6x + 2y - 2z = 8 \end{cases}$

Answer:
infinitely many solutions, dependent equations

EXAMPLE 6

Use matrices to solve the system: $\begin{cases} 2x + 3y - 4z = 6 \\ 4x + 6y - 8z = 12 \\ -6x - 9y + 12z = -18 \end{cases}$

Strategy We will represent the system with an augmented matrix and use a series of elementary row operations to produce an equivalent matrix in row echelon form.

WHY When the resulting row echelon form matrix is written as a system of three equations, we can determine whether the system is consistent or inconsistent and whether the equations are dependent or independent.

Solution

This system can be represented by the augmented matrix

$$\left[\begin{array}{ccc:c} 2 & 3 & -4 & 6 \\ 4 & 6 & -8 & 12 \\ -6 & -9 & 12 & -18 \end{array}\right]$$

To get a 1 in the top row of the first column, we multiply row 1 by $\frac{1}{2}$.

$$\left[\begin{array}{ccc:c} 1 & \frac{3}{2} & -2 & 3 \\ 4 & 6 & -8 & 12 \\ -6 & -9 & 12 & -18 \end{array}\right] \quad \tfrac{1}{2}R_1.$$

Next, we want to get 0's under the 1 in the first column. This can be achieved by multiplying row 1 by -4 and adding the results to row 2, and multiplying row 1 by 6 and adding the results to row 3.

$$\left[\begin{array}{ccc:c} 1 & \frac{3}{2} & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \\ -4R_1 + R_2. \\ 6R_1 + R_3. \end{array}$$

The last matrix represents the system

$$\begin{cases} x + \frac{3}{2}y - 2z = 3 \\ 0x + 0y + 0z = 0 \\ 0x + 0y + 0z = 0 \end{cases}$$

If we clear the first equation of fractions, we have the system

$$\begin{cases} 2x + 3y - 4z = 6 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

This system has dependent equations and infinitely many solutions.

ANSWERS TO SELF CHECKS

1. a. $\left[\begin{array}{cc|c} 2 & -4 & 9 \\ 5 & -1 & -2 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 1 & -1 & -4 \\ 0 & -2 & 7 & 0 \\ 10 & 8 & -4 & 5 \end{array}\right]$ **2. a.** $\left[\begin{array}{cc|c} 4 & -8 & 0 \\ 1 & -1 & 2 \end{array}\right]$

b. $\left[\begin{array}{cc|c} 0 & 20 & -3 \\ 1 & -8 & 0 \end{array}\right]$ **c.** $\left[\begin{array}{ccc|c} 2 & 1 & -8 & 4 \\ 0 & 0 & -6 & 24 \\ 0 & 1 & 4 & -2 \end{array}\right]$ **3.** $(3, 7)$ **4.** $(1, -1, 2)$

5. no solution, ∅; inconsistent system **6.** infinitely many solutions, dependent equations

SECTION 3.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. A matrix is a rectangular array of numbers.

2. The numbers in a matrix are called its elements.

3. A 3×4 matrix has 3 rows and 4 columns.

4. Elementary row operations are used to produce new matrices that lead to the solution of a system.

5. A matrix that represents the equations of a system is called an augmented matrix.

6. The augmented matrix $\left[\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & 1 & 4 \end{array}\right]$ has 1's down its main diagonal.

CONCEPTS

For each matrix, tell the number of rows and the number of columns.

7. $\left[\begin{array}{cc|c} 4 & 6 & -1 \\ \frac{1}{2} & 9 & -3 \end{array}\right]$ 2×3

8. $\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$ 3×4

9. Write the system of equations represented by the augmented matrix. Then use back substitution to find the solution.

$$\left[\begin{array}{cc|c} 1 & -1 & -10 \\ 0 & 1 & 6 \end{array}\right] \quad \begin{cases} x - y = -10 \\ y = 6 \end{cases} \quad (-4, 6)$$

10. Write the system of equations represented by the augmented matrix. Then use back substitution to find the solution.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -16 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad \begin{cases} x - 2y + z = -16 \\ y + 2z = 8 \\ z = 4 \end{cases}$$

$(-20, 0, 4)$

11. Matrices were used to solve a system of two linear equations. The final matrix is shown here. Explain what the result tells about the system.

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 0 & 2 \end{array}\right]$$

It has no solution. The system is inconsistent.

12. Matrices were used to solve a system of two linear equations. The final matrix is shown here. What does the result tell about the equations?

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 0 & 0 \end{array}\right]$$

They are dependent. The system has infinitely many solutions.

NOTATION

13. Consider the matrix: $A = \left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ 1 & 5 & -2 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

a. Explain what is meant by $\frac{1}{3}R_1$. Then perform the operation on matrix A.

multiply row 1 by $\frac{1}{3}$, $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 5 & -2 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

b. Explain what is meant by $-R_1 + R_2$. Then perform the operation on the answer to part a.

add -1 times row 1 to row 2, $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 3 & 1 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

Selected exercises available online at **www.webassign.net/brookscole**

14. Consider the matrix: $B = \left[\begin{array}{cc|c} -3 & 1 & -6 \\ 1 & -4 & 4 \end{array}\right]$

a. Explain what is meant by $R_1 \leftrightarrow R_2$. Then perform the operation on matrix B.

interchange rows 1 and 2, $\left[\begin{array}{cc|c} 1 & -4 & 4 \\ -3 & 1 & -6 \end{array}\right]$

b. Explain what is meant by $3R_1 + R_2$. Then perform the operation on the answer to part a.

add 3 times row 1 to row 2, $\left[\begin{array}{cc|c} 1 & -4 & 4 \\ 0 & -11 & 6 \end{array}\right]$

Complete each solution to solve the system.

15. Solve: $\begin{cases} 4x - y = 14 \\ x + y = 6 \end{cases}$

$$\left[\begin{array}{cc|c} 4 & -1 & 14 \\ 1 & 1 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 4 & -1 & 14 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -5 & -10 \end{array}\right] \quad -4R_1 + R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & 1 & 2 \end{array}\right] \quad -\tfrac{1}{5}R_2$$

This matrix represents the system

$$\begin{cases} x + y = 6 \\ y = 2 \end{cases}$$

The solution is $(4, 2)$.

16. Solve: $\begin{cases} 2x + 2y = 18 \\ x - y = 5 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 2 & 18 \\ 1 & -1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 9 \\ 1 & -1 & 5 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & -2 & -4 \end{array}\right] \quad -R_1 + R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & 1 & 2 \end{array}\right] \quad -\tfrac{1}{2}R_2$$

This matrix represents the system

$$\begin{cases} x + y = 9 \\ y = 2 \end{cases}$$

The solution is $(7, 2)$.

GUIDED PRACTICE

Represent each system with an augmented matrix. See Example 1.

17. $\begin{cases} x + 2y = 6 \\ 3x - y = -10 \end{cases}$ $\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 3 & -1 & -10 \end{array}\right]$

18. $\begin{cases} x + y + z = 4 \\ 2x + y - z = 1 \\ 2x - 3y = 1 \end{cases}$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 1 & -1 & 1 \\ 2 & -3 & 0 & 1 \end{array}\right]$

For each augmented matrix, give the system of equations that it represents.

19. $\left[\begin{array}{cc|c} 1 & 6 & 7 \\ 0 & 1 & 4 \end{array}\right]$ $\begin{cases} x + 6y = 7 \\ y = 4 \end{cases}$

20. $\left[\begin{array}{ccc|c} 2 & -2 & 9 & 1 \\ 3 & 1 & 1 & 0 \\ 2 & -6 & 8 & -7 \end{array}\right]$ $\begin{cases} 2x - 2y + 9z = 1 \\ 3x + y + z = 0 \\ 2x - 6y + 8z = -7 \end{cases}$

Perform each of the following elementary row operations on the augmented matrix $\left[\begin{array}{cc|c} -3 & 1 & -6 \\ 1 & -4 & 4 \end{array}\right]$. See Example 2.

21. $R_1 \leftrightarrow R_2$ $\left[\begin{array}{cc|c} 1 & -4 & 4 \\ -3 & 1 & -6 \end{array}\right]$

22. $5R_2$ $\left[\begin{array}{cc|c} -3 & 1 & -6 \\ 5 & -20 & 20 \end{array}\right]$

23. $-\dfrac{1}{3}R_1$ $\left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 2 \\ 1 & -4 & 4 \end{array}\right]$

24. $3R_2 + R_1$ $\left[\begin{array}{cc|c} 0 & -11 & 6 \\ 1 & -4 & 4 \end{array}\right]$

Perform each of the following elementary row operations on the augmented matrix $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ 1 & 5 & -2 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$. See Example 2.

25. $R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ -2 & 2 & -2 & 5 \\ 1 & 5 & -2 & 1 \end{array}\right]$

26. $-\dfrac{1}{2}R_3$ $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ 1 & 5 & -2 & 1 \\ 1 & -1 & 1 & -\frac{5}{2} \end{array}\right]$

27. $-R_1 + R_2$ $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ -2 & -1 & 7 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

28. $2R_2 + R_3$ $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ 1 & 5 & -2 & 1 \\ 0 & 12 & -6 & 7 \end{array}\right]$

Use matrices to solve each system. See Example 3.

29. $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$ $(1, 1)$

30. $\begin{cases} x + y = 3 \\ x - y = -1 \end{cases}$ $(1, 2)$

31. $\begin{cases} 2x + y = 1 \\ x + 2y = -4 \end{cases}$ $(2, -3)$

32. $\begin{cases} 5x - 4y = 10 \\ x - 7y = 2 \end{cases}$ $(2, 0)$

Use matrices to solve each system. See Example 4.

33. $\begin{cases} x + y + z = 6 \\ x + 2y + z = 8 \\ x + y + 2z = 9 \end{cases}$ (1, 2, 3)

34. $\begin{cases} x - y + z = 2 \\ x + 2y - z = 6 \\ 2x - y - z = 3 \end{cases}$ (3, 2, 1)

35. $\begin{cases} 3x + y - 3z = 5 \\ x - 2y + 4z = 10 \\ x + y + z = 13 \end{cases}$ (4, 5, 4)

36. $\begin{cases} 2x + y - 3z = -1 \\ 3x - 2y - z = -5 \\ x - 3y - 2z = -12 \end{cases}$ (1, 3, 2)

Use matrices to solve each system. If the equations of a system are dependent or if a system is inconsistent, so indicate. See Examples 5–6.

37. $\begin{cases} x - 3y = 9 \\ -2x + 6y = 18 \end{cases}$ no solution, inconsistent system

38. $\begin{cases} -6x + 12y = 10 \\ 2x - 4y = 8 \end{cases}$ no solution, inconsistent system

39. $\begin{cases} 4x + 4y = 12 \\ -x - y = -3 \end{cases}$ infinitely many solutions, dependent equations

40. $\begin{cases} 5x - 15y = 10 \\ 2x - 6y = 4 \end{cases}$ infinitely many solutions, dependent equations

41. $\begin{cases} 2x + y - z = 1 \\ x + 2y + 2z = 2 \\ 4x + 5y + 3z = 3 \end{cases}$ no solution, inconsistent system

42. $\begin{cases} x - 3y + 4z = 2 \\ 2x + y + 2z = 3 \\ 4x - 5y + 10z = 7 \end{cases}$ infinitely many solutions, dependent equations

43. $\begin{cases} x - y = 1 \\ 2x - z = 0 \\ 2y - z = -2 \end{cases}$ infinitely many solutions, dependent equations

44. $\begin{cases} x + y - 3z = 4 \\ 2x + 2y - 6z = 5 \\ -3x + y - z = 2 \end{cases}$ no solution, inconsistent system

TRY IT YOURSELF

Use matrices to solve each system of equations. If the equations of a system are dependent or if the system is inconsistent, so state.

45. $\begin{cases} 2x - y = -1 \\ x - 2y = 1 \end{cases}$ (−1, −1)

46. $\begin{cases} 2x - y = 0 \\ x + y = 3 \end{cases}$ (1, 2)

47. $\begin{cases} 3x + 4y = -12 \\ 9x - 2y = 6 \end{cases}$ (0, −3)

48. $\begin{cases} 2x - 3y = 16 \\ -4x + y = -22 \end{cases}$ (5, −2)

49. $\begin{cases} 3x - 2y + 4z = 4 \\ x + y + z = 3 \\ 6x - 2y - 3z = 10 \end{cases}$ (2, 1, 0)

50. $\begin{cases} 2x + 3y - z = -8 \\ x - y - z = -2 \\ -4x + 3y + z = 6 \end{cases}$ (−2, −1, 1)

51. $\begin{cases} 2a + b + 3c = 3 \\ -2a - b + c = 5 \\ 4a - 2b + 2c = 2 \end{cases}$ (−1, −1, 2)

52. $\begin{cases} 3a + 2b + c = 8 \\ 6a - b + 2c = 16 \\ -9a + b - c = -20 \end{cases}$ (2, 0, 2)

53. $\begin{cases} 6x + y - z = -2 \\ x + 2y + z = 5 \\ 5y - z = 2 \end{cases}$ (0, 1, 3)

54. $\begin{cases} 2x + 3y - 2z = 18 \\ 5x - 6y + z = 21 \\ 4y - 2z = 6 \end{cases}$ (8, 4, 5)

55. $\begin{cases} 5x + 3y = 4 \\ 3y - 4z = 4 \\ x + z = 1 \end{cases}$ (−4, 8, 5)

56. $\begin{cases} y + 2z = -2 \\ x + y = 1 \\ 2x - z = 0 \end{cases}$ (−1, 2, −2)

57. $\begin{cases} 2a - b + 4c = -2 \\ 5a + 8b + 7c = -8 \\ a + 3b + c = -3 \end{cases}$ no solution, inconsistent system

58. $\begin{cases} 2a + 3b + 4c = 6 \\ 2a - 3b - 4c = -4 \\ 4a + 6b + 8c = 12 \end{cases}$ dependent equations, infinitely many solutions

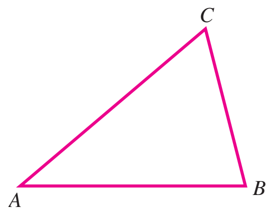
APPLICATIONS

Remember these facts from geometry: The sum of the measures of complementary angles is 90°, the sum of the measures of supplementary angles is 180°, and the sum of the measures of the interior angles of a triangle is 180°.

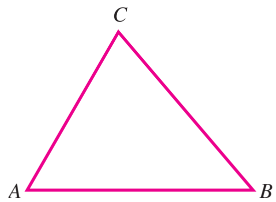
59. One angle measures 46° more than the measure of its complement. Find the measure of each angle. 22, 68

60. One angle measures 14° more than the measure of its supplement. Find the measure of each angle. 97, 83

61. In the illustration, $\angle B$ measures 25° more than the measure of $\angle A$, and the measure of $\angle C$ is 5° less than twice the measure of $\angle A$. Find the measure of each angle of the triangle. 40, 65, 75

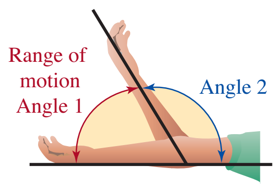


62. In the illustration, $\angle A$ measures 10° less than the measure of $\angle B$, and the measure of $\angle B$ is 10° less than the measure of $\angle C$. Find the measure of each angle of the triangle. 50, 60, 70

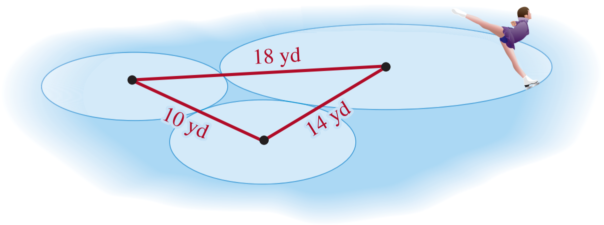


Write a system of equations to solve each problem. Then use matrices to solve it.

63. PHYSICAL THERAPY After an elbow injury, a volleyball player has restricted movement of her arm. Her range of motion (angle 1) is 28° less than angle 2. Find the measure of each angle. 76, 104



64. ICE SKATING Three circles were traced out by a figure skater during her performance. If the centers of the circles are the given distances apart, find the radius of each circle. 3 yd, 7 yd, 11 yd



WRITING

65. Explain what is meant by the phrase *back substitution.*

66. Explain how a type 3 row operation is similar to the addition method of solving a system of equations.

REVIEW

67. What is the formula used to find the slope of a line, given two points on the line? $m = \frac{y_2 - y_1}{x_2 - x_1}$ $(x_2 \neq x_1)$

68. What is the form of the equation of a horizontal line? Of a vertical line? $y = b, x = a$

69. What is the point–slope form of the equation of a line? $y - y_1 = m(x - x_1)$

70. What is the slope–intercept form of the equation of a line? $y = mx + b$

Objectives

1. Evaluate 2×2 and 3×3 determinants.
2. Use Cramer's rule to solve systems of two equations.
3. Use Cramer's rule to solve systems of three equations.

SECTION 3.7 Solving Systems of Equations Using Determinants

In this section, we will discuss another method for solving systems of linear equations. With this method, called *Cramer's rule,* we work with combinations of the coefficients and the constants of the equations written as *determinants.*

1 Evaluate 2 × 2 and 3 × 3 determinants.

An idea closely related to the concept of matrix is the **determinant.** A determinant is a number that is associated with a **square matrix,** a matrix that has the same number of rows and columns. For any square matrix A, the symbol $|A|$ represents the determinant of A. To write a determinant, we put the elements of a square matrix between two vertical lines.

Brackets: $\begin{bmatrix} 3 & 2 \\ 6 & 9 \end{bmatrix}$ Matrix

Vertical lines: $\begin{vmatrix} 3 & 2 \\ 6 & 9 \end{vmatrix}$ Determinant

Brackets: $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ Matrix

Vertical lines: $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$ Determinant

Like matrices, determinants are classified according to the number of rows and columns they contain. The determinant on the left is a 2×2 determinant. The other is a 3×3 determinant.

The determinant of a 2×2 matrix is the number that is equal to the product of the numbers on the main diagonal minus the product of the numbers on the other diagonal.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ Main diagonal

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ Other diagonal

Value of a 2 × 2 Determinant

If a, b, c, and d represent numbers, the **determinant** of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE 1

Evaluate: **a.** $\begin{vmatrix} 3 & 2 \\ 6 & 9 \end{vmatrix}$ **b.** $\begin{vmatrix} -5 & \frac{1}{2} \\ -1 & 0 \end{vmatrix}$

Strategy We will multiply the two numbers on the main diagonal and then multiply the numbers along the other diagonal and subtract the results.

WHY The value of a determinant of the form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$.

Solution

From the product of the numbers along the main diagonal, we subtract the product of the numbers along the other diagonal.

This is always minus.

a. $\begin{vmatrix} 3 & 2 \\ 6 & 9 \end{vmatrix} = 3(9) - 2(6)$

$= 27 - 12$

$= 15$

b. $\begin{vmatrix} -5 & \frac{1}{2} \\ -1 & 0 \end{vmatrix} = -5(0) - \frac{1}{2}(-1)$

$= 0 + \frac{1}{2}$

$= \frac{1}{2}$

Self Check 1

Evaluate: $\begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix}$ 10

Now Try **Problems 18 and 22**

Teaching Example 1 Evaluate: $\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix}$

Answer:
-5

A 3 × 3 determinant is evaluated by **expanding by minors.**

Value of a 3 × 3 Determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{Minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{Minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{Minor of } c_1}$$

To find the minor of a_1, we cross out the elements of the determinant that are in the same row and column as a_1:

$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$ The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.

To find the minor of b_1, we cross out the elements of the determinant that are in the same row and column as b_1:

$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & \cancel{b_2} & c_2 \\ a_3 & \cancel{b_3} & c_3 \end{vmatrix}$ The minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

To find the minor of c_1, we cross out the elements of the determinant that are in the same row and column as c_1:

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ a_2 & b_2 & \cancel{c_2} \\ a_3 & b_3 & \cancel{c_3} \end{vmatrix} \qquad \text{The minor of } c_1 \text{ is } \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Self Check 2

Evaluate: $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix}$ 0

Now Try **Problem 26**

Teaching Example 2 Evaluate:

$\begin{vmatrix} -2 & 3 & 2 \\ -1 & 4 & 5 \\ 1 & 7 & 6 \end{vmatrix}$

Answer:
33

EXAMPLE 2

Evaluate: $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$

Strategy We will expand the determinant along the first row using the numbers in the first row and their corresponding minors.

WHY We can then evaluate the resulting 2×2 determinants and simplify.

Solution

We evaluate this determinant by expanding by minors along the first row of the determinant.

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \underbrace{1\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}}_{\text{Minor of 1}} - \underbrace{3\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}}_{\text{Minor of 3}} + \underbrace{(-2)\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}}_{\text{Minor of } -2}$$

$$= 1(3 - 6) - 3(6 - 3) - 2(4 - 1)$$ Evaluate each 2×2 determinant.

$$= 1(-3) - 3(3) - 2(3)$$
$$= -3 - 9 - 6$$
$$= -18$$

We can evaluate a 3×3 determinant by expanding it along any row or column. To determine the signs between the terms of the expansion of a 3×3 determinant, we use the following array of signs.

Array of Signs for a 3×3 Determinant

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

This array of signs is often called the checkerboard pattern.

To remember the sign pattern, note that there is a + sign in the upper left position and the signs alternate for all the positions that follow.

Self Check 3

Evaluate $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$ by expanding along the last column. -18

Now Try **Problem 30**

EXAMPLE 3

Evaluate the determinant $\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix}$ by expanding along the middle column.

Strategy We will expand the determinant along the middle column using the numbers in the middle column and their corresponding minors.

WHY We can then evaluate the resulting 2×2 determinants and simplify.

Solution

This is the determinant of Example 2. To expand it along the middle column, we use the signs of the middle column of the array of signs:

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{vmatrix} = -3\underbrace{\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}}_{\text{Minor of 3}} + 1\underbrace{\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}}_{\text{Minor of 1}} - 2\underbrace{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}}_{\text{Minor of 2}}$$

$$= -3(6 - 3) + 1[3 - (-2)] - 2[3 - (-4)]$$
$$= -3(3) + 1(5) - 2(7)$$
$$= -9 + 5 - 14$$
$$= -18$$

Use the middle column of the checkerboard pattern:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Evaluate each 2×2 determinant.

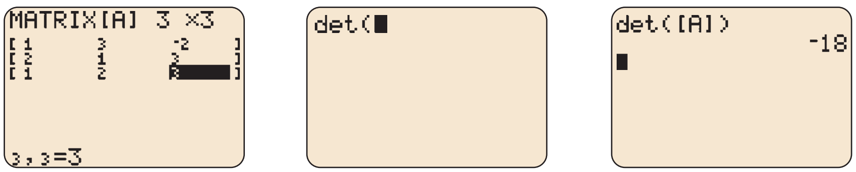
As expected, we get the same value as in Example 2.

Teaching Example 3 Evaluate $\begin{vmatrix} 1 & 4 & -3 \\ 2 & -1 & 6 \\ 5 & 0 & 9 \end{vmatrix}$ by expanding along the last row.

Answer: 24

Using Your CALCULATOR **Evaluating Determinants**

It is possible to use a graphing calculator to evaluate determinants. For example, to evaluate the determinant in Example 3, we first enter the matrix by pressing the $\boxed{\text{MATRIX}}$ key, selecting EDIT, and pressing the $\boxed{\text{ENTER}}$ key. Next, we enter the dimensions and the elements of the matrix to get figure (a). We then press $\boxed{\text{2nd}}$ $\boxed{\text{QUIT}}$ to clear the screen, press $\boxed{\text{MATRIX}}$, select MATH, and press 1 to get figure (b). We then press $\boxed{\text{MATRIX}}$, select NAMES, press 1, and press $\boxed{)}$ and $\boxed{\text{ENTER}}$ to get the value of the determinant. Figure (c) shows that the value of the determinant is −18.



(a) (b) (c)

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2 Use Cramer's rule to solve systems of two equations.

The method of using determinants to solve systems of linear equations is called **Cramer's rule,** named after the 18th-century mathematician Gabriel Cramer. To develop Cramer's rule, we consider the system

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

where x and y are variables and a, b, c, d, e, and f are constants.

If we multiply both sides of the first equation by d and multiply both sides of the second equation by $-b$, we can add the equations and eliminate y:

$$\begin{aligned} adx + bdy &= ed && \text{This is } d(ax + by) = d(e). \\ -bcx - bdy &= -bf && \text{This is } -b(cx + dy) = -b(f). \\ \hline adx - bcx \qquad &= ed - bf \end{aligned}$$

To solve for x, we use the distributive property to write $adx - bcx$ as $(ad - bc)x$ on the left-hand side and divide each side by $ad - bc$:

$$(ad - bc)x = ed - bf$$

$$x = \frac{ed - bf}{ad - bc} \quad \text{where } ad - bc \neq 0$$

We can find y in a similar manner. After eliminating the variable x, we get

$$y = \frac{af - ec}{ad - bc} \quad \text{where } ad - bc \neq 0$$

Determinants provide an easy way of remembering these formulas. Note that the denominator for both x and y is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The numerators can be expressed as determinants also:

$$x = \frac{ed - bf}{ad - bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{af - ec}{ad - bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

If we compare these formulas with the original system

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

we note that in the expressions for x and y on the preceding page, the denominator determinant is formed by using the coefficients a, b, c, and d of the variables in the equations. The numerator determinants are the same as the denominator determinant, except that the column of coefficients of the variable for which we are solving is replaced with the column of constants e and f.

Cramer's Rule for Two Equations in Two Variables

The solution of the system $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ is given by

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

If every determinant is 0, the system is consistent, but the equations are dependent. The system has infinitely many solutions.

If $D = 0$ and D_x or D_y is nonzero, the system is inconsistent and does not have a solution.

Self Check 4

Use Cramer's rule to solve:
$\begin{cases} 2x - 3y = -16 \\ 3x + 5y = 14 \end{cases}$ $(-2, 4)$

Now Try **Problem 34**

EXAMPLE 4

Use Cramer's rule to solve: $\begin{cases} 4x - 3y = 6 \\ -2x + 5y = 4 \end{cases}$

Strategy We will evaluate three determinants, D, D_x, and D_y.

WHY The x-value of the solution of the system is the quotient of D_x and D, and the y-value of the solution is the quotient of the two determinants D_y and D.

Solution

The value of x is the quotient of two determinants, D and D_x. The denominator determinant D is made up of the coefficients of x and y:

$$D = \begin{vmatrix} 4 & -3 \\ -2 & 5 \end{vmatrix}$$

To solve for x, we form the numerator determinant D_x from D by replacing its first column (the coefficients of x) with the column of constants (6 and 4).

To solve for y, we form the numerator determinant D_y from D by replacing its second column (the coefficients of y) with the column of constants (6 and 4).

To find the values of x and y, we evaluate each determinant:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 6 & -3 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ -2 & 5 \end{vmatrix}} = \frac{6(5) - (-3)(4)}{4(5) - (-3)(-2)} = \frac{30 + 12}{20 - 6} = \frac{42}{14} = 3$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 4 & 6 \\ -2 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ -2 & 5 \end{vmatrix}} = \frac{4(4) - 6(-2)}{14} = \frac{16 + 12}{14} = \frac{28}{14} = 2$$

The solution of this system is $(3, 2)$. Verify that $x = 3$ and $y = 2$ satisfy both equations.

Teaching Example 4 Use Cramer's rule to solve: $\begin{cases} 5x - 2y = 6 \\ 3x + 4y = -2 \end{cases}$

Answer: $\left(\frac{10}{13}, \frac{-14}{13}\right)$

EXAMPLE 5

Use Cramer's rule to solve: $\begin{cases} 7x = 8 - 4y \\ 2y = 3 - \frac{7}{2}x \end{cases}$

Strategy We will evaluate three determinants, D, D_x, and D_y.

WHY The x-value of the solution of the system is the quotient of D_x and D, and the y-value of the solution is the quotient of the two determinants D_y and D.

Solution

We multiply both sides of the second equation by 2 to eliminate the fraction and write the system in the form

$$\begin{cases} 7x + 4y = 8 \\ 7x + 4y = 6 \end{cases}$$

When we attempt to use Cramer's rule to solve this system for x, we obtain

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 8 & 4 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 7 & 4 \\ 7 & 4 \end{vmatrix}} = \frac{8}{0} \qquad \text{which is undefined.}$$

Since the denominator determinant D is 0 and the numerator determinant D_x is not 0, the system is inconsistent. It has no solutions.

We can see directly from the system that it is inconsistent. For any values of x and y, it is impossible that 7 times x plus 4 times y could be both 8 and 6.

Self Check 5

Use Cramer's rule to solve: $\begin{cases} 3x = 8 - 4y \\ y = \frac{5}{2} - \frac{3}{4}x \end{cases}$

Now Try **Problem 38**

Self Check 5 Answer
no solutions, $\varnothing$; inconsistent

Teaching Example 5 Use Cramer's rule to solve: $\begin{cases} 2x = 7 + y \\ y = 2x + 5 \end{cases}$

Answer:
no solution, $\varnothing$; inconsistent

3 Use Cramer's rule to solve systems of three equations.

Cramer's rule can be extended to solve systems of three linear equations in three variables.

Cramer's Rule for Three Equations in Three Variables

The solution of the system $\begin{cases} ax + by + cz = j \\ dx + ey + fz = k \\ gx + hy + iz = l \end{cases}$ is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad \text{and} \quad z = \frac{D_z}{D}$$

where

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ Only the coefficients of the variables.

$$D_x = \begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}$$ Replace the *x*-term coefficients with the constants.

$$D_y = \begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}$$ Replace the *y*-term coefficients with the constants.

$$D_z = \begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}$$ Replace the *z*-term coefficients with the constants.

If every determinant is 0, the system is consistent, but the equations are dependent. The system has infinitely many solutions.

If $D = 0$ and D_x or D_y or D_z is nonzero, the system is inconsistent and does not have a solution.

Self Check 6

Use Cramer's rule to solve:
$\begin{cases} x + y + 2z = 6 \\ 2x - y + z = 9 \\ x + y - 2z = -6 \end{cases}$ (2, −2, 3)

Now Try **Problem 42**

Teaching Example 6 Use Cramer's rule to solve: $\begin{cases} 6x + y - z = -2 \\ 5y - z = 2 \\ x + 2y + z = 5 \end{cases}$

Answer: (0, 1, 3)

EXAMPLE 6

Use Cramer's rule to solve: $\begin{cases} 2x + y + 4z = 12 \\ x + 2y + 2z = 9 \\ 3x - 3y - 2z = 1 \end{cases}$

Strategy We will evaluate four determinants, D, D_x, D_y, and D_z.

WHY The x-value of the solution of the system is the quotient of D_x and D, the y-value of the solution is the quotient of the two determinants D_y and D, and the z-value of the solution is the quotient of the two determinants D_z and D.

Solution

The denominator determinant D is the determinant formed by the coefficients of the variables. The numerator determinants, D_x, D_y, and D_z, are formed by replacing the coefficients of the variable being solved for by the column of constants. We form the quotients for x, y, and z and evaluate each determinant by expanding by minors about the first row:

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 12 & 1 & 4 \\ 9 & 2 & 2 \\ 1 & -3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 2 \\ 3 & -3 & -2 \end{vmatrix}}$$

$$= \frac{12\begin{vmatrix} 2 & 2 \\ -3 & -2 \end{vmatrix} - 1\begin{vmatrix} 9 & 2 \\ 1 & -2 \end{vmatrix} + 4\begin{vmatrix} 9 & 2 \\ 1 & -3 \end{vmatrix}}{2\begin{vmatrix} 2 & 2 \\ -3 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}}$$

$$= \frac{12(2) - 1(-20) + 4(-29)}{2(2) - 1(-8) + 4(-9)}$$

$$= \frac{-72}{-24}$$

$$= 3$$

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 12 & 4 \\ 1 & 9 & 2 \\ 3 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 2 \\ 3 & -3 & -2 \end{vmatrix}}$$

$$= \frac{2\begin{vmatrix} 9 & 2 \\ 1 & -2 \end{vmatrix} - 12\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} + 4\begin{vmatrix} 1 & 9 \\ 3 & 1 \end{vmatrix}}{-24}$$

$$= \frac{2(-20) - 12(-8) + 4(-26)}{-24}$$

$$= \frac{-48}{-24}$$

$$= 2$$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \\ 3 & -3 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 4 \\ 1 & 2 & 2 \\ 3 & -3 & -2 \end{vmatrix}}$$

$$= \frac{2\begin{vmatrix} 2 & 9 \\ -3 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 9 \\ 3 & 1 \end{vmatrix} + 12\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix}}{-24}$$

$$= \frac{2(29) - 1(-26) + 12(-9)}{-24}$$

$$= \frac{-24}{-24}$$

$$= 1$$

The solution of this system is (3, 2, 1).

ANSWERS TO SELF CHECKS

1. 10 **2.** 0 **3.** −18 **4.** (−2, 4) **5.** no solution, ∅; the system is inconsistent **6.** (2, −2, 3)

SECTION 3.7 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\begin{vmatrix} 2 & 1 \\ -6 & 1 \end{vmatrix}$ is a 2 × 2 <u>determinant</u>.

2. A <u>square</u> matrix has the same number of rows and columns.

3. The <u>minor</u> of b_1 in $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

4. In $\begin{vmatrix} 7 & -3 \\ 1 & 2 \end{vmatrix}$, 7 and 2 lie along the main <u>diagonal</u>.

5. A 3 × 3 determinant has 3 <u>rows</u> and 3 <u>columns</u>.

6. <u>Cramer's</u> rule uses determinants to solve systems of linear equations.

CONCEPTS

Fill in the blanks.

7. If the denominator determinant D for a system of equations is zero, the equations of the system are <u>dependent</u> or the system is <u>inconsistent</u>.

Selected exercises available online at
www.webassign.net/brookscole

8. To find the minor of 5, we cross out the elements of the determinant that are in the <u>same</u> row and column as 5.

$$\begin{vmatrix} 3 & 5 & 1 \\ 6 & -2 & 2 \\ 8 & -1 & 4 \end{vmatrix}$$

9. What is the value of $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$? $ad - bc$

10. $\begin{vmatrix} 5 & 1 & -1 \\ 8 & 7 & 4 \\ 9 & 7 & 6 \end{vmatrix} = -1\begin{vmatrix} 8 & 7 \\ 9 & 7 \end{vmatrix} - 4\begin{vmatrix} 5 & 1 \\ 9 & 7 \end{vmatrix} + 6\begin{vmatrix} 5 & 1 \\ 8 & 7 \end{vmatrix}$
In evaluating this determinant, about what row or column was it expanded? the third column

11. What is the denominator determinant D for the system $\begin{cases} 3x + 4y = 7 \\ 2x - 3y = 5 \end{cases}$? $\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}$

▶ **12.** What is the denominator determinant D for the system $\begin{cases} x + 2y = -8 \\ 3x + y - z = -2 \\ 8x + 4y - z = 6 \end{cases}$? $\begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \\ 8 & 4 & -1 \end{vmatrix}$

13. For the system $\begin{cases} 3x + 2y = 1 \\ 4x - y = 3 \end{cases}$, $D_x = -7$, $D_y = 5$, and $D = -11$. Find the solution of the system. $\left(\frac{7}{11}, -\frac{5}{11}\right)$

▶ **14.** For the system $\begin{cases} 2x + 3y - z = -8 \\ x - y - z = -2 \\ -4x + 3y + z = 6 \end{cases}$, $D_x = -28$, $D_y = -14$, $D_z = 14$ and $D = 14$. Find the solution. $(-2, -1, 1)$

NOTATION

Complete the evaluation of each determinant.

15.
$$\begin{aligned} \begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix} &= 5(6) - (-2)(-2) \\ &= 30 - 4 \\ &= 26 \end{aligned}$$

▶ **16.**
$$\begin{aligned} \begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 5 & 3 \end{vmatrix} &= 2\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} \\ &= 2(12 - 10) - 1(9 - 2) + 3(15 - 4) \\ &= 2(2) - 1(7) + 3(11) \\ &= 4 - 7 + 33 \\ &= 30 \end{aligned}$$

GUIDED PRACTICE

Evaluate each determinant. See Example 1.

17. $\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$ 8

▶ **18.** $\begin{vmatrix} 3 & -2 \\ -2 & 4 \end{vmatrix}$ 8

▶ **19.** $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$ -2

20. $\begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix}$ -2

21. $\begin{vmatrix} 10 & 0 \\ 1 & 20 \end{vmatrix}$ 200

22. $\begin{vmatrix} 1 & 15 \\ 15 & 0 \end{vmatrix}$ -225

23. $\begin{vmatrix} -6 & -2 \\ 15 & 4 \end{vmatrix}$ 6

▶ **24.** $\begin{vmatrix} 3 & -2 \\ 12 & -8 \end{vmatrix}$ 0

Evaluate each determinant. See Examples 2–3.

25. $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$ 1

26. $\begin{vmatrix} -1 & 2 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$ -13

27. $\begin{vmatrix} 1 & -2 & 3 \\ -2 & 1 & 1 \\ -3 & -2 & 1 \end{vmatrix}$ 26

28. $\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 3 \end{vmatrix}$ -9

29. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$ 0

▶ **30.** $\begin{vmatrix} 3 & 5 & 1 \\ 6 & -2 & 2 \\ 8 & -1 & 4 \end{vmatrix}$ -48

▶ **31.** $\begin{vmatrix} 1 & 2 & 1 \\ -3 & 7 & 3 \\ -4 & 3 & -5 \end{vmatrix}$ -79

▶ **32.** $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$ 0

Use Cramer's rule to solve each system of equations. See Example 4.

33. $\begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$ $(4, 2)$

34. $\begin{cases} x - y = 4 \\ 2x + y = 5 \end{cases}$ $(3, -1)$

35. $\begin{cases} 2x + 3y = 0 \\ 4x - 6y = -4 \end{cases}$ $\left(-\frac{1}{2}, \frac{1}{3}\right)$

▶ **36.** $\begin{cases} 4x - 3y = -1 \\ 8x + 3y = 4 \end{cases}$ $\left(\frac{1}{4}, \frac{2}{3}\right)$

Use Cramer's rule to solve each system of equations. If a system is inconsistent or if the equations are dependent, so indicate. See Example 5.

37. $\begin{cases} 3x + 2y = 11 \\ 6x + 4y = 11 \end{cases}$ no solution, inconsistent system

▶ **38.** $\begin{cases} 5x - 4y = 20 \\ 10x - 8y = 30 \end{cases}$ no solution, inconsistent system

▶ **39.** $\begin{cases} \frac{5}{6}x = 2 - y \\ 10x + 12y = 24 \end{cases}$ infinitely many solutions, dependent equations

40. $\begin{cases} 16x - 8y = 32 \\ x - 2 = \frac{y}{2} \end{cases}$ infinitely many solutions, dependent equations

Use Cramer's rule to solve each system of equations. See Example 6.

41. $\begin{cases} x + y + z = 4 \\ x + y - z = 0 \\ x - y + z = 2 \end{cases}$
(1, 1, 2)

42. $\begin{cases} x + y + z = 4 \\ x - y + z = 2 \\ x - y - z = 0 \end{cases}$
(2, 1, 1)

43. $\begin{cases} x + y + 2z = 7 \\ x + 2y + z = 8 \\ 2x + y + z = 9 \end{cases}$
(3, 2, 1)

▶ **44.** $\begin{cases} x + 2y + 2z = 10 \\ 2x + y + 2z = 9 \\ 2x + 2y + z = 1 \end{cases}$
(−2, −1, 7)

TRY IT YOURSELF

45. $\begin{cases} y = \dfrac{-2x + 1}{3} \\ 3x - 2y = 8 \end{cases}$
(2, −1)

▶ **46.** $\begin{cases} 2x + 3y = -1 \\ x = \dfrac{y - 9}{4} \end{cases}$
(−2, 1)

47. $\begin{cases} 2x + y + z = 5 \\ x - 2y + 3z = 10 \\ x + y - 4z = -3 \end{cases}$
(3, −2, 1)

▶ **48.** $\begin{cases} 3x + 2y - z = -8 \\ 2x - y + 7z = 10 \\ 2x + 2y - 3z = -10 \end{cases}$
(−2, 0, 2)

49. $\begin{cases} 4x - 3y = 1 \\ 6x - 8z = 1 \\ 2y - 4z = 0 \end{cases}$
$\left(-\frac{1}{2}, -1, -\frac{1}{2}\right)$

50. $\begin{cases} 4x + 3z = 4 \\ 2y - 6z = -1 \\ 8x + 4y + 3z = 9 \end{cases}$
$\left(\frac{3}{4}, \frac{1}{2}, \frac{1}{3}\right)$

51. $\begin{cases} 2x + 3y + 4z = 6 \\ 2x - 3y - 4z = -4 \\ 4x + 6y + 8z = 12 \end{cases}$
infinitely many solutions, dependent equations

52. $\begin{cases} x - 3y + 4z - 2 = 0 \\ 2x + y + 2z - 3 = 0 \\ 4x - 5y + 10z - 7 = 0 \end{cases}$
infinitely many solutions, dependent equations

53. $\begin{cases} x + y = 1 \\ \frac{1}{2}y + z = \frac{5}{2} \\ x - z = -3 \end{cases}$
(−2, 3, 1)

▶ **54.** $\begin{cases} \frac{1}{2}x + y + z + \frac{3}{2} = 0 \\ x + \frac{1}{2}y + z - \frac{1}{2} = 0 \\ x + y + \frac{1}{2}z + \frac{1}{2} = 0 \end{cases}$
$\left(\frac{9}{5}, -\frac{11}{5}, -\frac{1}{5}\right)$

55. $\begin{cases} 2x + y - z - 1 = 0 \\ x + 2y + 2z - 2 = 0 \\ 4x + 5y + 3z - 3 = 0 \end{cases}$ no solutions, inconsistent system

▶ **56.** $\begin{cases} 2x - y + 4z + 2 = 0 \\ 5x + 8y + 7z = -8 \\ x + 3y + z + 3 = 0 \end{cases}$ no solutions, inconsistent system

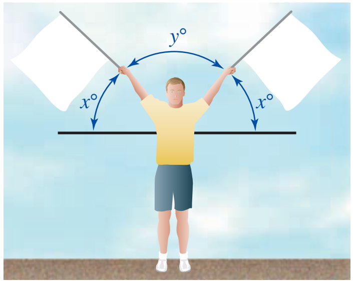
APPLICATIONS

Write a system of equations to solve each problem. Then use Cramer's rule to solve it.

57. INVENTORIES The table in the next column shows an end-of-the-year inventory report for a warehouse that supplies electronics stores. If the warehouse stocks two models of cordless telephones, one valued at \$67 and the other at \$100, how many of each model of phone did the warehouse have at the time of the inventory? 200 of the \$67 phones, 160 of the \$100 phones

Item	Number	Merchandise value
Televisions	800	\$1,005,450
Radios	200	\$15,785
Cordless phones	360	\$29,400

▶ **58.** SIGNALING A system of sending signals uses two flags held in various positions to represent letters of the alphabet. The illustration shows how the letter U is signaled. Find x and y, if y is to be 30° more than x. 50°, 80°



59. INVESTING A student wants to average a 6.6% return in the first year by investing \$20,000 in the three stocks listed in the table. Because HiTech is a high-risk investment, he wants to invest three times as much in SaveTel and OilCo combined as he invests in HiTech. How much should he invest in each stock? \$5,000 in HiTech, \$8,000 in SaveTel, \$7,000 in OilCo

Stock	Rate of return
HiTech	10%
SaveTel	5%
OilCo	6%

▶ **60.** INVESTING A woman wants to average a $7\frac{1}{3}$% return in the first year by investing \$30,000 in three certificates of deposit. She wants to invest five times as much in the 8% CD as in the 6% CD. How much should she invest in each CD?
\$2,500 in a 12-month CD, \$15,000 in a 24-month CD, \$12,500 in a 36-month CD

Type of CD	Rate of return
12-month	6%
24-month	7%
36-month	8%

Use a calculator with matrix capabilities to evaluate each determinant.

61. $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix}$ −23

▶ **62.** $\begin{vmatrix} -3 & 2 & -5 \\ 3 & -2 & 6 \\ 1 & -3 & 4 \end{vmatrix}$ −7

63. $\begin{vmatrix} 2 & 1 & -3 \\ -2 & 2 & 4 \\ 1 & -2 & 2 \end{vmatrix}$ 26

▶ **64.** $\begin{vmatrix} 4 & 2 & -3 \\ 2 & -5 & 6 \\ 2 & 5 & -2 \end{vmatrix}$ -108

WRITING

65. Explain how to find the minor of an element of a determinant.

66. Explain how to find x when solving a system of three linear equations by Cramer's rule. Use the words *coefficients* and *constants* in your explanation.

REVIEW

67. Are the lines $y = 2x - 7$ and $x - 2y = 7$ perpendicular? no

▶ **68.** Are the lines $y = 2x - 7$ and $2x - y = 10$ parallel? yes

69. Are the equations $y = 2x - 7$ and $f(x) = 2x - 7$ the same? yes

70. How are the graphs of $f(x) = x^2$ and $g(x) = x^2 - 2$ related? The graph of g is 2 units below the graph of f.

71. For the linear function $y = 2x - 7$, what variable is associated with the domain? x

72. Is the graph of a circle the graph of a function? no

73. The graph of a line passes through $(0, -3)$. Is this the x-intercept or the y-intercept of the line? y-intercept

74. What is the name of the function $f(x) = |x|$? the absolute value function

75. For $y = 2x^2 + 6x + 1$, what is the independent variable and what is the dependent variable? x, y

▶ **76.** If $f(x) = x^3 - x$, find $f(-1)$. 0

STUDY SKILLS CHECKLIST

Preparing for the Chapter 3 Test

In Chapter 3 you learned five methods to solve a system of linear equations. You also learned how to solve problems using systems of equations. As you prepare for the exam over this material, make sure you also review the following checklist.

☐ To check a proposed solution of a system of equations, be sure the coordinates of the ordered pair satisfies *both* equations.

Is $(3, -2)$ a solution of the system $\begin{cases} 3x + 4y = 1 \\ x + 2y = -1 \end{cases}$?

$$\begin{aligned} 3(3) + 4(-2) &= 1 \\ 9 - 8 &= 1 \\ 1 &= 1 \quad \text{True} \end{aligned} \qquad \begin{aligned} 3 + 2(-2) &= -1 \\ 3 - 4 &= -1 \\ -1 &= -1 \quad \text{True} \end{aligned}$$

Yes, $(3, -2)$ is a solution of the system.

☐ When using the substitution or the addition (elimination) method, remember to find the value of *both* the variables.

For the system of linear equations $\begin{cases} x = 2y - 3 \\ x + 4y = 3 \end{cases}$, the y-coordinate of the solution is $y = 1$.

To find the x-value, substitute 1 for y in either equation:

$$\begin{aligned} x &= 2y - 3 \\ x &= 2(1) - 3 \\ x &= 2 - 3 \\ x &= -1 \end{aligned}$$

The solution is $(-1, 1)$.

☐ The equations of a system must be written in standard form before the corresponding augmented matrix can be written.

$$\begin{cases} x = y + 4 & \text{Subtract } y \\ y = -2x + 5 & \text{Add } 2x \end{cases}$$

$$\begin{cases} x - y = 4 \\ 2x + y = 5 \end{cases} \quad \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 2 & 1 & 5 \end{array}\right]$$

☐ To evaluate a 2×2 determinant, multiply the numbers on the main diagonal *minus* multiply the numbers on the other diagonal and simplify.

$$\begin{aligned} \begin{vmatrix} 3 & -2 \\ 5 & -4 \end{vmatrix} &= 3(-4) - (-2)(5) \\ &= -12 - (-10) \\ &= -12 + 10 \\ &= -2 \end{aligned}$$

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

CHAPTER 3 SUMMARY AND REVIEW

SECTION 3.1 Solving Systems of Equations by Graphing

DEFINITIONS AND CONCEPTS | **EXAMPLES**

When two equations are considered at the same time, we say that they form a **system of equations.**

A **solution of a system** of equations in two variables is an ordered pair that satisfies both equations of the system.

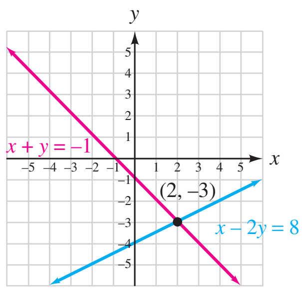
The ordered pair $(2, -3)$ is a solution of the system $\begin{cases} x + y = -1 \\ x - 2y = 8 \end{cases}$ because its coordinates $x = 2$ and $y = -3$ satisfy both equations of the system.

$$\begin{aligned} x + y &= -1 && \text{First equation} \\ 2 + (-3) &\stackrel{?}{=} -1 \\ -1 &= -1 && \text{True} \end{aligned} \qquad \begin{aligned} x - 2y &= 8 && \text{Second equation} \\ 2 - 2(-3) &\stackrel{?}{=} 8 \\ 8 &= 8 && \text{True} \end{aligned}$$

To **solve a system graphically:**

1. Graph each equation on the same rectangular coordinate system.
2. Find the coordinates of the point (or points) where the graphs intersect. These ordered pairs give the solution.
3. If the graphs have no point in common, the system has no solution. If the graphs are the same line, there are infinitely many solutions.
4. Check the proposed solution in each equation of the original system.

To solve the system $\begin{cases} x + y = -1 \\ x - 2y = 8 \end{cases}$ graphically, graph each equation as shown in the illustration.

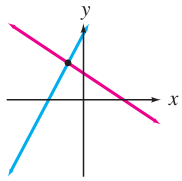


The graphs appear to intersect at the point $(2, -3)$. The check shown above verifies that $(2, -3)$ is the solution of the system.

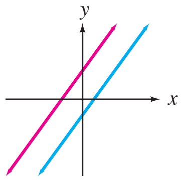
A system of equations that has at least one solution is called a **consistent system.** If the graphs are parallel lines, the system has no solution and is called an **inconsistent system.**

Equations with different graphs are called **independent equations.** If the graphs are the same line, the system has infinitely many solutions. The equations are called **dependent equations.**

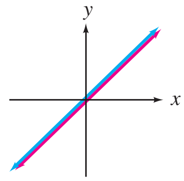
Since the system shown above has a solution, it is a *consistent system.* Since the graphs are different, the equations are *independent.*



Consistent System
Independent Equations



Inconsistent System
Independent Equations



Consistent System
Dependent Equations

REVIEW EXERCISES

Determine whether the ordered pair is a solution of the system of equations.

1. $\left(-1, \frac{1}{2}\right)$, $\begin{cases} x + 2y = 0 \\ x + 4y = 1 \end{cases}$ yes

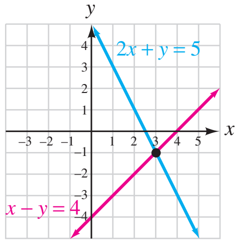
2. $(13, 23)$, $\begin{cases} 3a - 2b + 7 = 0 \\ -2a + b = -4 \end{cases}$ no

See the illustration.

3. Give three points that satisfy the equation $2x + y = 5$. $(1, 3), (2, 1), (4, -3)$ (answers may vary)

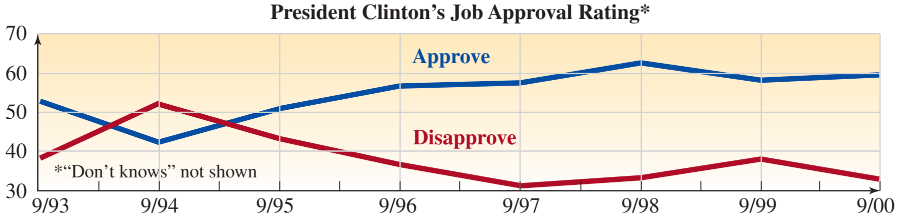
4. Give three points that satisfy the equation $x - y = 4$. $(0, -4), (2, -2), (4, 0)$ (answers may vary)

5. Find the solution of: $\begin{cases} 2x + y = 5 \\ x - y = 4 \end{cases}$ $(3, -1)$



6. POLITICS Explain the importance of the points of intersection of the graphs shown below.
President Clinton's job approval and disapproval ratings were the same: approximately 47% in 5/94 and approximately 48% in 5/95.

President Clinton's Job Approval Rating*

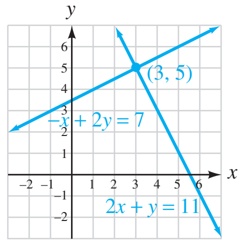


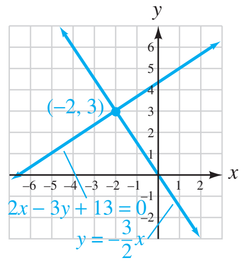
Source: *Los Angeles Times* (September 12, 1997)

Solve each system by the graphing method, if possible. If a system is inconsistent or if the equations are dependent, so indicate.

7. $\begin{cases} 2x + y = 11 \\ -x + 2y = 7 \end{cases}$

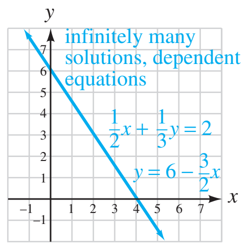
8. $\begin{cases} y = -\frac{3}{2}x \\ 2x - 3y + 13 = 0 \end{cases}$

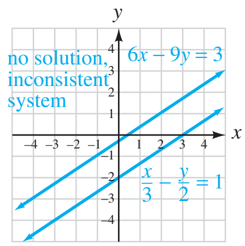




9. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 2 \\ y = 6 - \frac{3}{2}x \end{cases}$

10. $\begin{cases} \frac{x}{3} - \frac{y}{2} = 1 \\ 6x - 9y = 3 \end{cases}$





SECTION 3.2 Solving Systems of Equations Algebraically

DEFINITIONS AND CONCEPTS	EXAMPLES
To solve a system of two linear equations in x and y by the **substitution method:**	To solve the system $\begin{cases} x + y = -1 \\ x - 2y = 8 \end{cases}$ by substitution, follow these steps:
1. Solve one equation for either x or y to get the *substitution equation.*	**1.** Solve $x + y = -1$ for y. $y = -x - 1$ This is the substitution equation.

2. Substitute the resulting expression for that variable into the other equation and solve it.
3. Substitute the value of the variable found in step 2 into the substitution equation and solve that equation.
4. Check the proposed solution in each of the original equations. Write the solutions as an ordered pair.

If in step 2 the variable drops out and a false statement results, the system has **no solution.** If a true statement results, the system has **infinitely many solutions.**

2. Substitute $-x - 1$ for y in the second equation and solve for x.

$$\begin{aligned} x - 2(\mathbf{-x - 1}) &= 8 \\ x + 2x + 2 &= 8 && \text{Distribute.} \\ 3x + 2 &= 8 && \text{Combine like terms.} \\ 3x &= 6 && \text{Subtract 2 from both sides.} \\ x &= 2 && \text{Divide both sides by 3.} \end{aligned}$$

3. Substitute 2 for x in the substitution equation and solve for y.

$$\begin{aligned} y &= -\mathbf{x} - 1 \\ y &= -\mathbf{2} - 1 \\ y &= -3 \end{aligned}$$

4. The solution is $(2, -3)$. Verify this by checking it in each of the original equations.

To solve a system of two linear equations in x and y by the **addition (elimination) method:**

1. Write both equations in general form: $Ax + By = C$.
2. If necessary, multiply one or both of the equations by a nonzero number to make the coefficients of x (or y) opposites.
3. Add the equations to eliminate the terms involving x (or y).
4. Solve the equation resulting from step 3.
5. Find the value of the remaining variable by substituting the solution found in step 4 into any equation containing both variables. Or, repeat steps 2–4 to eliminate the other variable.
6. Check the proposed solution in each of the original equations. Write the solutions as an ordered pair.

To use addition to solve the system $\begin{cases} x + y = -1 \\ x - 2y = 8 \end{cases}$, follow these steps:

1. Since both equations are in general form, we move to step 2.
2. Multiply both sides of the first equation by 2 to get the coefficients of y to be opposites.

$$\begin{matrix} \textbf{(1)} \\ \textbf{(2)} \end{matrix} \begin{cases} x + y = -1 \\ x - 2y = 8 \end{cases} \begin{matrix} \xrightarrow{\text{Multiply by 2}} \\ \xrightarrow[\text{Unchanged}]{} \end{matrix} \begin{cases} 2x + 2y = -2 \\ x - 2y = 8 \end{cases} \begin{matrix} \textbf{(3)} \\ \textbf{(2)} \end{matrix}$$

3. Add equations 3 and 2 to eliminate y.

$$\begin{aligned} 2x + 2y &= -2 \\ x - 2y &= 8 \\ \hline 3x &= 6 && \text{Add the terms, column-by-column.} \end{aligned}$$

4. Since the resulting equation has only one variable, solve it for x.

$$\begin{aligned} 3x &= 6 \\ x &= 2 && \text{Divide both sides by 3.} \end{aligned}$$

5. To find y, substitute 2 for x in equation 1.

$$\begin{aligned} \mathbf{x} + y &= -1 \\ \mathbf{2} + y &= -1 \\ y &= -3 && \text{Subtract 2 from both sides.} \end{aligned}$$

6. The solution is $(2, -3)$. Verify this by checking it in each of the original equations.

REVIEW EXERCISES

Solve each system using the substitution method. If a system is inconsistent or if the equations are dependent, so indicate.

11. $\begin{cases} x = y - 4 \\ 2x + 3y = 7 \end{cases}$ $(-1, 3)$

12. $\begin{cases} y = 2x + 5 \\ 3x - 5y = -4 \end{cases}$ $(-3, -1)$

13. $\begin{cases} 0.1x + 0.2y = 1.1 \\ 2x - y = 2 \end{cases}$ $(3, 4)$

14. $\begin{cases} x = -2 - 3y \\ -2x - 6y = 4 \end{cases}$ infinitely many solutions, dependent equations

Solve each system using the addition method. If a system is inconsistent or if the equations are dependent, so indicate.

15. $\begin{cases} x + y = -2 \\ 2x + 3y = -3 \end{cases}$
$(-3, 1)$

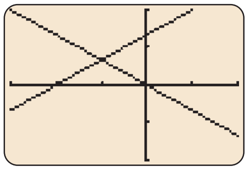
16. $\begin{cases} 2x - 3y = 5 \\ 2x - 3y = 8 \end{cases}$
no solution, inconsistent system

17. $\begin{cases} x + \frac{1}{2}y = 7 \\ -2x = 3y - 6 \end{cases}$ $(9, -4)$

18. $\begin{cases} y = \dfrac{x - 3}{2} \\ x = \dfrac{2y + 7}{2} \end{cases}$ $\left(4, \frac{1}{2}\right)$

19. To solve $\begin{cases} 5x - 2y = 19 \\ 3x + 4y = 1 \end{cases}$, which method, addition or substitution, would you use? Explain.
Using the addition method, the computations are easier.

20. Estimate the solution of the system $\begin{cases} y = -\frac{2}{3}x \\ 2x - 3y = -4 \end{cases}$ from the graphs in the illustration. Then solve the system algebraically.
$(-1, 0.7)$ (answers may vary); $\left(-1, \frac{2}{3}\right)$



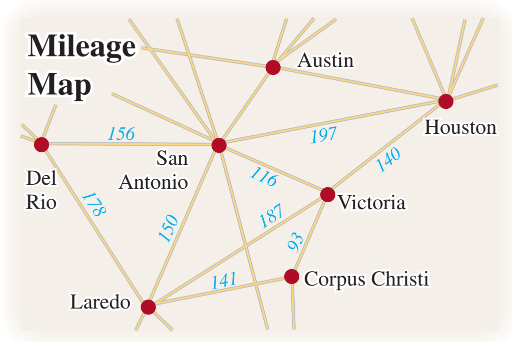
SECTION 3.3 Problem Solving Using Systems of Two Equations

DEFINITIONS AND CONCEPTS	EXAMPLES
To solve problems involving two unknown quantities, use the following **problem-solving strategy:** **1. Analyze** the facts of the problem. **2.** Pick different variables to represent the two unknown quantities. **Form two equations** involving those variables. **3. Solve** the system of equations by graphing, substitution, or elimination. **4. State** the conclusion. **5. Check** the result in the words of the problem.	RETIREMENT INCOME A retired office manager invested \$10,000 in two accounts, one paying 5% annual interest and the other 6%. If the interest earned for the first year was \$540, how much did she invest at each rate? **Analyze** A total of \$10,000 is invested at two different rates for 1 year. The total interest earned was \$540. **Form** Let $x =$ the number of dollars invested at 5% and $y =$ the number of dollars invested at 6%. We can use the formula $I = Prt$ to determine that the interest earned on the 5% investment is \$$0.05x$ and the interest earned on the 6% investment is \$$0.06y$. This information leads to two equations: $\begin{cases} x + y = 10{,}000 \\ 0.05x + 0.06y = 540 \end{cases}$ *The sum of the two amounts invested is \$10,000.* *The sum of the two amounts of interest earned is \$540.* **Solve** To solve by elimination we can multiply both sides of the first equation by -6, multiply the second equation by 100, and add the equations. $-6x - 6y = -60{,}000$ *This is $-6(x + y) = -6(10{,}000)$.* $5x + 6y = 54{,}000$ *This is $100(0.05x + 0.06y) = 100(540)$.* $-x = -6{,}000$ $x = 6{,}000$ **State** The amount invested at 5% was \$6,000 and the amount invested at 6% was \$10,000 − \$6,000, or \$4,000. **Check** The sum of \$6,000 and \$4,000 is \$10,000. \$6,000 invested at 5% for 1 year would earn \$6,000 · 0.05 = \$300. \$4,000 invested at 6% for 1 year would earn \$4,000 · 0.06 = \$240. The total interest earned is \$300 + \$240 = \$540. The results check.

REVIEW EXERCISES

Use two equations to solve each problem.

21. MAPS The distance between Austin and Houston is 4 miles less than twice the distance between Austin and San Antonio. The round trip from Houston to Austin to San Antonio and back to Houston is 442 miles. Determine the mileages between Austin and Houston and between Austin and San Antonio. 162 mi, 83 mi



22. RIVERBOATS A Mississippi riverboat travels 30 miles downstream in 3 hours and then makes the return trip upstream in 5 hours. Find the speed of the riverboat in still water and the speed of the current. 8 mph, 2 mph

23. MIXING SOLUTIONS How many fluid ounces of 6% sucrose solution must be mixed with 18% sucrose solution to make 750 ounces of a 10% sucrose solution? 500 oz of 6%, 250 oz of 18%

24. INVESTING One year, a couple invested a total of \$10,000 in two projects. Their first investment, a mini-mall, made a 6% profit. Their other investment, a skateboard park, made a 12% profit. If their combined income was \$960, how much did they invest at each rate? \$4,000 at 6%, \$6,000 at 12%

25. COOKING Two teaspoons and five tablespoons is equivalent to 85 milliliters of liquid. Five teaspoons and two tablespoons is equivalent to 55 milliliters of liquid. Find the number of milliliters in one teaspoon and the number of milliliters in one tablespoon. teaspoon: 5 ml, tablespoon: 15 ml

26. BREAK POINTS A bottling company is considering purchasing a new piece of equipment for their production line. The machine they currently use has a setup cost of \$250 and a cost of \$0.04 per bottle. The new machine has a setup cost of \$600 and a cost of \$0.02 per bottle. Find the break point. 17,500 bottles

SECTION 3.4 Solving Systems of Equations in Three Variables

DEFINITIONS AND CONCEPTS	EXAMPLES
The graph of an equation of the form $Ax + By + Cz = D$ is a flat surface, called a **plane.** A **solution of a system of three linear equations** in three variables is an **ordered triple** that satisfies each equation of the system.	The ordered triple $(4, 0, -3)$ is a solution of $\begin{cases} x + y - z = 7 \\ x - y + z = 1 \\ 2x + y + z = 5 \end{cases}$ because its coordinates, $x = 4$, $y = 0$, and $z = -3$, satisfy each equation: $x + y - z = 7$; $4 + 0 - (-3) \stackrel{?}{=} 7$; $7 = 7$ True $x - y + z = 1$; $4 - 0 + (-3) \stackrel{?}{=} 1$; $1 = 1$ True $2x + y + z = 5$; $2(4) + 0 + (-3) \stackrel{?}{=} 5$; $5 = 5$ True
To **solve a system of three linear equations with three variables:** **1.** Write each equation in standard form $Ax + By + Cz = D$ and clear any decimal or fractions.	Solve the system: $\begin{cases} x + 2y - z = 1 & \textbf{(1)} \\ 2x - y + z = 6 & \textbf{(2)} \\ x + 3y - z = 2 & \textbf{(3)} \end{cases}$ ***Step 1*** Each equation is written in standard form.

2. Pick any two equations and eliminate a variable.
3. Pick a different pair of equations and eliminate the same variable as in step 1.
4. Solve the resulting pair of two equations, each with two variables.
5. To find the value of the third variable, substitute the values of the two variables found in step 4 into any equation containing all three variables and solve the equation.
6. Check the proposed solution in all three of the original equations. Write the solution as an ordered triple.

If at any time in the elimination process the variables drop out and a false statement results, the system has **no solution.** If a true statement results, the system has **infinitely many solutions.**

Step 2 To eliminate z, we add equations 1 and 2.

$$\begin{aligned} \textbf{(1)}\quad x + 2y - z &= 1 \\ \textbf{(2)}\quad \underline{2x - y + z} &\underline{= 6} \\ \textbf{(4)}\quad 3x + y \qquad &= 7 \end{aligned}$$

Step 3 To eliminate z again, we add equations 2 and 3.

$$\begin{aligned} \textbf{(2)}\quad 2x - y + z &= 6 \\ \textbf{(3)}\quad \underline{x + 3y - z} &\underline{= 2} \\ \textbf{(5)}\quad 3x + 2y \qquad &= 8 \end{aligned}$$

Step 4 Equations 4 and 5 form a system of two equations in x and y. To solve this system, we multiply equation 4 by -1 and add the resulting equation to equation 5 to eliminate x.

$$\begin{aligned} -3x - y &= -7 && \text{This is } -1(3x + y) = -1(7). \\ \textbf{(5)}\quad \underline{3x + 2y} &\underline{= 8} \\ y &= 1 \end{aligned}$$

To find x, we substitute 1 for y in any equation containing x and y (such as equation 4) and solve for x:

$$\begin{aligned} 3x + \mathbf{y} &= 7 && \text{This is equation 4.} \\ 3x + \mathbf{1} &= 7 && \text{Substitute 1 for y.} \\ x &= 2 && \text{Solve for x.} \end{aligned}$$

Step 5 To find z, we substitute 2 for x and 1 for y in any equation containing x, y, and z (such as equation 2) and solve for z:

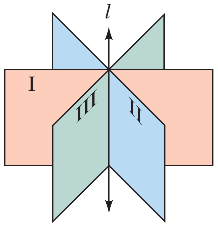
$$\begin{aligned} 2\mathbf{x} - \mathbf{y} + z &= 6 && \text{This is equation 2.} \\ 2(\mathbf{2}) - \mathbf{1} + z &= 6 && \text{Substitute for x and y.} \\ 4 - 1 + z &= 6 \\ z &= 3 && \text{Solve for z.} \end{aligned}$$

Step 6 The solution is (2, 1, 3). Verify this by checking it in each of the original equations.

REVIEW EXERCISES

27. Determine whether $(2, -1, 1)$ is a solution of the system: $\begin{cases} x - y + z = 4 \\ x + 2y - z = -1 \\ x + y - 3z = -1 \end{cases}$ no

28. A system of three linear equations in three variables is graphed on the right. Does the equation have a solution? If so, how many solutions does it have?
yes; infinitely many solutions



Solve each system. If a system is inconsistent or if the equations are dependent, so indicate.

29. $\begin{cases} x - 2y + 3z = -7 \\ -x + 3y + 2z = -8 \\ 2x - y - z = 7 \end{cases}$
$(2, 0, -3)$

30. $\begin{cases} x + y + z = 4 \\ x - 2y - z = 1 \\ 2x - y - 2z = -1 \end{cases}$
$(2, -1, 3)$

31. $\begin{cases} 2x + y + z = -1 \\ 6x - 3y - 2z = 3 \\ 4x - y - z = 4 \end{cases}$
$\left(\frac{1}{2}, 4, -6\right)$

32. $\begin{cases} 2x + 3y + z = -5 \\ -x + 2y - z = -6 \\ 3x + y + 2z = 4 \end{cases}$
no solution, inconsistent system

33. $\begin{cases} x + y - z = -3 \\ x + z = 2 \\ 2x - y + 2z = 3 \end{cases}$
$(-1, 1, 3)$

34. $\begin{cases} 3x + 3y + 6z = -6 \\ -x - y - 2z = 2 \\ 2x + 2y + 4z = -4 \end{cases}$
infinitely many solutions, dependent equations

SECTION 3.5 Problem Solving Using Systems of Three Equations

DEFINITIONS AND CONCEPTS	EXAMPLES
Problems that involve **three unknown quantities** can be solved using a strategy similar to that for solving problems involving two unknowns.	BATTERIES A hardware store sells three types of batteries: AA size for \$1 each, C size for \$1.50 each, and D size for \$2.00 each. One Saturday, the store sold 25 batteries for a total of \$34. If the number of C batteries that were sold was four less than the number of AA batteries that were sold, how many of each size battery were sold?

Analyze To find the three unknowns we will write a system of three equations in three variables.

Form Let A = the number of AA batteries sold, C = the number of C batteries sold, and D = the number of D batteries sold. The given information leads to three equations:

$$\begin{cases} A + C + D = 25 \\ 1A + 1.50C + 2D = 34 \\ C = A - 4 \end{cases}$$

The total number of batteries sold was 25.
The total value of the batteries sold was \$34.
The number of C batteries sold was 4 less than AA batteries sold.

If we multiply the second equation by 10 to clear the decimal and write the third equation in standard form, we have the system:

$$\begin{array}{l} \textbf{(1)} \\ \textbf{(2)} \\ \textbf{(3)} \end{array} \begin{cases} A + C + D = 25 \\ 10A + 15C + 20D = 340 \\ -A + C = -4 \end{cases}$$

Solve Since equation 3 does not contain a D-term, we will find another equation that does not contain a D-term. If each side of equation 1 is multiplied by -20 and the resulting equation is added to the equation 2, D is eliminated, and we obtain

$$\begin{array}{rrcl} & -20A - 20C - 20D &=& -500 \\ \textbf{(2)} & \underline{10A + 15C + 20D} &=& \underline{340} \\ \textbf{(4)} & -10A - 5C \qquad\quad &=& -160 \end{array}$$

This is $-20(A + C + D) = -20(25)$.

Equations 3 and 4 form a system of two equations in A and C that can be solved in the usual manner. (The remaining work is left to the reader.)

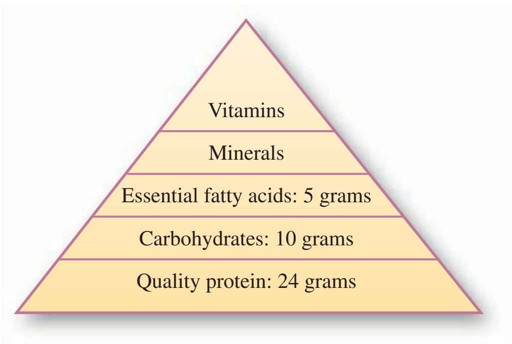
$$\begin{cases} -10A - 5C = -160 \\ -A + C = -4 \end{cases}$$

State There were 12 AA batteries, 8 C batteries, and 5 D batteries sold.

Check Verify that these results are correct by checking them in the words of the problem.

REVIEW EXERCISES

35. VETERINARY MEDICINE The daily requirements of a balanced diet for an animal are shown in the nutritional pyramid below. The number of grams per cup of nutrients in three food mixes are shown in the table. How many cups of each mix should be used to meet the daily requirements for protein, carbohydrates, and essential fatty acids in the animal's diet? To answer this problem, write a system of three equations and solve it using Cramer's rule. 2 cups mix A, 1 cup mix B, 1 cup mix C



Grams per cup			
	Protein	**Carbohydrates**	**Fatty acids**
Mix A	5	2	1
Mix B	6	3	2
Mix C	8	3	1

36. TEDDY BEARS A toy company produces three sizes of teddy bears. Each day, the total cost to produce the bears is \$850, the total time needed to stuff them is 480 minutes, and the total time needed to sew them is 1,260 minutes. Use the information in the table to determine how many of each type of bear is produced daily.
50 small bears, 60 medium bears, 40 large bears

Size of teddy bear	Production cost	Stuffing time	Sewing time
Small	\$3	2 min	6 min
Medium	\$5	3 min	8 min
Large	\$10	5 min	12 min

37. FINANCIAL PLANNING A financial planner invested \$22,000 in three accounts, paying 5%, 6%, and 7% annual interest. She invested \$2,000 more at 6% than she did at 5%. If the total interest earned in one year was \$1,370, how much was invested at each rate?
\$5,000 at 5%, \$7,000 at 6%, and \$10,000 at 7%

38. BALLISTICS The path of a thrown object is a parabola with an equation of $y = ax^2 + bx + c$. The parabola passes through the points (0, 0), (8, 12), and (12, 15). Find the equation of the parabola.
$y = -\frac{1}{16}x^2 + 2x$

SECTION 3.6 Solving Systems of Equations Using Matrices

DEFINITIONS AND CONCEPTS	EXAMPLES
A **matrix** is a rectangular array of numbers. Each number in a matrix is called an **element** or an **entry** of the matrix. A matrix with m rows and n columns has **order** $m \times n$.	A 2×3 matrix: $\begin{bmatrix} 2 & -7 & 5 \\ -3 & 4 & 1 \end{bmatrix}$ A 3×3 matrix: $\begin{bmatrix} 5 & -3 & 12 \\ 4 & 7 & -5 \\ 1 & -4 & 2 \end{bmatrix}$
A system of linear equations can be represented by an **augmented matrix.** Each row of the augmented matrix represents one equation of the system.	The system of equations $\begin{cases} 3x + 5y = 12 \\ 2x - 7y = -5 \end{cases}$ can be represented by the augmented matrix $\left[\begin{array}{cc:c} 3 & 5 & 12 \\ 2 & -7 & -5 \end{array}\right]$.
Systems of linear equations can be solved using **Gaussian elimination** and **elementary row operations:** **1.** Any two rows can be interchanged. **2.** Any row can be multiplied by a nonzero constant.	To solve the system $\begin{cases} 2x - 3y = 0 \\ x + 2y = 7 \end{cases}$, proceed as follows: This system is represented by the augmented matrix $\left[\begin{array}{cc:c} 2 & -3 & 0 \\ 1 & 2 & 7 \end{array}\right]$

3. Any row can be changed by adding a nonzero constant multiple of another row to it. To solve a system of two linear equations in two variables using matrices, we transform the augmented matrix into an equivalent matrix that has 1's down its main diagonal and a 0 below the 1 in the first column. The resulting matrix represents a system that can be solved by back substitution. Matrices can also be used to solve systems of three linear equations containing three variables.	We can use elementary row operations to transform the matrix into the following forms: $\left[\begin{array}{rr\|r} 1 & 2 & 7 \\ 2 & -3 & 0 \end{array}\right]$ Interchange row 1 and row 2. In symbols: $R_1 \leftrightarrow R_2$. $\left[\begin{array}{rr\|r} 1 & 2 & 7 \\ 0 & -7 & -14 \end{array}\right]$ Multiply row 1 by -2 and add to row 2. In symbols: $-2R_1 + R_2$. $\left[\begin{array}{rr\|r} 1 & 2 & 7 \\ 0 & 1 & 2 \end{array}\right]$ Multiply row 2 by $-\frac{1}{7}$. In symbols: $-\frac{1}{7}R_2$. This augmented matrix represents the system: $\begin{cases} x + 2y = 7 \\ y = 2 \end{cases}$ Thus, $y = 2$, and, by back substitution, $x = 3$. The solution of the system is $(3, 2)$.

REVIEW EXERCISES

Represent each system of equations using an augmented matrix.

39. $\begin{cases} 5x + 4y = 3 \\ x - y = -3 \end{cases}$ $\left[\begin{array}{rr\|r} 5 & 4 & 3 \\ 1 & -1 & -3 \end{array}\right]$

40. $\begin{cases} x + 2y + 3z = 6 \\ x - 3y - z = 4 \\ 6x + y - 2z = -1 \end{cases}$ $\left[\begin{array}{rrr\|r} 1 & 2 & 3 & 6 \\ 1 & -3 & -1 & 4 \\ 6 & 1 & -2 & -1 \end{array}\right]$

Solve each system using matrices, if possible.

41. $\begin{cases} x - y = 4 \\ 3x + 7y = -18 \end{cases}$ $(1, -3)$

42. $\begin{cases} x + 2y - 3z = 5 \\ x + y + z = 0 \\ 3x + 4y + 2z = -1 \end{cases}$ $(5, -3, -2)$

43. $\begin{cases} 16x - 8y = 32 \\ -2x + y = -4 \end{cases}$ infinitely many solutions, dependent equations

44. $\begin{cases} x + 2y + 2z = 2 \\ 4x + 5y + 3z = 3 \\ 2x + y - z = 1 \end{cases}$ no solution, inconsistent system

SECTION 3.7 Solving Systems of Equations Using Determinants

DEFINITIONS AND CONCEPTS	EXAMPLES
A **determinant of a square matrix** is a number.	A 2×2 determinant: $\begin{vmatrix} 3 & -3 \\ -4 & 5 \end{vmatrix}$ A 3×3 determinant: $\begin{vmatrix} 3 & 8 & 3 \\ 7 & 2 & 2 \\ 1 & 5 & 1 \end{vmatrix}$
To **evaluate** a 2×2 determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$	Evaluate: $\begin{vmatrix} 3 & -3 \\ -4 & 5 \end{vmatrix} = 3(5) - (-3)(-4) = 15 - 12 = 3$
To evaluate a 3×3 determinant, we **expand it by minors** along any row or column.	Evaluate: Minor of 3, Minor of 8, Minor of 3 $\begin{vmatrix} 3 & 8 & 3 \\ 7 & 2 & 2 \\ 1 & 5 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} - 8\begin{vmatrix} 7 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 2 \\ 1 & 5 \end{vmatrix}$ $= 3(-8) - 8(5) + 3(33) = -24 - 40 + 99 = 35$

Cramer's rule can be used to solve systems of linear equations.

Use Cramer's rule to solve: $\begin{cases} 2x - 3y = 0 \\ x + 2y = 7 \end{cases}$

The denominator determinant is D: $\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 4 - (-3) = 7$

The numerator determinant for x is D_x: $\begin{vmatrix} 0 & -3 \\ 7 & 2 \end{vmatrix} = 0 - (-3)(7) = 21$

The numerator determinant for y is D_y: $\begin{vmatrix} 2 & 0 \\ 1 & 7 \end{vmatrix} = 14 - 0 = 14$

Thus, we have

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 0 & -3 \\ 7 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = \frac{21}{7} = 3 \qquad y = \frac{D_y}{D} = \frac{\begin{vmatrix} 2 & 0 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}} = \frac{14}{7} = 2$$

The solution of the system is (3, 2).

REVIEW EXERCISES

Evaluate each determinant.

45. $\begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix}$ 18

46. $\begin{vmatrix} -3 & -4 \\ 5 & -6 \end{vmatrix}$ 38

47. $\begin{vmatrix} -1 & 2 & -1 \\ 2 & -1 & 3 \\ 1 & -2 & 2 \end{vmatrix}$ −3

48. $\begin{vmatrix} 3 & -2 & 2 \\ 1 & -2 & -2 \\ 2 & 1 & -1 \end{vmatrix}$ 28

Use Cramer's rule to solve each system, if possible.

49. $\begin{cases} 3x + 4y = 10 \\ 2x - 3y = 1 \end{cases}$
(2, 1)

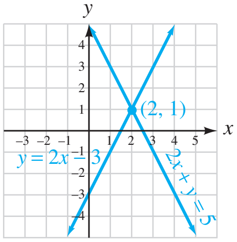
50. $\begin{cases} -6x - 4y = -6 \\ 3x + 2y = 5 \end{cases}$
no solution, inconsistent system

51. $\begin{cases} x + 2y + z = 0 \\ 2x + y + z = 3 \\ x + y + 2z = 5 \end{cases}$
(1, −2, 3)

52. $\begin{cases} 2x + 3y + z = 2 \\ x + 3y + 2z = 7 \\ x - y - z = -7 \end{cases}$
(−3, 2, 2)

CHAPTER 3 TEST

1. Solve by graphing: $\begin{cases} 2x + y = 5 \\ y = 2x - 3 \end{cases}$



2. Use substitution to solve: $\begin{cases} 2x - 4y = 14 \\ x + 2y = 7 \end{cases}$ (7, 0)

3. Use addition (elimination) to solve: $\begin{cases} 2x + 3y = -5 \\ 3x - 2y = 12 \end{cases}$
(2, −3)

4. Are the equations of the following system dependent or independent?
$$\begin{cases} 3(x + y) = x - 3 \\ -y = \dfrac{2x + 3}{3} \end{cases}$$ dependent

5. Is $\left(-1, -\frac{1}{2}, 5\right)$ a solution of $\begin{cases} x - 2y + z = 5 \\ 2x + 4y = -4 \\ -6y + 4z = 22 \end{cases}$? no

6. Solve the following system using the addition method.
$$\begin{cases} x + y + z = 4 \\ x + y - z = 6 \\ 2x - 3y + z = -1 \end{cases}$$ (3, 2, −1)

Write a system of equations to solve each problem.

7. In the EXIT sign, find x and y, if y is 15 more than x. 55, 70

8. ANTIFREEZE How much of a 40% antifreeze solution must a mechanic mix with an 80% antifreeze solution if 20 gallons of a 50% antifreeze solution are needed? 15 gal 40%, 5 gal 80%

Use matrices to solve each system.

9. $\begin{cases} x + y = 4 \\ 2x - y = 2 \end{cases}$
(2, 2)

10. $\begin{cases} x + y + 2z = -1 \\ x + 3y - 6z = 7 \\ 2x - y + 2z = 0 \end{cases}$
(1, 0, −1)

Evaluate each determinant.

11. $\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$ 22

12. $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 1 & -2 & 2 \end{vmatrix}$ 4

For Problems 13–16, consider the system $\begin{cases} x - y = -6 \\ 3x + y = -6 \end{cases}$, which is to be solved using Cramer's rule.

13. When solving for x, what is the numerator determinant D_x? **(Don't evaluate it.)**
$\begin{vmatrix} -6 & -1 \\ -6 & 1 \end{vmatrix}$

14. When solving for y, what is the denominator determinant D? **(Don't evaluate it.)**
$\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$

15. Solve the system for x. -3

16. Solve the system for y. 3

17. Solve the following system for z only, using Cramer's rule. -1

$$\begin{cases} x + y + z = 4 \\ x + y - z = 6 \\ 2x - 3y + z = -1 \end{cases}$$

18. MOVIE TICKETS The receipts for one showing of a movie were \$410 for an audience of 100 people. The ticket prices are given in the table. If twice as many children's tickets as general admission tickets were purchased, how many of each type of ticket were sold? C: 60, GA: 30, S: 10

	Ticket prices
Children	\$3.00
General Admission	\$6.00
Seniors	\$5.00

19. MIXING NUTS The owner of a produce store wanted to mix peanuts selling for \$3 per pound, cashews selling for \$9 per pound, and Brazil nuts selling for \$9 per pound to get 50 pounds of a mixture that would sell for \$6 per pound. She used 15 fewer pounds of cashews than peanuts. How many pounds of each did she use?
25 lb peanuts, 10 lb cashews, 15 lb Brazil nuts

20. Which method, substitution or addition, would you use to solve the following system? Explain.

$$\begin{cases} \frac{x}{2} - \frac{y}{4} = -4 \\ y = -2 - x \end{cases}$$

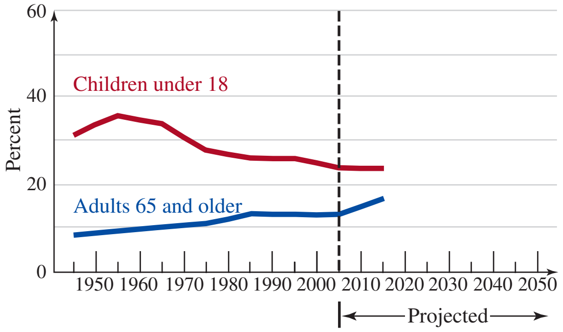
Substitution, the second equation is solved for y.

21. What does it mean to say that a system of two linear equations in two variables is an *inconsistent system?*
The system has no solution.

22. Suppose that two variables are used to solve an application problem. Why must two equations be written to solve the problem?

23. POPULATION PROJECTIONS If the population trends for the years 2005–2015 continue as projected, estimate the point of intersection of the graphs. Interpret your answer. (2035, 25)

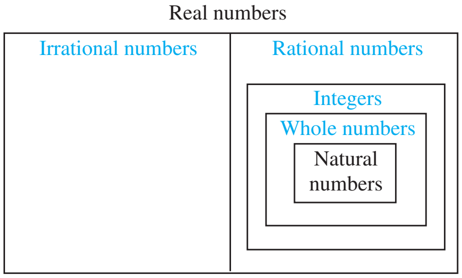
Children under age 18 and adults 65 and older as a percent of the U.S. population



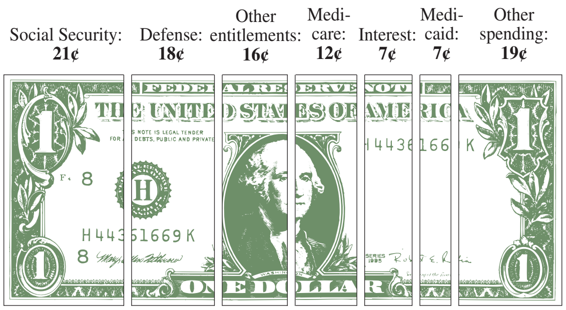
Source: U.S. Bureau of the Census

CHAPTERS 1–3 CUMULATIVE REVIEW

1. Complete the illustration by labeling the rational numbers, irrational numbers, integers, whole numbers, and natural numbers. [Section 1.2]



2. FEDERAL BUDGET The President's proposed budget for the fiscal year 2005 was \$2.4 trillion. The illustration shows how a typical dollar of the budget was to be spent. Determine the amount he proposed to spend on Social Security. [Section 1.8] \$504,000,000,000



Source: Budget of the United States Government FY 2005

Evaluate each expression for $a = -3$ and $b = -5$.

3. $-|b| - ab^2$ [Section 1.3] 70

4. $\dfrac{14 + 2[2a - (b - a)]}{-b - 2}$ [Section 1.3] 2

Simplify each expression.

5. $0.5x^2 - 6(2.1x^2 - x) + 6.7x$ [Section 1.4] $-12.1x^2 + 12.7x$

6. $-(c + 2) - (2 - c)$ [Section 1.4] -4

7. COMMUTING Use the following facts to determine a commuter's average speed when she drives to work. [Section 1.8]

- If she drives her car, it takes a quarter of an hour to get to work.
- If she rides the bus, it takes half an hour to get to work.
- When she drives, her average speed is 10 miles per hour faster than that of the bus. 20 mph

8. DRIED FRUITS Dried apple slices cost \$4.60 per pound, and dried banana chips sell for \$3.40 per pound. How many pounds of each should be used to create a 10-pound mixture that sells for \$4 per pound? [Section 1.8] 5 lb apple slices, 5 lb banana chips

Solve each equation, if possible. If an equation is an identity, so indicate.

9. $\frac{3}{4}x + 1.5 = -19.5$ [Section 1.5] -28

10. $7 - x - x - x = 8$ [Section 1.5] $-\frac{1}{3}$

11. $\dfrac{x + 7}{3} = \dfrac{x - 2}{5} - \dfrac{x}{15} + \dfrac{7}{3}$ [Section 1.5] -2

12. $3p - 6 = 4(p - 2) + 2 - p$ [Section 1.5] all real numbers, identity

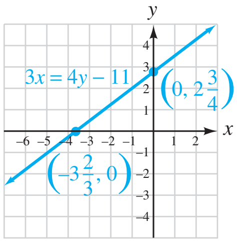
Solve each equation for the indicated variable.

13. $C = Ax + AB$ for B [Section 1.6] $B = \frac{C - Ax}{A}$

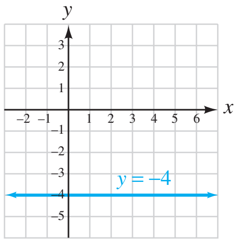
14. $l = a + (n - 1)d$ for n [Section 1.6] $n = \frac{l - a + d}{d}$ or $n = \frac{l - a}{d} + 1$

Graph each equation.

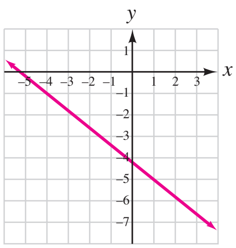
15. $3x = 4y - 11$ [Section 2.2]



16. $y = -4$ [Section 2.2]



17. Find the slope of the line in the graph. [Section 2.3] $-\frac{4}{5}$



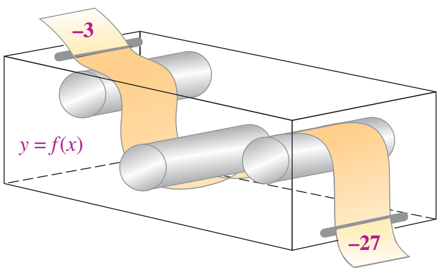
18. Write an equation of the line that passes through (4, 5) and is parallel to the graph of $y = -3x$. Answer in slope–intercept form. [Section 2.4] $y = -3x + 17$

If $f(x) = -x^2 - \frac{x}{2}$, find each value.

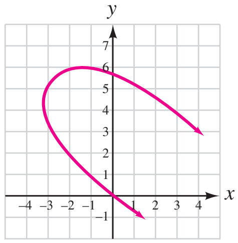
19. $f(10)$ [Section 2.5] -105

20. $f(-10)$ [Section 2.5] -95

21. We can think of a function as a machine. Write a function that turns the given input in the illustration into the given output. [Section 2.5]
$f(x) = x^3$ (answer may vary)



22. Determine whether the following graph is the graph of a function. Explain. [Section 2.5] no



23. Does $x = y^2$ define a function? [Section 2.5] no

24. COLLECTIBLES A collector buys the Hummel figurine shown in the illustration, anticipating that it will be worth \$650 in 20 years. Assuming straight-line appreciation, write an equation that gives the value v of the figurine x years after it is purchased. [Section 2.4]
$v = 17.5x + 300$

25. Solve: $\begin{cases} y = \dfrac{-2x + 1}{3} \\ 3x - 2y = 8 \end{cases}$ [Section 3.2] $(2, -1)$

26. Solve: $\begin{cases} -x + 3y + 2z = 5 \\ 3x + 2y + z = -1 \\ 2x - y + 3z = 4 \end{cases}$ [Section 3.4] $(-1, 0, 2)$

Evaluate each determinant.

27. $\begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix}$

[Section 3.7] 26

28. $\begin{vmatrix} 2 & 1 & -3 \\ -2 & 2 & 4 \\ 1 & -2 & 2 \end{vmatrix}$

[Section 3.7] 26

4

Inequalities

from Campus to Careers

Heating, Ventilation, and Air Conditioning Technician

HVAC technicians make sure that we are warm in the winter and cool in the summer. They install, maintain, and repair heating and cooling systems in residential, commercial, and industrial buildings. HVAC technicians constantly work with numbers as they take measurements, read blueprints, and prepare work estimates. The installation instructions and diagrams they follow require strong mathematical skills in algebra and geometry.

HVAC technicians adjust heating /cooling system controls to recommended temperature settings. In **Problem 84** of **Study Set 4.2,** you will use interval notation to express the temperature range for a room by interpreting the settings on a thermostat.

JOB TITLE: Heating, Ventilation, and Air Conditioning Technician

EDUCATION: Technical school training or completion of an apprenticeship.

JOB OUTLOOK: Employment opportunities are projected to increase by 18% to 26% through the year 2014.

ANNUAL EARNINGS: Starting salaries range from $25,000 to $35,000.

FOR MORE INFORMATION: www.bls.gov/oco/ocos192.htm

Objectives

1. Read and interpret inequality symbols.
2. Graph intervals and use interval and set-builder notation.
3. Solve linear inequalities using properties of inequality.
4. Use linear inequalities to solve problems.

SECTION 4.1

Solving Linear Inequalities in One Variable

Traffic signs often appear in front of schools. From the figure a motorist knows that

- A speed *greater than* 25 miles per hour breaks the law and could possibly result in a ticket for speeding.
- A speed *less than or equal to* 25 miles per hour is within the posted speed limit.

Statements such as these can be expressed using *inequality symbols*.

1 Read and interpret inequality symbols.

Inequalities are statements indicating that two quantities are unequal. Inequalities contain one or more of the following symbols.

Inequality Symbols

$a \neq b$	means	"a is not equal to b."
$a < b$	means	"a is less than b."
$a > b$	means	"a is greater than b."
$a \leq b$	means	"a is less than or equal to b."
$a \geq b$	means	"a is greater than or equal to b."

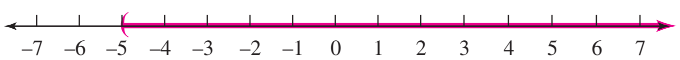
By definition, $a < b$ means that "a is less than b," but it also means that $b > a$. Furthermore, if a is to the left of b on a number line, then $a < b$. If a is to the right of b on a number line, then $a > b$.

By definition, $a \leq b$ is true if a is less than b or if a is equal to b. For example, the inequality $-2 \leq 4$ is true, and so is $4 \leq 4$.

We can use a variable and inequality symbols to describe the warning that the traffic sign above gives to drivers. If x represents the motorist's speed in miles per hour, he or she is in danger of receiving a speeding ticket if $x > 25$, and he or she is observing the posted speed limit if $x \leq 25$.

2 Graph intervals and use interval and set-builder notation.

The graph of a set of real numbers that is a portion of a number line is called an **interval.** The graph shown below represents all real numbers that are greater than -5. This interval contains numbers that satisfy the inequality $x > -5$, such as $-4.99, -3, -1.8, 0, 2\frac{3}{4}, \pi$, and 1,050. The left **parenthesis** at -5 indicates that -5 is not included in the interval.



We can also express this interval in **interval notation** as $(-5, \infty)$, where ∞ (read as **positive infinity**) indicates that the interval extends indefinitely to the right. The left **parenthesis** is used to show that the endpoint -5 is not included.

Set-builder notation is another way of describing the set of real numbers graphed on the previous page. With this notation, the condition for membership in the set is specified using a variable. For example, the set of real numbers greater than -5 is written in set-builder notation as

$$\{x \mid x > -5\}$$

the set of all real numbers x such that x is greater than -5

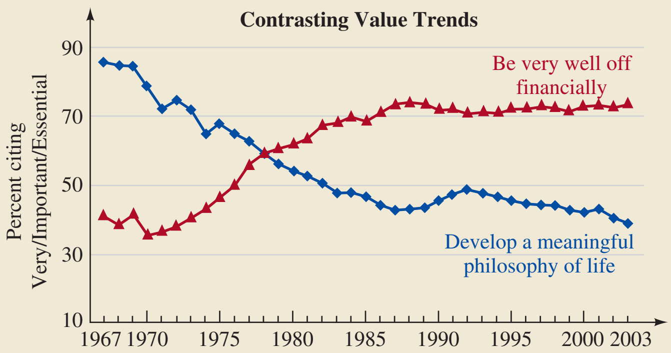
THINK IT THROUGH *Changing Values*

College students are more interested in making money than ever before and less interested in developing a personal take on life. The rising trend to be financially sound is attributed to several major changes, including a steady rise in students' desire to raise a family.

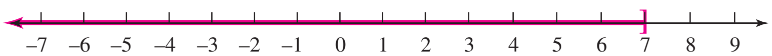
Lindsey Bowers, The Daily Cougar *(U. Houston), 2004*

The graph below shows the results of the annual survey conducted by the Higher Education Research Institute. The study has been conducted for 38 years and is the longest-running survey of its kind. A total of 267,449 incoming freshmen at 413 colleges and universities were asked about their attitudes and priorities. For which years is the following true? from 1979 to 2003

The percent citing that being well off financially is very important	>	The percent citing that developing a meaningful philosophy of life is very important



The interval shown in the figure below is the graph of the real numbers less than or equal to 7. This interval contains numbers that satisfy the inequality $x \le 7$. The right **bracket** at 7 indicates that 7 is included in the interval. To express this interval in interval notation, we write $(-\infty, 7]$, where $-\infty$ (read as **negative infinity**) indicates that the interval extends indefinitely to the left. The bracket is used to show that 7 is included in the interval. To describe the interval using set-builder notation, we write $\{x \mid x \le 7\}$.



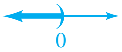
Caution! The symbol ∞ does not represent a number. It indicates that an interval extends indefinitely to the right. We always use a parenthesis after the symbol ∞. For similar reasons, a parenthesis is always used in front of the symbol $-\infty$ when writing interval notation.

Self Check 1

Represent the set of negative real numbers using interval notation, with a graph, and using set-builder notation.

Now Try **Problem 38**

Self Check 1 Answer

$(-\infty, 0)$ 

, $\{x \mid x < 0\}$

Teaching Example 1 Represent the set of real numbers greater than -2 using interval notation, with a graph, and using set-builder notation.

Answer:

$(-2, \infty)$ $\{x \mid x > -2\}$

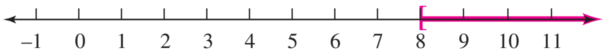
EXAMPLE 1 Represent the set of real numbers greater than or equal to 8 using interval notation, with a graph, and using set-builder notation.

Strategy To graph the set of numbers on a number line, we will first determine whether the endpoint is included. Then we will determine whether the real numbers to the right or the left of the endpoint should be shaded.

WHY We draw the graph first because the corresponding interval notation and set-builder notation follow directly from it.

Solution

The set of all real numbers that are greater than or equal to 8 is graphed below. Because the set includes 8, a bracket is used at 8 on the number line. The numbers to the right of 8 are shaded because they are greater than 8.



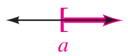
The interval is written as $[8, \infty)$ and the set-builder notation is written as $\{x \mid x \geq 8\}$.

If an interval extends forever in one direction, as in the previous examples, it is called an **unbounded interval.** The following chart illustrates the types of unbounded intervals and shows how they are described using an inequality and a graph.

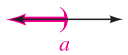
Unbounded Intervals

The interval (a, ∞) includes all real numbers x such that $x > a$.

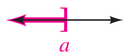
The interval $[a, \infty)$ includes all real numbers x such that $x \geq a$.



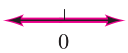
The interval $(-\infty, a)$ includes all real numbers x such that $x < a$.



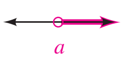
The interval $(-\infty, a]$ includes all real numbers x such that $x \leq a$.

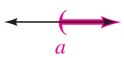


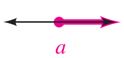
The interval $(-\infty, \infty)$ includes all real numbers x. The graph of this interval is the entire number line.

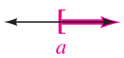


When graphing intervals, an open circle can be used to show that a point is not included in a graph, and a solid circle can be used to show that a point *is* included. For example,



is equivalent to




is equivalent to


In this book, we will use parentheses and brackets when graphing intervals, because they are consistent with interval notation.

3 Solve linear inequalities using properties of inequality.

In this section, we will work with **linear inequalities** in one variable.

Linear Inequalities

A **linear inequality** in one variable (say, x) is any inequality that can be expressed in one of the following forms, where a, b, and c represent real numbers and $a \neq 0$.

$$ax + b < c \qquad ax + b \leq c \qquad ax + b > c \quad \text{or} \quad ax + b \geq c$$

Some examples of linear inequalities are

$$3x < 0, \qquad 3(2x - 9) < 9, \qquad \text{and} \qquad -12x - 18 \geq 16$$

To **solve a linear inequality** means to find all the values that, when substituted for the variable, make the inequality true. The set of all solutions of an inequality is called its **solution set.** Most of the inequalities we will solve have infinitely many solutions. We will use the following properties to solve inequalities.

Addition and Subtraction Properties of Inequality

Adding the same number to, or subtracting the same number from, both sides of an inequality does not change the solutions.

For any real numbers a, b, and c,

If $a < b$, then $a + c < b + c$.
If $a < b$, then $a - c < b - c$.

Similar statements can be made for the symbols $\leq$, $>$, or $\geq$.

As with equations, there are properties for multiplying and dividing both sides of an inequality by the same number. To develop what is called the **multiplication property of inequality,** consider the true statement $2 < 5$. If both sides are multiplied by a positive number, such as 3, another true inequality results.

$2 < 5$
$\mathbf{3} \cdot 2 < \mathbf{3} \cdot 5$ Multiply both sides by 3.
$6 < 15$ This is a true inequality.

However, if we multiply both sides of $2 < 5$ by a negative number, such as -3, the direction of the inequality symbol must be reversed to produce another true inequality.

$2 < 5$
$\mathbf{-3} \cdot 2 > \mathbf{-3} \cdot 5$ Multiply both sides by the negative number -3 and reverse the direction of the inequality.
$-6 > -15$ This is a true inequality.

The inequality $-6 > -15$ is true because -6 is to the right of -15 on the number line.

Dividing both sides of an inequality by the same negative number also requires that the direction of the inequality symbol be reversed.

$-4 < 6$ This is a true inequality.
$\dfrac{-4}{\mathbf{-2}} > \dfrac{6}{\mathbf{-2}}$ Divide both sides by -2 and change $<$ to $>$.
$2 > -3$ This is a true inequality.

These examples illustrate the multiplication and division properties of inequality.

Multiplication and Division Properties of Inequality

Multiplying or dividing both sides of an inequality by the same positive number does not change the solutions.

For any real numbers a, b, and c, where c is positive,

If $a < b$, then $ac < bc$.
If $a < b$, then $\frac{a}{c} < \frac{b}{c}$.

If we multiply or divide both sides of an inequality by a negative number, the direction of the inequality symbol must be reversed for the inequalities to have the same solutions.

For any real numbers a, b, and c, where c is negative,

If $a < b$, then $ac > bc$.
If $a < b$, then $\frac{a}{c} > \frac{b}{c}$.

Similar statements can be made for the symbols $\le$, $>$, or $\ge$.

After applying one of the properties of inequality, the resulting inequality is equivalent to the original one. Like equivalent equations, **equivalent inequalities** have the same solution set.

Self Check 2

Solve $2(3x + 2) > -44$. Write the solution set in interval notation and then graph it.

Now Try **Problem 49**

Self Check 2 Answer
$(-8, \infty)$

Teaching Example 2 Solve $4(2x - 3) > 4$. Write the solution set in interval notation and then graph it.
Answer:
$(2, \infty)$

EXAMPLE 2

Solve $3(2x - 9) < 9$. Write the solution set in interval notation and then graph it.

Strategy We will use the distributive property to remove the parentheses and use the properties of inequality to isolate the variable on one side.

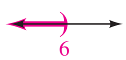
WHY Once we have obtained an equivalent inequality with the variable isolated on one side, the solution set is obvious.

Solution

To isolate x on the left-hand side of the inequality, we use the same strategy as we used to solve equations.

$3(2x - 9) < 9$	This is the inequality to solve.
$6x - 27 < 9$	Distribute the multiplication by 3.
$6x - 27 + 27 < 9 + 27$	To undo the subtraction of 27, add 27 to both sides.
$6x < 36$	Simplify each side.
$\frac{6x}{6} < \frac{36}{6}$	To undo the multiplication by 6, divide both sides by 6.
$x < 6$	Simplify each side.

The solution set is the interval $(-\infty, 6)$, whose graph is shown to the right. We can also write the solution set using set-builder notation: $\{x \mid x < 6\}$.



The solution set contains infinitely many real numbers. We cannot check to see whether all of them satisfy the original inequality. As an informal check, we pick one number in the graph, such as 4, and see whether it satisfies the inequality.

Check:	$3(2x - 9) < 9$	This is the original inequality.
	$3[2(4) - 9] \overset{?}{<} 9$	Substitute 4 for x. Read $\overset{?}{<}$ as "is possibly less than."
	$3(8 - 9) \overset{?}{<} 9$	
	$3(-1) \overset{?}{<} 9$	
	$-3 < 9$	This is a true statement.

Since $-3 < 9$, 4 satisfies the inequality. The solution appears to be correct.

EXAMPLE 3 Solve $-12x - 8 \le 16$. Write the solution set in interval notation and then graph it.

Strategy We will use the properties of inequality to isolate the variable on one side.

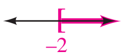
WHY Once we have obtained an equivalent inequality with the variable isolated on one side, the solution set is obvious.

Solution

To solve this inequality, we need to isolate x.

$-12x - 8 \le 16$	This is the inequality to solve.
$-12x - 8 + 8 \le 16 + 8$	To undo the subtraction of 8, add 8 to both sides.
$-12x \le 24$	Simplify each side.
$\dfrac{-12x}{-12} \ge \dfrac{24}{-12}$	To undo the multiplication by -12, divide both sides by -12. Because we are dividing by a negative number, we reverse the $\le$ symbol.
$x \ge -2$	Simplify each side.

The solution set is $\{x \mid x \ge -2\}$ or the interval $[-2, \infty)$, whose graph is shown to the right.



Self Check 3

Solve $-6x + 6 \le 0$. Write the solution set in interval notation and then graph it.

Now Try **Problem 56**

Self Check 3 Answer

$[1, \infty)$

Teaching Example 3 Solve $-15x + 3 \ge 33$. Write the solution set in interval notation and then graph it.

Answer:

$(-\infty, -2]$

EXAMPLE 4 Solve $\dfrac{2}{3}(x + 2) > \dfrac{4}{5}(x - 3)$. Write the solution set in interval notation and then graph it.

Strategy we will clear the inequality of fractions by multiplying both sides by the LCD of $\frac{2}{3}$ and $\frac{4}{5}$.

WHY It is easier to solve an inequality that involves only integers.

Solution

$\dfrac{2}{3}(x + 2) > \dfrac{4}{5}(x - 3)$	This is the inequality to solve.
$15 \cdot \dfrac{2}{3}(x + 2) > 15 \cdot \dfrac{4}{5}(x - 3)$	Multiply both sides by the LCD of $\frac{2}{3}$ and $\frac{4}{5}$, which is 15.
$10(x + 2) > 12(x - 3)$	Simplify: $15 \cdot \frac{2}{3} = 10$ and $15 \cdot \frac{4}{5} = 12$.
$10x + 20 > 12x - 36$	Distribute the multiplication by 10 and 12.
$-2x + 20 > -36$	To eliminate $12x$ on the right-hand side, subtract $12x$ from both sides.
$-2x > -56$	Subtract 20 from both sides.
$\dfrac{-2x}{-2} < \dfrac{-56}{-2}$	Divide both sides by -2 and reverse the $>$ symbol.
$x < 28$	Simplify each side.

The solution set is the interval $(-\infty, 28)$, whose graph is shown to the right.

Self Check 4

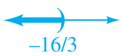
Solve $\dfrac{3}{2}(x + 2) < \dfrac{3}{5}(x - 3)$.

Write the solution set in interval notation and then graph it.

Now Try **Problem 61**

Self Check 4 Answer

$\left(-\infty, -\frac{16}{3}\right)$



Teaching Example 4 Solve $\frac{3}{7}(x + 3) > \frac{3}{14}(x + 2)$. Write the solution set in interval notation and then graph it.

Answer:

$(-4, \infty)$

Success Tip When solving an inequality, the variable sometimes ends up on the right-hand side. For instance, suppose we solve an inequality and obtain $-3 < x$. This inequality can be expressed in the equivalent form $x > -3$, which most students find easier to graph and to express in interval notation.

When solving equations, we have seen that some are true for all real numbers while others have no solution. Similar situations can occur when solving inequalities.

Self Check 5

Solve each inequality. Graph the solution set and write it using interval notation.

a. $-8n + 10 \geq 1 - 2(4n - 2)$

b. $\dfrac{4d - 5}{-10} > \dfrac{2(2d - 3)}{-10}$

Now Try **Problems 65 and 66**

Self Check 5 Answer

a. $(-\infty, \infty)$, $\mathbb{R}$ 

b. no solution, $\varnothing$

Teaching Example 5 Solve each inequality. Graph the solution set and write it using interval notation.

a. $4(a - 5) \leq 4a + 2$

b. $\dfrac{2x + 3}{2} < \dfrac{3x + 1}{3}$

Answer:

a. $(-\infty, \infty)$ 

b. no solution, $\varnothing$

EXAMPLE 5 Solve each inequality. Graph the solution set and write it using interval notation.

a. $\dfrac{3a - 4}{-5} > \dfrac{3a + 15}{-5}$ **b.** $1 - 2a \geq 2(1 - a)$

Strategy We will use properties of inequality to isolate the variable on one side.

WHY Once we have obtained an equivalent inequality, with the variable isolated on one side, the solution set is obvious.

Solution

a.

$$\frac{3a - 4}{-5} > \frac{3a + 15}{-5}$$ This is the inequality to solve.

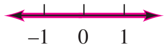
$$-5\left(\frac{3a - 4}{-5}\right) < -5\left(\frac{3a + 15}{-5}\right)$$ To clear the inequality of fractions, multiply both sides by -5. Since we are multiplying both sides by a negative number, reverse the direction of the inequality.

$$3a - 4 < 3a + 15$$ Simplify each side.

$$3a - 4 - 3a < 3a + 15 - 3a$$ Subtract $3a$ from both sides.

$$-4 < 15$$ This is a true statement.

The terms involving the variable a drop out. The resulting true statement indicates that the original inequality is true for all values of a. Therefore, the solution set is the set of real numbers, denoted $(-\infty, \infty)$ or $\mathbb{R}$, and its graph is as shown.



b.

$$1 - 2a \geq 2(1 - a)$$ This is the inequality to solve.

$$1 - 2a \geq 2 - 2a$$ Distribute the multiplication by 2.

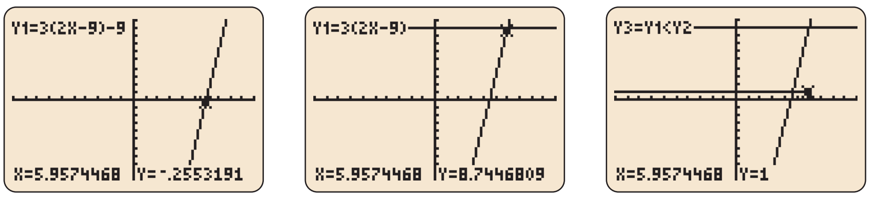
$$1 - 2a + 2a \geq 2 - 2a + 2a$$ Add $2a$ to both sides.

$$1 \geq 2$$ This is a false statement.

The terms involving the variable a drop out. The resulting false statement indicates that the original inequality is false for all values of a. Therefore, the inequality has no solution. The solution set has no elements and is denoted $\varnothing$.

Using Your CALCULATOR **Solving Linear Inequalities in One Variable**

There are several ways to solve linear inequalities graphically. For example, to solve $3(2x - 9) < 9$ we can subtract 9 from both sides and solve instead the equivalent inequality $3(2x - 9) - 9 < 0$. Using standard window settings of $[-10, 10]$ for x and $[-10, 10]$ for y, we graph $y = 3(2x - 9) - 9$ and then use TRACE. Moving the cursor closer and closer to the x-axis, as shown in figure (a), we see that the graph is below the x-axis for x-values in the interval $(-\infty, 6)$. This interval is the solution, because in this interval, $3(2x - 9) - 9 < 0$.



(a) (b) (c)

Another way to solve $3(2x - 9) < 9$ is to graph $y = 3(2x - 9)$ and $y = 9$. We can then trace to see that the graph of $y = 3(2x - 9)$ is below the graph of $y = 9$ for x-values in the interval $(-\infty, 6)$. See figure (b). This interval is the solution, because in this interval, $3(2x - 9) < 9$.

A third approach is to enter and then graph

$Y_1 = 3(2x - 9)$

$Y_2 = 9$

$Y_3 = Y_1 < Y_2$ To do this, use the VARS key. Consult your owner's manual for the specific directions.

The graphs of $y = 3(2x - 9)$, $y = 9$, and a horizontal line 1 unit above the x-axis will be displayed, as shown in figure (c). In the TRACE mode, we then move the cursor to the rightmost endpoint of the horizontal line to determine that the interval $(-\infty, 6)$ is the solution of $3(2x - 9) < 9$.

4 Use linear inequalities to solve problems.

In previous chapters, we have used a five-step problem-solving strategy to solve problems. This process involved writing and then solving equations. We will now show how inequalities can be used to solve problems. To decide whether to use an equation or an inequality to solve a problem, you must look for key words and phrases. Here are some common statements that translate to inequalities.

The statement	Translates to	The statement	Translates to
a does not exceed b.	$a \le b$	a is at least b.	$a \ge b$
a is at most b.	$a \le b$	a is not less than b.	$a \ge b$
a is no more than b.	$a < b$	a will exceed b.	$a > b$

EXAMPLE 6 Translate the sentence to mathematical symbols: *The instructor said that the test would take no more than 50 minutes.*

Strategy We will look for key words or phrases.

WHY Key phrases can be translated into mathematical symbols.

Solution

Since the test will take no more than 50 minutes, it will take 50 minutes or less to complete. If we let t represent the time it takes to complete the test, then $t \le 50$.

Self Check 6

Translate the sentence to mathematical symbols: *A PG-13 movie rating means that you must be at least 13 years old to see the movie.* $a \ge 13$

Now Try **Problem 70**

Teaching Example 6 Translate the sentence to mathematical symbols: You must be at least 18 years old to vote.
Answer: $a \ge 18$

EXAMPLE 7 ***Political Contributions*** Some volunteers are making long-distance telephone calls to solicit contributions for their candidate. The calls are billed at the rate of 25¢ for the first three minutes and 7¢ for each additional minute or part thereof. If the campaign chairperson has ordered that the cost of each call is not to exceed \$1.00, for how many minutes can a volunteer talk to a prospective donor on the phone?

Self Check 7

MOONBOUNCE RENTAL A rental company charges \$50.00 for the first 2 hours of renting a moonbounce and \$9.95 per hour for each additional hour or part thereof. How long can the moonbounce be rented if \$132.00 is budgeted for this expense? 10 hr

Now Try **Problem 91**

Teaching Example 7 PARKING A parking garage charges \$10.00 for the first 3 hours and \$1.95 for each additional hour or part thereof. How long can the owner park her car if \$15.00 was budgeted for parking?
Answer:
5 hr

Analyze We are given the rate at which a call is billed. Since the cost of a call is not to exceed \$1.00, the cost must be *less than or equal to* \$1.00. This phrase indicates that we should write an inequality to find how long a volunteer can talk to a prospective donor.

Form We will let x = the total number of minutes that a call can last. Then the cost of a call will be 25¢ for the first three minutes plus 7¢ times the number of additional minutes, where the number of *additional* minutes is $x - 3$ (the total number of minutes minus the first 3 minutes). With this information, we can form an inequality.

The cost of the first three minutes	plus	the cost of the additional minutes	is not to exceed	\$1.00.
0.25	+	$0.07(x - 3)$	$\leq$	1

Solve To simplify the computations, we first clear the inequality of decimals.

$$0.25 + 0.07(x - 3) \leq 1$$

$$25 + 7(x - 3) \leq 100 \quad \text{To eliminate the decimals, multiply both sides by 100.}$$

$$25 + 7x - 21 \leq 100 \quad \text{Distribute the multiplication by 7.}$$

$$7x + 4 \leq 100 \quad \text{Combine like terms.}$$

$$7x \leq 96 \quad \text{Subtract 4 from both sides.}$$

$$x \leq 13.\overline{714285} \quad \text{Divide both sides by 7.}$$

State Since the phone company doesn't bill for part of a minute, the longest time a call can last is 13 minutes. If a call lasts for $13.\overline{714285}$ minutes, it will be charged as a 14-minute call, and the cost will be \$0.25 + \$0.07(11) = \$1.02.

Check If the call lasts 13 minutes, the cost will be \$0.25 + \$0.07(10) = \$0.95. This is less than \$1.00. The result checks.

ANSWERS TO SELF CHECKS

1. $(-\infty, 0)$ 

, $\{x \mid x < 0\}$ **2.** $(-8, \infty)$ 

3. $[1, \infty)$ 

4. $\left(-\infty, -\frac{16}{3}\right)$ 

5. a. $(-\infty, \infty)$ 

b. no solution, $\varnothing$ **6.** $a \geq 13$ **7.** 10 hr

SECTION 4.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\neq, <, >, \leq$, and $\geq$ are inequality symbols.
2. $(-\infty, 5)$ is an example of an unbounded interval.
3. The parenthesis on the right of the interval notation $(-\infty, 5)$ indicates that 5 is not included in the interval.
4. To solve an inequality means to find all values of the variable that make the inequality true.
5. $3x + 2 \geq 7$ is an example of a linear inequality.
6. ∞ is a symbol representing positive infinity.
7. The symbol for "is less than" is $<$. The symbol for "is greater than or equal to" is $\geq$.
8. Equivalent inequalities are inequalities that have the same solution set.

CONCEPTS

Classify each of the following as an equation, an expression, or an inequality.

9. $-6 - 5x = 8$
equation
10. $5 - 2x$
expression
11. $7x - 5x > -4x$
inequality
12. $-(7x - 9)$
expression

Selected exercises available online at
www.webassign.net/brookscole

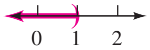
13. $\frac{x}{2} + 1 \le 3(x + 7)$ inequality

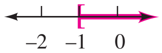
14. Match each interval with its graph.

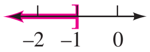
a. $(-\infty, -1]$ iii

a. $(-\infty, 1)$ i

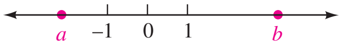
c. $[-1, \infty)$ ii

i. 

ii. 

iii. 

In the illustration, determine which of the following are true and which are false.



15. $b > 0$ true

16. $a - b < 0$ true

17. $ab > 0$ false

18. $b - a > 0$ true

19. $ab < 0$ true

20. $b - a < 0$ false

Perform each step listed below on the inequality $4 > -2$. Do not reverse the inequality symbol. Determine whether the resulting statement is true.

21. Add 2 to both sides. yes

22. Subtract 4 from both sides. yes

23. Multiply both sides by 4. yes

24. Divide both sides by -2. no

Consider the linear inequality $3x + 6 \le 6$. Determine whether each value is a solution of the inequality.

25. 0 yes

26. $\frac{2}{3}$ no

27. -10 yes

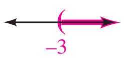
28. 1.5 no

The solution set of a linear inequality in x is graphed in the illustration. For that inequality, determine whether a true or false statement results when

29. -4 is substituted for x. false

30. -3 is substituted for x. false

31. 0 is substituted for x. true



NOTATION

Complete each solution to solve the inequality.

32.
$$-5x - 1 \ge -11$$
$$-5x \ge \boxed{-10}$$
$$\frac{-5x}{-5} \boxed{\le} \frac{-10}{-5}$$
$$x \le \boxed{2}$$

Using interval notation, the result is $(\boxed{-\infty}, 2]$.

Using set-builder notation, the result is $\{x \mid \boxed{x \le 2}\}$.

33.
$$3 - 6x \boxed{<} 17 + x$$
$$3 - \boxed{7x} < 17$$
$$-7x < \boxed{14}$$
$$\frac{-7x}{-7} \boxed{>} \frac{14}{-7}$$
$$x > \boxed{-2}$$

Using interval notation, the result is $(-2, \infty)$.

Using set-builder notation, the result is $\{x \mid x > -2\}$.

Write each inequality with the variable on the left side.

▶ **34.** $-10 > x$ $x < -10$

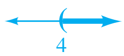
35. $\frac{7}{8} < x$ $x > \frac{7}{8}$

36. $0 \le x$ $x \ge 0$

GUIDED PRACTICE

Represent each set of real numbers using interval notation, with a graph, and using set-builder notation. See Example 1.

37. All real numbers greater than 4



$(4, \infty)$; $\{x \mid x > 4\}$

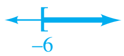
38. All real numbers less than -4

$(-\infty, -4)$; $\{x \mid x < -4\}$

39. All real numbers less than or equal to 4

$(-\infty, 4]$; $\{x \mid x \le 4\}$

▶ **40.** All real numbers greater than or equal to -6



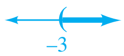
$[-6, \infty)$ $\{x \mid x \ge -6\}$

Solve each inequality. Write the solution set in interval notation and then graph it. See Example 2.

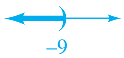
41. $x + 4 < 5$ $(-\infty, 1)$

42. $x - 5 > 2$ $(7, \infty)$

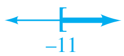
43. $3x > -9$ $(-3, \infty)$



▶ **44.** $4x < -36$ $(-\infty, -9)$

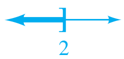


45. $2x - 7 \ge -29$ $[-11, \infty)$

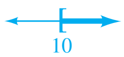


46. $6x + 8 \le -16$ $(-\infty, -4]$

47. $9a + 11 \le 29$ $(-\infty, 2]$

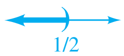


▶ **48.** $3b - 26 \ge 4$ $[10, \infty)$

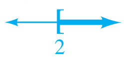


49. $3(4x + 1) < 15$ $(-\infty, 1)$

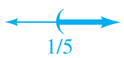
50. $2(3x + 4) \le 11$ $\left(-\infty, \frac{1}{2}\right)$



51. $3(5x - 7) \ge 9$ $[2, \infty)$



52. $5(x - 1) > -4$ $\left(\frac{1}{5}, \infty\right)$



Solve each inequality. Write the solution set in interval notation and then graph it. See Example 3.

53. $-30y \le -600$ $[20, \infty)$

54. $-6y \ge -600$ $(-\infty, 100]$

55. $-5t + 3 \le 5$ $\left[-\frac{2}{5}, \infty\right)$

56. $-9t + 6 \ge 16$ $\left(-\infty, -\frac{10}{9}\right]$

57. $-3x - 1 \le 5$ $[-2, \infty)$

58. $-2y + 6 < 16$ $(-5, \infty)$

▶ **59.** $t + 1 - 3t \ge t - 20$ $(-\infty, 7]$

60. $a + 4 - 10a > a - 16$ $(-\infty, 2)$

Solve each inequality. Write the solution set in interval notation and then graph it. See Example 4.

61. $2(2b + 2) > -\frac{11}{2}(2 - b)$ $(-\infty, 10)$

62. $2(4 - h) \ge -\frac{9}{4}(h - 3) + \frac{1}{2}h$ $(-\infty, 5]$

63. $\frac{1}{3}(y - 12) \le \frac{1}{2}y + 2$ $[-36, \infty)$

▶ **64.** $\frac{1}{4}\left(x - \frac{4}{3}\right) \le \frac{1}{3}(3x + 6)$ $\left[-\frac{28}{9}, \infty\right)$

Solve each inequality. Write the solution set in interval notation and then graph it. See Example 5.

65. $\frac{5a + 2}{-4} > \frac{5a + 1}{-4}$ no solution, $\varnothing$

▶ **66.** $2(5x - 6) > 4x - 15 + 6x$ $(-\infty, \infty)$,

67. $3(4x - 2) > 14x - 7 - 2x$ $(-\infty, \infty)$,

68. $\frac{7 - n}{-6} > \frac{1 - n}{-6}$ no solution, $\varnothing$

Translate each sentence into mathematical symbols. See Example 6.

69. As many as 16 people were seriously injured.
the number of seriously injured ≤ 16

70. There are no fewer than 10 references to carpools in the speech.
the number of references to carpools ≥ 10

71. The car is at least 25 years old.
the age of the car ≥ 25

▶ **72.** The age of the house does not exceed 90 years.
the age of the house ≤ 90

TRY IT YOURSELF

Solve each inequality. Write the solution set in interval notation and then graph it.

73. $0.6x \ge 36$ $[60, \infty)$

74. $0.2x < 8$ $(-\infty, 40)$

75. $3 > -\frac{9}{10}x$ $\left(-\frac{10}{3}, \infty\right)$

▶ **76.** $-\frac{2}{5} < -\frac{4}{5}x$ $\left(-\infty, \frac{1}{2}\right)$

77. $7 < \frac{5}{3}a - 3$ $(6, \infty)$

78. $5 > \frac{7}{2}a - 9$ $(-\infty, 4)$

79. $0.4x + 0.4 \le 0.1x + 0.85$ $(-\infty, 1.5]$

▶ **80.** $0.05 - 0.5x \le -0.7 - 0.8x$ $(-\infty, -2.5]$

81. $3(z - 2) \le 2(z + 7)$ $(-\infty, 20]$

▶ **82.** $5(3 + z) > -3(z + 3)$ $(-3, \infty)$

83. $-\frac{5x}{4} > \frac{3 - 5x}{4}$ no solution, $\varnothing$

84. $5(2n + 2) - n > 3n - 3(1 - 2n)$ $(-\infty, \infty)$,

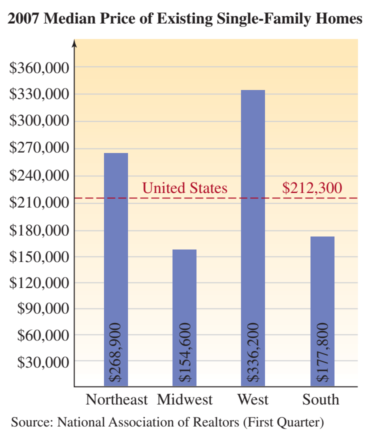
85. $\frac{2}{3}x + \frac{3}{2}(x - 5) \le x$ $\left(-\infty, \frac{45}{7}\right]$

▶ **86.** $\frac{5}{9}(x + 3) - \frac{4}{3}(x - 3) \ge x - 1$ $\left(-\infty, \frac{15}{4}\right]$

APPLICATIONS

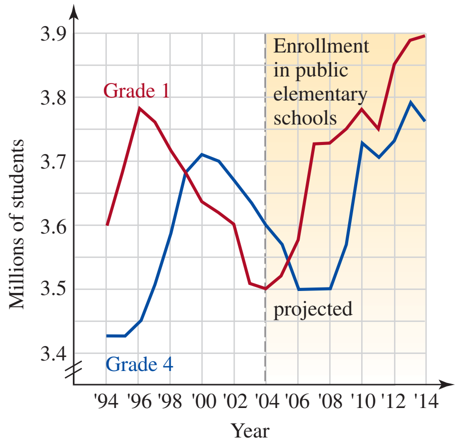
87. REAL ESTATE Refer to the graph below. For which regions of the country was the following inequality true in the year 2007? Midwest, South

Median sales price $<$ U.S. median price



▶ **88.** PUBLIC EDUCATION Refer to the illustration. For which years is the following inequality true?
1999–2005

Enrollment in grade 4 $\geq$ Enrollment in grade 1

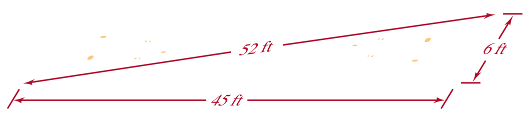


Source: National Center for Education Statistics

89. GEOMETRY The **triangle inequality** states an important relationship between the sides of any triangle:

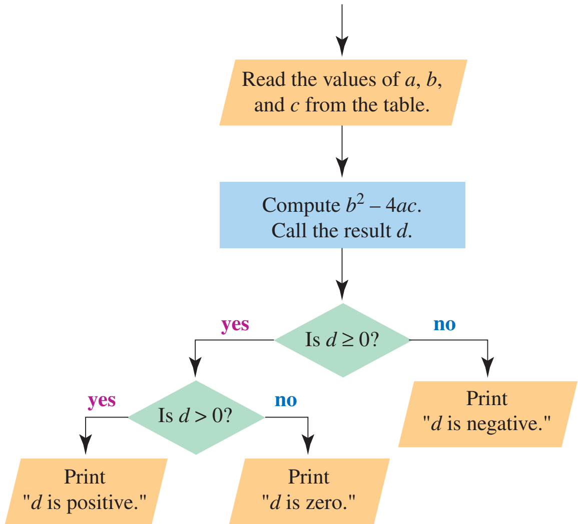
The sum of the lenths of two sides of a triangle $>$ the length of the third side.

Use the triangle inequality to show that the dimensions of the shuffleboard court shown in the next column must be mislabeled. $6 + 45 > 52$



▶ **90.** COMPUTER PROGRAMMING Flowcharts like the one below are used by programmers to show the step-by-step instructions of a computer program. For row 1 in the table, work through the steps of the flow chart using the values of a, b, and c, and determine what the computer printout would be. Do the same for row 2 and for row 3.
d is negative, d is zero, d is positive

	a	b	c
Row 1	1	1	1
Row 2	9	−12	4
Row 3	11	−25	−24



▶ **91.** FUNDRAISING A school PTA wants to rent a dunking tank for its annual school fundraising carnival. The cost is \$85.00 for the first three hours and then \$19.50 for each additional hour or part thereof. How long can the tank be rented if up to \$185 is budgeted for this expense? 8 hr

▶ **92.** INVESTMENTS If a woman has invested \$10,000 at 8% annual interest, how much more must she invest at 9% so that her annual income will exceed \$1,250? more than \$5,000

▶ **93.** GRADES A student has scores of 70, 77, and 85 on three government exams. What score does she need on a fourth exam to give her an average of 80 or better? 88 or higher

94. WIKIPEDIA The Web-based encyclopedia called Wikipedia was launched in January of 2001. The size of the English language edition can be modeled by the equation $a = 0.56t + 0.40$, where a is the number of articles in millions and t is the number of years since 2005. If the current trend continues, when will the number of articles exceed 6 million. (Source: Wikipedia article: *Size of Wikipedia*) 2015

95. WORK SCHEDULES A student works two part-time jobs. He earns \$7 an hour for working at the college library and \$12 an hour for construction work. To save time for study, he limits his work to 20 hours a week. If he enjoys working at the library more, how many hours can he work at the library and still earn at least \$175 a week? 13 hr

96. VIDEO GAME SYSTEMS A student who can afford to spend up to \$1,000 sees the ad shown in the illustration. If she decides to buy the video game system, find the greatest number of video games that she can also purchase. Disregard sales tax. 11

▶ **97.** SCHEDULING EQUIPMENT An excavating company charges \$300 an hour for the use of a backhoe and \$500 an hour for the use of a bulldozer. (Part of an hour counts as a full hour.) The company employs one operator for 40 hours per week to operate the machinery. If the company wants to bring in at least \$18,500 each week from equipment rental, how many hours per week can it schedule the operator to use a backhoe? 7 hr

▶ **98.** MEDICAL PLANS To save costs, a college raised the employee deductible as shown below. For what size hospital bills is Plan 2 better for the employee than Plan 1? (*Hint:* The cost to the employee includes both the deductible payment and the employee's coinsurance payment.) anything over \$1,800

Plan 1	Plan 2
Employee pays \$200	Employee pays \$400
Plan pays 70% of the rest	Plan pays 80% of the rest

Use a graphing calculator to solve each inequality.

99. $2x + 3 < 5$ $x < 1$

100. $3x - 2 > 4$ $x > 2$

101. $5x + 2 \geq 4x - 2$ $x \geq -4$

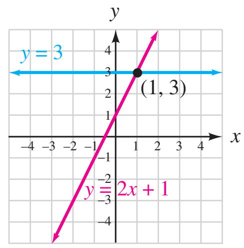
102. $3x - 4 \leq 2x + 4$ $x \leq 8$

WRITING

103. The techniques for solving linear equations and linear inequalities are similar, yet different. Explain.

▶ **104.** Explain how the symbol ∞ is used in this section. Is ∞ a real number?

105. Explain how to use the following graph to solve $2x + 1 < 3$.



106. Explain each property of inequalities.

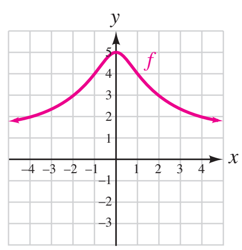
a. If $a < b$, and c is any real number, then $a + c < b + c$.

b. If $a < b$, and c is any negative real number, then $ac > bc$.

REVIEW

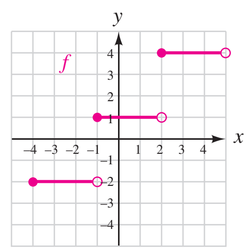
Use the graph to find $f(-1)$, $f(0)$, and $f(2)$.

107.



4, 5, 3

108.



1, 1, 4

Complete each table.

109. $f(x) = x - x^3$

Input	Output
−2	6
2	−6

110. $g(t) = \dfrac{t^2 - 1}{5}$

Input	Output
−6	7
4	3

SECTION 4.2

Solving Compound Inequalities

Objectives

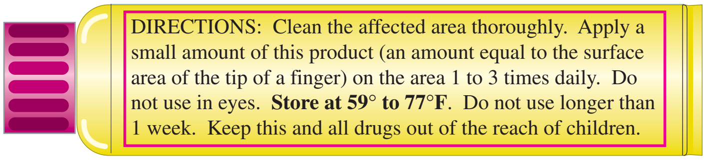
1. Find the intersection and the union of two sets.
2. Solve compound inequalities containing the word *and*.
3. Solve double linear inequalities.
4. Solve compound inequalities containing the word *or*.

The label on the tube of antibiotic ointment, shown in the figure below, advises the user about the temperature at which the medication should be stored. A careful reading of the statement reveals that the storage instruction consists of two parts:

The storage temperature should be at least 59°F.

and

The storage temperature should be at most 77°F.



When the words *and* or *or* are used to connect pairs of inequalities, we call these statements *compound inequalities.* To solve compound inequalities, we need to know how to find the *intersection* and *union* of two sets.

1 Find the intersection and the union of two sets.

Just as operations such as addition and multiplication are performed on real numbers, operations can also be performed on sets. The operation of intersection of two sets produces a new third set that consists of all of the elements that the two given sets have in common.

The Intersection of Two Sets

The **intersection of set A and set B,** written $A \cap B$, is the set of all elements that are common to set A and set B.

The Language of Algebra The *intersection* of two sets is the collection of elements that they have in common. When two streets cross, we call the area of pavement that they have in common an *intersection.*

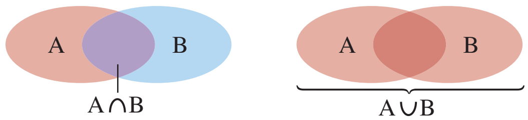
The operation of union of two sets produces a third set that is a combination of all of the elements of the two given sets.

The Union of Two Sets

The **union of set A and set B,** written $A \cup B$, is the set of elements that belong to set A or set B or both.

The Language of Algebra The *union* of two sets is the collection of elements that belong to either set. The concept is similar to that of a family *reunion,* which brings together the members of several families.

Venn diagrams can be used to illustrate the intersection and union of sets. The area shown in purple in figure (a) represents $A \cap B$ and the area shown in both shades of red in figure (b) represents $A \cup B$.



(a) (b)

Self Check 1

Let $C = \{8, 9, 10, 11\}$ and $D = \{3, 6, 9, 12, 15\}$.

a. Find $C \cap D$.
b. Find $C \cup D$.

Now Try **Problems 32 and 36**

Self Check 1 Answers
a. $\{9\}$
b. $\{3, 6, 8, 9, 10, 11, 12, 15\}$

Teaching Example 1 Let $A = \{-5, -3, -1, 1, 3, 5\}$ and $B = \{0, 1, 2, 3\}$
a. Find $A \cap B$
b. Find $A \cup B$
Answers:
a. $\{1, 3\}$
b. $\{-5, -3, -1, 0, 1, 2, 3, 5\}$

EXAMPLE 1 Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ and $B = \{-4, -2, 0, 2, 4\}$.

a. Find $A \cap B$. **b.** Find $A \cup B$.

Strategy In part a, we will find the elements that sets A and B have in common, and in part b, we will find the elements that are in one set or the other.

WHY The symbol $\cap$ means intersection, and the symbol $\cup$ means union.

Solution

a. Since the numbers 0, 2, and 4 are common to both sets A and B, we have

$$A \cap B = \{0, 2, 4\}$$

b. Since the numbers in either or both sets are $-4, -2, 0, 1, 2, 3, 4, 5$, and 6, we have

$$A \cup B = \{-4, -2, 0, 1, 2, 3, 4, 5, 6\}$$

2 Solve compound inequalities containing the word *and*.

When two inequalities are joined with the word *and,* we call the statement a **compound inequality.** Some examples are

$$x \geq -3 \quad \text{and} \quad x \leq 6$$

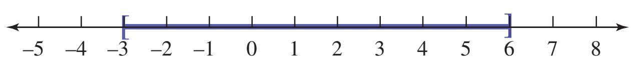
$$\frac{x}{2} + 1 > 0 \quad \text{and} \quad 2x - 3 < 5$$

$$x + 3 \leq 2x - 1 \quad \text{and} \quad 3x - 2 < 5x - 4$$

The solution set of these inequalities contains the numbers that make *both* of the inequalities true. For example, we can find the solution set of the compound inequality $x \geq -3$ and $x \leq 6$ by first graphing the solution sets of each inequality on the same number line and then looking for the numbers common to both graphs.

In figure (a) below, the graph of the solution set of $x \geq -3$ is shown in red, and the graph of the solution set of $x \leq 6$ is shown in blue. Figure (b) shows the graph of the solution of $x \geq -3$ and $x \leq 6$. The purple shaded interval in figure (b) is where the red and blue graphs overlap in figure (a). It represents the numbers common to the graphs of $x \geq -3$ and $x \leq 6$.

(a)



(b)

The solution set of $x \geq -3$ and $x \leq 6$ can be denoted by the **bounded interval** $[-3, 6]$, where the brackets indicate that the endpoints, -3 and 6, are included. It represents all real numbers between -3 and 6, including -3 and 6. Intervals such as this, which contain both endpoints, are called **closed intervals.**

When solving a compound inequality containing *and,* the solution set is the *intersection* of the solution sets of the two inequalities. The **intersection** of two sets is the set of elements that are common to both sets. We can denote the intersection of two sets using the symbol $\cap$, which is read as "intersection." For the compound inequality $x \geq -3$ and $x \leq 6$, we can write

$$[-3, \infty) \cap (-\infty, 6] = [-3, 6]$$

The solution set of the compound inequality $x \geq -3$ and $x \leq 6$ can be expressed in several ways:

1. *As a graph:* 

2. *In interval notation:* $[-3, 6]$

3. *In words:* all real numbers between -3 and 6, including -3 and 6

4. *Using set-builder notation:* $\{x \mid x \geq -3 \text{ and } x \leq 6\}$

EXAMPLE 2

Solve $\frac{x}{2} + 1 > 0$ and $2x - 3 < 5$. Graph the solution set and write it using interval notation.

Strategy We will solve each inequality separately. Then we will graph the two solution sets on the same number line and determine their intersection.

WHY The solution set of a compound inequality containing the word *and* is the intersection of the solution sets of the two inequalities.

Solution

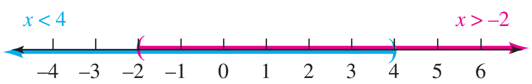
In each case, we can use properties of inequality to isolate the variable on one side of the inequality.

$$\frac{x}{2} + 1 > 0 \quad \text{and} \quad 2x - 3 < 5$$

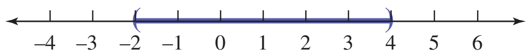
$$\frac{x}{2} > -1 \quad \Big| \quad 2x < 8$$

$$x > -2 \quad \Big| \quad x < 4$$

Next, we graph the solutions of each inequality on the same number line and determine their intersection.



The intersection of the graphs in the figure is the set of all real numbers between -2 and 4. Using interval notation, the solution set is the interval $(-2, 4)$, whose graph is shown in purple below. This bounded interval, which does not include either endpoint, is called an **open interval.**



Self Check 2

Solve $3x > -18$ and $\frac{x}{5} - 1 \leq 1$. Graph the solution set and write it using interval notation.

Now Try **Problem 40**

Self Check 2 Answer

$(-6, 10]$ 

Teaching Example 2 Solve $2x + 3 \geq 5$ and $\frac{x}{3} - 2 < 1$. Graph the solution set and write it using interval notation.

Answer:

$[1, 9)$ 

The Language of Algebra When graphing on a number line, $(-2, 4)$ represents an *interval* when graphing on a rectangular coordinate system, $(-2, 4)$ is an *ordered pair* that gives the coordinates of a point.

The solution of the compound inequality in the Self Check of Example 2 is the interval $(-6, 10]$. A bounded interval such as this, which includes only one endpoint, is called a **half-open interval.** The following chart shows the various types of bounded intervals, along with the inequalities and interval notation that describe them.

Intervals

Open intervals The interval (a, b) includes all real numbers x such that $a < x < b$.

Half-open intervals The interval $[a, b)$ includes all real numbers x such that $a \le x < b$.

The interval $(a, b]$ includes all real numbers x such that $a < x \le b$.

Closed intervals The interval $[a, b]$ includes all real numbers x such that $a \le x \le b$.

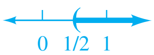
Self Check 3

Solve $2x + 3 < 4x + 2$ and $3x + 1 < 5x + 3$. Graph the solution set and write it using interval notation.

Now Try **Problem 44**

Self Check 3 Answer

$\left(\frac{1}{2}, \infty\right)$



Teaching Example 3 Solve $3x + 4 \le 2x + 6$ and $5x - 2 > 7x + 2$. Graph the solution set and write it using interval notation.

Answer:

$(-\infty, -2)$

EXAMPLE 3 Solve $x + 3 \le 2x - 1$ and $3x - 2 < 5x - 4$. Graph the solution set and write it using interval notation.

Strategy We will solve each inequality separately. Then we will graph the two solution sets on the same number line and determine their intersection.

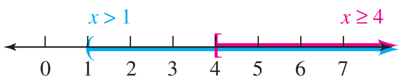
WHY The solution set of a compound inequality containing the word *and* is the intersection of the solution sets of the two inequalities.

Solution

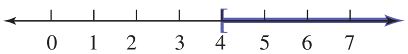
In each case, we can use properties of inequality to isolate the variable on one side.

$$\begin{array}{rcl|l} x + 3 \le 2x - 1 & \text{and} & 3x - 2 < 5x - 4 & \text{This is the compound inequality to solve.} \\ 4 \le x & & 2 < 2x & \\ x \ge 4 & & 1 < x & \\ & & x > 1 & \end{array}$$

The graph of $x \ge 4$ is shown below in red and the graph of $x > 1$ is shown below in blue.



Only those x where $x \ge 4$ and $x > 1$ are in the solution set of the compound inequality. Since all numbers greater than or equal to 4 are also greater than 1, the solutions are the numbers x where $x \ge 4$. The solution set is the interval $[4, \infty)$, whose graph is shown below.



EXAMPLE 4 Solve $x - 1 > -3$ and $2x < -8$, if possible.

Strategy We will solve each inequality separately. Then we will graph the two solution sets on the same number line and determine their intersection.

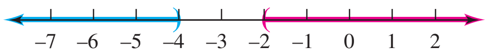
WHY The solution set of a compound inequality containing the word *and* is the intersection of the solution sets of the two inequalities.

Solution

In each case, we can use properties of inequality to isolate the variable on one side.

$$x - 1 > -3 \quad \text{and} \quad 2x < -8$$
$$x > -2 \quad \big| \quad x < -4$$

We note that the graphs of the solution sets shown below do not intersect.



This means that there are no numbers that make both parts of the original compound inequality true. So the solution set is the empty set, which can be denoted as $\varnothing$.

3 Solve double linear inequalities.

Inequalities that contain two inequality symbols are called **double inequalities.** An example of a double inequality is

$-3 \le 2x + 5 < 7$ Read as "-3 is less than or equal to $2x + 5$ and $2x + 5$ is less than 7."

Any double linear inequality can be rewritten as a compound inequality containing the word *and.* In general, the following is true.

Double Linear Inequalities

The compound inequality $c < x < d$ is equivalent to $c < x$ and $x < d$.

EXAMPLE 5 Solve $-3 \le 2x + 5 < 7$. Graph the solution set.

Strategy We will solve the double inequality by applying properties of inequality to *all three of its parts* to isolate x in the middle.

WHY This double inequality $-3 \le 2x + 5 < 7$ means that $-3 \le 2x + 5$ and $2x + 5 < 7$. We can solve this double inequality more easily by leaving it in its original form.

Solution

$$-3 \le 2x + 5 < 7$$

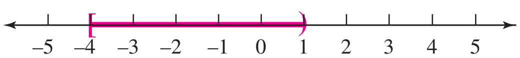
$-3 - 5 \le 2x + 5 - 5 < 7 - 5$ To undo the addition of 5, subtract 5 from all three parts.

$-8 \le 2x < 2$ Perform the subtractions.

$\dfrac{-8}{2} \le \dfrac{2x}{2} < \dfrac{2}{2}$ To undo the multiplication by 2, divide all three parts by 2.

$-4 \le x < 1$ Perform the divisions.

The solution set is the half-open interval $[-4, 1)$, whose graph is shown below.



Self Check 4

Solve $2x - 3 < x - 2$ and $0 < x - 3.5$, if possible.

Now Try Problem 48

Self Check 4 Answer

no solution, $\varnothing$

Teaching Example 4 Solve $6x - 3 < 6$ and $2x + 1 > 7$, if possible.

Answer:

no solution, $\varnothing$

Self Check 5

Solve $-5 \le 3x - 8 \le 7$. Graph the solution set and write it in interval notation.

Now Try Problem 51

Self Check 5 Answer

$[1, 5]$

Teaching Example 5 Solve $2 < 3x - 1 \le 8$. Graph the solution set and write it in interval notation.

Answer:

$(1, 3]$

> ***Caution!*** When multiplying or dividing all three parts of a double inequality by a negative number, don't forget to reverse the direction of *both* inequalities. As an example, we will solve $-15 < -5x \le 25$.

$-15 < -5x \le 25$	This is the inequality to solve.
$\dfrac{-15}{-5} > \dfrac{-5x}{-5} \ge \dfrac{25}{-5}$	Divide all three parts by -5 to isolate x in the middle. Reverse both inequality signs.
$3 > x \ge -5$	Perform the divisions.
$-5 \le x < 3$	Write an equivalent compound inequality with the smaller number, -5, on the left.

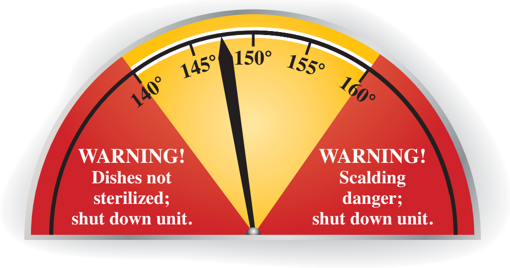
4 Solve compound inequalities containing the word *or*.

A warning on the water temperature gauge of a commercial dishwasher cautions the operator to shut down the unit if

The water temperature goes below 140°

or

The water temperature goes above 160°

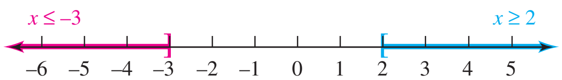


When two inequalities are joined with the word *or,* we also call the statement a compound inequality. Some examples are

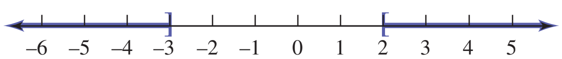
$$x < 140 \quad \text{or} \quad x > 160$$
$$x \le -3 \quad \text{or} \quad x \ge 2$$
$$\frac{x}{3} > \frac{2}{3} \quad \text{or} \quad -(x-2) > 3$$

The solution set of a compound inequality containing the word *or* contains the numbers that make *one or the other or both* inequalities true. For example, we can find the solution set of the compound inequality $x \le -3$ or $x \ge 2$ by putting the graphs of each inequality on the same number line.

In figure (a), the graph of the solution set of $x \le -3$ is shown in red, and the graph of the solution set of $x \ge 2$ is shown in blue. Figure (b) shows the graph of the solution set of $x \le -3$ or $x \ge 2$. This graph is a combination of the graph of $x \le -3$ with the graph of $x \ge 2$.



(a)

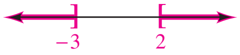


(b)

When solving a compound inequality containing *or,* the solution set is the *union* of the solution sets of the two inequalities. The **union** of two sets is the set of elements that are in either of the sets or both. We can denote the union of two sets using the symbol $\cup$, which is read as "union." For the compound inequality $x \le -3$ or $x \ge 2$, we can write the solution set using interval notation:

$(-\infty, -3] \cup [2, \infty)$

We can express the solution set of the compound inequality $x \le -3$ or $x \ge 2$ in several ways:

1. *As a graph:* 

2. *In interval notation:* $(-\infty, -3] \cup [2, \infty)$
3. *In words:* all real numbers less than or equal to -3 or greater than or equal to 2
4. *In set-builder notation:* $\{x \mid x \le -3 \text{ or } x \ge 2\}$

Caution! In the statement $x \le -3$ or $x \ge 2$, it is incorrect to string the inequalities together as $2 \le x \le -3$, because that would imply that $2 \le -3$, which is false.

EXAMPLE 6

Solve $\frac{x}{3} > \frac{2}{3}$ or $-(x - 2) > 3$. Graph the solution set.

Strategy We will solve each inequality separately. Then we will graph the two solution sets on the same number line to show their union.

WHY The solution set of a compound inequality containing the word *or* is the union of the solution sets of the two inequalities.

Solution

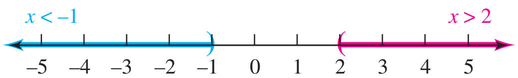
To solve each inequality, we proceed as follows:

$$\frac{x}{3} > \frac{2}{3} \quad \text{or} \quad -(x - 2) > 3$$

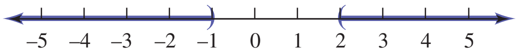
This is the compound inequality to solve.

$$x > 2 \qquad\qquad -x + 2 > 3$$
$$-x > 1$$
$$x < -1$$

The graph of the solution set is obtained by graphing the solution sets of each inequality on the same number line, as shown.



The union of the two solution sets consists of all real numbers less than -1 or greater than 2. Using interval notation, the solution set is the interval $(-\infty, -1) \cup (2, \infty)$. Its graph appears below.



Self Check 6

Solve $\frac{x}{2} > 2$ or $-3(x - 2) > 0$. Graph the solution set and write it in interval notation.

Now Try **Problem 58**

Self Check 6 Answer

$(-\infty, 2) \cup (4, \infty)$

Teaching Example 6 Solve $\frac{x}{6} \ge \frac{1}{2}$ or $-2(x + 1) > 4$. Graph the solution set and write it in interval notation.

Answer:

$(-\infty, -3) \cup [3, \infty)$ 

Self Check 7

Solve $x - 1 < 5$ or $-2x \le 10$. Graph the solution set.

Now Try **Problem 62**

Self Check 7 Answer

$(-\infty, \infty)$

Teaching Example 7 Solve: $-3x \le 12$ or $5x + 1 < 11$

Answer:

$(-\infty, \infty)$

EXAMPLE 7

Solve $x + 3 \ge -3$ or $-x > 0$.

Strategy We will solve each inequality separately. Then we will graph the two solution sets on the same number line to show their union.

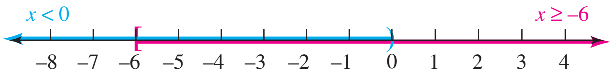
WHY The solution set of a compound inequality containing the word *or* is the union of the solution sets of the two inequalities.

Solution

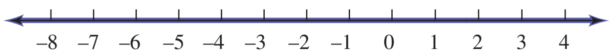
We solve each inequality separately.

$$x + 3 \ge -3 \quad \text{or} \quad -x > 0 \qquad \text{This is the compound inequality to solve.}$$
$$x \ge -6 \quad \Big| \quad x < 0$$

We graph the solution set of each inequality on the same number line.



The entire number line is shaded, which indicates that all real numbers satisfy the original compound inequality. Using interval notation, the solution set is denoted $(-\infty, \infty)$. Its graph is shown below.



THINK IT THROUGH *Study Abroad*

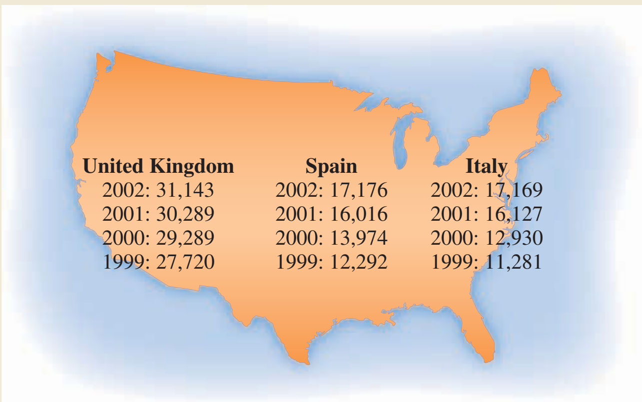
"American students are studying abroad in growing numbers despite economic and security concerns post-Sept 11."

Open Doors Report 2003, Institute of International Education

Recent figures released by Institute of International Education show that the United Kingdom, Spain, and Italy are the top three destinations for American students studying abroad. For what year, or years, was

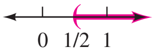
1. the number of American students studying in the United Kingdom greater than 30,000 *and* the number of American students studying in Spain greater than 13,000? 2001, 2002
2. the number of American students studying in Spain greater than 15,000 *or* the number of American students studying in Italy less than 12,000? 1999, 2001, 2002
3. the number of American students studying in Spain greater than the number studying in Italy *and* the number of American students studying in the United Kingdom at least twice the number studying in Italy? 1999, 2000

American Students Studying in . . .

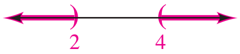


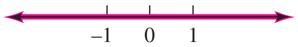
ANSWERS TO SELF CHECKS

1. a. $\{9\}$ **b.** $\{3, 6, 8, 9, 10, 11, 12, 15\}$ **2.** $(-6, 10]$ 

3. $\left(\frac{1}{2}, \infty\right)$ 

4. no solution, $\varnothing$ **5.** $[1, 5]$ 

6. $(-\infty, 2) \cup (4, \infty)$ 

7. $(-\infty, \infty)$ 

SECTION 4.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. The intersection of two sets is the set of elements that are common to both sets. The union of two sets is the set of elements that are in one set, or the other, or both.

2. $x \ge 3$ and $x \le 4$ is a compound inequality.

3. $-6 < x + 1 \le 1$ is a double linear inequality.

4. $(2, 8)$ is an example of an open interval.

5. The bounded interval $[-4, 0]$ is an example of a closed interval.

6. $(0, 9]$ is an example of a half-open interval.

CONCEPTS

Fill in the blanks.

7. The word *and* between two inequality statements requires that both of the inequalities must be true for the entire statement to be true.

8. The word *or* between two inequality statements requires that only one of the inequalities must be true for the entire statement to be true.

9. If the three parts of a double inequality are divided by a negative number, the direction of both inequality symbols must be reversed.

10. The double inequality $-2 < 3x + 4 < 10$ can be written as $-2 < 3x + 4$ and $3x + 4 < 10$.

11. In each case, determine whether -3 is a solution of the compound inequality.

a. $\frac{x}{3} + 1 \ge 0$ and $2x - 3 < -10$ no

b. $2x \le 0$ or $-3x < -5$ yes

12. In each case, determine whether -3 is a solution of the double linear inequality.

a. $-1 < -3x + 4 < 12$ no

b. $-1 < -3x + 4 < 14$ yes

13. Give the solution set of each inequality in interval notation, if possible.

a. $x < -3$ and $x > 3$ no solution

b. $x < 3$ or $x > -3$ $(-\infty, \infty)$

14. Give the solution set of each inequality in interval notation, if possible.

a. $x < 0$ or $x \ge 0$ $(-\infty, \infty)$

b. $x < 0$ and $x > 0$ no solution

15. Match each interval with its corresponding graph.

a. $[2, 3)$ ii

b. $(2, 3)$ iii

c. $[2, 3]$ i

i. 

ii. 

iii. 

Give the interval notation that describes each set. Then graph it.

16. The real numbers between -3 and 3
$(-3, 3)$ 

17. The real numbers less than -3 or greater than 3
$(-\infty, -3) \cup (3, \infty)$ 

18. The real numbers between -3 and 3, including 3
$(-3, 3]$ 

19. $\{x \mid x \ge -3 \text{ and } x \le 3\}$
$[-3, 3]$ 

20. $\{x \mid x \le -3 \text{ or } x \ge 3\}$
$(-\infty, -3] \cup [3, \infty)$ 

NOTATION

21. The intersection of set A and set B is denoted as $A \cap B$.

22. The union of set A and set B is denoted as $A \cup B$.

23. Graph: $(-\infty, 2) \cup [3, \infty)$ 

24. Graph: $(-\infty, 3) \cap [-2, \infty)$ 

25. Graph: $(0, 8) \cap (2, 10)$ 

26. Graph: $(-5, 7] \cup [2, 9]$ 

▶ Selected exercises available online at
www.webassign.net/brookscole

Classify each interval as open, half-open, or closed.

27. $(-2, 15]$ half-open

28. $[-2, 15]$ closed

29. $(-2, 15)$ open

30. $[-2, 15)$ half-open

GUIDED PRACTICE

Let $A = \{0, 1, 2, 3, 4, 5, 6\}$, $B = \{4, 6, 8, 10\}$, $C = \{-3, -1, 0, 1, 2\}$, *and* $D = \{-3, 1, 2, 5, 8\}$. *Find each set.* See Example 1.

31. $A \cap B$
$\{4, 6\}$

32. $A \cap D$
$\{1, 2, 5\}$

▶ **33.** $C \cap D$
$\{-3, 1, 2\}$

34. $B \cap C$
$\varnothing$

35. $B \cup C$
$\{-3, -1, 0, 1, 2, 4, 6, 8, 10\}$

36. $A \cup C$
$\{-3, -1, 0, 1, 2, 3, 4, 5, 6\}$

37. $A \cup D$
$\{-3, 0, 1, 2, 3, 4, 5, 6, 8\}$

▶ **38.** $C \cup D$
$\{-3, -1, 0, 1, 2, 5, 8\}$

Solve each compound inequality. Write the solution set in interval notation and then graph it. See Example 2.

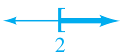
39. $x > -2$ and $x \le 5$ $(-2, 5]$ 

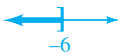
40. $x \le -4$ and $x \ge -7$ $[-7, -4]$ 

▶ **41.** $\frac{1}{2}x \le 2$ and $0.75x \ge -6$ $[-8, 4]$ 

42. $4x \ge -x + 5$ and $6 \ge 4x - 3$ $\left[1, \frac{9}{4}\right]$ 

Solve each compound inequality. Write the solution set in interval notation and then graph it. See Example 3.

43. $5(x - 2) \ge 0$ and $-3x < 9$ $[2, \infty)$ 

▶ **44.** $6x + 1 < 5x - 3$ and $\frac{x}{2} + 9 \le 6$ $(-\infty, -6]$ 

45. $x - 1 \le 2(x + 2)$ and $x \le 2x - 5$ $[5, \infty)$ 

46. $5(x + 1) \le 4(x + 3)$ and $x + 12 < -3$
$(-\infty, -15)$ 

Find the solution set of each compound inequality. See Example 4.

47. $x + 2 < -\frac{1}{3}x$ and $-6x < 9x$ $\varnothing$

48. $-x < -2x$ and $3x > 2x$ $\varnothing$

▶ **49.** $\frac{3}{2}x + \frac{1}{5} < 5$ and $2x + 1 > 9$ $\varnothing$

50. $-\frac{x}{4} > -2.5$ and $9x > 2(4x + 5)$ $\varnothing$

Solve each double inequality. Write the solution set in interval notation and then graph it. See Example 5.

51. $4 \le x + 3 \le 7$ $[1, 4]$ 

▶ **52.** $15 > 2x - 7 > 9$ $(8, 11)$ 

53. $-6 \le \frac{1}{3}a + 1 < 0$ $[-21, -3)$ 

54. $0 \le \frac{4 - x}{3} \le 2$ $[-2, 4]$ 

Solve each compound inequality. Write the solution set (if one exists) in interval notation and then graph it. See Examples 6–7.

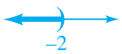
55. $x \le -2$ or $x > 6$
$(-\infty, -2] \cup (6, \infty)$ 

56. $x \ge -1$ or $x \le -3$
$(-\infty, -3] \cup [-1, \infty)$ 

57. $x - 3 < -4$ or $x - 2 > 0$
$(-\infty, -1) \cup (2, \infty)$ 

▶ **58.** $4x < -12$ or $\frac{x}{2} > 4$
$(-\infty, -3) \cup (8, \infty)$ 

59. $-4(x + 2) \ge 12$ or $3x + 8 < 11$
$(-\infty, 1)$ 

▶ **60.** $4.5x - 1 < -10$ or $6 - 2x \ge 12$
$(-\infty, -2)$ 

61. $4.5x - 2 > 2.5$ or $\frac{1}{2}x \le 1$
$(-\infty, \infty)$ 

▶ **62.** $0 < x$ or $3x - 5 > 4x - 7$
$(-\infty, \infty)$ 

TRY IT YOURSELF

Solve each compound inequality and write the solution set in interval notation. If the solution set is nonempty, graph it.

63. $2x < -18$ and $-4x < 40$
$(-10, -9)$ 

64. $x + 3 < 3x - 1$ and $4x - 3 \le 3x$
$(2, 3]$ 

65. $-5.3 < x - 2.3 < -1.3$
$(-3, 1)$ 

66. $25 > 3x - 2 > 7$
$(3, 9)$


67. $-2 < -b + 3 < 5$
$(-2, 5)$


68. $2 < -t - 2 < 9$
$(-11, -4)$


69. $-6 < -3(x - 4) \leq 24$
$[-4, 6)$


70. $-4 \leq -2(x + 8) < 8$
$(-12, -6]$


▶ **71.** $2x + 1 \geq 5$ and $-3(x + 1) \geq -9$
$[2, 2]$


72. $2(-2) \leq 3x - 1$ and $3x - 1 \leq -1 - 3$
$[-1, -1]$


73. $\dfrac{x}{0.7} + 5 > 4$ and $-4.8 \leq \dfrac{3x}{-0.125}$
$(-0.7, 0.2]$


74. $-4 > \dfrac{2}{3}x - 2 > -6$
$(-6, -3)$


▶ **75.** $-2 \leq \dfrac{5 - 3x}{2} \leq 2$
$\left[\frac{1}{3}, 3\right]$


76. $3x + 2 < 8$ or $2x - 3 > 11$
$(-\infty, 2) \cup (7, \infty)$


77. $3x + 4 < -2$ or $3x + 4 > 10$
$(-\infty, -2) \cup (2, \infty)$


78. $x > 3$ or $x < 5$
$(-\infty, \infty)$

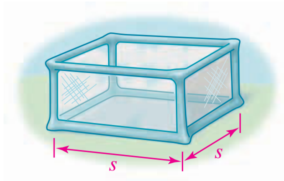

79. $x < -15$ or $x > -100$
$(-\infty, \infty)$


80. $3x + 4 < 3$ and $2x - 3 \geq 12$ $\varnothing$

APPLICATIONS

▶ **81.** BABY FURNITURE A company manufactures various sizes of playpens having perimeters between 128 and 192 inches, inclusive. See next column.

a. Complete the double inequality that mathematically describes the range of the perimeters of the playpens.



$128 \leq 4s \leq 192$

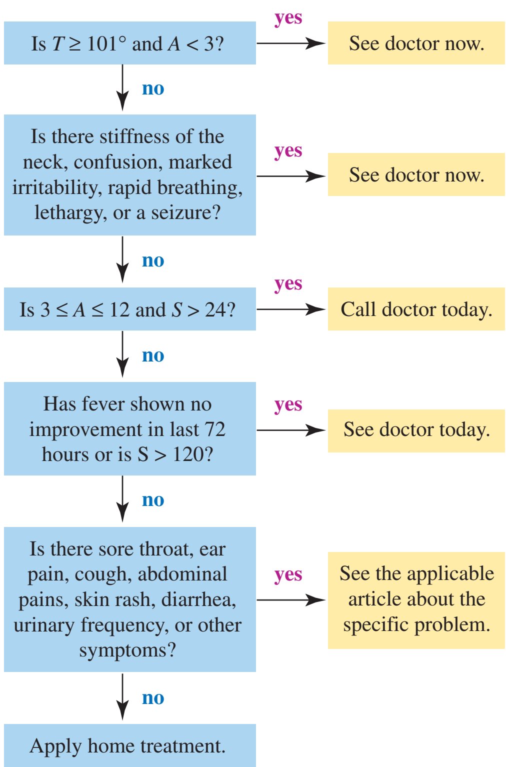
b. Solve the double inequality to find the range of the side lengths of the playpens. $32 \leq s \leq 48$

▶ **82.** TRUCKING The distance that a truck can travel in 8 hours, at a constant rate of r mph, is given by $8r$. A trucker wants to travel at least 350 miles, and company regulations don't allow him to exceed 450 miles in one 8-hour shift.

a. Complete the double inequality that describes the mileage range of the truck. $350 \leq 8r \leq 450$

b. Solve the double inequality to find the range of the average rate (speed) of the truck for the 8-hour trip. $43.75 \leq r \leq 56.25$

▶ **83.** TREATING A FEVER Use the following flowchart to determine what action should be taken for a 13-month-old child who has had a 99.8° temperature for 3 days and is not suffering any other symptoms. T represents the child's temperature, A the child's age in months, and S the number of hours the child has experienced the symptoms. See doctor today.



Based on information from *Take Care of Yourself* (Addison-Wesley, 1993)

84. THERMOSTATS The *Temp range* control on the thermostat shown below directs the heater to come on when the room temperature gets 5 degrees below the *Temp setting;* it directs the air conditioner to come on when the room temperature gets 5 degrees above the *Temp setting.* Use interval notation to describe

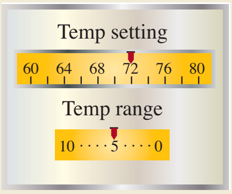
from Campus to Careers

Heating, Ventilation, and Air Conditioning Technician

© Andrew Brooks/Corbis

a. the temperature range for the room when neither the heater nor the air conditioner will be on. (67, 77)

b. the temperature range for the room when either the heater or the air conditioner will be on. (*Note:* The lowest temperature theoretically possible is −460°F, called *absolute zero.*) (−460, 67] ∪ [77, ∞)



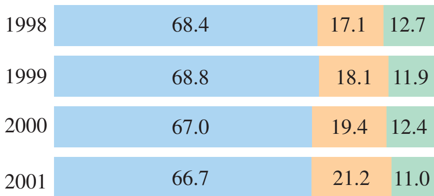
85. HEALTH CARE Refer to the illustration. Let P represent the percent of children covered by private insurance, M the percent covered by Medicaid, and N the percent not covered. For what years are the following true?

a. $P \geq 68$ and $M \geq 18$ 1999

b. $P \geq 68$ or $M \geq 18$ 1998–2001

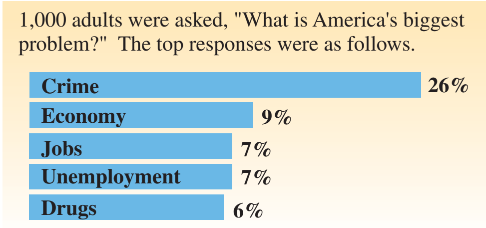
c. $P \geq 67$ and $N \geq 12.5$ 1999, 2000

d. $P \geq 67$ or $N \geq 12.5$ 1998–2000



Source: U.S. Department of Health and Human Services

86. POLLS For each response to the poll question shown in the illustration, the *margin of error* is +/− (read as "plus or minus") 3.2%. This means that for the statistical methods used to do the polling, the actual response could be as much as 3.2 points more or 3.2 points less than shown. Use interval notation to describe the possible interval (in percent) for each response.
crime: [22.8, 29.2], economy: [5.8, 12.2], jobs: [3.8, 10.2], unemployment: [3.8, 10.2], drugs: [2.8, 9.2]

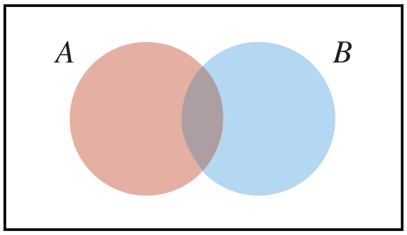


WRITING

87. Explain how to find the union and how to find the intersection of $(-\infty, 5)$ and $(-2, \infty)$ graphically.

88. Explain why the double inequality $2 < x < 8$ can be written in the form $2 < x$ and $x < 8$.

89. Each of the shaded regions in the **Venn diagram** in the illustration below represents a set. Describe the intersection of set A and set B.



90. See Exercise 89. Describe the union of set A and set B.

REVIEW

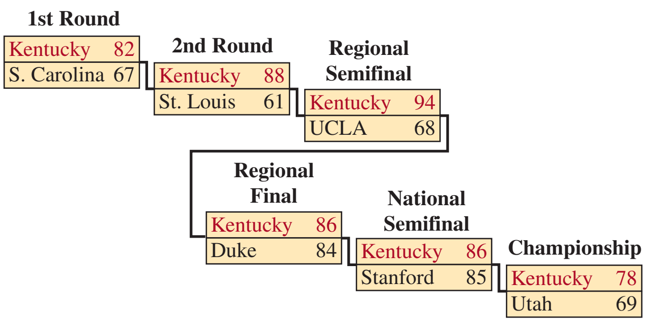
Refer to the illustration below, which shows the results of each of the games of the eventual champion, the University of Kentucky, in the 1998 NCAA Men's Basketball Tournament. Round to the nearest tenth when necessary.

91. What are the mean, median, and mode of the set of Kentucky scores? 85.7, 86, 86

92. What are the mean and the median of the set of scores of Kentucky's opponents? 72.3, 68.5

93. Find the margin of victory for Kentucky in each of its games. Then find the average (mean) margin of victory for Kentucky in the tournament. 13.3 pts/game

94. What was the average (mean) combined score for Kentucky and its opponents in the tournament? 158



SECTION 4.3

Solving Absolute Value Equations and Inequalities

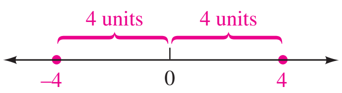
Objectives

1. Find absolute values.
2. Solve equations of the form $|x| = k$.
3. Solve equations with two absolute values.
4. Solve inequalities of the form $|x| < k$.
5. Solve inequalities of the form $|x| > k$.

Many quantities in mathematics, science, and engineering are expressed as positive numbers. To guarantee that a quantity is positive, we often use absolute value. In this section, we will work with equations and inequalities involving the absolute value of algebraic expressions. Using the definition of absolute value, we will develop procedures to solve absolute value equations and absolute value inequalities.

1 Find absolute values.

Recall that the absolute value of any real number is the distance between the number and zero on the number line. For example, the points shown to the right with coordinates of 4 and -4 both lie 4 units from 0. Thus, $|4| = |-4| = 4$.



The absolute value of a real number can be defined more formally as follows.

Absolute Value

If $x \geq 0$, then $|x| = x$.

If $x < 0$, then $|x| = -x$.

This definition gives a way to associate a nonnegative real number with any real number.

- If $x \geq 0$, then x (which is positive or 0) is its own absolute value.
- If $x < 0$, then $-x$ (which is positive) is the absolute value.

Either way, $|x|$ is positive or 0. That is, $|x| \geq 0$ for all real numbers x.

EXAMPLE 1

Find: **a.** $|9|$ **b.** $|-5.68|$ **c.** $|0|$ and **d.** $2|-8|$

Strategy We will use the definition of absolute value.

WHY The absolute value definition gives another way to find the absolute value of a number.

Solution

a. Since $9 \geq 0$, the number 9 is its own absolute value: $|9| = 9$.

b. Since $-5.68 < 0$, the opposite (negative) of -5.68 is the absolute value:

$$|-5.68| = -(-5.68) = 5.68$$

c. Since $0 \geq 0$, 0 is its own absolute value: $|0| = 0$.

d.
$$\begin{aligned} 2|-8| &= 2 \cdot |-8| \\ &= 2 \cdot 8 \\ &= 16 \end{aligned}$$

Self Check 1

Find:

a. $|-3|$ **b.** $|100.99|$

c. $|-2\pi|$ **d.** $\frac{1}{3}|-6|$

Now Try **Problems 26 and 36**

Self Check 1 Answers

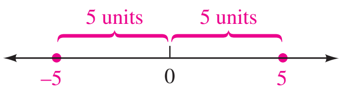
a. 3 **b.** 100.99 **c.** 2π **d.** 2

Teaching Example 1 Find:
a. $|-\pi|$ **b.** $|27.54|$ **c.** $-2|-7|$
Answers:
a. π **b.** 27.54 **c.** -14

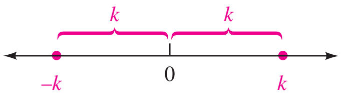
Caution! The placement of a $-$ sign in an expression containing an absolute value symbol is important. For example, $|-19| = 19$, but $-|19| = -19$.

2 Solve equations of the form $|x| = k$.

The absolute value of a real number represents the distance on a number line from a point to the origin. To solve the **absolute value equation** $|x| = 5$, we must find the coordinates of all points on a number line that are exactly 5 units from zero. See figure (a). The only two points that satisfy this condition have coordinates 5 and -5. That is, $x = 5$ or $x = -5$.



(a)



(b)

In general, the solution set of the absolute value equation $|x| = k$, where $k \geq 0$, includes the coordinates of the points on the number line that are k units from the origin. See figure (b).

Absolute Value Equations

If $k \geq 0$, then

$$|x| = k \quad \text{is equivalent to} \quad x = k \text{ or } x = -k$$

Self Check 2

Solve:

a. $|y| = 24$ **b.** $|x| = \dfrac{1}{2}$

c. $|a| = -1.1$

Now Try Problem 38

Self Check 2 Answers

a. $24, -24$ **b.** $\frac{1}{2}, -\frac{1}{2}$

c. no solution

Teaching Example 2 Solve:

a. $|x| = 7$ **b.** $|y| = -2$

c. $|r| = \frac{2}{5}$

Answers:

a. $7, -7$ **b.** no solution

c. $-\frac{2}{5}, \frac{2}{5}$

EXAMPLE 2 Solve: **a.** $|x| = 8$ **b.** $|s| = 0.003$ **c.** $|c| = -15$

Strategy To solve each of these absolute value equations, we will write and solve an equivalent compound equation. For part (c), we will solve the equation by inspection.

WHY We can use this approach because an equation of the form $|x| = k$, where k is not negative, is equivalent to $x = k$ or $x = -k$.

Solution

a. If $|x| = 8$, then $x = 8$ or $x = -8$.

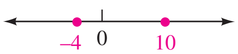
b. If $|s| = 0.003$, then $s = 0.003$ or $s = -0.003$.

c. The absolute value of a number is either positive or zero but never negative. Therefore, there is no value for c for which $|c| = -15$. This equation has no solution.

The equation $|x - 3| = 7$ indicates that a point on a number line with a coordinate of $x - 3$ is 7 units from the origin. Thus, $x - 3$ can be either 7 or -7.

$$\begin{array}{rcl} x - 3 = 7 & \text{or} & x - 3 = -7 \\ x = 10 & | & x = -4 \end{array}$$

The solutions of the absolute value equation are 10 and -4. We can graph them on a number line, as shown in the figure. If either of these numbers is substituted for x in $|x - 3| = 7$, the equation is satisfied.



Check:

$$\begin{array}{rcl} |x - 3| = 7 & & |x - 3| = 7 \\ |10 - 3| \stackrel{?}{=} 7 & & |-4 - 3| \stackrel{?}{=} 7 \\ |7| \stackrel{?}{=} 7 & & |-7| \stackrel{?}{=} 7 \\ 7 = 7 & & 7 = 7 \end{array}$$

EXAMPLE 3 Solve: **a.** $|3x - 2| = 5$ **b.** $|5 - x| = -10$

Strategy To solve the first absolute value equation, we will write and then solve an equivalent compound equation. We will solve the second equation by inspection.

WHY Both equations are of the form $|x| = k$. However, the standard method for solving absolute value equations cannot be applied to $|5 - x| = -10$ because k is negative.

Solution

a. We can solve $|3x - 2| = 5$ by writing and then solving an equivalent compound equation:

$$3x - 2 = 5 \quad \text{or} \quad 3x - 2 = -5$$

Now we solve each equation for x.

$$3x - 2 = 5 \quad \text{or} \quad 3x - 2 = -5 \qquad \text{This is the compound equation to solve.}$$
$$3x = 7 \qquad\qquad 3x = -3$$
$$x = \frac{7}{3} \qquad\qquad x = -1$$

Verify that both solutions, $\frac{7}{3}$ and -1, check.

b. For any real number x, $|5 - x|$ will be nonnegative. For this reason, $|5 - x| = -10$ has no solution.

Self Check 3

Solve:

a. $|2x - 3| = 7$ $5, -2$

b. $\left|\frac{x}{4} - 1\right| = -3$ no solution

Now Try **Problem 41**

Teaching Example 3 Solve:
a. $|6 - 2x| = 12$ **b.** $\left|\frac{x+1}{2}\right| = -4$
Answers:
a. $-3, 9$ **b.** $-9, 7$

Success Tip Since the absolute value of a quantity cannot be negative, equations such as $\left|7x + \frac{1}{2}\right| = -4$ have no solution. Their solution sets are empty.

When solving absolute value equations, we want the absolute value isolated on one side. To isolate an absolute value, we can use the equation-solving procedures studied earlier.

EXAMPLE 4 Solve: $\left|\frac{2}{3}x + 3\right| + 4 = 10$

Strategy We will isolate $\left|\frac{2}{3}x + 3\right|$ on the left side of the equation and then write and solve an equivalent compound equation.

WHY After isolating the absolute value expression on the left, the resulting equation will have the desired form $|x| = k$.

Solution

We can isolate $\left|\frac{2}{3}x + 3\right|$ on the left-hand side of the equation by subtracting 4 from both sides.

$$\left|\frac{2}{3}x + 3\right| + 4 = 10 \qquad \text{This is the equation to solve.}$$

$$\left|\frac{2}{3}x + 3\right| = 6 \qquad \text{Subtract 4 from both sides.}$$

With the absolute value now isolated, we can solve $\left|\frac{2}{3}x + 3\right| = 6$ by writing and then solving an equivalent compound equation.

$$\frac{2}{3}x + 3 = 6 \quad \text{or} \quad \frac{2}{3}x + 3 = -6$$

Self Check 4

Solve:
$|0.4x - 2| - 0.6 = 0.4$

Now Try **Problem 50**

Self Check 4 Answer
7.5, 2.5

Teaching Example 4 Solve:
$\left|\frac{3x}{2} + 4\right| - 3 = -2$
Answer:
$-\frac{10}{3}, -2$

Now we solve each equation for x:

$$\frac{2}{3}x + 3 = 6 \quad \text{or} \quad \frac{2}{3}x + 3 = -6$$
$$\frac{2}{3}x = 3 \qquad\qquad \frac{2}{3}x = -9$$
$$2x = 9 \qquad\qquad 2x = -27$$
$$x = \frac{9}{2} \qquad\qquad x = -\frac{27}{2}$$

Verify that both solutions, $\frac{9}{2}$ and $-\frac{27}{2}$, check by substituting them into the original equation.

Self Check 5

Solve: $-5\left|\frac{2x}{3} + 4\right| + 1 = 1$

Now Try **Problem 54**

Self Check 5 Answer
-6

Teaching Example 5 Solve: $-2\left|\frac{1}{3}x + 5\right| + 6 = 6$
Answer:
-15

EXAMPLE 5 Solve: $3\left|\frac{1}{2}x - 5\right| - 4 = -4$

Strategy We will isolate $\left|\frac{1}{2}x - 5\right|$ on the left side of the equation and then write and solve an equivalent compound equation.

WHY After isolating the absolute value expression on the left, the resulting equation will have the desired form $|x| = k$.

Solution

$$3\left|\frac{1}{2}x - 5\right| - 4 = -4 \quad \text{This is the equation to solve.}$$
$$3\left|\frac{1}{2}x - 5\right| = 0 \quad \text{Add 4 to both sides.}$$
$$\left|\frac{1}{2}x - 5\right| = 0 \quad \text{Divide both sides by 3.}$$

Since 0 is the only number whose absolute value is 0, the expression $\frac{1}{2}x - 5$ must be 0, and we have

$$\frac{1}{2}x - 5 = 0$$
$$\frac{1}{2}x = 5 \quad \text{Add 5 to both sides.}$$
$$x = 10 \quad \text{Multiply both sides by 2.}$$

The solution is 10. Verify that it satisfies the original equation.

3 Solve equations with two absolute values.

The equation $|a| = |b|$ is true when $a = b$ or when $a = -b$. For example,

$$|3| = |3| \quad \text{or} \quad |3| = |-3|$$

These are the same number. These numbers are opposites.

In general, the following statement is true.

Solving Equations with Two Absolute Values

If X and Y represent algebraic expressions, the equation $|X| = |Y|$ is equivalent to

$$X = Y \quad \text{or} \quad X = -Y$$

EXAMPLE 6 Solve: $|5x + 3| = |3x + 25|$

Strategy We will write and solve an equivalent compound equation.

WHY We can use this approach because the equation is of the form $|X| = |Y|$.

Solution

This equation is true when $5x + 3 = 3x + 25$, or when $5x + 3 = -(3x + 25)$. We solve each equation for x.

$$\begin{aligned} 5x + 3 &= 3x + 25 \\ 2x &= 22 \\ x &= 11 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 5x + 3 &= -(3x + 25) \\ 5x + 3 &= -3x - 25 \\ 8x &= -28 \\ x &= -\frac{28}{8} \\ x &= -\frac{7}{2} \end{aligned}$$

Verify that both solutions, 11 and $-\frac{7}{2}$, check by substituting them into the original equation.

Self Check 6

Solve: $|2x - 3| = |4x + 9|$

Now Try Problem 58

Self Check 6 Answer

$-1, -6$

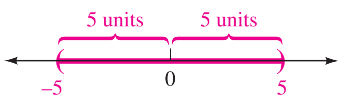
Teaching Example 6 Solve: $|4x + 2| = |x - 4|$

Answer:

$-2, \frac{2}{5}$

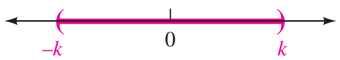
4 Solve inequalities of the form $|x| < k$.

To solve the **absolute value inequality** $|x| < 5$, we must find the coordinates of all points on a number line that are less than 5 units from the origin. See the figure. Thus, x is between -5 and 5, and



$$|x| < 5 \quad \text{is equivalent to} \quad -5 < x < 5$$

In general, the solution set of the absolute value inequality $|x| < k$ where $k > 0$ includes the coordinates of the points on the number line that are less than k units from the origin. See the figure.



Solving $|x| < k$ and $|x| \le k$

$\lvert x\rvert < k$	is equivalent to	$-k < x < k$	where $k > 0$
$\lvert x\rvert \le k$	is equivalent to	$-k \le x \le k$	where $k \ge 0$

EXAMPLE 7 Solve $|2x - 3| < 9$ and graph the solution set.

Strategy To solve this absolute value inequality, we will write and solve an equivalent double inequality.

WHY We can use this approach because the inequality is of the form $|x| < k$, and k is positive.

Solution

The absolute value inequality $|2x - 3| < 9$ is equivalent to the double inequality.

$$-9 < 2x - 3 < 9$$

Now we solve for x.

$$\begin{aligned} -9 &< 2x - 3 < 9 \\ -6 &< 2x < 12 && \text{Add 3 to all three parts.} \\ -3 &< x < 6 && \text{Divide all parts by 2.} \end{aligned}$$

Any number between -3 and 6 is in the solution set. This is the interval $(-3, 6)$; its graph is shown in the figure to the right.

Self Check 7

Solve $|3x + 2| < 4$ and graph the solution set.

Now Try Problem 68

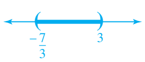
Self Check 7 Answer

$\left(-2, \frac{2}{3}\right)$

Teaching Example 7 Solve $|3x - 1| < 8$ and graph the solution set.

Answer:

$\left(-\frac{7}{3}, 3\right)$



Self Check 8

TOLERANCES Refer to Example 8. Find the tolerance range if the tolerance is ± 0.0015.

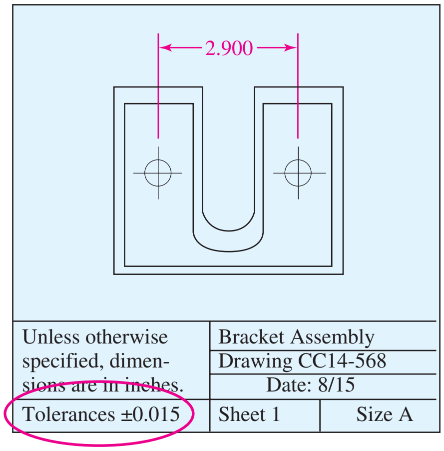
Now Try **Problem 103**

Self Check 8 Answer
[2.8985, 2.9015]

Teaching Example 8 TOLERANCES
Refer to Example 8. Find the tolerance range if the tolerance is ± 0.00015.
Answer:
[2.89985, 2.90015]

EXAMPLE 8 *Tolerances*

When manufactured parts are inspected by a quality control engineer, they are classified as acceptable if each dimension falls within a given *tolerance range* of the dimensions listed on the blueprint. For the bracket shown in the diagram below, the distance between the two drilled holes is given as 2.900 inches. Because the tolerance is ± 0.015 inch, this distance can be as much as 0.015 inch longer or 0.015 inch shorter, and the part will be considered acceptable. The acceptable distance d between holes can be represented by the absolute value inequality $|d - 2.900| \le 0.015$. Solve the inequality and explain the result.



Strategy To solve $|d - 2.900| \le 0.015$, we will write and solve an equivalent double inequality.

WHY We can use this approach because the inequality is of the form $|x| < k$, and k is positive.

Solution

$$|d - 2.900| \le 0.015 \quad \text{is equivalent to} \quad -0.015 \le d - 2.900 \le 0.015$$

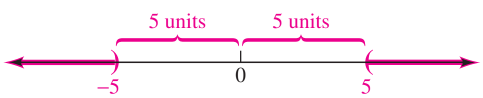
Now we solve for d.

$$-0.015 \le d - 2.900 \le 0.015 \qquad \text{This is the double inequality to solve.}$$
$$2.885 \le d \le 2.915 \qquad \text{To isolate } d \text{, add 2.900 to all three parts.}$$

The solution set is the interval [2.885, 2.915]. This means that the distance between the two holes should be between 2.885 and 2.915 inches, inclusive. If the distance is less than 2.885 inches or more than 2.915 inches, the part should be rejected.

5 Solve inequalities of the form $|x| > k$.

To solve the absolute value inequality $|x| > 5$, we must find the coordinates of all points on a number line that are more than 5 units from the origin. These numbers are left of -5 or right of 5.



The solution set can be written as the compound inequality $x < -5$ or $x > 5$. In interval notation the solution set can be written as $(-\infty, -5) \cup (5, \infty)$.

In general, the solution set of $|x| > k$ includes the coordinates of the points on the number line that are more than k units from the origin. See the figure. Thus,

$|x| > k$ is equivalent to $x < -k$ or $x > k$

The word *or* indicates an either/or situation. It is only necessary that x satisfy one of the two conditions to be in the solution set.

Solving $|x| > k$ and $|x| \geq k$

If $k \geq 0$, then

$|x| > k$ is equivalent to $x < -k$ or $x > k$

$|x| \geq k$ is equivalent to $x \leq -k$ or $x \geq k$

EXAMPLE 9

Solve $\left|\frac{3-x}{5}\right| \geq 6$ and graph the solution set.

Strategy To solve this absolute value inequality, we will write and solve an equivalent compound inequality.

WHY We can use this approach because the inequality is of the form $|x| \geq k$, and k is positive.

Solution

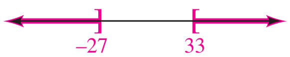
The absolute value inequality $\left|\frac{3-x}{5}\right| \geq 6$ is equivalent to

$$\frac{3-x}{5} \leq -6 \quad \text{or} \quad \frac{3-x}{5} \geq 6$$

Now we solve each inequality for x.

$$\begin{array}{rcrl}
\frac{3-x}{5} \leq -6 & \text{or} & \frac{3-x}{5} \geq 6 & \\
3 - x \leq -30 & & 3 - x \geq 30 & \text{Multiply both sides by 5.} \\
-x \leq -33 & & -x \geq 27 & \text{Subtract 3 from both sides.} \\
x \geq 33 & & x \leq -27 & \text{Divide both sides by } -1 \text{ and reverse the direction of the inequality symbol.}
\end{array}$$

The solution set is the interval $(-\infty, -27] \cup [33, \infty)$. Its graph appears in the figure to the right.



Self Check 9

Solve $\left|\frac{2-x}{4}\right| \geq 1$ and graph the solution set.

Now Try **Problem 77**

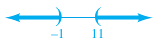
Self Check 9 Answer

$(-\infty, -2] \cup [6, \infty)$

Teaching Example 9 Solve $\left|\frac{5-x}{2}\right| > 3$ and graph the solution set.

Answer:

$(-\infty, -1) \cup (11, \infty)$



EXAMPLE 10

Solve $\left|\frac{2}{3}x - 2\right| - 3 > 6$ and graph the solution set.

Strategy We will first isolate $\left|\frac{2}{3}x - 2\right|$ on the left side of the inequality and write and solve an equivalent compound inequality.

WHY After isolating the absolute value expression, the resulting inequality will have the form $|x| > k$, and k is positive.

Solution

We add 3 to both sides to isolate the absolute value on the left-hand side.

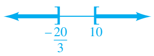
Self Check 10

Solve $\left|\frac{3}{4}x + 2\right| - 1 > 3$ and graph the solution set.

Now Try **Problem 78**

Self Check 10 Answer

$(-\infty, -8) \cup \left(\frac{8}{3}, \infty\right)$

Teaching Example 10 Solve $\left|\frac{3}{5}x - 1\right| + 2 \geq 7$ and graph the solution set.
Answer:
$\left(-\infty, -\frac{20}{3}\right] \cup [10, \infty)$ 

$$\left|\frac{2}{3}x - 2\right| - 3 > 6 \quad \text{This is the inequality to solve.}$$

$$\left|\frac{2}{3}x - 2\right| > 9 \quad \text{Add 3 to both sides to isolate the absolute value.}$$

We then proceed as follows:

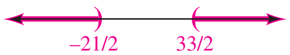
$$\frac{2}{3}x - 2 < -9 \quad \text{or} \quad \frac{2}{3}x - 2 > 9$$

$$\frac{2}{3}x < -7 \qquad \frac{2}{3}x > 11 \quad \text{Add 2 to both sides.}$$

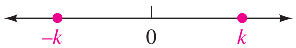
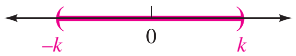
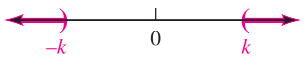
$$2x < -21 \qquad 2x > 33 \quad \text{Multiply both sides by 3.}$$

$$x < -\frac{21}{2} \qquad x > \frac{33}{2} \quad \text{Divide both sides by 2.}$$

The solution set is $\left(-\infty, -\frac{21}{2}\right) \cup \left(\frac{33}{2}, \infty\right)$. Its graph appears to the right.

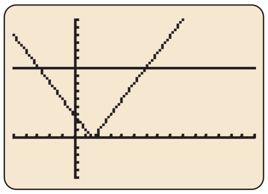


The following summary shows how we can interpret absolute value in three ways. Assume k is positive.

	Geometric description	Graphic description	Algebraic description
1.	$\lvert x \rvert = k$ means that x is k units from 0 on the number line.		$\lvert x \rvert = k$ is equivalent to $x = k$ or $x = -k$.
2.	$\lvert x \rvert < k$ means that x is less than k units from 0 on the number line.		$\lvert x \rvert < k$ is equivalent to $-k < x < k$.
3.	$\lvert x \rvert > k$ means that x is more than k units from 0 on the number line.		$\lvert x \rvert > k$ is equivalent to $x > k$ or $x < -k$.

Using Your CALCULATOR Solving Absolute Value Inequalities

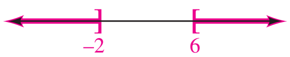
We can also solve absolute value inequalities using a graphing calculator. For example, to solve $|2x - 3| < 9$, we graph the equations $y = |2x - 3|$ and $y = 9$ on the same coordinate system. If we use settings of $[-5, 15]$ for x and $[-5, 15]$ for y, we get the graph shown to the right.



The inequality $|2x - 3| < 9$ will be true for all x-coordinates of points that lie on the graph of $y = |2x - 3|$ and below the graph of $y = 9$. Using the TRACE or INTERSECT feature, we can see that these values of x are in the interval $(-3, 6)$.

ANSWERS TO SELF CHECKS

1. a. 3 **b.** 100.99 **c.** 2π **d.** 2 **2. a.** 24, -24 **b.** $\frac{1}{2}, -\frac{1}{2}$ **c.** no solution **3. a.** 5, -2 **b.** no solution **4.** 7.5, 2.5 **5.** -6 **6.** $-1, -6$ **7.** $\left(-2, \frac{2}{3}\right)$ 

8. [2.8985, 2.9015] **9.** $(-\infty, -2] \cup [6, \infty)$ 

10. $(-\infty, -8) \cup \left(\frac{8}{3}, \infty\right)$ 

SECTION 4.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. $|2x - 1| = 10$ is an absolute value equation.

2. $|2x - 1| > 10$ is an absolute value inequality.

CONCEPTS

Fill in the blanks.

3. To isolate the absolute value in $|3 - x| - 4 = 5$, we add 4 to both sides.

4. $|x| = 2$ is equivalent to $x = 2$ or $x = -2$.

5. $|x| \geq 0$ for all real numbers x.

6. If $x < 0$, $|x| = -x$.

7. To solve $|x| > 5$, we must find the coordinates of all points on a number line that are more than 5 units from 0.

8. To solve $|x| = 5$, we must find the coordinates of all points on a number line that are 5 units from 0.

9. To solve $|x| < 5$, we must find the coordinates of all points on a number line that are less than 5 units from 0.

10. The equation $|a| = |b|$ is true when $a = b$ or when $a = -b$.

Determine whether −3 is a solution of the given equation or inequality.

11. $|x - 1| = 4$ yes

12. $|x - 1| > 4$ no

13. $|x - 1| \leq 4$ yes

14. $|5 - x| = |x + 12|$ no

For each absolute value equation or inequality, write an equivalent compound equation or inequality.

15. $|x| = 8$
$x = 8$ or $x = -8$

16. $|x| \geq 8$
$x \leq -8$ or $x \geq 8$

17. $|x| \leq 8$
$-8 \leq x \leq 8$

18. $|5x - 1| = |x + 3|$
$5x - 1 = x + 3$ or $5x - 1 = -(x + 3)$

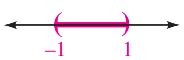
NOTATION

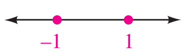
19. Match each equation or inequality with its graph.

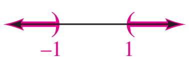
a. $|x| = 1$ ii

b. $|x| > 1$ iii

c. $|x| < 1$ i

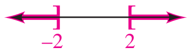
i. 

ii. 

iii. 

20. Match each graph with its corresponding equation or inequality.

a.  iii

b.  i

c.  ii

i. $|x| \geq 2$

ii. $|x| \leq 2$

iii. $|x| = 2$

Write each compound inequality as an inequality containing absolute value symbols.

21. $-4 < x < 4$ $|x| < 4$

22. $x < -4$ or $x > 4$ $|x| > 4$

23. $x + 3 < -6$ or $x + 3 > 6$ $|x + 3| > 6$

24. $-5 \leq x - 3 \leq 5$ $|x - 3| \leq 5$

GUIDED PRACTICE

Find the value of each expression. See Example 1.

25. $|8|$ 8

26. $|-18|$ 18

27. $-|0.02|$ -0.02

28. $-|-3.14|$ -3.14

29. $-\left|-\frac{31}{16}\right|$ $-\frac{31}{16}$

30. $-\left|\frac{25}{4}\right|$ $-\frac{25}{4}$

31. $|\pi|$ π

32. $\left|-\frac{\pi}{2}\right|$ $\frac{\pi}{2}$

33. $5|-5|$ 25

34. $9|-1|$ 9

35. $-\frac{1}{2}|-4|$ -2

36. $-16\left|-\frac{1}{4}\right|$ -4

Solve each equation. See Example 2.

37. $|x| = 23$ 23, -23

38. $|x| = 90$ 90, -90

39. $|t| = -5$ $\varnothing$

40. $|m| = -7$ $\varnothing$

Solve each equation. See Example 3.

41. $|3x + 2| = 16$ $\frac{14}{3}$, -6

42. $|5x - 3| = 22$ 5, $-\frac{19}{5}$

43. $\left|\frac{7}{2}x + 3\right| = -5$ $\varnothing$

44. $\left|\frac{2x}{3} + 10\right| = 0$ -15

45. $|3 - 4x| = 52$ $-\frac{49}{4}$, $\frac{55}{4}$

46. $|8 - 5x| = 18$ $\frac{26}{5}$, -2

47. $\left|\frac{1}{2}x - 2\right| = 0$ 4

48. $|4x + 3| = -2$ $\varnothing$

Solve each equation. See Examples 4–5.

49. $|x + 3| + 7 = 10$
0, -6

50. $|x - 2| + 3 = 5$
0, 4

51. $\left|\frac{3x + 48}{3}\right| - 4 = 8$
-4, -28

52. $\left|\frac{4x - 64}{4}\right| + 2 = 34$
48, -16

53. $|3x + 1| + 2 = 6$ $-\frac{5}{3}, 1$

54. $2\left|\frac{1}{2}x - 1\right| + 1 = 5$ $6, -2$

55. $2|3x + 24| = 0$ -8

56. $5|x - 21| = -8$ $\varnothing$

Solve each equation. See Example 6.

57. $|2x + 1| = |3x + 3|$ $-2, -\frac{4}{5}$

58. $|5x - 7| = |4x + 1|$ $8, \frac{2}{3}$

59. $|2 - x| = |3x + 2|$ $0, -2$

60. $|4x + 3| = |9 - 2x|$ $1, -6$

61. $\left|\frac{x}{2} + 2\right| = \left|\frac{x}{2} - 2\right|$ 0

62. $|5x - 3| = |5 - 3x|$ $1, -1$

63. $\left|x + \frac{1}{3}\right| = |x - 3|$ $\frac{4}{3}$

64. $\left|x - \frac{1}{4}\right| = |x + 4|$ $-\frac{15}{8}$

Solve each inequality. Write the solution set in interval notation and if it is nonempty, graph it. See Examples 7–8.

65. $|x| < 4$ $(-4, 4)$

66. $|x| < 9$ $(-9, 9)$

67. $|x + 9| \le 12$ $[-21, 3]$

68. $|x - 8| \le 12$ $[-4, 20]$

69. $|3x - 2| < 10$ $\left(-\frac{8}{3}, 4\right)$

70. $|4 - 3x| \le 13$ $\left[-3, \frac{17}{3}\right]$

71. $|3x + 2| \le -3$ $\varnothing$

72. $|5x - 12| < -5$ $\varnothing$

Solve each inequality. Write the solution set in interval notation and if it is nonempty, graph it. See Examples 9–10.

73. $|x| > 3$ $(-\infty, -3) \cup (3, \infty)$

74. $|x| + 2 > 9$ $(-\infty, -7) \cup (7, \infty)$

75. $|x - 12| > 24$ $(-\infty, -12) \cup (36, \infty)$

76. $|x + 5| \ge 7$ $(-\infty, -12] \cup [2, \infty)$

77. $\left|\frac{3x + 2}{2}\right| > 7$ $\left(-\infty, -\frac{16}{3}\right) \cup (4, \infty)$

78. $\left|\frac{2x - 5}{5}\right| - 2 > 3$ $(-\infty, -10) \cup (15, \infty)$

79. $|4x + 3| > -5$ $(-\infty, \infty)$

80. $|7x + 2| > -8$ $(-\infty, \infty)$

TRY IT YOURSELF

Solve each equation or inequality. For the inequalities, write the solution set in interval notation and graph it.

81. $|x - 3.1| = 6$ $9.1, -2.9$

82. $|x + 4.3| = 8.9$ $4.6, -13.2$

83. $8 = -1 + |0.3x - 3|$ $40, -20$

84. $-1 = 1 - |0.1x + 8|$ $-60, -100$

85. $|2 - 3x| \ge 8$ $(-\infty, -2] \cup \left[\frac{10}{3}, \infty\right)$

86. $|-1 - 2x| > 5$ $(-\infty, -3) \cup (2, \infty)$

87. $-|2x - 3| < -7$ $(-\infty, -2) \cup (5, \infty)$

88. $-|3x + 1| < -8$ $(-\infty, -3) \cup \left(\frac{7}{3}, \infty\right)$

89. $-7 = 2 - |0.3x - 3|$ $40, -20$

90. $-2 = 3 - |0.2x + 4|$ $-5, -45$

91. $\left|\frac{x - 2}{3}\right| \le 4$ $[-10, 14]$

92. $\left|\frac{x - 2}{3}\right| > 4$ $(-\infty, -10) \cup (14, \infty)$

93. $|7x + 12| = |x - 6|$ $-3, -\frac{3}{4}$

94. $|8 - x| = |x + 2|$ 3

95. $\left|\frac{1}{3}x + 7\right| + 5 > 6$ $(-\infty, -24) \cup (-18, \infty)$

96. $-2|3x - 4| < 16$ $(-\infty, \infty)$

97. $-14 = |x - 3|$ $\varnothing$

98. $-70 = 5 + |x + 4|$ $\varnothing$

99. $|0.5x + 1| + 2 \le 0$ $\varnothing$

100. $15 \ge 7 - |1.4x + 9|$ $(-\infty, \infty)$

APPLICATIONS

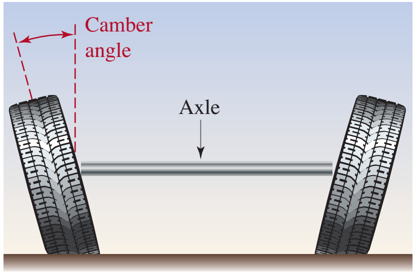
101. TEMPERATURE RANGES The temperatures on a summer day satisfied the inequality $|t - 78°| \le 8°$, where t is a temperature in degrees Fahrenheit. Solve this inequality and express the range of temperatures as a double inequality. $70° \le t \le 86°$

102. OPERATING TEMPERATURES A car CD player has an operating temperature of $|t - 40°| < 80°$, where t is a temperature in degrees Fahrenheit. Solve the inequality and express this range of temperatures as an interval. $(-40°, 120°)$

103. AUTO MECHANICS On most cars, the bottoms of the front wheels are closer together than the tops, creating a *camber angle*. This lessens road shock to the steering system. The specifications for a certain car state that the camber angle c of its wheels should be $0.6° \pm 0.5°$.

a. Express the range with an inequality containing absolute value symbols. $|c - 0.6°| \le 0.5°$

b. Solve the inequality and express this range of camber angles as an interval. $[0.1°, 1.1°]$



104. STEEL PRODUCTION A sheet of steel is to be 0.250 inch thick with a tolerance of 0.025 inch.

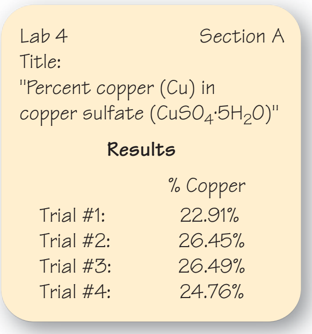
a. Express this specification with an inequality containing absolute value symbols, using x to represent the thickness of a sheet of steel. $|x - 0.250| \le 0.025$

b. Solve the inequality and express the range of thickness as an interval. $[0.225, 0.275]$

105. ERROR ANALYSIS In a lab, students measured the percent of copper p in a sample of copper sulfate. The students know that copper sulfate is actually 25.46% copper by mass. They are to compare their results to the actual value and find the amount of *experimental error.*

a. Which measurements shown satisfy the absolute value inequality $|p - 25.46| \le 1.00$? 26.45%, 24.76%

b. What can be said about the amount of error for each of the trials listed in part a? It is less than or equal to 1%.


Lab 4 Section A
Title:
"Percent copper (Cu) in copper sulfate ($CuSO_4 \cdot 5H_2O$)"

Results

	% Copper
Trial #1:	22.91%
Trial #2:	26.45%
Trial #3:	26.49%
Trial #4:	24.76%

106. ERROR ANALYSIS See Exercise 105.

a. Which measurements satisfy the absolute value inequality $|p - 25.46| > 1.00$? 22.91%, 26.49%

b. What can be said about the amount of error for each of the trials listed in part a? It is more than 1%.

WRITING

107. Explain how to find the absolute value of a given number.

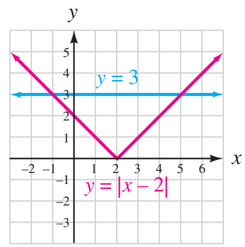
108. Explain why the equation $|x - 4| = -5$ has no solutions.

109. Explain the use of parentheses and brackets when graphing inequalities.

110. Explain the differences between the solution set of $|x| < 8$ and the solution set of $|x| > 8$.

111. Explain how to use the graph to solve $|x - 2| < 3$.

112. Explain how to use the graph to solve $|x - 2| \ge 3$.



REVIEW

For problems 113 and 114 refer to the illustration.

113. RAILROAD CROSSINGS The warning sign in the illustration is to be painted on the street in front of a railroad crossing. If y is 30° more than twice x, find x and y. 50°, 130°

114. GEOMETRY Refer to the illustration. Find $2x + 2y$. 360°

Objectives

1. Graph linear inequalities.
2. Graph inequalities with a boundary through the origin.
3. Graph inequalities having horizontal and vertical boundary lines.
4. Solve applied problems involving linear inequalities in two variables.

SECTION 4.4

Linear Inequalities in Two Variables

In the first three sections of this chapter, we have worked with linear inequalities in one variable. Some examples are

$$x \geq -7, \qquad 5 < \frac{7}{2}a - 9, \qquad \text{and} \qquad 5(3 + z) > -3(z + 3)$$

These inequalities have infinitely many solutions. When their solutions are graphed on a real number line, we obtain an interval.

In this section, we will discuss **linear inequalities in two variables.** Some examples are

$$y > 3x + 2, \qquad 2x - 3y \leq 6, \qquad \text{and} \qquad y < 2x$$

Linear Inequalities in Two Variables

A **linear inequality** in x and y is any inequality that can be written in the form

$$Ax + By < C \quad \text{or} \quad Ax + By > C \quad \text{or} \quad Ax + By \leq C \quad \text{or} \quad Ax + By \geq C$$

where A, B, and C represent real numbers and A and B are not both 0.

The solutions of these inequalities are ordered pairs whose coordinates satisfy the inequality. We can graph their solutions on a rectangular coordinate system.

1 Graph linear inequalities.

The **graph of a linear inequality** in x and y is the graph of all ordered pairs (x, y) that **satisfy** the inequality.

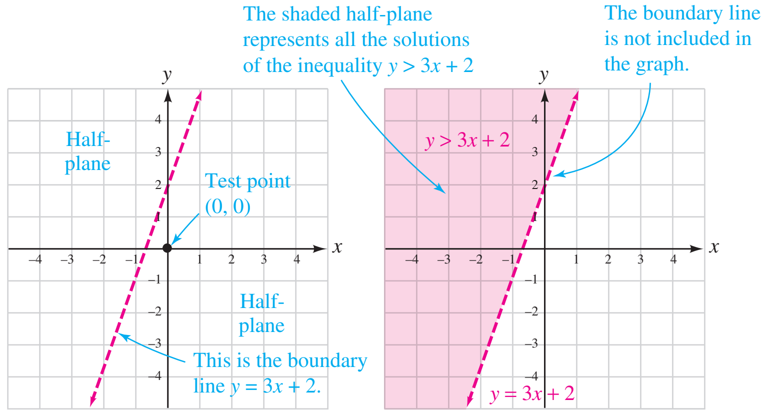
To graph the linear inequality $y > 3x + 2$, we begin by graphing the linear equation $y = 3x + 2$. Its graph, shown in figure (a) on the next page, is a **boundary line** that separates the rectangular coordinate plane into two regions called **half-planes.** It is drawn with a dashed line to show that it is not part of the graph of $y > 3x + 2$.

To find which half-plane is the graph of $y > 3x + 2$, we can substitute the coordinates of any point in either half-plane. We will choose the origin as the **test point** because its coordinates, (0, 0), make the computations easy. We substitute 0 for x and 0 for y into the inequality and simplify.

***Check the test point* (0, 0):**

$y > 3x + 2$	This is the original inequality.
$0 \overset{?}{>} 3(0) + 2$	Substitute 0 for y and 0 for x.
$0 > 2$	This statement is false.

Since the coordinates of the origin do not satisfy $y > 3x + 2$, the origin and all the other points on that side of the boundary are not part of the graph of the inequality. Thus, the half-plane on the other side of the dashed line is the graph. We shade that region, as shown in figure (b).



(a) (b)

EXAMPLE 1 Graph: $2x - 3y \leq 6$

Strategy We will graph the *equation* $2x - 3y = 6$ to establish a boundary line between two regions of the coordinate plane. Since the inequality symbol $\leq$ includes an equal symbol, the graph of the boundary is drawn solid to show that it is part of the solution. Then we will determine which region contains points whose coordinates satisfy $2x - 3y \leq 6$.

WHY To *graph a linear inequality* in two variables means to draw a "picture" of the ordered pairs (x, y) that make the inequality true.

Solution

This inequality is the combination of the inequality $2x - 3y < 6$ and the equation $2x - 3y = 6$.

We begin by graphing $2x - 3y = 6$ to find the boundary line that separates the two half-planes. We do so by noting that the line's x-intercept is $(3, 0)$ and its y-intercept is $(0, -2)$. This time, we draw the solid line shown in figure (a), because equality is permitted by the symbol $\leq$. To decide which half-plane to shade, we check to see whether the coordinates of the origin satisfy the inequality.

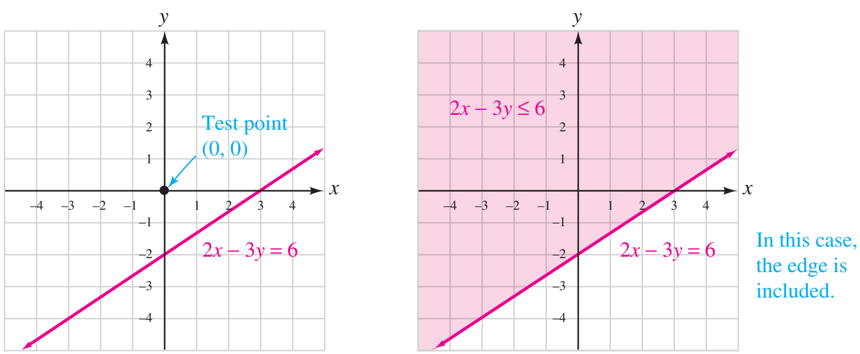
***Check the test point* (0, 0):**

$2x - 3y \leq 6$ This is the original inequality.

$2(0) - 3(0) \overset{?}{\leq} 6$ Substitute 0 for x and 0 for y.

$0 \leq 6$ This statement is true.

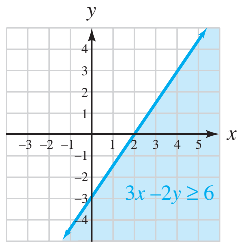
The coordinates of the origin satisfy the inequality. In fact, the coordinates of every point on the same side of the boundary line as the origin satisfy the inequality. We then shade that half-plane to complete the graph of $2x - 3y \leq 6$, shown in figure (b).



(a) (b)

Self Check 1

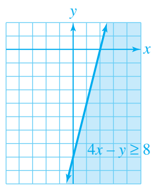
Graph: $3x - 2y \geq 6$



Now Try **Problem 24**

Teaching Example 1 Graph: $4x - y \geq 8$

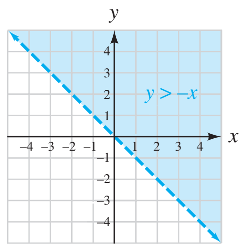
Answer:



2 Graph inequalities with a boundary through the origin.

Self Check 2

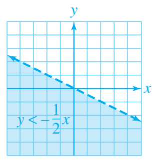
Graph: $y > -x$



Now Try **Problem 31**

Teaching Example 2 Graph: $y < -\frac{1}{2}x$

Answer:



EXAMPLE 2 Graph: $y < 2x$

Strategy We will graph the *equation* $y = 2x$ to establish a boundary line. Since the inequality symbol $<$ does not include an equal symbol, the graph of the boundary is drawn dashed to show that it is not part of the solution. Then we will determine which region contains points whose coordinates satisfy $y < 2x$.

WHY To *graph a linear inequality* in two variables means to draw a "picture" of the ordered pairs (x, y) that make the inequality true.

Solution

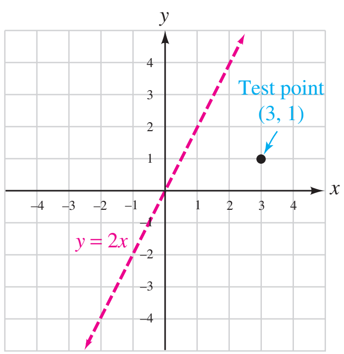
To graph $y = 2x$, we use the fact that the equation is in slope–intercept form and that $m = 2 = \frac{2}{1}$ and $b = 0$. Since the symbol $<$ does not include an equal symbol, the points on the graph of $y = 2x$ are not on the graph of $y < 2x$. We draw the boundary line as a dashed line to show this, as in figure (a) below.

To decide which half-plane is the graph of $y < 2x$, we check to see whether the coordinates of some fixed point satisfy the inequality. We cannot use the origin as a test point, because the boundary line passes through the origin. However, we can choose a different point—say, (3, 1).

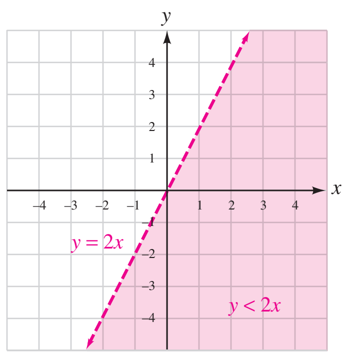
***Check the test point* (3, 1):**

$y < 2x$	This is the original inequality.
$1 \overset{?}{<} 2(3)$	Substitute 1 for y and 3 for x.
$1 < 6$	This is a true statement.

Since $1 < 6$ is a true inequality, the point (3, 1) satisfies the inequality and is in the graph of $y < 2x$. We then shade the half-plane containing (3, 1), as shown in figure (b).



(a)



In this case, the edge is not included.

(b)

The following is a summary of the procedure for graphing linear inequalities.

Graphing Linear Inequalities in Two Variables

1. Graph the boundary line of the region. If the inequality allows the possibility of equality (the symbol is either $\le$ or $\ge$), draw the boundary line as a solid line. If equality is not allowed ($<$ or $>$), draw the boundary line as a dashed line.
2. Pick a test point that is on one side of the boundary line. (Use the origin if possible.) Replace x and y in the original inequality with the coordinates of that point. If the inequality is satisfied, shade the side that contains that point. If the inequality is not satisfied, shade the other side.

3 Graph inequalities having horizontal and vertical boundary lines.

Recall from Chapter 2 that the graph of the equation $x = a$ is a vertical line with x-intercept at $(a, 0)$, and the graph of the equation $y = b$ is a horizontal line with y-intercept at $(0, b)$.

EXAMPLE 3 Graph: $x \geq -1$

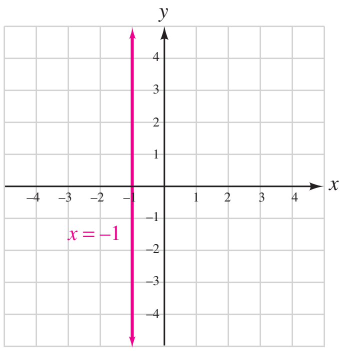
Strategy We will graph the *equation* $x = -1$ to establish the boundary line. Then we will determine which region contains points whose coordinates satisfy the inequality $x \geq -1$.

WHY To *graph a linear inequality* in two variables means to draw a "picture" of the ordered pairs (x, y) that make the inequality true.

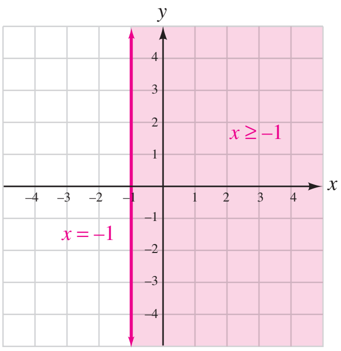
Solution

The graph of the boundary $x = -1$ is a vertical line passing through $(-1, 0)$. We draw the boundary as a solid line to show that it is part of the solution. See figure (a).

In this case, we need not pick a test point. The inequality $x \geq -1$ is satisfied by points with an x-coordinate greater than or equal to -1. Points satisfying this condition lie to the right of the boundary. We shade that half-plane, as shown in figure (b), to complete the graph of $x \geq -1$.



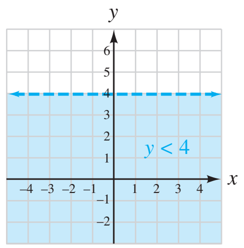
(a)



(b)

Self Check 3

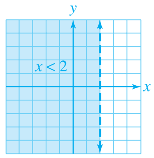
Graph: $y < 4$



Now Try **Problem 36**

Teaching Example 3 Graph: $x < 2$

Answer:



4 Solve applied problems involving linear inequalities in two variables.

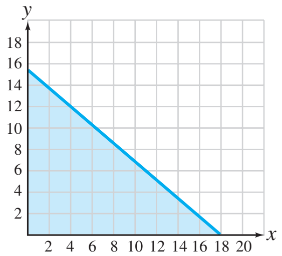
In the next example, we will solve a problem by writing a linear inequality in two variables to model a situation mathematically.

EXAMPLE 4 ***Social Security*** Retirees, ages 62–65, can earn as much as \$11,640 and still receive their full Social Security benefits. If their annual earnings exceed \$11,640, their benefits are reduced. A 64-year-old retired woman receiving Social Security works two part-time jobs: one at the library, paying \$485 per week, and another at a pet store, paying \$388 per week. Write an inequality representing the number of weeks the woman can work at each job during the year without losing any of her benefits.

Analyze We need to find the various combinations of weeks she can work at the library and at the pet store so that her annual income is less than or equal to \$11,640.

Self Check 4

BABYSITTING One babysitter charges \$10 per hour and another charges \$12 per hour. Marie can afford no more than \$180 per month for babysitting. Write an inequality representing the number of hours Marie can hire the first babysitter, (x) and the number of hours that she can hire the second babysitter (y).



Now Try **Problem 58**

Self Check 4 Answer

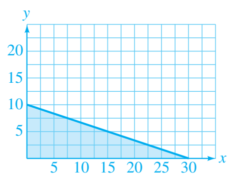
$10x + 12y \le 180$

Teaching Example 4 GARDENING

A nursery sells annuals for \$1.00 per plant and perennials for \$3.00 per plant. Beverly has budgeted no more than \$30.00 on garden plants. Write an inequality that describes the possible combinations of number of annuals (x) and perennials (y) that she can purchase. Graph the inequality.

Answer:

$x + 3y \le 30$



Form If we let x = the number of weeks she works at the library, she will earn \$485x annually. If we let y = the number of weeks she works at the pet store, she will earn \$388y annually. Combining the income from these jobs, the total is not to exceed \$11,640.

The weekly rate on the library job	·	the weeks worked on the library job	plus	the weekly rate on the pet store job	·	the weeks worked on the pet store job	should not exceed	\$11,640.
\$485	·	x	+	\$388	·	y	≤	\$11,640

Solve The graph of $485x + 388y \le 11{,}640$ is shown in the figure below. Since she cannot work a negative number of weeks, the graph has no meaning when x or y is negative, so only the first quadrant is used.

State Any point in the shaded region indicates a way that she can schedule her work weeks and earn \$11,640 or less annually. For example, if she works 8 weeks at the library and 16 weeks at the pet store, represented by the ordered pair (8, 16), she will earn

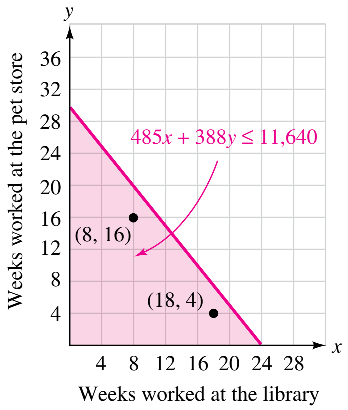
$$\begin{aligned} \$485(8) + \$388(16) &= \$3{,}880 + \$6{,}208 \\ &= \$10{,}088 \end{aligned}$$

Check If she works 18 weeks at the library and 4 weeks at the pet store, represented by (18, 4), she will earn

$$\begin{aligned} \$485(18) + \$388(4) &= \$8{,}730 + \$1{,}552 \\ &= \$10{,}282 \end{aligned}$$

which is less than \$11,640.

As an informal check, we will consider the earnings represented by a point that lies in the graph.



Using Your CALCULATOR Graphing Inequalities

Some graphing calculators (such as the TI-83 PLUS) have a graph style icon in the Y = editor. Some of the different graph styles are

\	line	A straight line or curved graph is shown.	$\backslash Y_1 =$
◥	above	Shading covers the area above a graph.	◥ $Y_1 =$
◣	below	Shading covers the area below a graph.	◣ $Y_1 =$

We can change the icon in front of Y_1 by placing the cursor on it and pressing the $\boxed{\text{ENTER}}$ key.

To graph $2x - 3y \le 6$ of Example 1, we first write it in an equivalent form, with y isolated on the left-hand side.

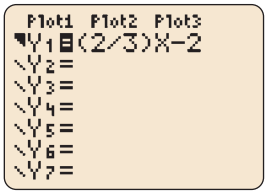
$$2x - 3y \le 6$$

$$-3y \le -2x + 6 \quad \text{Subtract } 2x \text{ from both sides.}$$

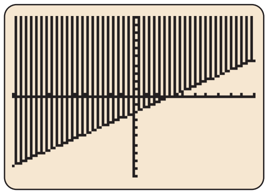
$$y \ge \frac{2}{3}x - 2 \quad \text{Divide both sides by } -3. \text{ Change the direction of the inequality symbol.}$$

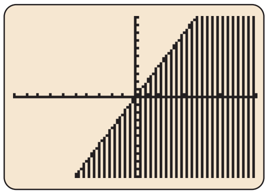
We then change the graph style icon to above (◥), because the inequality $y \geq \frac{2}{3}x - 2$ contains a $\geq$ symbol. Using window settings of $[-10, 10]$ for x and $[-10, 10]$ for y, we enter the boundary equation $y = \frac{2}{3}x - 2$. See figure (a). Finally, we press the $\boxed{\text{GRAPH}}$ key to get figure (b).

To graph $y < 2x$ from Example 2, we change the graph style icon to below (◣), because the inequality contains a $<$ symbol. Using window settings of $[-10, 10]$ for x and $[-10, 10]$ for y, we enter the boundary equation $y = 2x$ and press the $\boxed{\text{GRAPH}}$ key to get figure (c).



(a)


(b)

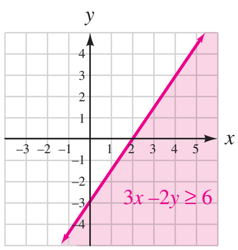

(c)

If your calculator does not have a graph style icon, you can graph linear inequalities with a SHADE feature. To do so, consult your owner's manual.

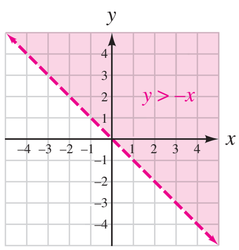
It is important to note that graphing calculators do not distinguish between solid and broken lines to show whether or not the edge of a region is included within the graph.

ANSWERS TO SELF CHECKS

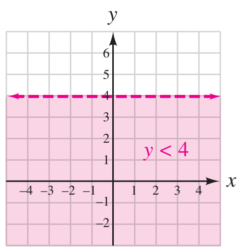
1.



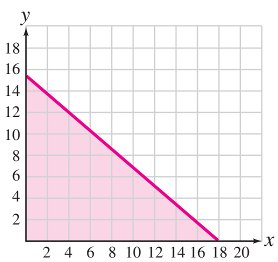
2.



3.



4.



SECTION 4.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. $4x - 2y \geq -8$ is an example of a linear inequality in two variables.

▶ **2.** Graphs of linear inequalities are half-planes.

3. The boundary line of a half-plane is called an edge.

4. The graph of a linear inequality in x and y contains the points (x, y) whose coordinates satisfy the inequality.

▶ Selected exercises available online at
www.webassign.net/brookscole

CONCEPTS

Determine whether each ordered pair is a solution of $3x - 2y \geq 5$.

5. $(3, 1)$ yes

6. $(0, 3)$ no

7. $(-1, -4)$ yes

8. $\left(1, \frac{1}{2}\right)$ no

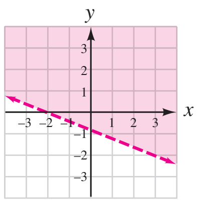
Refer to the graph of a linear inequality shown below. Determine whether each point satisfies the inequality.

9. $(-1, 4)$ yes

10. $(3, -2)$ no

11. $(0, 0)$ yes

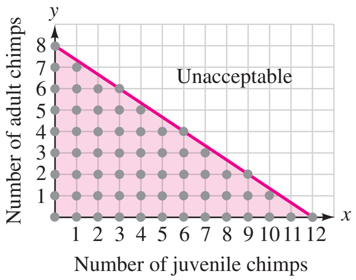
12. $(-3, -3)$ no



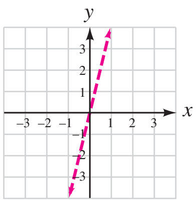
13. To graph the inequality $y > 3x - 1$, we begin by graphing the boundary line $y = 3x - 1$ Find the slope m of the line and find its y-intercept. $m = 3, (0, -1)$

14. To graph the inequality $2x + 3y \leq -6$, we begin by graphing the boundary line $2x + 3y = -6$. Find its x- and y-intercepts. $(-3, 0), (0, -2)$

15. ZOOS To determine the allowable number of juvenile chimpanzees x and adult chimpanzees y that can live in an enclosure, a zookeeper refers to the following graph. Can 7 juvenile and 4 adult chimps be kept in the enclosure? no

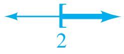


16. The dashed boundary for the graph of a linear inequality is shown. Why can't the origin be used as a test point to determine which side to shade?
The test point must be on one side of the boundary.

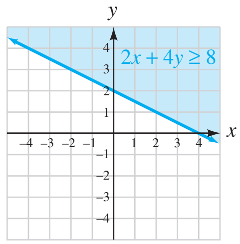


NOTATION

17. Solve $2x + 4 \geq 8$ (an inequality in one variable) and graph its solution set.



18. Graph $2x + 4y \geq 8$ (an inequality in two variables).



Determine whether the graph of each inequality includes the boundary line. In each case, would the boundary be a solid or a dashed line?

19. $y < 3x - 1$ no, dashed

20. $2x + 3y \geq -6$ yes, solid

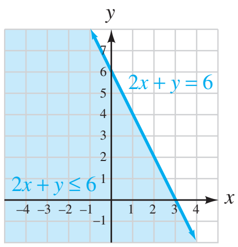
21. $y \leq -10$ yes, solid

22. $x > 1$ no, dashed

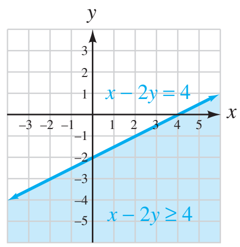
GUIDED PRACTICE

Graph each inequality. See Example 1.

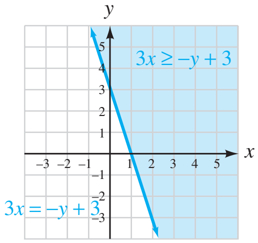
23. $2x + y \leq 6$



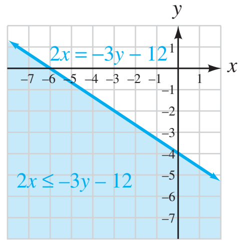
24. $x - 2y \geq 4$



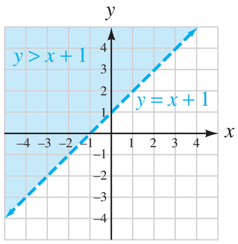
25. $3x \geq -y + 3$



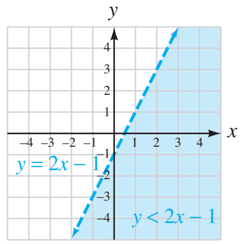
26. $2x \leq -3y - 12$



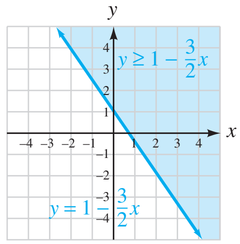
27. $y > x + 1$



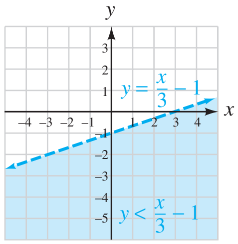
28. $y < 2x - 1$



29. $y \geq 1 - \frac{3}{2}x$

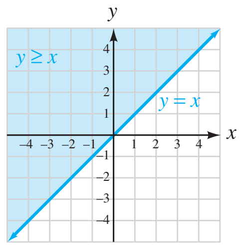


▶ **30.** $y < \frac{x}{3} - 1$

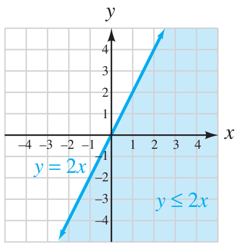


Graph each inequality. See Example 2.

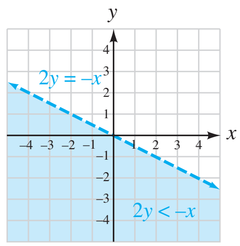
31. $y \geq x$



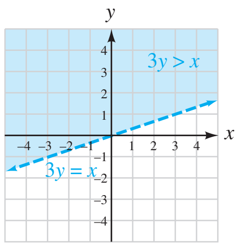
▶ **32.** $y \leq 2x$



33. $2y < -x$

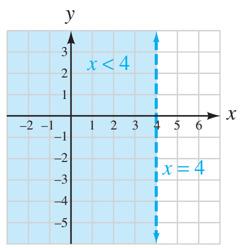


34. $3y > x$

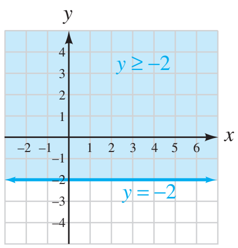


Graph each inequality. See Example 3.

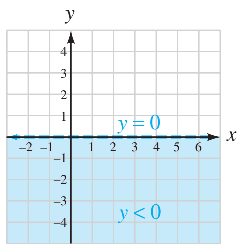
35. $x < 4$



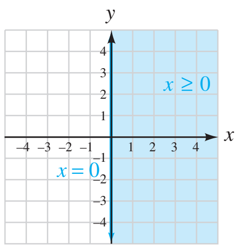
▶ **36.** $y \geq -2$



37. $y < 0$



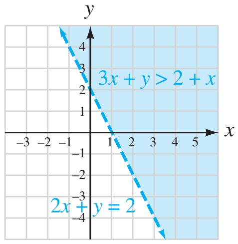
▶ **38.** $x \geq 0$



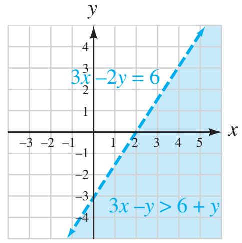
TRY IT YOURSELF

Simplify each expression and graph the resulting inequality.

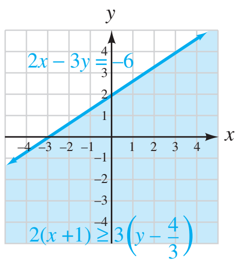
39. $3x + y > 2 + x$



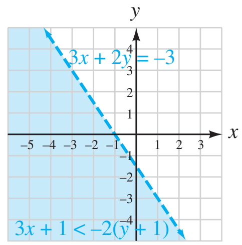
▶ **40.** $3x - y > 6 + y$



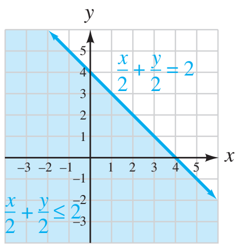
41. $2(x + 1) \geq 3\left(y - \frac{4}{3}\right)$



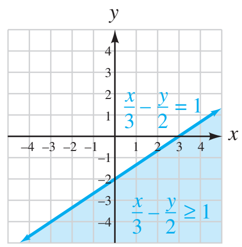
42. $3x + 1 < -2(y + 1)$



43. $\frac{x}{2} + \frac{y}{2} \leq 2$

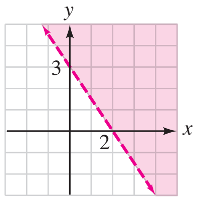


▶ **44.** $\frac{x}{3} - \frac{y}{2} \geq 1$



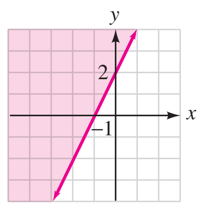
Find the equation of the boundary line and find the inequality whose graph is shown.

45.



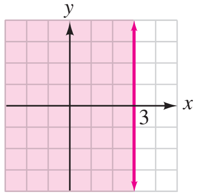
$3x + 2y > 6$

▶ **46.**



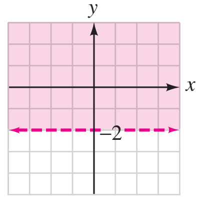
$y \geq 2x + 2$

47.



$x \leq 3$

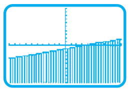
▶ **48.**



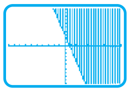
$y > -2$

Use a graphing calculator to graph each inequality.

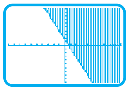
49. $y < 0.27x - 1$



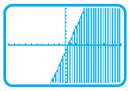
50. $y > -3.5x + 2.7$



51. $y \geq -2.37x + 1.5$



52. $y < 3.37x - 1.7$



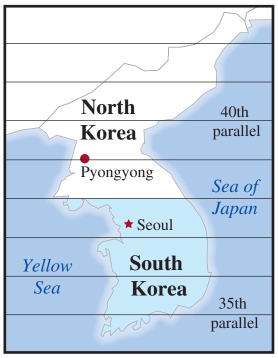
APPLICATIONS

53. GEOGRAPHY A region of the continental United States is shaded in the following map.

a. What is the boundary that separates the shaded and unshaded regions? the Mississippi River

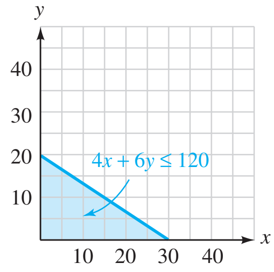
b. In words, describe the shaded area with respect to the boundary.
the area of the U.S. west of the Mississippi River

54. THE KOREAN WAR After World War II, the 38th parallel of north latitude was established as the boundary between North Korea and South Korea. The Korean War began on June 25, 1950, when the North Korean army crossed this line and invaded South Korea. In the illustration, shade the region of the Korean Peninsula south of the 38th parallel.

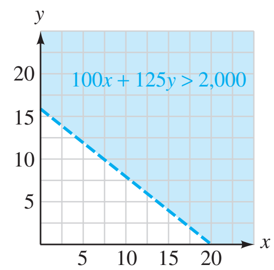


Write a linear inequality that models the situation. Then graph each inequality for nonnegative values of x and y and give three ordered pairs that satisfy the inequality.

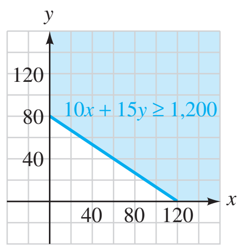
55. RESTAURANT SEATING As part of a remodeling project, a restaurant owner will install new booths that seat 4 persons, and new tables that seat 6 persons. The overall seating must conform to the sign shown below. Write an inequality that describes the possible combinations of booths (x) and tables (y) that the owner can install. (5, 15), (15, 10), (20, 5); answers may vary



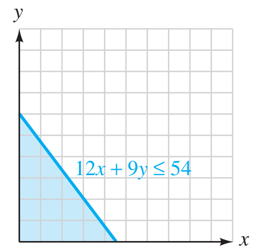
56. GARDENING During an Arbor Day sale, a garden store sold more than \$2,000 worth of maple and pine trees. If a 6-foot maple costs \$100 and a 5-foot pine costs \$125, write an inequality that shows the possible ways that maple trees (x) and pine trees (y) were sold. (0, 17), (5, 20), (15, 10); answers may vary



57. SPORTING GOODS A sporting goods manufacturer allocates at least 1,200 units of time per day to make fishing rods and reels. If it takes 10 units of time to make a rod and 15 units of time to make a reel, write an inequality that describes the possible ways to schedule the time to make rods (x) and reels (y).
(40, 80), (80, 80), (120, 40); answers may vary



58. HOUSEKEEPING One housekeeper charges \$12 per hour, and another charges \$9 per hour. If Sarah can afford no more than \$54 per month to clean her house, write an inequality that describes the possible number of hours that she can hire the first housekeeper (x) and the second housekeeper (y).



$12x + 9y \le 54$; (2, 3), (3, 2), (1, 4); answers may vary

WRITING

59. Explain how to determine where to draw the boundary of the graph of a linear inequality, and whether to draw it as a solid or a dashed line.

60. Explain how to determine which side of the boundary of the graph of a linear inequality should be shaded.

REVIEW

Determine whether the ordered pair (−4, 3) is a solution of the system of linear equations.

61. $\begin{cases} 4x - y = -19 \\ 3x + 2y = -6 \end{cases}$ yes

62. $\begin{cases} y = 2x + 11 \\ \frac{x}{2} + y = 0 \end{cases}$ no

Solve each system of equations.

63. $\begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$ (3, 1)

64. $\begin{cases} x - \frac{y}{2} = -2 \\ 0.01x + 0.02y = 0.03 \end{cases}$ (−1, 2)

SECTION 4.5
Systems of Linear Inequalities

Objectives

1. Solve systems of linear inequalities.
2. Graph compound inequalities.
3. Solve problems involving systems of linear inequalities.

We have discussed how to solve systems of linear equations by the graphing method. For example, to solve

$$\begin{cases} y = -x + 1 \\ 2x - y = 2 \end{cases}$$

we graph both equations on the same set of coordinate axes and find the coordinates of the point of intersection of the straight lines.

In this section, we will discuss how to **solve systems of linear inequalities** graphically such as

$$\begin{cases} y \le -x + 1 \\ 2x - y > 2 \end{cases}$$

1 Solve systems of linear inequalities.

When the solution of a linear inequality in x and y is graphed, the result is a half-plane. To solve a system of linear inequalities, we graph each of the inequalities on one set of coordinate axes and look for the intersection, or overlap, of the shaded half-planes.

EXAMPLE 1 Graph the solution set of: $\begin{cases} y \le -x + 1 \\ 2x - y > 2 \end{cases}$

Strategy We will graph the solutions of $y \le -x + 1$ in one color and the solutions of $2x - y > 2$ in another color on the same rectangular coordinate system.

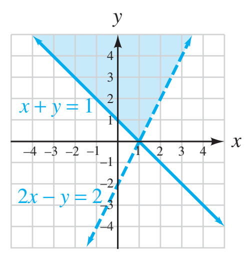
WHY We can then see where the graphs of the two inequalities intersect. This is the solution set on the system.

Solution

To graph $y \le -x + 1$, we graph the boundary $y = -x + 1$, as shown in figure (a) on the next page. Since the edge is to be included, we draw it as a solid line. To determine which half-plane to shade, we use the origin as a test point. Because the

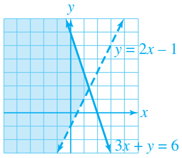
Self Check 1

Graph: $\begin{cases} x + y \ge 1 \\ 2x - y < 2 \end{cases}$



Now Try Problem 22

Teaching Example 1 Graph:
$\begin{cases} y > 2x - 1 \\ 3x + y \le 6 \end{cases}$
Answer:



coordinates of the origin satisfy $y \le -x + 1$, we shade (in red) the half-plane containing the origin.

In figure (b), we superimpose the graph of $2x - y > 2$ on the graph of $y \le -x + 1$ so that we can determine the points that the graphs have in common. To graph $2x - y > 2$, we graph the boundary $2x - y = 2$ as a dashed line. Since the test point (0, 0) does not satisfy $2x - y > 2$, we then shade (in blue) the half-plane that does not contain (0, 0).

The area that is shaded twice represents the solutions of the given system. Any point in the doubly shaded region in purple (including the purple portion of one of the boundaries) has coordinates that satisfy both inequalities.

$y = -x + 1$
We graph this line using the slope and y-intercept.

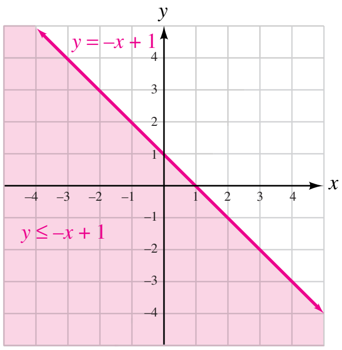
$$m = -1 = -\frac{1}{1}$$

$b = 1$

y-intercept: (0, 1)

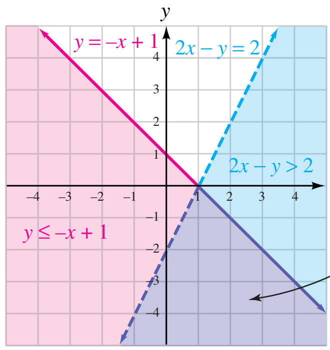
$2x - y = 2$
We graph this line using the intercept method.

x	y	(x, y)
0	−2	(0, −2)
1	0	(1, 0)



The graph of $y \le -x + 1$ is shaded in red.

(a)



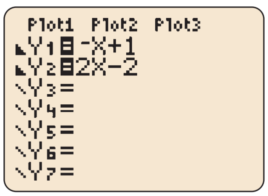
The solutions of the system are shaded in purple. The purple region is the intersection or overlap of the red and blue regions.

The graph of $2x - y > 2$ is shaded in blue. It is drawn over the graph of $y \le -x + 1$.

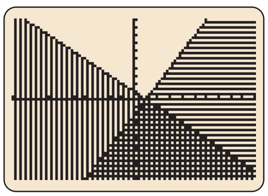
(b)

Using Your CALCULATOR Solving Systems of Inequalities

To solve the system of Example 1 with a graphing calculator, we can use window settings of $x = [-10, 10]$ and $y = [-10, 10]$. To graph $y \le -x + 1$, we enter the boundary equation $y = -x + 1$ and change the graph style icon to below (◣). See figure (a). To graph $2x - y > 2$, we first write it in equivalent form as $y < 2x - 2$. Then we enter the boundary equation $y = 2x - 2$ and change the graph style icon to below (◣). See figure (a). Finally, we press the $\boxed{\text{GRAPH}}$ key to obtain figure (b).



(a)



(b)

In general, to solve systems of linear inequalities, we will follow these steps.

Solving Systems of Linear Inequalities

1. Graph each inequality on the same rectangular coordinate system.
2. Use shading to highlight the intersection of the graphs (the region where the graphs overlap). The points in this region are the solutions of the system.
3. As an informal check, pick a point from the region and verify that its coordinates satisfy each inequality of the original system.

EXAMPLE 2

Graph the solution set of: $\begin{cases} x \geq 1 \\ y \geq x \\ 4x + 5y < 20 \end{cases}$

Strategy We will graph the solutions of $x \geq 1$, $y \geq x$, and $4x + 5y < 20$ on the same coordinate system.

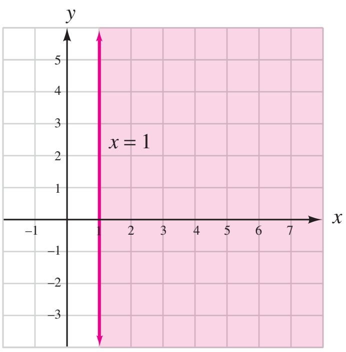
WHY We can then see where the graphs of the three inequalities intersect. This is the solution set of the system.

Solution

We will find the graph of the solution set of the system in stages, using three graphs.

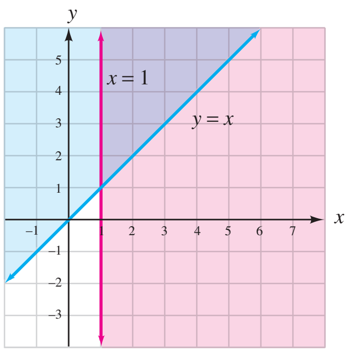
The graph of $x \geq 1$ includes the points that lie on the graph of $x = 1$ and to the right, as shown in red in figure (a).

Figure (b) shows the graph of $x \geq 1$ and the graph of $y \geq x$. The graph of $y \geq x$, in blue, includes the points that lie on the graph of the boundary $y = x$ and above it.



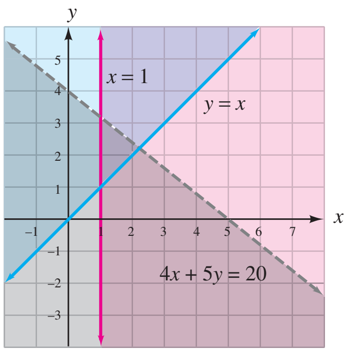
The graph of $x \geq 1$ is shaded in red.

(a)



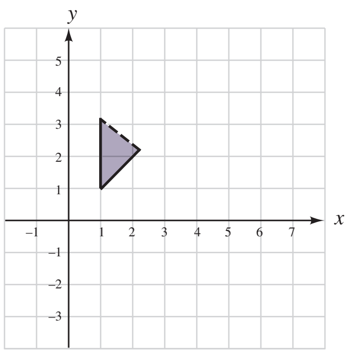
The graph of $y \geq x$ is shaded in blue.

(b)



The graph of $4x + 5y < 20$ is shaded in grey.

(c)

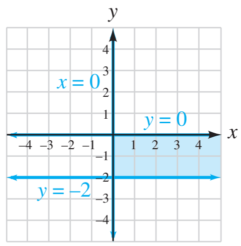


This is the solution of the system.

(d)

Self Check 2

Graph the solution set of: $\begin{cases} x \geq 0 \\ y \leq 0 \\ y \geq -2 \end{cases}$



Now Try **Problem 26**

Teaching Example 2 Graph the solution set of: $\begin{cases} x \geq 0 \\ y \geq -x + 3 \\ 2x - 3y < 6 \end{cases}$

Answer:

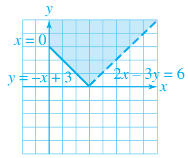


Figure (c) shows the graph of $x \geq 1$, $y \geq x$, and $4x + 5y < 20$. The graph of $4x + 5y < 20$ includes the points that lie below the graph of the boundary $4x + 5y = 20$.

The graph of the solution of the system includes the points that lie within the shaded triangle together with the points on the two sides of the triangle that are drawn with solid line segments, as shown in figure (d).

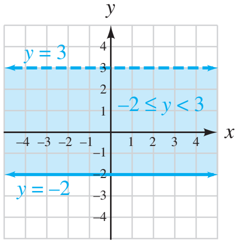
Check: Pick a point in the shaded region, such as (1.5, 2), and show that it satisfies each inequality of the system.

2 Graph compound inequalities.

We have graphed the solution set of double linear inequalities, such as $2 < x \leq 5$, on a number line. These inequalities contained only one variable. In the next example, we will graph the solution set of $2 \leq x \leq 5$ in the context of two variables. In this case, we use the rectangular coordinate system.

Self Check 3

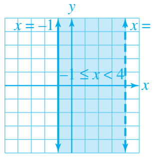
Graph: $-2 \leq y < 3$



Now Try **Problem 30**

Teaching Example 3 Graph $-1 \leq x < 4$ on a rectangular coordinate plane

Answer:



EXAMPLE 3 Graph $2 < x \leq 5$ on the rectangular coordinate plane.

Strategy We will write the double inequality $2 < x \leq 5$ as an equivalent system of inequalities. Then we will graph the system.

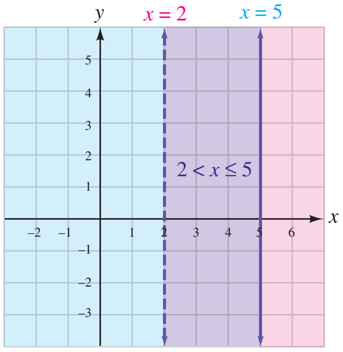
WHY The compound inequality $2 < x \leq 5$ is equivalent to $2 < x$ and $x \leq 5$.

Solution

The compound inequality $2 < x \leq 5$ is equivalent to the following system of two linear inequalities:

$$\begin{cases} 2 < x \\ x \leq 5 \end{cases}$$

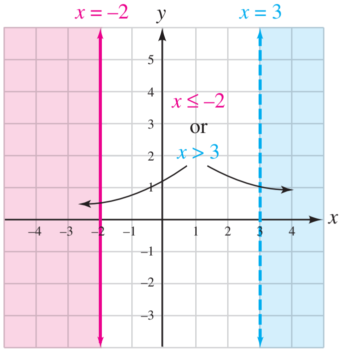
The graph of $2 < x$, shown in the figure in red, is the half-plane to the right of the vertical line $x = 2$. The graph of $x \leq 5$, shown in the figure in blue, includes the line $x = 5$ and the half-plane to its left. The graph of $2 < x \leq 5$ will contain all points in the plane that satisfy the inequalities $2 \leq x$ and $x \leq 5$ simultaneously (at the same time). These points are in the purple-shaded region of the figure.



To graph a compound inequality containing the word *or* in the rectangular coordinate system, we sketch the *union* of the solution sets of the inequalities involved. For example, the figure shows the graph of the compound inequality

$$x \leq -2 \text{ or } x > 3$$

in the rectangular coordinate system.



3 Solve problems involving systems of linear inequalities.

EXAMPLE 4 ***Landscaping*** A homeowner has a budget of \$300 to \$600 for trees and bushes to landscape his yard. After some shopping, he finds that good trees cost \$150 and mature bushes cost \$75. What combinations of trees and bushes can he afford to buy?

Analyze We need to find the number of trees and the number of bushes that the homeowner can afford. This suggests we should use two variables. We know that he is willing to spend *at least* \$300 and *at most* \$600 for trees and bushes. These phrases suggest that we should write two inequalities that model the situation.

Form If $x =$ the number of trees purchased, then \$150x will be the cost of the trees. If $y =$ the number of bushes purchased, then \$75y will be the cost of the bushes. We know that the homeowner wants the sum of these costs to be from \$300 to \$600. We can then form the following system of linear inequalities.

The cost of a tree	·	the number of trees purchased	plus	the cost of a bush	·	the number of bushes purchased	should be at least	\$300.
\$150	·	x	+	\$75	·	y	$\geq$	\$300

The cost of a tree	·	the number of trees purchased	plus	the cost of a bush	·	the number of bushes purchased	should be at most	\$600.
\$150	·	x	+	\$75	·	y	$\leq$	\$600

Solve We graph the system

$$\begin{cases} 150x + 75y \geq 300 \\ 150x + 75y \leq 600 \end{cases}$$

as in the figure. The coordinates of each point shown in the red-shaded graph give a possible combination of trees (x) and bushes (y) that can be purchased.

State The possible combinations of trees and bushes that can be purchased are given by

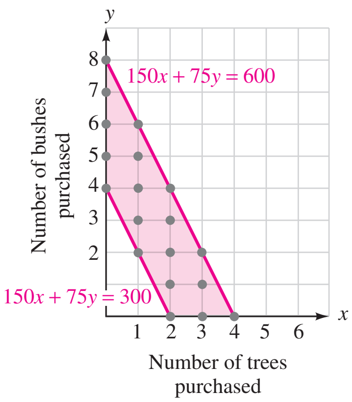
(0, 4), (0, 5), (0, 6), (0, 7), (0, 8)
(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 0), (2, 1), (2, 2), (2, 3), (2, 4)
(3, 0), (3, 1), (3, 2), (4, 0)

The ordered pair (1, 6), for example, indicates that the homeowner can afford 1 tree and 6 bushes.

Only these points can be used, because the homeowner cannot buy a portion of a tree or a bush.

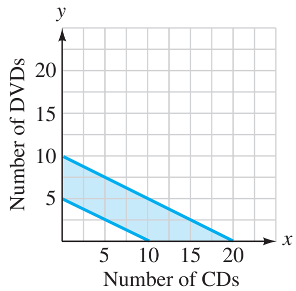
Check Check some of the ordered pairs to verify that they satisfy both inequalities.

Because the homeowner cannot buy a negative number of trees or bushes, we graph the system only for $x \geq 0$ and $y \geq 0$.



Self Check 4

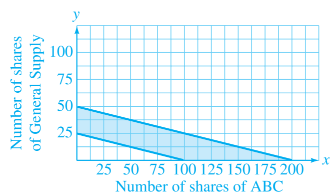
BUYING MEDIA An electronics store sells CDs for \$10 and DVDs for \$20. Laurie wants to spend at least \$100 but no more than \$200 on (x) CDs and (y) DVDs. What combinations of CDs and DVDs can she afford to buy?



Now Try **Problem 51**

Self Check 4 Answer
Some possible combinations are (2, 5), (3, 5); answers may vary

Teaching Example 4 INVESTING
Marcia has at least \$500 but no more than \$1,000 to invest in stocks from two companies. Each share of ABC stock costs \$5 per share, and each share of General Supply costs \$20 per share. What combinations of shares of ABC and General Supply can she afford to buy?
Answer:
Some possible combinations are (25, 25) (30, 25); answers may vary



ANSWERS TO SELF CHECKS

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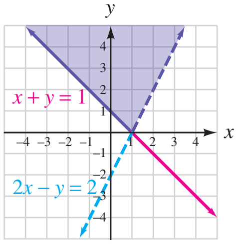

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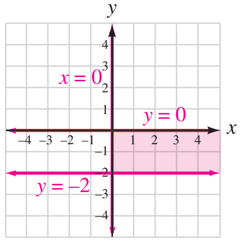

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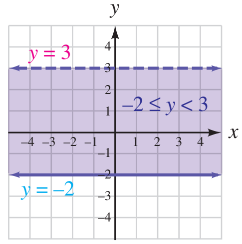

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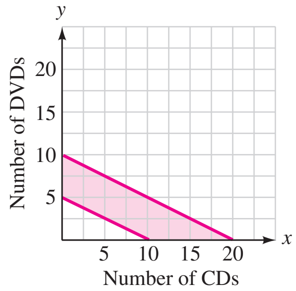


SECTION 4.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\begin{cases} x + y \le 2 \\ x - 3y > 10 \end{cases}$ is a system of linear inequalities.
2. If a boundary line is included in the graph of an inequality, we draw it as a solid line.
3. To solve a system of inequalities by graphing, we graph each inequality. The solution is the region where the graphs overlap or intersect.
4. To determine which half-plane to shade when graphing a linear inequality, we see whether the coordinates of a test point satisfy the inequality.

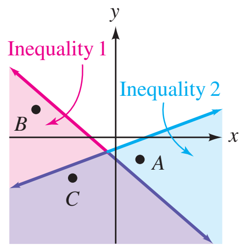
CONCEPTS

Determine whether each ordered pair satisfies the system of linear inequalities $\begin{cases} x + y \le 2 \\ x - 3y > 10. \end{cases}$

5. $(2, -3)$ yes
6. $(12, -1)$ no
7. $(0, -3)$ no
8. $(-0.5, -5)$ yes
9. Determine whether $(-3, 10)$ satisfies the compound inequality $-5 < x \le 8$ in the rectangular coordinate system. yes
10. Determine whether $(-3, 3)$ satisfies the compound inequality $y \le 0$ or $y > 4$ in the rectangular coordinate system. no

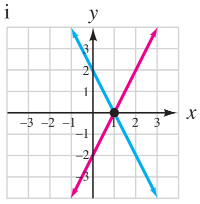
In the illustration, the solution of one linear inequality is shaded in red, and the solution of a second linear inequality is shaded in blue. The intersection of these two regions is shaded in purple. Determine whether a true or false statement results if the coordinates of each point are substituted into the given inequality.

11. A, inequality 1 false
12. A, inequality 2 true
13. B, inequality 1 true
14. B, inequality 2 false
15. C, inequality 1 true
16. C, inequality 2 true



Match each equation, inequality, or system with the graph of its solution.

17. $2x + y = 2$ iii
18. $2x + y \ge 2$ ii
19. $\begin{cases} 2x + y = 2 \\ 2x - y = 2 \end{cases}$ i
20. $\begin{cases} 2x + y \ge 2 \\ 2x - y \le 2 \end{cases}$ iv

i


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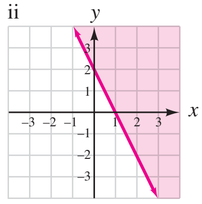

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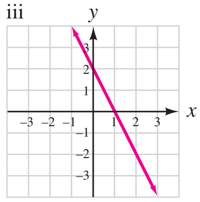

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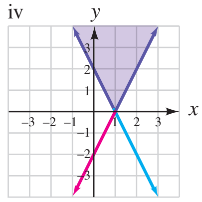


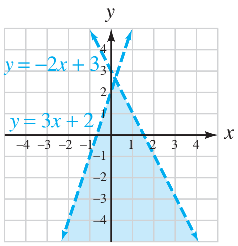
Selected exercises available online at **www.webassign.net/brookscole**

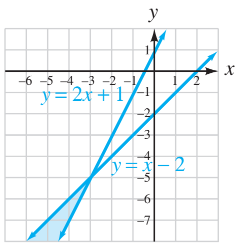
GUIDED PRACTICE

Graph the solution set of each system. See Example 1.

21. $\begin{cases} y < 3x + 2 \\ y < -2x + 3 \end{cases}$

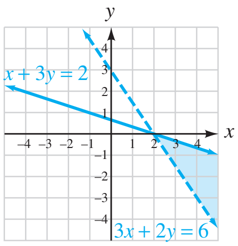
▶ **22.** $\begin{cases} y \le x - 2 \\ y \ge 2x + 1 \end{cases}$

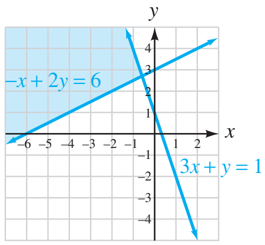




23. $\begin{cases} 3x + 2y > 6 \\ x + 3y \le 2 \end{cases}$

24. $\begin{cases} 3x + y \le 1 \\ -x + 2y \ge 6 \end{cases}$

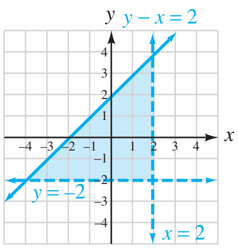


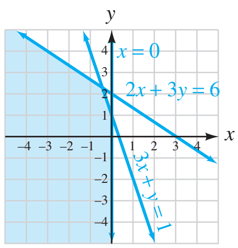


Graph the solution set of each system. See Example 2.

▶ **25.** $\begin{cases} y - x \le 2 \\ y > -2 \\ x < 2 \end{cases}$

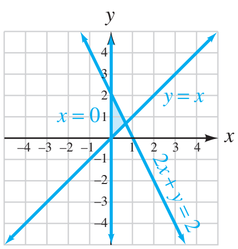
26. $\begin{cases} 2x + 3y \le 6 \\ 3x + y \le 1 \\ x \le 0 \end{cases}$

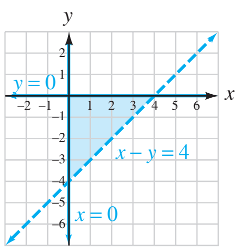




▶ **27.** $\begin{cases} 2x + y \le 2 \\ y \ge x \\ x \ge 0 \end{cases}$

28. $\begin{cases} x - y < 4 \\ y \le 0 \\ x \ge 0 \end{cases}$

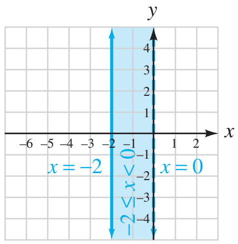


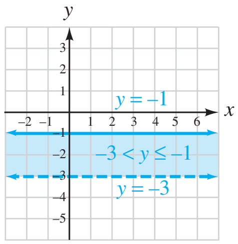


Graph the solution set of each system. See Example 3.

29. $-2 \le x < 0$

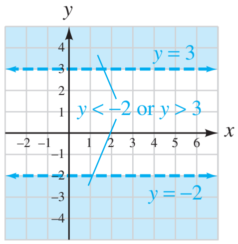
▶ **30.** $-3 < y \le -1$

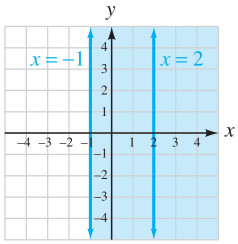




31. $y < -2$ or $y > 3$

▶ **32.** $-x \le 1$ or $x \ge 2$



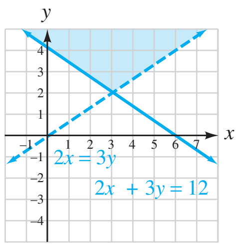


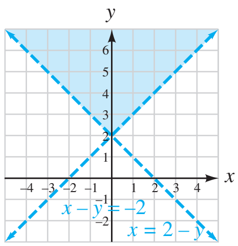
TRY IT YOURSELF

Graph the solution set of each system of inequalities.

33. $\begin{cases} 2x < 3y \\ 2x + 3y \ge 12 \end{cases}$

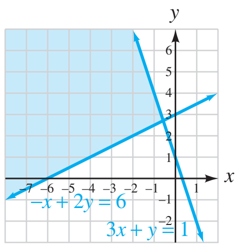
34. $\begin{cases} x > 2 - y \\ x - y < -2 \end{cases}$

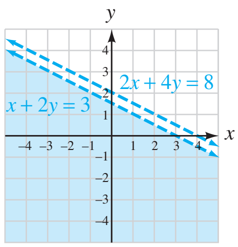




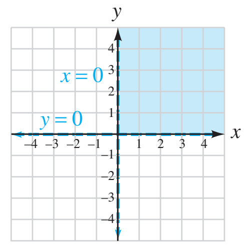
▶ **35.** $\begin{cases} -x + 2y \ge 6 \\ 3x + y \le 1 \end{cases}$

▶ **36.** $\begin{cases} x + 2y < 3 \\ 2x + 4y < 8 \end{cases}$

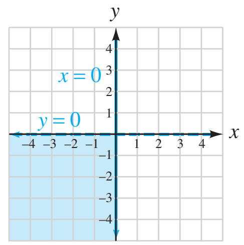




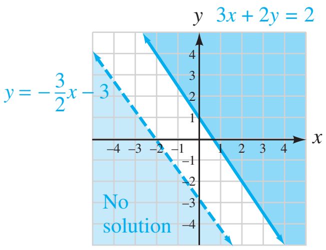
37. $\begin{cases} x > 0 \\ y > 0 \end{cases}$



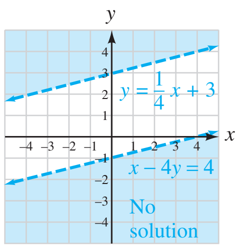
38. $\begin{cases} x \le 0 \\ y < 0 \end{cases}$



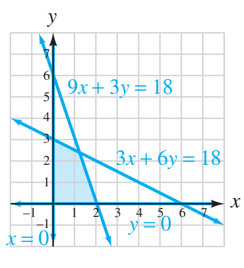
▶ **39.** $\begin{cases} y < -\frac{3}{2}x - 3 \\ 3x + 2y \ge 2 \end{cases}$



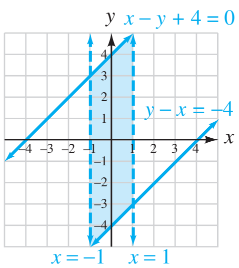
40. $\begin{cases} y > \frac{1}{4}x + 3 \\ x - 4y > 4 \end{cases}$



▶ **41.** $\begin{cases} x \ge 0 \\ y \ge 0 \\ 9x + 3y \le 18 \\ 3x + 6y \le 18 \end{cases}$

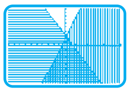


42. $\begin{cases} x < 1 \\ x > -1 \\ x - y + 4 \ge 0 \\ y - x > -4 \end{cases}$

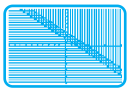


Use a graphing calculator to solve each system.

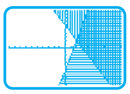
43. $\begin{cases} y < 3x + 2 \\ y < -2x + 3 \end{cases}$



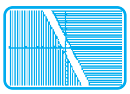
44. $\begin{cases} y > -x + 2 \\ y < -x + 4 \end{cases}$



45. $\begin{cases} 2x + y \ge 6 \\ y \le 2(2x - 3) \end{cases}$



▶ **46.** $\begin{cases} 3x + y < -2 \\ y > 3(1 - x) \end{cases}$

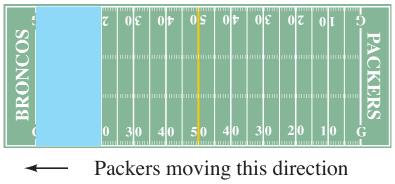


APPLICATIONS

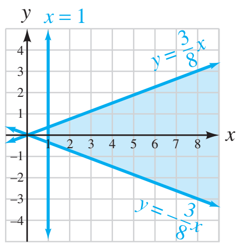
▶ **47.** FOOTBALL In 2003, the Green Bay Packers scored either a touchdown or a field goal 65.4% of the time when their offense was in the *red zone*. This was the best record in the NFL. If x represents the yard line the football is on, a team's red zone is an area on their opponent's half of the field that can be described by the system

$$\begin{cases} x > 0 \\ x \le 20 \end{cases}$$

Shade the red zone on the field shown below.



▶ **48.** TRACK AND FIELD In the shot put, the solid metal ball must land in a marked sector for it to be a fair throw. In the illustration, graph the system of inequalities that describes the region outside of the ring in which a shot must land.



$$\begin{cases} y \le \frac{3}{8}x \\ y \ge -\frac{3}{8}x \\ x \ge 1 \end{cases}$$

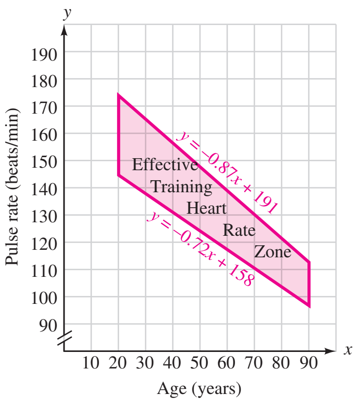
49. NO-FLY ZONES After the Gulf War, U.S. and Allied forces enforced northern and southern no-fly zones over Iraq. Iraqi aircraft were prohibited from flying in this air space. If y represents the latitude parallel measurement, the no-fly zones can be described by

$$y \ge 36 \quad \text{or} \quad y \le 33$$

On the map below, shade the regions of Iraq over which there was a no-fly zone.

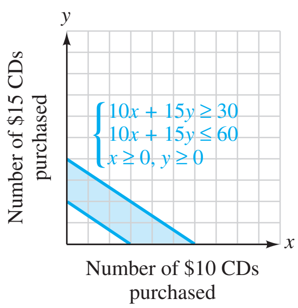
50. CARDIOVASCULAR FITNESS The following graph shows the range of pulse rates that persons ages 20–90 should maintain during aerobic exercise to get the most benefit from the training. The shaded region "Effective Training Heart Rate Zone" can be described by a system of linear inequalities. Determine what inequality symbol should be inserted in each blank.

$$\begin{cases} x \geq 20 \\ x \leq 90 \\ y \leq -0.87x + 191 \\ y \geq -0.72x + 158 \end{cases}$$

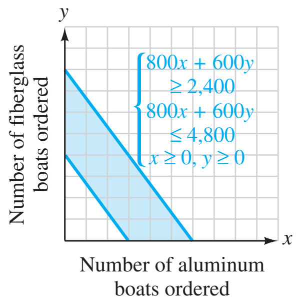


Graph each system and give two possible solutions.

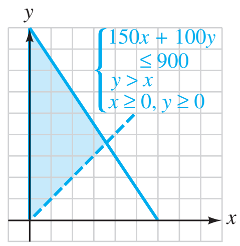
51. COMPACT DISCS Melodic Music has compact discs on sale for either \$10 or \$15. If a customer wants to spend at least \$30 but no more than \$60 on CDs, use the illustration to graph a system of inequalities that will show the possible ways a customer can buy \$10 CDs (x) and \$15 CDs (y).
1 \$10 CD and 2 \$15 CDs, 4 \$10 CDs and 1 \$15 CD



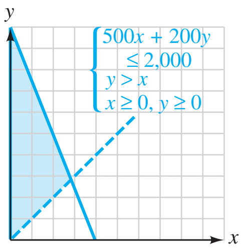
52. BOAT SALES Dry Boat Works wholesales aluminum boats for \$800 and fiberglass boats for \$600. Northland Marina wants to order at least \$2,400 worth but no more than \$4,800 worth of boats. Use the illustration in the next column to graph a system of inequalities that will show the possible combination of aluminum boats (x) and fiberglass boats (y) that can be ordered.
4 alum. and 1 fiberglass, 1 alum. and 4 fiberglass



53. FURNITURE SALES A distributor wholesales desk chairs for \$150 and side chairs for \$100. Best Furniture wants to order no more than \$900 worth of chairs, including more side chairs than desk chairs. Use the illustration to graph a system of inequalities that will show the possible combinations of desk chairs (x) and side chairs (y) that can be ordered.
2 desk chairs and 4 side chairs, 1 desk chair and 5 side chairs



54. FURNACE EQUIPMENT J. Bolden Heating Company wants to order no more than \$2,000 worth of electronic air cleaners and humidifiers from a wholesaler that charges \$500 for air cleaners and \$200 for humidifiers. If Bolden wants more humidifiers than air cleaners, use the illustration to graph a system of inequalities that will show the possible combinations of air cleaners (x) and humidifiers (y) that can be ordered.
1 air cleaner and 2 humidifiers, 2 air cleaners and 3 humidifiers



WRITING

55. Explain how to solve a system of two linear inequalities graphically.

56. Explain how a system of two linear inequalities might have no solution.

REVIEW

Use the given conditions to determine in which quadrant of a rectangular coordinate system each point (x, y) is located.

57. $x > 0$ and $y < 0$ IV

58. $x < 0$ and $y < 0$ III

59. $x < 0$ and $y > 0$ II

60. $x > 0$ and $y > 0$ I

STUDY SKILLS CHECKLIST

Preparing for the Chapter 4 Test

In Chapter 4 you learned three methods to solve a system of linear equations. You also learned how to graph the solutions of a system of linear inequalities. As you prepare for the exam over this material, make sure you also review the following checklist.

☐ To check a proposed solution of a system of equations, be sure the coordinates of the ordered pair satisfies *both* equations.

Is $(3, -2)$ a solution of the system $\begin{cases} 3x + 4y = 1 \\ x + 2y = -1 \end{cases}$?

$$\begin{aligned} 3(3) + 4(-2) &= 1 \\ 9 - 8 &= 1 \\ 1 &= 1 \quad \text{True} \end{aligned} \qquad \begin{aligned} 3 + 2(-2) &= -1 \\ 3 - 4 &= -1 \\ -1 &= -1 \quad \text{True} \end{aligned}$$

Yes, $(3, -2)$ is a solution of the system.

☐ When solving a system of equations by graphing, you must determine the coordinates of the point in intersection of the graphs. That ordered pair is the solution of the system.

☐ When using the substitution or the addition (elimination) method, remember to find the value of *both* the variables.

For the system of linear equations $\begin{cases} x = 2y - 3 \\ x + 4y = 3 \end{cases}$, the y-coordinate of the solution is $y = 1$. To find the x-value, substitute 1 for y in either equation:

$$\begin{aligned} x &= 2y - 3 \\ x &= 2(1) - 3 \\ x &= 2 - 3 \\ x &= -1 \end{aligned}$$

The solution is $(-1, 1)$.

☐ If the substitution method of solving a system is being used, make sure you use parentheses when plugging the expression from the substitution equations in the remaining equation.

$$\begin{cases} 2x - 3y = 11 \\ y = 3x - 13 \end{cases}$$

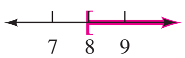
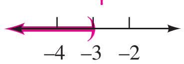
$2x - 3y = 11$	This is equation 1.
$2x - 3(3x - 13) = 11$	Substitute $3x - 13$ for y.
$2x - 9x + 39 = 11$	Use the distributive property.
$-7x + 39 = 11$	Combine like terms.
$-7x = -28$	Subtract 39 from both sides.
$x = 4$	Divide both sides by -7.
$y = 3x - 13$	This is the substitution equation.
$y = 3(4) - 13$	Substitute 4 for x.
$y = 12 - 13$	Multiply $3(4) = 12$.
$y = -1$	This is the y-value of the solution.

The solution is $(4, -1)$.

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

CHAPTER 4 SUMMARY AND REVIEW

SECTION 4.1 Solving Linear Inequalities in One Variable

DEFINITIONS AND CONCEPTS	EXAMPLES
Inequalities are statements that contain one or more **inequality symbols.**	Each of the following inequalities is true: $5 \neq 7$ 5 is not equal to 7. $-1 < 9$ −1 is less than 9. $8 > 5$ 8 is greater than 5. $3 \leq 6$ 3 is less than or equal to 6. $4 \geq 4$ 4 is greater than or equal to 4.
A **linear inequality in one variable** is any inequality that can be written in the form $ax + b < c$, where a, b, and c represent real numbers and $a \neq 0$. The inequality symbols $>$, $\leq$, and $\geq$ can also be used.	Linear inequalities in one variable: $2x + 3 < 6$ $8x > 2x - 7$ $5(x + 9) \leq 6(x - 6)$
The **solution of a linear inequality** is a number that satisfies the inequality.	The number 5 is a solution of $12 < 3x + 11$ because 5 satisfies the inequality. $12 < 3x + 11$ $12 \overset{?}{<} 3(5) + 11$ Substitute 5 for x. $12 \overset{?}{<} 15 + 11$ $12 < 26$ True
To **solve a linear inequality** in one variable we use **properties of inequality** to find the values of its variable that make the inequality true. Adding the same number to, or subtracting the same number from, both sides of an inequality does not change the solutions. Multiplying or dividing both sides of an inequality by the same positive number does not change the solutions. The set of all solutions of an inequality is called its **solution set.** If we multiply or divide both sides of an inequality by a negative number, the direction of the **inequality symbol must be reversed** for the inequalities to have the same solutions.	Solve: $3x - 12 \geq 12$ $3x - 12 + 12 \geq 12 + 12$ Add 12 to both sides. $3x \geq 24$ $\frac{3x}{3} \geq \frac{24}{3}$ Divide both sides by 3. $x \geq 8$ The solution set can be expressed in three ways: Graph:  Interval notation: $[8, \infty)$ Set-builder notation: $\{x \mid x \geq 8\}$ Solve: $-5x + 7 > 22$ $-5x + 7 - 7 > 22 - 7$ Subtract 7 from both sides. $-5x > 15$ $\frac{-5x}{-5} < \frac{15}{-5}$ Divide both sides by −5 and reverse the direction of the inequality symbol. $x < -3$ The solution set is: Graph:  Interval notation: $(-\infty, -3)$ Set-builder notation: $\{x \mid x < -3\}$

REVIEW EXERCISES

1. Determine whether -2 is a solution of each inequality.

   a. $3x - 6 < x - 10$ no

   b. $\frac{x}{2} - 3 \geq 4(x + 1)$ yes

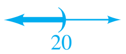
2. Represent the set of real numbers greater than or equal to -5 with a graph, using interval notation, and using set-builder notation.
   $[-5, \infty)$, $\{x \mid x \geq -5\}$

Solve each inequality. Graph the solution set and write it using interval notation and set-builder notation.

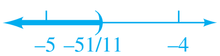
3. $5(x - 2) \leq 5$
   $(-\infty, 3]$, $\{x \mid x \leq 3\}$

4. $0.3x - 0.4 \geq 1.2 - 0.1x$
   $[4, \infty)$, $\{x \mid x \geq 4\}$

5. $-16 < -\frac{4}{5}t$
   $(-\infty, 20)$, $\{t \mid t < 20\}$



6. $\frac{7}{4}(x + 3) < \frac{3}{8}(x - 3)$
   $\left(-\infty, -\frac{51}{11}\right)$, $\left\{x \mid x < -\frac{51}{11}\right\}$



7. $7 - [6t - 5(t - 3)] > 2(t - 3) - 3(t + 1)$
   $(-\infty, \infty)$, $\mathbb{R}$

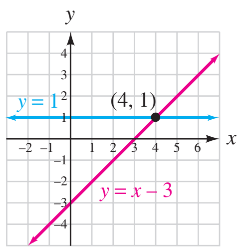
8. $\frac{2b + 7}{2} \leq \frac{3b - 1}{3}$ no solution, $\varnothing$

9. INVESTMENTS A woman has invested $10,000 at 6% annual interest. How much more must she invest at 7% so that her annual income is at least $2,000? $20,000 or more

10. ICE SKATING For the free-skating portion of a competition, an ice skater received scores of 5.3, 4.8, 4.7, 4.9, and 5.1 from the first five judges. What score must she receive from the sixth and final judge to average better than 5.0 for her performance?
    She needs to receive a score that is greater than 5.2.

11. LAWYERS A lawyer earns $200 an hour for telephone consultations and $300 an hour for office consultations with clients. To save time for court appearances, she limits her consulting to 15 hours a week. What is the greatest number of hours that she can spend on the phone and still earn at least $4,000 in consulting fees a week? 5 hr

12. Explain how to use the graphs of $y = 1$ and $y = x - 3$ to solve $x - 3 \leq 1$.



SECTION 4.2 Solving Compound Inequalities

DEFINITIONS AND CONCEPTS	EXAMPLES
The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements that are common to set A and set B.	Let $A = \{-2, 0, 3, 5\}$ and $B = \{-3, 0, 5, 7\}$. $A \cap B = \{0, 5\}$ *The intersection contains the elements that the sets have in common.*
The **union** of two sets A and B, written $A \cup B$, is the set of all elements that are in set A, set B, or both.	$A \cup B = \{-3, -2, 0, 3, 5, 7\}$ *The union contains the elements that are in one or the other set, or both.*

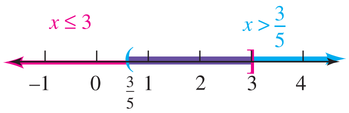
When the word *and* or the word *or* is used to connect pairs of inequalities, we call the statement a **compound inequality.**

The solution set of a **compound inequality containing the word *and*** includes all numbers that make both of the inequalities true. That is, it is the intersection of their solution sets.

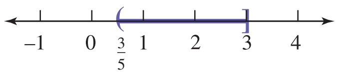
Solve: $2x - 1 \le 5$ and $5x + 1 > 4$

We solve each inequality separately. Then we graph the two solution sets on the same number line and determine their intersection.

$$\begin{array}{rcr} 2x - 1 \le 5 & \text{and} & 5x + 1 > 4 \\ 2x \le 6 & & 5x > 3 \\ x \le 3 & & x > \frac{3}{5} \end{array}$$



The purple-shaded interval is where the red and blue graphs overlap. Thus, the solution set is:



Interval notation: $\left(\frac{3}{5}, 3\right]$

Inequalities that contain exactly two inequality symbols are called **double inequalities.** Any double linear inequality can be written as a compound inequality containing the word *and.* For example:

$c < x < d$ is equivalent to $c < x$ and $x < d$

Solve: $-7 \le 3x - 1 < 5$

We apply properties of inequality to *all three of its parts* to isolate x in the middle.

$$-7 \le 3x - 1 < 5$$

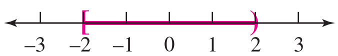
$$-7 + 1 \le 3x - 1 + 1 < 5 + 1 \quad \text{Add 1 to all three parts.}$$

$$-6 \le 3x < 6$$

$$\frac{-6}{3} \le \frac{3x}{3} < \frac{6}{3} \quad \text{Divide each part by 3.}$$

$$-2 \le x < 2$$

The solution set is:



Interval notation: $[-2, 2)$

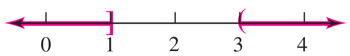
The solution set of a **compound inequality containing the word *or*** includes all numbers that make one or the other, or both, inequalities true. That is, it is the union of their solution sets.

Solve: $2x - 1 > 5$ or $-(5x - 7) \ge 2$

We solve each inequality separately. Then we graph the two solution sets on the same number line to show their union.

$$\begin{array}{rcr} 2x - 1 > 5 & \text{or} & -(5x - 7) \ge 2 \\ 2x > 6 & & -5x + 7 \ge 2 \\ x > 3 & & -5x \ge -5 \\ & & x \le 1 \end{array}$$

The solution set is:



Interval notation: $(-\infty, 1] \cup (3, \infty)$
This is the union of two intervals.

REVIEW EXERCISES

Let $A = \{-6, -3, 0, 3, 6\}$ and $B = \{-5, -3, 3, 8\}$.

13. Find $A \cap B$. $\{-3, 3\}$

14. Find $A \cup B$. $\{-6, -5, -3, 0, 3, 6, 8\}$

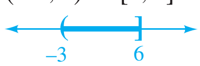
Determine whether -4 is a solution of the compound inequality.

15. $x < 0$ and $x > -5$ yes

16. $x + 3 < -3x - 1$ and $4x - 3 > 3x$ no

Graph each set.

17. $(-3, 3) \cup [1, 6]$



18. $(-\infty, 2] \cap [1, 4)$

Solve each compound inequality. Graph the solution set and write it using interval notation.

19. $-2x > 8$ and $x + 4 \geq -6$
$[-10, -4)$


20. $5(x + 2) \leq 4(x + 1)$ and $11 + x < 0$
$(-\infty, -11)$


21. $\frac{2}{5}x - 2 < -\frac{4}{5}$ and $\frac{x}{-3} < -1$
no solution; $\varnothing$

22. $4\left(x - \frac{1}{4}\right) \leq 3x - 1$ and $x \geq 0$
$[0, 0]$


Solve each double inequality. Graph the solution set and write it using interval notation.

23. $3 < 3x + 4 < 10$
$\left(-\frac{1}{3}, 2\right)$


24. $-2 \leq \frac{5 - x}{2} \leq 2$
$[1, 9]$


Determine whether −4 is a solution of the compound inequality.

25. $x < 1.6$ or $x > -3.9$ yes

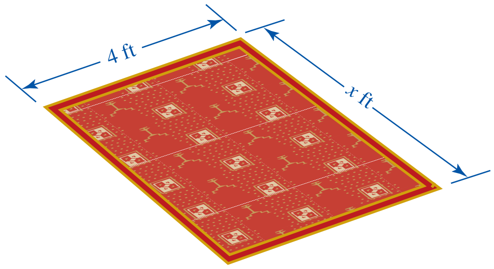
26. $x + 1 < 2x - 1$ or $4x - 3 > 3x$ no

Solve each compound inequality. Graph the solution set and write it using interval notation.

27. $x + 1 < -4$ or $x - 4 > 0$
$(-\infty, -5) \cup (4, \infty)$


28. $\frac{x}{2} + 3 > -2$ or $4 - x > 4$
$(-\infty, \infty)$


29. RUGS A manufacturer makes a line of decorator rugs that are 4 feet wide and of varying lengths x (in feet). The floor area covered by the rugs ranges from 17 ft^2 to 25 ft^2. Write and then solve a double linear inequality to find the range of the lengths of the rugs.
$17 \leq 4x \leq 25$, $4.25 \text{ ft} \leq x \leq 6.25 \text{ ft}$, $[4.25, 6.25]$



30. Match each word in Column I with *two* associated items in Column II.

Column I	***Column II***
a. or ii, iv	**i.** $\cap$
	ii. $\cup$
b. and i, iii	**iii.** intersection
	iv. union

SECTION 4.3 Solving Absolute Value Equations and Inequalities

DEFINITIONS AND CONCEPTS	EXAMPLES
To **solve absolute value equations** of the form $\lvert x\rvert = k$, where $k > 0$, solve the equivalent **compound equation** $x = k$ or $x = -k$	Solve: $\lvert 2x + 1\rvert = 7$ This absolute value equation is equivalent to the following compound equation, which we can solve: $2x + 1 = 7$ or $2x + 1 = -7$ $2x = 6$ | $2x = -8$ $x = 3$ | $x = -4$ This equation has two solutions: 3 and -4. The solution set is $\{-4, 3\}$.
If k is negative, then $\lvert x\rvert = k$ has no solution.	Solve: $\lvert 4x - 5\rvert = -3$ Since an absolute value can never be negative, there are no real numbers x that make $\lvert 4x - 5\rvert = -3$ true. The equation has no solution and the solution set is $\varnothing$.

To **solve absolute value equations** of the form $|X| = |Y|$, solve the compound equation

$$X = Y \quad \text{or} \quad X = -Y$$

Solve: $|3x - 2| = |2x + 4|$

This equation is equivalent to the following compound equation, which we can solve:

$$\begin{array}{rcl|rcl} 3x - 2 &=& 2x + 4 & 3x - 2 &=& -(2x + 4) \\ x - 2 &=& 4 & 3x - 2 &=& -2x - 4 \\ x &=& 6 & 5x - 2 &=& -4 \\ & & & 5x &=& -2 \\ & & & x &=& -\frac{2}{5} \end{array}$$

(or)

This equation has two solutions: 6 and $-\frac{2}{5}$. The solution set is $\left\{-\frac{2}{5}, 6\right\}$.

To **solve absolute value inequalities** of the form $|X| < k$, where $k > 0$, solve the equivalent double inequality $-k < X < k$.

Use a similar approach to solve $|X| \le k$.

Solve: $|4x - 3| < 9$

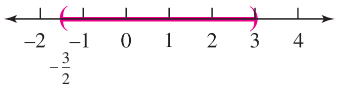
This inequality is equivalent to the following double inequality which we can solve:

$$-9 < 4x - 3 < 9$$

$$-6 < 4x < 12 \qquad \textit{Add 3 to all three parts.}$$

$$-\frac{3}{2} < x < 3 \qquad \textit{Divide each part by 4 and simplify.}$$

The solution set is:



Interval notation: $\left(-\frac{3}{2}, 3\right)$

To **solve absolute value inequalities** of the form $|X| \ge k$, where $k > 0$, solve the equivalent compound inequality $X \le -k$ or $X \ge k$.

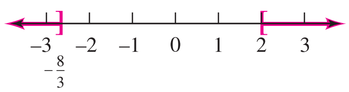
Use a similar approach to solve $X > k$.

Solve: $|3x + 1| \ge 7$

This inequality is equivalent to the following compound inequality, which we can solve:

$$\begin{array}{rcl|rcl} 3x + 1 &\le& -7 & 3x + 1 &\ge& 7 \\ 3x &\le& -8 & 3x &\ge& 6 \\ x &\le& -\frac{8}{3} & x &\ge& 2 \end{array}$$

(or)



Interval notation: $\left(-\infty, -\frac{8}{3}\right] \cup [2, \infty)$

This is the union of two intervals.

REVIEW EXERCISES

Solve each absolute value equation.

31. $|4x| = 8$ $\quad 2, -2$

32. $2|3x + 1| - 1 = 19$ $\quad 3, -\frac{11}{3}$

33. $\left|\frac{3}{2}x - 4\right| - 10 = -1$ $\quad \frac{26}{3}, -\frac{10}{3}$

34. $\left|\frac{2 - x}{3}\right| = -4$ $\quad$ no solution, $\varnothing$

35. $|-4(2x - 6)| = 0$ $\quad 3$

36. $\left|\frac{3}{8} + \frac{x}{3}\right| = \frac{5}{12}$ $\quad \frac{1}{8}, -\frac{19}{8}$

37. $|3x + 2| = |2x - 3|$ $\quad \frac{1}{5}, -5$

38. $\left|\frac{2(1 - x) + 1}{2}\right| = \left|\frac{3x - 2}{3}\right|$ $\quad \frac{13}{12}$

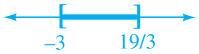
Solve each absolute value inequality. Graph the solution set and write it using interval notation.

39. $|x| \le 3$ $\quad [-3, 3]$

40. $|2x + 7| < 3$ $\quad (-5, -2)$

41. $2|5 - 3x| \le 28$

$\left[-3, \frac{19}{3}\right]$

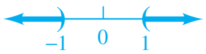


42. $\left|\frac{2}{3}x + 14\right| + 6 < 6$

no solution, $\varnothing$

43. $|x| > 1$

$(-\infty, -1) \cup (1, \infty)$



44. $\left|\frac{1 - 5x}{3}\right| \ge 7$

$(-\infty, -4] \cup \left[\frac{22}{5}, \infty\right)$

45. $|3x - 8| - 4 > 0$

$\left(-\infty, \frac{4}{3}\right) \cup (4, \infty)$

46. $\left|\frac{3}{2}x - 14\right| \ge 0$

$(-\infty, \infty)$, $\mathbb{R}$

47. Explain why $|0.04x - 8.8| < -2$ has no solution.
Since $|0.04x - 8.8|$ is always greater than or equal to 0 for any real number x, this absolute value inequality has no solution.

48. Explain why the solution set of $\left|\frac{3x}{50} + \frac{1}{45}\right| \ge -\frac{4}{5}$ is the set of all real numbers.
Since $\left|\frac{3x}{50} + \frac{1}{45}\right|$ is always greater than or equal to 0 for any real number x, this absolute value inequality is true for all real numbers.

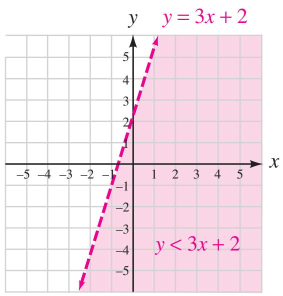
49. PRODUCE Before packing, a farmer weighs freshly picked tomatoes on the scale shown. Tomatoes having a weight w (in ounces) that falls within the highlighted range are sold to grocery stores.

a. Complete the following absolute value inequality that expresses the acceptable weight range:
$|w - \boxed{8}| \le \boxed{2}$

b. Solve the inequality from part a and express the acceptable weight range using interval notation.
$[6, 10]$

50. Let $f(x) = \frac{1}{3}|6x| - 1$. For what value(s) of x is $f(x) = 5$? $3, -3$

SECTION 4.4 Linear Inequalities In Two Variables

DEFINITIONS AND CONCEPTS	EXAMPLES
The graph of a **linear inequality in x and y** is the graph of all ordered pairs (x, y) whose coordinates satisfy the inequality. To **graph a linear inequality in x and y,** graph the **boundary line.** Draw a solid boundary line if the inequality has $\le$ or $\ge$. Draw a dashed line if the inequality has $<$ or $>$. Then use a **test point** to decide which side of the boundary should be shaded. If the inequality is satisfied, shade the side that contains the test point. If the inequality is not satisfied, shade the other side of the boundary.	Graph: $y < 3x + 2$ To find the boundary line, we graph $y = 3x + 2$. Since the inequality symbol $<$ does not include an $=$ symbol, points on the boundary line are not included in the graph, and we draw a dashed boundary line. To find which half-plane is the graph of $y < 3x + 2$, we choose the origin $(0, 0)$ as the test point and see whether its coordinates satisfy the inequality. $y < 3x + 2$ This is the original inequality. $0 < 3(0) + 2$ Substitute 0 for x and 0 for y. $0 < 2$ True Since the coordinates of the origin satisfy the inequality, the origin is in the graph. In fact, the coordinates of every point on the same side of the boundary line as the origin satisfy the inequality. We then shade that half-plane to complete the graph. 

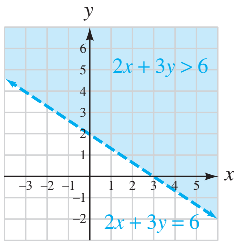
REVIEW EXERCISES

51. Determine whether $(-1, -4)$ is a solution of the linear inequality $6x - 4y \geq 15$. no

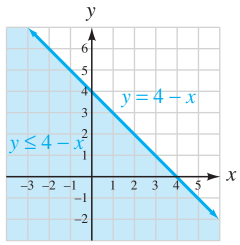
52. Does the graph of $6x - 4y \geq 15$ include the boundary line? yes

Graph each inequality in the rectangular coordinate system.

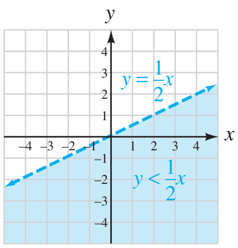
53. $2x + 3y > 6$



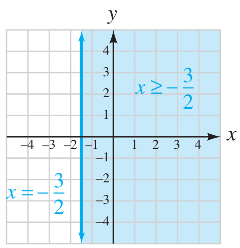
54. $y \leq 4 - x$



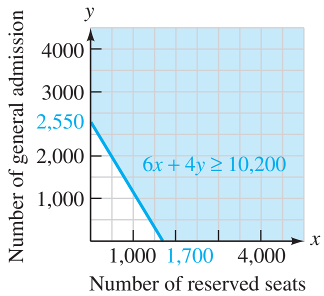
55. $y < \frac{1}{2}x$



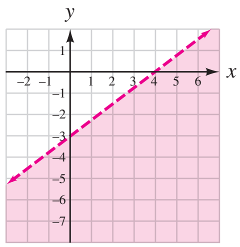
56. $x \geq -\frac{3}{2}$



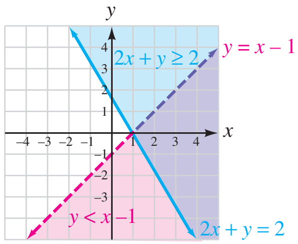
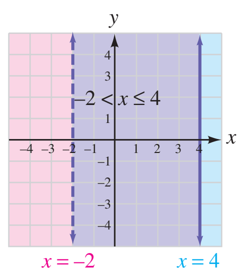
57. CONCERT TICKETS Tickets to a concert cost \$6 for reserved seats and \$4 for general admission. If receipts must be at least \$10,200 to meet expenses, find an inequality that shows the possible combinations of the number of reserved seats (x) and the number of general admission tickets (y) that the box office can sell. Then graph the inequality for nonnegative values of x and y and give three ordered pairs that satisfy the inequality.
$6x + 4y \geq 10{,}200$; (1,800, 0), (1,000, 1,500), (2,000, 2,000)



58. Find the equation of the boundary line. Then give the inequality whose graph is shown. $3x - 4y > 12$

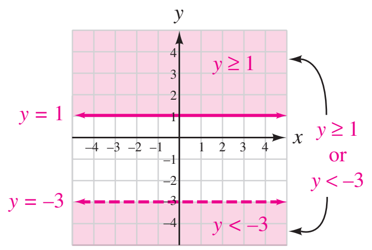


SECTION 4.5 Systems of Linear Inequalities

DEFINITIONS AND CONCEPTS	EXAMPLES
To **solve a system of linear inequalities,** graph each of the inequalities on the same rectangular coordinate system and look for the intersection of the shaded half-planes. The area that is shaded twice represents the solutions of the given system.	Graph the solution set of: $\begin{cases} y < x - 1 \\ 2x + y \geq 2 \end{cases}$ The graph of $y < x - 1$ is shaded in red. The graph of $2x + y \geq 2$ is shaded in blue. Any point in the doubly shaded region has coordinates that satisfy both inequalities. 
Compound inequalities can be graphed in the rectangular coordinate system.	Graph: $-2 < x \leq 4$ The graph of this double inequality contains all points in the plane that satisfy the inequalities $-2 < x$ and $x \leq 4$. These points are in the purple-shaded region of the figure. 

Graph: $y \geq 1$ or $y < -3$

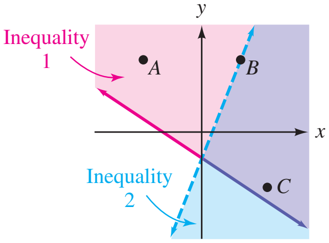
The graph of this compound inequality contains all the points in the plane that satisfy one or the other inequalities.



REVIEW EXERCISES

59. Determine whether $(1, -2)$ is a solution of the system of linear inequalities $\begin{cases} y \leq -x + 1 \\ 2x - y > 2 \end{cases}$. yes

60. In the illustration, the solution of one linear inequality is shaded in red, and the solution of a second is shaded in blue. Determine whether a true or false statement results if the coordinates of the given point are substituted into the given inequality.

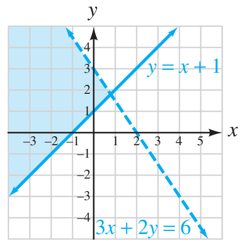


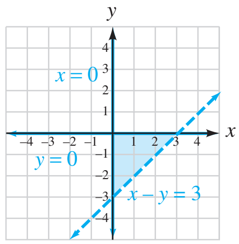
a. A, inequality 1 true
b. A, inequality 2 false
c. B, inequality 1 true
d. B, inequality 2 false
e. C, inequality 1 true
f. C, inequality 2 true

Graph the solution set of each system of inequalities.

61. $\begin{cases} y \geq x + 1 \\ 3x + 2y < 6 \end{cases}$

62. $\begin{cases} x - y < 3 \\ y \leq 0 \\ x \geq 0 \end{cases}$

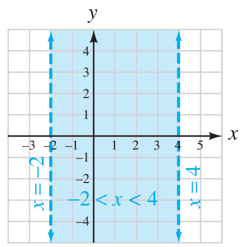


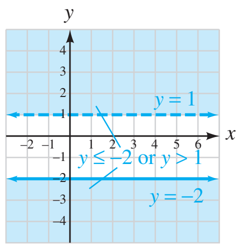


Graph each compound inequality in the rectangular coordinate system.

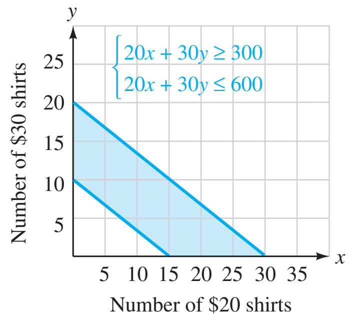
63. $-2 < x < 4$

64. $y \leq -2$ or $y > 1$



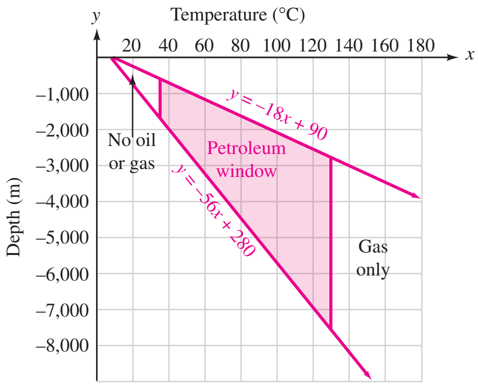


65. INVENTORY A men's clothing store carries shirts that sell for \$20 and some that sell for \$30. The store manager wants to stock at least \$300 but no more than \$600 in shirts at the store. Graph a system of inequalities that will show the possible combinations of the number of \$20 shirts (x) and the number of \$30 shirts (y) that the store can stock. Give two possible solutions.



5 \$20 shirts and 15 \$30 shirts, 15 \$20 shirts and 10 \$30 shirts

66. PETROLEUM EXPLORATION Organic matter converts to oil and gas within a specific range of temperature and depth called the *petroleum window*. The petroleum window shown can be described by a system of linear inequalities, where x is the temperature in °C of the soil at a depth of y meters. Determine what inequality symbol should be inserted in each blank.



Based on data from *The Blue Planet* (Wiley, 1995)

$$\begin{cases} x \geq 35 \\ x \leq 130 \\ y \geq -56x + 280 \\ y \leq -18x + 90 \end{cases}$$

CHAPTER 4 TEST

1. Fill in the blanks.
   a. $<, >, \leq,$ and $\geq$ are inequality symbols.
   b. ∞ is a symbol representing infinity.
   c. $x + 1 > 2$ or $2x - 3 \leq 8$ is a compound inequality.
   d. We read $\cup$ as union and $\cap$ as intersection.
   e. $\begin{cases} x + y > 10 \\ 3x - 2y \geq 4 \end{cases}$ is a system of linear inequalities in two variables.
2. Determine whether -2 is a solution of the inequality:

   $3(x - 2) \leq 2(x + 7)$ yes

Solve each inequality. Graph the solution set and write it using interval notation.

3. $\frac{2}{3}t - 1 > 7$

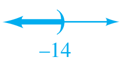
   $(12, \infty)$

4. $-2(2x + 3) \geq 14$

   $(-\infty, -5]$

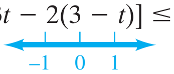
5. $\frac{x}{4} - \frac{1}{3} > \frac{5}{6} + \frac{x}{3}$

   $(-\infty, -14)$



6. $4 - 4[3t - 2(3 - t)] \leq -15t - (5t - 28)$

   $(-\infty, \infty)$



7. AVERAGING GRADES Use the information from the gradebook to determine what score Karen Nelson-Sims needs on the fifth exam so that her exam average exceeds 80. more than 78

Sociology 101 8:00-10:00 pm MW	Exam 1	Exam 2	Exam 3	Exam 4	Exam 5
Nelson-Sims, Karen	70	79	85	88	

8. INSURANCE COVERAGE A fire damage restoration crew charges \$175 for the first hour of cleanup and \$80 for each additional hour or part thereof. For how long can the crew work at a home with smoke and water damage if the homeowner's insurance policy will only cover up to \$1,000 of the cleanup cost? 11 hr
9. Determine whether 4 is a solution of the compound inequality $x + 6 \geq 10$ and $3x - 8 > 4$. no

Let $A = \{-4, 0, 7, 8, 9, 11\}$ and $B = \{-5, -4, 0, 10, 11\}$.

10. Find $A \cap B$. $\{-4, 0, 11\}$
11. Find $A \cup B$. $\{-5, -4, 0, 7, 8, 9, 10, 11\}$
12. Graph each set.
    a. $(-3, 6) \cup [5, \infty)$
    b. $[-2, 7] \cap (-\infty, 1)$

Solve each compound inequality. Give the result in interval notation, if possible, and graph the solution set.

13. $3x \geq -2x + 5$ and $7 \geq 4x - 2$

    $\left[1, \frac{9}{4}\right]$

14. $3x < -9$ or $-\frac{x}{4} < -2$

    $(-\infty, -3) \cup (8, \infty)$

15. $-2 < \frac{x - 4}{3} < 4$

    $(-2, 16)$

16. $\frac{4}{5}(x + 1) > 1$ and $-(0.3x + 1.5) > 2.9 - 0.2x$

    no solution, $\varnothing$

Solve each equation.

17. $|4 - 3x| = 19$ $-5, \frac{23}{3}$
18. $|3x + 4| = |x + 12|$ $4, -4$
19. $10 = 4\left|\frac{3x}{8} - \frac{3x}{2}\right| + 6$ $\frac{8}{9}, -\frac{8}{9}$
20. $|16x| = -16$ no solution, $\varnothing$
21. $5|20 - 2x| = 0$ 10
22. ALUMINUM PRODUCTION A sheet of aluminum is to be 0.0625 inch thick, with a tolerance of 0.0015 inch. Write and then solve an absolute value inequality that describes this specification, using x to represent the thickness of a sheet of aluminum. $|x - 0.0625| \leq 0.0015$; $[0.0610, 0.0640]$

Solve each inequality. Graph the solution set and write it using interval notation.

23. $|x + 3| \leq 4$

    $[-7, 1]$

24. $\left|\frac{x - 2}{2}\right| > 5.5$

    $(-\infty, -9) \cup (13, \infty)$

25. $|4 - 2x| + 48 > 50$
$(-\infty, 1) \cup (3, \infty)$

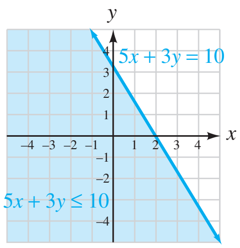
26. $2|3(x - 2)| \le 4$
$\left[\frac{4}{3}, \frac{8}{3}\right]$

27. $|4.5x - 0.9| \ge -0.7$
$(-\infty, \infty)$, $\mathbb{R}$

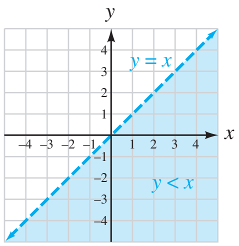
28. Let $f(x) = |2x + 9|$. For what value(s) of x is $f(x) < 3$? $(-6, -3)$

Graph each solution set on a rectangular coordinate system.

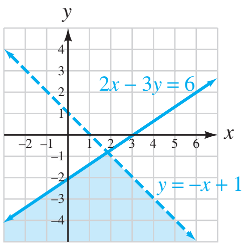
29. $5x + 3y \le 10$



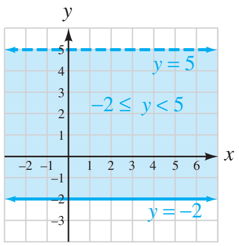
30. $y < x$



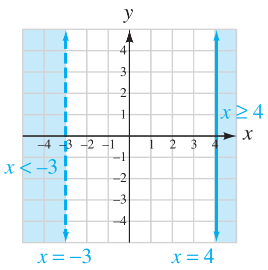
31. $\begin{cases} 2x - 3y \ge 6 \\ y < -x + 1 \end{cases}$



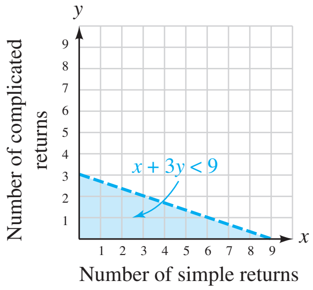
32. $-2 \le y < 5$



33. $x < -3$ or $x \ge 4$

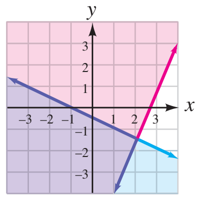


34. ACCOUNTING On average, it takes an accountant 1 hour to complete a simple tax return and 3 hours to complete a complicated return. If the accountant wants to work less than 9 hours per day, find an inequality that shows the possible combinations of the number of simple returns (x) and the number of complicated returns (y) that can be completed each day. Then graph the inequality and give three ordered pairs that satisfy it. (1, 1), (2, 1), (2, 2); answers may vary



35. Two linear inequalities are graphed on the same coordinate axes in the illustration. The solution set of the first inequality is shaded in red, and the solution set of the second in blue.

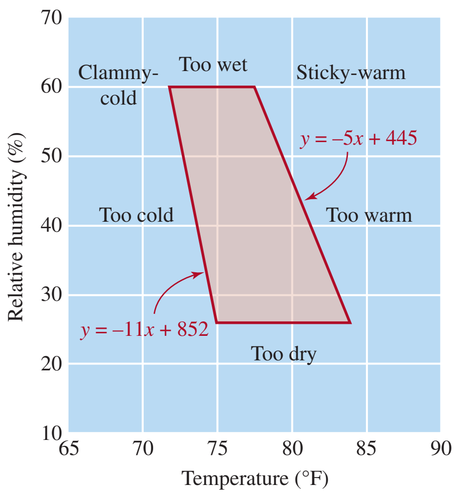
a. Determine from the graph whether $(3, -4)$ is a solution of either inequality.
$(3, -4)$ is a solution of inequality 2.



b. Is $(3, -4)$ a solution of the system of two linear inequalities? Explain your answer.
No, it does not lie in the doubly shaded region.

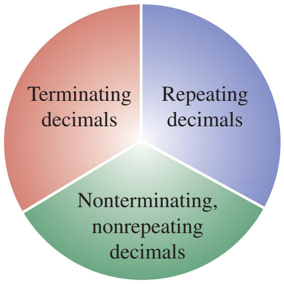
36. INDOOR CLIMATES The general zone of comfort acceptable to most people when working in an office can be described by a system of linear inequalities where x is the dry bulb temperature and y is the percent relative humidity. See the illustration. Determine what inequality symbol should be inserted in each blank.

$$\begin{cases} y \ \boxed{\le}\ 60 \\ y \ \boxed{\ge}\ 27 \\ y \ \boxed{\ge}\ -11x + 852 \\ y \ \boxed{\le}\ -5x + 445 \end{cases}$$

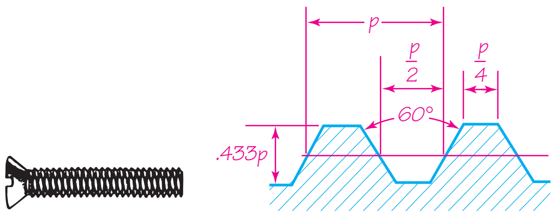


CHAPTERS 1–4 CUMULATIVE REVIEW

1. The diagram shows the sets that compose the set of real numbers. Which of the indicated sets make up the *rational numbers* and the *irrational numbers?* [Section 1.2] rational numbers: terminating and repeating decimals; irrational numbers: nonterminating, nonrepeating decimals



2. HARDWARE The thread profile of a screw is determined by the distance between threads. This distance, indicated by the letter p, is known as the *pitch.* If $p = 0.125$, find each of the dimensions labeled in the illustration. [Section 1.3] 0.125, 0.0625, 0.03125, 0.054125



Evaluate each expression for $x = 2$ and $y = -4$.

3. $|x| - xy$ [Section 1.3] 10

4. $\dfrac{x^2 - y^2}{3x + y}$ [Section 1.3] -6

Simplify each expression.

5. $-(a + 2) - (a - b)$ [Section 1.4] $-2a + b - 2$

6. $36\left(\dfrac{2}{9}t - \dfrac{3}{4}\right) + 36\left(\dfrac{1}{2}\right)$ [Section 1.4] $8t - 9$

Solve each equation, if possible.

7. $6(x - 1) = 2(x + 3)$ [Section 1.5] 3

8. $\dfrac{5b}{2} - 10 = \dfrac{b}{3} + 3$ [Section 1.5] 6

9. $2a - 5 = -2a + 4(a - 2) + 1$ [Section 1.5] no solution, $\varnothing$

10. $\dfrac{2z + 3}{3} + \dfrac{3z - 4}{6} = \dfrac{z - 2}{2}$ [Section 1.5] -2

11. Solve $l = a + (n - 1)d$ for d. [Section 1.5] $d = \frac{l - a}{n - 1}$

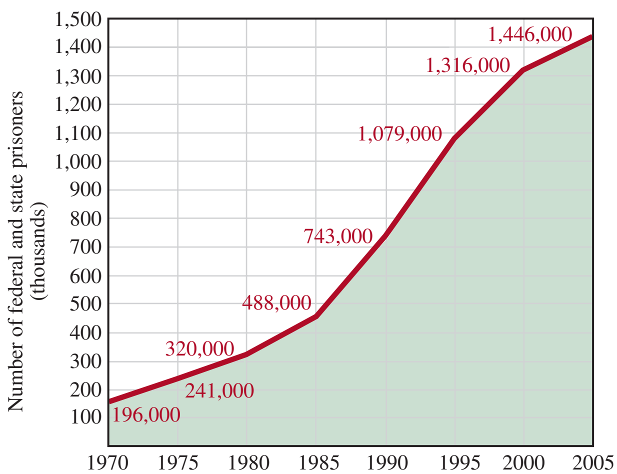
12. PLASTIC WRAP Estimate the number of *square feet* of plastic wrap on a roll if the dimensions printed on the box describe the roll as 205 feet long by $11\frac{3}{4}$ inches wide. [Section 1.6] 201 ft^2

13. INVESTMENTS Find the amount of money that was invested at $8\frac{7}{8}\%$ if it earned \$1,775 in simple interest in one year. [Section 1.6] \$20,000

14. MARATHONS Two marathon runners leave the starting gate, one running 12 mph and the other 10 mph. If they maintain the pace, how long will it take for them to be one-half of a mile apart? [Section 1.8] $\frac{1}{4}$ hr

15. Find the slope of the line that passes through $(0, -8)$ and $(-5, 0)$. [Section 2.3] $-\frac{8}{5}$

16. PRISONS The following graph shows the growth of the U.S. prison population from 1970 to 2005.
   a. Find the rate of change in the prison population from 2000 to 2005. [Section 2.3] 26,000 prisoners/yr
   b. During what five-year period was the rate of change in the U.S. prison population the greatest? Find the rate of change. [Section 2.3] 1990–1995, 67,200 prisoners/yr



Source: U.S. Bureau of Justice Statistics

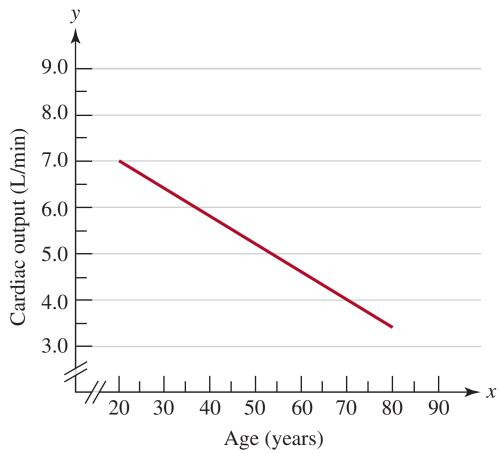
17. Determine whether the lines represented by the equations $3x = y + 4$ and $y = 3(x - 4) - 1$ are parallel, perpendicular, or neither. [Section 2.4] parallel

18. Find an equation of the line that passes through $(-2, 3)$ and is perpendicular to the graph of $3x + y = 8$. Write the equation in slope–intercept form. [Section 2.4] $y = \frac{1}{3}x + \frac{11}{3}$

19. AGING The graph below shows the effects of aging on cardiac output (the amount of blood that the heart can pump in one minute).

a. Write the equation of the line.
[Section 2.4] $y = -0.06x + 8.2$

b. Use your answer to part a to determine the cardiac output at age 90.
[Section 2.4] 2.8 L/min



Based on data from *Cardiopulmonary Anatomy and Physiology, Essentials for Respiratory Care*, 2nd ed. (Delmar Publishers, 1994)

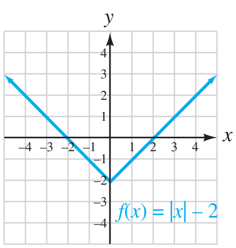
20. Find the domain and range of the relation:
$\{(5, 6), (-12, 4), (8, 6), (-6, -6), (5, 4)\}$
[Section 2.5] D: $\{-12, -6, 5, 8\}$, R: $\{-6, 4, 6\}$

Let $f(x) = 3x^2 - x$. Find each of the following.

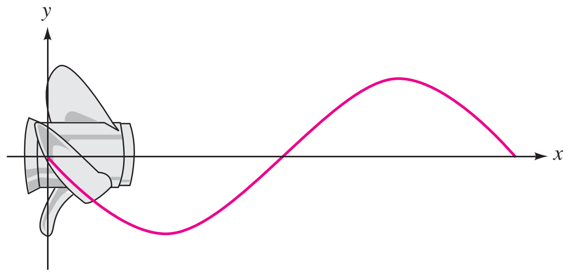
21. $f(-2)$ [Section 2.5] 14

22. $f(t)$ [Section 2.5] $3t^2 - t$

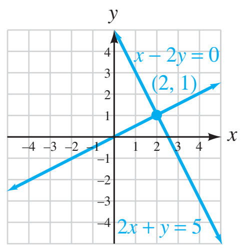
23. Graph $f(x) = |x| - 2$ and give its domain and range.
[Section 2.6]
D: the set of real numbers, R: the set of all real numbers greater than or equal to -2



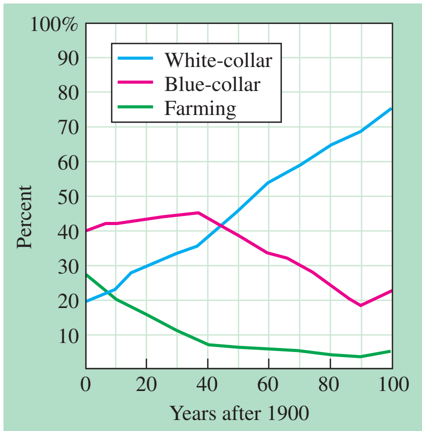
24. BOATING The graph in the illustration shows the vertical distance from a point on the tip of a propeller to the centerline as the propeller spins. Is this the graph of a function? [Section 2.6] yes



25. Use graphing to solve:
$$\begin{cases} 2x + y = 5 \\ x - 2y = 0 \end{cases}$$
[Section 3.1]
(2, 1)



26. U.S. WORKERS The graph below shows how the makeup of the U.S. workforce changed over the years 1900–2000. Estimate the coordinates of the points of intersection in the graph. Explain their significance.
[Section 3.1]
(7, 23), in 1907 the percent of U.S. workers in white-collar and farming jobs was the same (23%); (45, 42), in 1945 the percent of U.S. workers in white-collar and blue-collar jobs was the same (42%).



Source: *U.S. Statistical Abstract*

27. Use elimination to solve: $\begin{cases} \frac{x}{10} + \frac{y}{5} = \frac{1}{2} \\ \frac{x}{2} - \frac{y}{5} = \frac{13}{10} \end{cases}$ [Section 3.2] (3, 1)

28. Use substitution to solve: $\begin{cases} 3x = 4 - y \\ 4x - 3y = -1 + 2x \end{cases}$

[Section 3.2] (1, 1)

29. ENTREPRENEURS A person invests \$18,375 to set up a small business producing a piece of computer software that will sell for \$29.95. If each piece can be produced for \$5.45, how many pieces must be sold to break even? [Section 3.3] 750

30. Solve: $\begin{cases} x + y + z = 1 \\ 2x - y - z = -4 \\ x - 2y + z = 4 \end{cases}$ [Section 3.4] $(-1, -1, 3)$

31. CONCERT TICKETS Tickets for a concert cost \$5, \$3, and \$2. Twice as many \$5 tickets were sold as \$2 tickets. The receipts for 750 tickets were \$2,625. How many tickets were sold at each price?

[Section 3.5] 250 \$5 tickets, 375 \$3 tickets, 125 \$2 tickets

32. Use matrices to solve the system.

$$\begin{cases} 4x - 3y = -1 \\ 3x + 4y = -7 \end{cases}$$ [Section 3.6] $(-1, -1)$

33. Evaluate the determinant: $\begin{vmatrix} -9 & 7 \\ 4 & -2 \end{vmatrix}$

[Section 3.7] -10

34. Use Cramer's rule to solve the system.

$$\begin{cases} 5x + 2y = 11 \\ 7x + 6y = 9 \end{cases}$$ [Section 3.7] $(3, -2)$

Solve each inequality. Graph the solution set and write it using interval notation and set-builder notation.

35. $-3(x - 4) \geq x - 32$ [Section 4.1]

$(-\infty, 11]$, $\{x \mid x \leq 11\}$

36. $-8 < -3x + 1 < 10$ [Section 4.2]

$(-3, 3)$, $\{x \mid -3 < x < 3\}$

Solve each compound inequality. Graph the solution set and write it using interval notation.

37. $3x + 2 < 8$ or $2x - 3 > 11$ [Section 4.2]

$(-\infty, 2) \cup (7, \infty)$

38. $5x - 3 \geq 2$ and $6 \geq 4x - 3$ [Section 4.2]

$\left[1, \frac{9}{4}\right]$

Solve each equation.

39. $2|4x - 3| + 1 = 19$ [Section 4.3] $3, -\frac{3}{2}$

40. $|2x - 1| = |3x + 4|$ [Section 4.3] $-5, -\frac{3}{5}$

Solve each inequality. Graph the solution set and write it using interval notation.

41. $|3x - 2| \leq 4$ [Section 4.3]

$\left[-\frac{2}{3}, 2\right]$

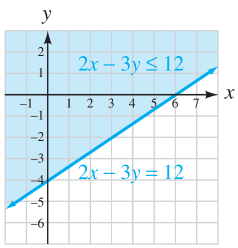
42. $|2x + 3| - 1 > 4$ [Section 4.3]

$(-\infty, -4) \cup (1, \infty)$

Graph the solution set.

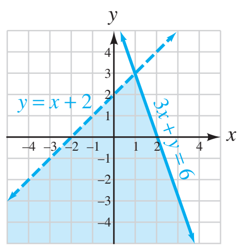
43. $2x - 3y \leq 12$

[Section 4.4]



44. $\begin{cases} y < x + 2 \\ 3x + y \leq 6 \end{cases}$

[Section 4.5]



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Exponents, Polynomials, and Polynomial Functions

5

from Campus to Careers

Landscape Architect

Whether it's a community park, a college campus, or simply someone's backyard, landscape architects are skilled at creating outdoor areas that are both functional and beautiful. They use algebra and geometry to prepare working drawings, design scale models, and estimate costs. Throughout the planning and construction phases, they make computations to find everything from drainage slopes and sunlight angles to walkway elevations.

In **Problem 65** of **Study Set 5.9,** you will determine the dimensions of a concrete walkway around a fountain.

JOB TITLE:
Landscape Architect
EDUCATION: A bachelor's degree in landscape architecture and some experience is required.
JOB OUTLOOK: Excellent; it is expected to increase 18% to 26% through 2014.
ANNUAL EARNINGS: The median salary in 2007 was $67,862.
FOR MORE INFORMATION:
www.bls.gov/oco/ocos039.htm

Objectives

1. Identify bases and exponents.
2. Use the product and power rules for exponents.
3. Use the zero and the negative integer exponent rules.
4. Use the quotient rule for exponents.
5. Simplify quotients raised to negative powers.

SECTION 5.1
Exponents

In Chapter 1, we evaluated exponential expressions having natural-number exponents. In this section, we will extend the definition of exponent to include negative-integer exponents, as in 3^{-2}, and zero exponents, as in 3^0. We will also develop several rules that can be used to simplify expressions.

1 Identify bases and exponents.

Exponents provide a way to write products of repeated factors in a concise form. For example,

$y \cdot y = y^2$ Read y^2 as "y to the second power" or "y squared."

$z \cdot z \cdot z = z^3$ Read z^3 as "z to the third power" or "z cubed."

$x \cdot x \cdot x \cdot x = x^4$ Read x^4 as "x to the fourth power."

These examples illustrate the following definition.

Natural-Number Exponents

If n represents a natural number, then

$$x^n = \overbrace{x \cdot x \cdot x \cdot \cdots \cdot x}^{n \text{ factors of } x}$$

The **exponential expression** x^n is called a **power of x,** and we read it as "x to the nth power." In this expression, x is called the **base,** and n is called the **exponent.**

Base $\rightarrow x^n \leftarrow$ Exponent

A natural-number exponent tells how many times the base of an exponential expression is to be used as a factor in a product. When $n = 1$, the exponent is usually omitted. For example, $x^1 = x$.

Self Check 1

Identify the base and the exponent in each expression:

a. $(-kt)^4$ **b.** πr^2 **c.** $-h^8$

Now Try **Problems 30 and 31**

Self Check 1 Answers

a. $-kt$, 4 **b.** r, 2 **c.** h, 8

Teaching Example 1 Identify the base and the exponent in each expression:

a. $(-rs)^3$ **b.** $9y^4$ **c.** $-a^7$

Answers:

a. $-rs$, 3 **b.** y, 4 **c.** a, 7

EXAMPLE 1 Identify the base and the exponent in each expression:

a. $(-a)^2$ **b.** $-a^2$ **c.** $5x^3$ **d.** $(5x)^3$

Strategy To identify the base and the exponent, we will look for the form $\square^{\square}$.

WHY The exponent is the small raised number to the right of the base.

Solution

a. For $(-a)^2$, $-a$ is the base and the exponent is 2: $(-a)^2 = (-a)(-a)$.

b. For $-a^2$, the base is a and the exponent is 2: $-a^2 = -(a \cdot a)$.

c. For $5x^3$, x is the base and the exponent is 3: $5x^3 = 5 \cdot x \cdot x \cdot x$.

d. For $(5x)^3$, the base is $5x$ and the exponent is 3: $(5x)^3 = (5x)(5x)(5x)$.

2 Use the product and power rules for exponents.

Several rules for exponents come from the definition of exponent. The first rule gives a way to find the result when multiplying exponential expressions that have the same base.

Since x^5 means that x is to be used as a factor five times, and since x^3 means that x is to be used as a factor three times, $x^5 \cdot x^3$ means that x will be used as a factor eight times.

$$x^5x^3 = \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{\text{5 factors of } x} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors of } x} = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{\text{8 factors of } x} = x^8$$

In general,

$$x^mx^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{m \text{ factors of } x} \cdot \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors of } x} = \underbrace{x \cdot x \cdot x \cdot x \cdot \cdots \cdot x}_{m + n \text{ factors of } x} = x^{m+n}$$

This result is called the **product rule for exponents.**

The Product Rule for Exponents

To multiply exponential expressions with the same base, keep the common base and add the exponents.

For any real number x and any natural numbers m and n,

$$x^mx^n = x^{m+n}$$

EXAMPLE 2

Simplify each expression:

a. $x^{11}x^5$ **b.** y^5y^4y **c.** $a^2b^3a^3b^2$ **d.** $-8x^4(x^3)$

Strategy Since there are products of the form x^mx^n in each of these expressions, we will use the product rule for exponents and keep the common base and add the exponents.

WHY We use the product rule to multiply exponential expressions with the same base.

Solution

a. $x^{11}x^5 = x^{11+5}$ Keep the common base x.

$= x^{16}$ Add the exponents.

b. $y^5y^4y = (y^5y^4)y$

$= y^9y^1$ $y = y^1$.

$= y^{10}$

c. $a^2b^3a^3b^2 = a^2a^3b^3b^2$

$= a^5b^5$

d. $-8x^4(x^3) = -8(x^4x^3)$

$= -8x^7$

Self Check 2

Simplify each expression:

a. 2^32^5 $2^8 = 256$

b. $k \cdot k^4$ k^5

c. $a^2b^3a^3b^4$ a^5b^7

d. $-8a^4(a^2b)$ $-8a^6b$

Now Try **Problems 36 and 40**

Teaching Example 2 Simplify each expression:

a. 7^37^2 **b.** $x \cdot x^8$

c. $r^2s^6r^5s^4$ **d.** $-6x^3(x^2y^4)$

Answers:

a. $7^5 = 16{,}807$ **b.** x^9

c. r^7s^{10} **d.** $-6x^5y^4$

Caution! The product rule for exponents applies only to exponential expressions with the same base. The expression x^5y^3, for example, cannot be simplified, because the bases of the exponential expressions are different.

To find another rule for exponents, we simplify $(x^4)^3$, which means x^4 cubed or $x^4 \cdot x^4 \cdot x^4$.

$$(x^4)^3 = x^4 \cdot x^4 \cdot x^4 = \underbrace{x \cdot x \cdot x \cdot x}_{x^4} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{x^4} \cdot \underbrace{x \cdot x \cdot x \cdot x}_{x^4} = x^{12}$$

In general, we have

$$(x^m)^n = \underbrace{x^m \cdot x^m \cdot x^m \cdot \cdots \cdot x^m}_{n \text{ factors of } x^m} = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot \cdots \cdot x}_{mn \text{ factors of } x} = x^{mn}$$

Power Rule for Exponents

To raise an exponential expression to a power, keep the base and multiply the exponents.

For any real number x and natural numbers m and n

$$(x^m)^n = x^{m \cdot n} = x^{mn}$$

Self Check 3

Simplify each expression:

a. $(a^5)^8$ a^{40}
b. $(6^3)^5$ 6^{15}
c. $(a^4a^3)^3$ a^{21}
d. $(a^3)^3(a^2)^3$ a^{15}

Now Try **Problems 48 and 49**

Teaching Example 3 Simplify each expression:
a. $(7^3)^8$ **b.** $(r^3)^5$
c. $(r^3r^4)^5$ **b.** $(rr^2)^4(r^3)^6$
Answers:
a. 7^{24} **b.** r^{15}
c. r^{35} **d.** r^{30}

EXAMPLE 3 Simplify each expression:
a. $(3^2)^3$ **b.** $(x^{11})^5$ **c.** $(x^2x^3)^6$ **d.** $(x^2)^4(x^3)^2$

Strategy Since there are powers of the form $(x^m)^n$ in each of these expressions, we will use the power rule for exponents and keep the base and multiply the exponents to simplify them.

WHY We use the power rule to raise an exponential expression to a power.

Solution

a. $(3^2)^3 = 3^{2\cdot 3}$ Keep the base 3.
$= 3^6$ Multiply the exponents.
$= 729$

b. $(x^{11})^5 = x^{11\cdot 5}$
$= x^{55}$

c. $(x^2x^3)^6 = (x^5)^6$
$= x^{30}$

d. $(x^2)^4(x^3)^2 = x^8x^6$
$= x^{14}$

To develop a third rule for exponents, we square $3x$ to get

$$(3x)^2 = (3x)(3x) = 3 \cdot 3 \cdot x \cdot x = 3^2x^2 = 9x^2$$

In general, we have

$$(xy)^n = \overbrace{(xy)(xy)(xy)\cdot\cdots\cdot(xy)}^{n \text{ factors of } xy} = \overbrace{xxx\cdot\cdots\cdot x}^{n \text{ factors of } x} \cdot \overbrace{yyy\cdot\cdots\cdot y}^{n \text{ factors of } y} = x^ny^n$$

To find a fourth rule for exponents, we cube $\frac{x}{3}$ to get

$$\left(\frac{x}{3}\right)^3 = \frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3} = \frac{x\cdot x\cdot x}{3\cdot 3\cdot 3} = \frac{x^3}{3^3} = \frac{x^3}{27}$$

In general, we have

$$\left(\frac{x}{y}\right)^n = \overbrace{\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\cdot\cdots\cdot\left(\frac{x}{y}\right)}^{n \text{ factors of } \frac{x}{y}} \quad (y \neq 0)$$

$$= \frac{\overbrace{xxx\cdot\cdots\cdot x}^{n \text{ factors of } x}}{\underbrace{yyy\cdot\cdots\cdot y}_{n \text{ factors of } y}} \quad \text{Multiply the numerators and multiply the denominators.}$$

$$= \frac{x^n}{y^n}$$

The previous results are called the *power of a product* and the *power of a quotient* rules.

Powers of a Product and a Quotient

To raise a product to a power, raise each factor of the product to that power. To raise a quotient to a power, raise the numerator and denominator to that power.

For any real numbers x and y, and any natural number n,

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \quad \text{where} \quad y \neq 0$$

EXAMPLE 4 Simplify each expression. Assume that no denominators are zero.

a. $(x^2y)^3$ **b.** $(x^3y^4)^4$ **c.** $\left(\frac{x}{y^2}\right)^4$ **d.** $\left(\frac{6x^3}{5y^4}\right)^2$

Strategy Since these expressions have the form $(xy)^n$ and $\left(\frac{x}{y}\right)^n$, we will use the power of a product rule and the power of a quotient rule to simplify them.

WHY We use the power of a product rule to raise a product to a power and the power of a quotient rule is used to raise a quotient to a power.

Solution

a. $(x^2y)^3 = (x^2)^3y^3$ Raise each factor of the product x^2y to the 3rd power.

$= x^6y^3$

b. $(x^3y^4)^4 = (x^3)^4(y^4)^4$

$= x^{12}y^{16}$

c. $\left(\frac{x}{y^2}\right)^4 = \frac{x^4}{(y^2)^4}$ Raise the numerator and denominator to the 4th power.

$= \frac{x^4}{y^8}$

d. $\left(\frac{6x^3}{5y^4}\right)^2 = \frac{6^2(x^3)^2}{5^2(y^4)^2}$

$= \frac{36x^6}{25y^8}$

Self Check 4

Simplify each expression:

a. $(a^4b^5)^2$ a^8b^{10}

b. $\left(\frac{-6a^5}{b^7}\right)^3$ $-\frac{216a^{15}}{b^{21}}$

Now Try **Problems 54 and 58**

Teaching Example 4 Simplify each expression:
a. $(x^4y^7)^3$ **b.** $(x^3y)^5$
c. $\left(\frac{-5x^2}{y^3}\right)^3$
Answers:
a. $x^{12}y^{21}$ **b.** $x^{15}y^5$ **c.** $-\frac{125x^6}{y^9}$

3 Use the zero and the negative integer exponent rules.

Since we want the rules for exponents to hold for exponents of 0, we have

$$x^0x^n = x^{0+n} = x^n = 1x^n$$

Because $x^0x^n = 1x^n$, it follows that $x^0 = 1$ where $x \neq 0$. In words, *a nonzero base raised to the 0 power is 1.*

Zero Exponents

A nonzero base raised to the 0 power is 1.

For any nonzero base x,

$$x^0 = 1$$

Caution! 0^0 is undefined.

Because of the previous definition, any nonzero base raised to the 0th power is 1. For example, if no variables are zero, then

$$3^0 = 1, \qquad (-7)^0 = 1, \qquad (3ax^3)^0 = 1, \qquad \left(\frac{1}{2}x^5y^7z^9\right)^0 = 1$$

Self Check 5

Simplify each expression:

a. $2xy^0$ $2x$

b. $-(xy)^0$ -1

Now Try **Problems 62 and 66**

Teaching Example 5 Simplify each expression:
a. $(ab)^0$ **b.** $-x^0$ **c.** $(-3)^0$ **d.** -3^0 **e.** $7x^0$
Answers:
a. 1 **b.** -1 **c.** 1 **d.** -1 **e.** 7

EXAMPLE 5 Simplify each expression: **a.** $(5x)^0$ **b.** $5x^0$ **c.** $-5x^0y$

Strategy Since there are factors of the form x^0 in each expression, we will use the zero exponent rule to simplify them.

WHY Any nonzero base raised to the 0 power is 1.

Solution

a. $(5x)^0 = 1$ The base is $5x$ and the exponent is 0.

b. $5x^0 = 5 \cdot x^0 = 5 \cdot 1 = 5$ The base is x and the exponent is 0.

c. $-5x^0y = -5 \cdot x^0 \cdot y = -5 \cdot 1 \cdot y = -5y$

Since the rules for exponents are true for negative-integer exponents, we have

$$x^{-n}x^n = x^{-n+n} = x^0 = 1 \quad \text{where} \quad x \neq 0$$

Because $x^{-n} \cdot x^n = 1$ and $\frac{1}{x^n} \cdot x^n = 1$, we define x^{-n} *to be the reciprocal of* x^n.

Negative Exponents

For any nonzero number x and any integer n

$$x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n$$

In words, x^{-n} is the reciprocal of x^n.

Using this definition, we can write expressions containing negative exponents as expressions without negative exponents. For example,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9} \qquad 10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} \qquad \frac{1}{4^{-4}} = 4^4 = 256$$

and if b, c, and x are not 0, we have

$$(2c)^{-3} = \frac{1}{(2c)^3} = \frac{1}{8c^3} \qquad 3x^{-1} = 3 \cdot \frac{1}{x^1} = \frac{3}{x} \qquad \frac{7}{b^{-2}} = 7 \cdot \frac{1}{b^{-2}} = 7b^2$$

Caution! A negative exponent does not indicate a negative number. It indicates a reciprocal.

Self Check 6

Write each expression without negative exponents:

a. $-3.14t^{-7}$ $-\frac{3.14}{t^7}$

b. 7^{-2} $\frac{1}{49}$

Now Try **Problems 67 and 70**

Teaching Example 6 Write each expression without negative exponents:
a. -4^{-2} **b.** $8x^{-3}$
Answers:
a. $-\frac{1}{16}$ **b.** $\frac{8}{x^3}$

EXAMPLE 6 Write each expression without negative exponents:
a. 7^{-2} **b.** $-2m^{-8}$

Strategy Since there are factors of the form x^{-n} in each expression, we will use the negative integer exponent rule to write equivalent expressions with positive exponents.

WHY This rule enables us to rid these expressions of negative exponents.

Solution

a. $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$

b. $-2m^{-8} = -2 \cdot \frac{1}{m^8} = -\frac{2}{m^8}$ The base is m. The original exponent is -8. $m^{-8} = \frac{1}{m^8}$

The Language of Algebra By the definition of negative exponents, a base cannot be 0. Thus, an expression such as 0^{-5} is undefined.

EXAMPLE 7 Simplify each expression: **a.** $x^{-5}x^3$ **b.** $(x^{-3})^{-2}$

Strategy We will use the product rule and the power rule for exponents to simplify these expressions.

WHY The first expression has the form $x^m x^n$ and the second has the form $(x^m)^n$.

Solution

a. $x^{-5}x^3 = x^{-5+3}$ Use the product rule: keep the common base x and add the exponents.

$= x^{-2}$

$= \dfrac{1}{x^2}$

b. $(x^{-3})^{-2} = x^{(-3)(-2)}$ Use the power rule: keep the base x and multiply the exponents.

$= x^6$

Self Check 7

Simplify each expression:

a. $a^{-7}a^3$ $\frac{1}{a^4}$

b. $(a^{-5})^{-3}$ a^{15}

Now Try **Problems 72 and 74**

Teaching Example 7 Simplify each expression:
a. y^2y^{-8} **b.** $(y^{-4})^{-3}$ **c.** $(y^2)^{-3}$
Answers:
a. $\frac{1}{y^6}$ **b.** y^{12} **c.** $\frac{1}{y^6}$

Negative exponents can appear in the numerator and/or the denominator of a fraction. To develop rules to apply to such situations, consider the following example.

$$\frac{x^{-4}}{y^{-3}} = \frac{\frac{1}{x^4}}{\frac{1}{y^3}} = \frac{1}{x^4} \cdot \frac{y^3}{1} = \frac{y^3}{x^4}$$

We can obtain this result in a simpler way. Beginning with $\frac{x^{-4}}{y^{-3}}$, move x^{-4} to the denominator and change the sign of its exponent. Then move y^{-3} to the numerator and change the sign of its exponent.

$$\frac{x^{-4}}{y^{-3}} = \frac{y^3}{x^4}$$

This example illustrates the following rules.

Changing from Negative to Positive Exponents

A factor can be moved from the denominator to the numerator or from the numerator to the denominator of a fraction if the sign of its exponent is changed.

For any nonzero real numbers x and y, and any integers m and n,

$$\frac{1}{x^{-n}} = x^n \quad \text{and} \quad \frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$$

These rules streamline the process when simplifying fractions involving negative exponents.

Self Check 8

Simplify each expression. Write answers using positive exponents.

a. $\frac{1}{t^{-9}}$ t^9

b. $\frac{5^{-2}}{4^{-3}}$ $\frac{64}{25}$

c. $-\frac{h^{-6}}{8r^{-7}}$ $-\frac{r^7}{8h^6}$

Now Try **Problems 75, 77, and 82**

Teaching Example 8 Simplify each expression. Write answers using positive exponents.

a. $\frac{5}{x^{-3}}$ **b.** $\frac{4^{-2}}{3^{-1}}$ **c.** $-\frac{x^{-2}}{4y^{-5}}$

Answers:

a. $5x^3$ **b.** $\frac{3}{16}$ **c.** $-\frac{y^5}{4x^2}$

EXAMPLE 8 Simplify each expression. Write answers using positive exponents.

a. $\frac{1}{c^{-10}}$ **b.** $\frac{2^{-3}}{3^{-4}}$ **c.** $-\frac{s^{-2}}{5t^{-9}}$

Strategy Since these expressions have the form $\frac{1}{x^{-n}}$ or $\frac{x^{-m}}{y^{-n}}$, we will use the rule for changing exponents from negative to positive to write equivalent expressions with positive exponents only.

WHY This rule enables us to rid these expressions of negative exponents by moving factors with negative exponents to the other side of the fraction bar and changing the sign of the exponent to positive.

Solution

a. $\frac{1}{c^{-10}} = c^{10}$ Move c^{-10} to the numerator and change the sign of the exponent.

b. $\frac{2^{-3}}{3^{-4}} = \frac{3^4}{2^3}$ Move 2^{-3} to the denominator and change the sign of the exponent. Move 3^{-4} to the numerator and change the sign of the exponent.

$= \frac{81}{8}$ Evaluate 3^4 and 2^3.

c. $-\frac{s^{-2}}{5t^{-9}} = -\frac{t^9}{5s^2}$ Move s^{-2} to the denominator and change the sign of the exponent. Since $5t^{-9}$ has no parentheses, t is the base. Move t^{-9} to the numerator and change the sign of the exponent.

Caution! This rule does not allow us to move *terms* that have negative exponents. For example,

$$\frac{3^{-2}+8}{5} \neq \frac{8}{3^2 \cdot 5}$$

4 Use the quotient rule for exponents.

To develop a rule for dividing exponential expressions, we proceed as follows:

$$\frac{x^m}{x^n} = x^m\left(\frac{1}{x^n}\right) = x^m x^{-n} = x^{m+(-n)} = x^{m-n}$$

The result is called the **quotient rule for exponents.**

The Quotient Rule for Exponents

To divide exponential expressions with the same nonzero base, keep the common base and subtract the exponents.

For any nonzero number x and any integers m and n,

$$\frac{x^m}{x^n} = x^{m-n}$$

EXAMPLE 9 Simplify each expression. Write answers using positive exponents. **a.** $\frac{a^5}{a^3}$ **b.** $\frac{2x^{-5}}{x^{11}}$

Strategy We will use the quotient rule for exponents to simplify these expressions.

WHY The expressions have the form $\frac{x^m}{x^n}$.

Solution

a. $\frac{a^5}{a^3} = a^{5-3}$ Keep the common base a. Subtract the exponents.

$= a^2$

b. $\frac{2x^{-5}}{x^{11}} = 2x^{-5-11}$

$= 2x^{-16}$

$= \frac{2}{x^{16}}$

Sucess Tip We can also simplify $\frac{2x^{-5}}{x^{11}}$ by moving x^{-5} to the denominator and changing the sign of the exponent.

$$\frac{2x^{-5}}{x^{11}} = \frac{2}{x^{11}x^5} = \frac{2}{x^{16}}$$

Self Check 9

Simplify each expression:

a. $\frac{b^7}{b^5}$ b^2

b. $\frac{3b^{-3}}{b^3}$ $\frac{3}{b^6}$

Now Try **Problems 84 and 86**

Teaching Example 9 Simplify each expression:

a. $\frac{x^9}{x^5}$ **b.** $\frac{5x^{-2}}{x^9}$

Answers:

a. x^4 **b.** $\frac{5}{x^{11}}$

EXAMPLE 10 Simplify each expression. Write answers using positive exponents. **a.** $\frac{x^4x^3}{x^{-5}}$ **b.** $\frac{(x^2)^3}{(x^3)^2}$ **c.** $\frac{x^2y^3}{xy^4}$ **d.** $\left(\frac{2a^{-2}b^3}{3a^5b^4}\right)^3$

Strategy To simplify these expressions, we must use more than one rule for exponents.

WHY The expressions involve products, powers, and quotients of exponential expressions with the same base as well as negative exponents.

Solution

a. $\frac{x^4x^3}{x^{-5}} = \frac{x^7}{x^{-5}}$

$= x^{7-(-5)}$

$= x^{12}$

b. $\frac{(x^2)^3}{(x^3)^2} = \frac{x^6}{x^6}$

$= x^{6-6}$

$= x^0$

$= 1$

c. $\frac{x^2y^3}{xy^4} = x^{2-1}y^{3-4}$

$= xy^{-1}$

$= x \cdot \frac{1}{y}$

$= \frac{x}{y}$

d. $\left(\frac{2a^{-2}b^3}{3a^5b^4}\right)^3 = \left(\frac{2a^{-2-5}b^{3-4}}{3}\right)^3$

$= \left(\frac{2a^{-7}b^{-1}}{3}\right)^3$

$= \left(\frac{2}{3a^7b}\right)^3$

$= \frac{8}{27a^{21}b^3}$

Self Check 10

Simplify each expression:

a. $\frac{(a^{-2})^3}{(a^2)^{-3}}$ 1

b. $\left(\frac{a^{-2}b^5}{b^8}\right)^{-3}$ a^6b^9

Now Try **Problems 88, 90, and 92**

Teaching Example 10 Simplify each expression:

a. $\frac{x^{-3}x^{-2}}{x^{-4}}$ **b.** $\frac{(x^{-3})^2}{(x^2)^4}$ **c.** $\left(\frac{5x^{-2}y^3}{2xy^{-3}}\right)^2$

Answers:

a. $\frac{1}{x}$ **b.** $\frac{1}{x^{14}}$ **c.** $\frac{25y^{12}}{4x^6}$

5 Simplify quotients raised to negative powers.

To illustrate another rule for exponents, we consider the following simplification of $\left(\frac{2}{3}\right)^{-4}$.

$$\left(\frac{2}{3}\right)^{-4} = \frac{1}{\left(\frac{2}{3}\right)^4} = \frac{1}{\frac{2^4}{3^4}} = 1 \div \frac{2^4}{3^4} = 1 \cdot \frac{3^4}{2^4} = \frac{3^4}{2^4} = \left(\frac{3}{2}\right)^4$$

The example just shown illustrates that to raise a fraction to a negative power, we can invert the fraction and raise it to a positive power.

Fractions to Negative Powers

A fraction raised to a negative power is equal to the reciprocal of the fraction raised to the positive power.

For any nonzero real numbers x and y, and any integer n,

$$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$$

Self Check 11

Write $\left(\frac{3a^3b^2}{2aa^5b^{-2}}\right)^{-5}$ without using parentheses or negative exponents.

Now Try **Problems 96, 98, and 100**

Self Check 11 Answer

$\frac{32a^{15}}{243b^{20}}$

Teaching Example 11 Write $\left(\frac{5x^{-2}y^3}{3x^2y^{-4}}\right)^{-2}$ without using parentheses or negative exponents.

Answer: $\frac{9x^8}{25y^{14}}$

EXAMPLE 11

Write each expression without using parentheses or negative exponents: **a.** $\left(\frac{2}{3}\right)^{-4}$ **b.** $\left(\frac{y^2}{x^3}\right)^{-3}$ **c.** $\left(\frac{2x^2}{3y^{-3}}\right)^{-4}$ **d.** $\left(\frac{a^{-2}b^3}{a^2a^3b^4}\right)^{-3}$

Strategy To simplify these expressions, we must use more than one rule for exponents.

WHY The expressions involve fractions to negative powers as well as products, powers, and quotients of exponential expressions with the same base.

Solution

a.
$$\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^{4} = \frac{3^4}{2^4} = \frac{81}{16}$$

b.
$$\left(\frac{y^2}{x^3}\right)^{-3} = \left(\frac{x^3}{y^2}\right)^{3} = \frac{x^9}{y^6}$$

c.
$$\left(\frac{2x^2}{3y^{-3}}\right)^{-4} = \left(\frac{3y^{-3}}{2x^2}\right)^{4} = \frac{3^4y^{-12}}{2^4x^8} = \frac{81}{16x^8y^{12}}$$

d.
$$\left(\frac{a^{-2}b^3}{a^2a^3b^4}\right)^{-3} = \left(\frac{a^2a^3b^4}{a^{-2}b^3}\right)^{3} = \left(\frac{a^5b^4}{a^{-2}b^3}\right)^{3} = (a^{5-(-2)}b^{4-3})^3 = (a^7b)^3 = a^{21}b^3$$

Summary of Rules for Exponents

If m and n represent integers and there are no divisions by 0, then

Product rule	***Power rule***	***Power of a product***
$x^m \cdot x^n = x^{m+n}$	$(x^m)^n = x^{mn}$	$(xy)^n = x^ny^n$
Quotient rule	***Power of a quotient***	***Exponents of 0 and 1***
$\frac{x^m}{x^n} = x^{m-n}$	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$x^0 = 1$ and $x^1 = x$

Negative exponent

$$x^{-n} = \frac{1}{x^n}$$

Negative exponents appearing in fractions

$$\frac{1}{x^{-n}} = x^n \qquad \frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m} \qquad \left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^{n}$$

ANSWERS TO SELF CHECKS

1. a. $-kt, 4$ **b.** $r, 2$ **c.** $h, 8$ **2. a.** 256 **b.** k^5 **c.** a^5b^7 **d.** $-8a^6b$ **3. a.** a^{40} **b.** 6^{15} **c.** a^{21} **d.** a^{15} **4. a.** a^8b^{10} **b.** $-\frac{216a^{15}}{b^{21}}$ **5. a.** $2x$ **b.** -1 **6. a.** $-\frac{3.14}{t^7}$ **b.** $\frac{1}{49}$ **7. a.** $\frac{1}{a^4}$ **b.** a^{15} **8. a.** t^9 **b.** $\frac{64}{25}$ **c.** $-\frac{r^7}{8h^6}$ **9. a.** b^2 **b.** $\frac{3}{b^6}$ **10. a.** 1 **b.** a^6b^9 **11.** $\frac{32a^{15}}{243b^{20}}$

SECTION 5.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. x^n is read as "x to the nth _power_."
2. In the exponential expression x^n, x is called the _base_, and n is called the _exponent_.
3. $3^4 \cdot 3^8$ is a _product_ of exponent expression with the same base, and $\frac{x^4}{x^2}$ is a _quotient_ of exponential expression with the same base.
4. The exponential expression 3^{-2} has a _negative_ exponent.

CONCEPTS

Find each value.

5. 3^2 9
6. 3^4 81
7. $\left(\frac{1}{2}\right)^4$ $\frac{1}{16}$
8. $\left(-\frac{2}{3}\right)^3$ $-\frac{8}{27}$
9. -3^2 -9
10. -3^4 -81
11. $(-3)^2$ 9
12. $(-3)^3$ -27

Complete the rules for exponents. Assume that $x \neq 0$ and $y \neq 0$.

13. $x^mx^n = \boxed{x^{m+n}}$
14. $(x^m)^n = \boxed{x^{mn}}$
15. $(xy)^n = \boxed{x^ny^n}$
16. $\left(\frac{x}{y}\right)^n = \boxed{\frac{x^n}{y^n}}$
17. $x^0 = \boxed{1}$
18. $x^{-n} = \boxed{\frac{1}{x^n}}$
19. $\frac{x^m}{x^n} = \boxed{x^{m-n}}$
20. $\left(\frac{x}{y}\right)^{-n} = \boxed{\left(\frac{y}{x}\right)^n}$

Explain the difference between the two expressions.

21. $2x$ and x^2
 $2x = 2 \cdot x; x^2 = x \cdot x$
22. $-2x$ and x^{-2}
 $-2x = -2 \cdot x; x^{-2} = \frac{1}{x^2}$
23. **a.** To multiply exponential expressions with the same base, keep the common base and _add_ the exponents.
 b. To divide exponential expressions with the same base, keep the common base and _subtract_ the exponents.
 c. To raise an exponential expression to a power, keep the base and _multiply_ the exponents.
24. **a.** To raise a product to a power, raise each _factor_ of the product to that power.
 b. To raise a quotient to a power, raise the _numerator_ and the _denominator_ to that power.
 c. Any nonzero base raised to the 0 power is _1_.
 d. x^{-n} is the _reciprocal_ of x^n.

NOTATION

Complete each simplification.

25. $$\frac{x^5x^4}{x^{-2}} = \frac{x^{\boxed{9}}}{x^{-2}} = x^{9\boxed{-(-2)}} = x^{\boxed{11}}$$
26. $$\left(\frac{a^{-4}}{a^3}\right)^2 = (a^{-4-3})^2 = (a^{\boxed{-7}})^2 = a^{\boxed{-14}} = \frac{1}{\boxed{a^{14}}}$$

GUIDED PRACTICE

Identify the base and the exponent. See Example 1.

27. 5^3 5; 3
28. -7^2 7; 2
29. $-x^5$ x; 5
30. $(-t)^4$ $-t$; 4
31. $2b^6$ b; 6
32. $(3xy)^5$ $3xy$; 5
33. $\left(\frac{n}{4}\right)^3$ $\frac{n}{4}$; 3
34. $(-pq)^2$ $-pq$; 2

Simplify each expression. See Example 2.

35. x^2x^3 x^5
36. y^3y^4 y^7
37. $x^2x^3x^5$ x^{10}
38. $y^3y^7y^2$ y^{12}
39. $2aba^3b^4$ $2a^4b^5$
40. $2x^2y^3x^3y^2$ $2x^5y^5$
41. $-5y^3(y^4)$ $-5y^7$
42. $-\frac{2}{3}a^4(a^2b^2)$ $-\frac{2}{3}a^6b^2$

Simplify each expression. See Example 3.

43. $(2^3)^2$ 64
44. $(3^2)^2$ 81
45. $(z^{12})^2$ z^{24}
46. $(x^4)^7$ x^{28}
47. $(a^2a^3)^4$ a^{20}
48. $(b^2b^4)^4$ b^{24}
49. $(b^2)^3(b^3)^2$ b^{12}
50. $(c^4)^2(c^2)^3$ c^{14}

Selected exercises available online at
www.webassign.net/brookscole

Simplify each expression. See Example 4.

51. $(3x^3y^4)^3$ $27x^9y^{12}$

52. $\left(\frac{1}{2}a^2b^5\right)^4$ $\frac{1}{16}a^8b^{20}$

53. $\left(-\frac{1}{3}mn^2\right)^6$ $\frac{1}{729}m^6n^{12}$

54. $(-3p^2q^3)^5$ $-243p^{10}q^{15}$

55. $\left(\frac{a^3}{b^2}\right)^5$ $\frac{a^{15}}{b^{10}}$

56. $\left(\frac{a^2}{b^3}\right)^4$ $\frac{a^8}{b^{12}}$

57. $\left(\frac{4a^2}{3b^3}\right)^2$ $\frac{16a^4}{9b^6}$

▶ **58.** $\left(\frac{3a^5}{2b^4}\right)^3$ $\frac{27a^{15}}{8b^{12}}$

Simplify each expression. See Example 5.

59. 8^0 1

▶ **60.** -9^0 -1

61. $(-8t)^0$ 1

62. $(-9m)^0$ 1

63. $3(4x)^0$ 3

64. $4\left(\frac{2}{3}y^2\right)^0$ 4

65. $-6a^0b$ $-6b$

▶ **66.** $14p^8b^0$ $14p^8$

Write each expression without negative exponents. See Example 6.

67. 5^{-2} $\frac{1}{25}$

68. -5^{-4} $-\frac{1}{625}$

69. $-3p^{-2}$ $-\frac{3}{p^2}$

▶ **70.** $\frac{3}{4}a^3b^{-2}$ $\frac{3a^3}{4b^2}$

Simplify each expression. See Example 7.

71. $y^{-6}y^4$ $\frac{1}{y^2}$

72. p^6p^{-4} p^2

73. $(r^{-4})^{-2}$ r^8

▶ **74.** $(s^4)^{-3}$ $\frac{1}{s^{12}}$

Simplify each expression. Write answers using positive exponents. See Example 8.

75. $\frac{2}{a^{-3}}$ $2a^3$

76. $\frac{-5b}{a^{-7}}$ $-5a^7b$

77. $\frac{3^{-2}}{2^{-3}}$ $\frac{8}{9}$

78. $\frac{4^{-2}a}{3^{-2}b}$ $\frac{9a}{16b}$

79. $-\frac{a^{-3}}{4b^{-5}}$ $-\frac{b^5}{4a^3}$

▶ **80.** $\frac{a^{-2}}{4b^{-4}}$ $\frac{b^4}{4a^2}$

81. $-\frac{p^3}{5q^{-3}}$ $-\frac{p^3q^3}{5}$

82. $-\frac{4p^{-3}}{q^3}$ $-\frac{4}{p^3q^3}$

Simplify each expression. Write answers without using negative exponents. See Example 9.

83. $\frac{p^7}{p^3}$ p^4

84. $\frac{q^5}{q^6}$ $\frac{1}{q}$

85. $\frac{3y^{-4}}{y^{12}}$ $\frac{3}{y^{16}}$

▶ **86.** $\frac{y^4}{4y^{-3}}$ $\frac{y^7}{4}$

Simplify each expression. Write answers without using negative exponents. See Example 10.

87. $\frac{a^2a^3}{a^{-2}}$ a^7

88. $\frac{b^{-3}b^{-5}}{b^{-2}}$ $\frac{1}{b^6}$

89. $\frac{(a^3)^4}{(a^2)^6}$ 1

90. $\frac{(b^4)^5}{(b^5)^4}$ 1

91. $\left(\frac{4a^{-2}b}{3ab^{-3}}\right)^3$ $\frac{64b^{12}}{27a^9}$

▶ **92.** $\left(\frac{2ab^{-3}}{3a^{-2}b^2}\right)^2$ $\frac{4a^6}{9b^{10}}$

93. $\left(\frac{-2a^4b}{a^{-3}b^2}\right)^3$ $-\frac{8a^{21}}{b^3}$

94. $\left(\frac{-3x^4y^2}{-9x^5y^{-2}}\right)^2$ $\frac{y^8}{9x^2}$

Simplify each expression without using parentheses or negative exponents. See Example 11.

95. $\left(\frac{2}{3}\right)^{-2}$ $\frac{9}{4}$

96. $\left(\frac{4}{5}\right)^{-3}$ $\frac{125}{64}$

97. $\left(\frac{a^3}{b^2}\right)^{-4}$ $\frac{b^8}{a^{12}}$

98. $\left(\frac{p^5}{q^2}\right)^{-3}$ $\frac{q^6}{p^{15}}$

99. $\left(\frac{3a^2}{2b^{-4}}\right)^{-2}$ $\frac{4}{9a^4b^8}$

100. $\left(\frac{4m^3}{3n^5}\right)^{-3}$ $\frac{27n^{15}}{64m^9}$

101. $\left(\frac{-3pqr^{-4}}{2p^2q^{-3}r^2}\right)^{-2}$ $\frac{4p^2r^{12}}{9q^8}$

▶ **102.** $\left(\frac{4a^2b^3z^{-4}}{3a^{-2}b^{-7}z^3}\right)^{-3}$ $\frac{27z^{21}}{64a^{12}b^{30}}$

TRY IT YOURSELF

Simplify each expression. Assume that no denominators are zero. Write each answer without using negative exponents.

103. 5^{-4} $\frac{1}{625}$

104. -5^{-2} $-\frac{1}{25}$

105. $(-5)^{-2}$ $\frac{1}{25}$

106. $(-5)^{-4}$ $\frac{1}{625}$

107. $(-2x)^5$ $-32x^5$

108. $(-3a)^3$ $-27a^3$

109. k^0k^7 k^7

110. x^8x^{11} x^{19}

111. p^9pp^0 p^{10}

112. z^7z^0z z^8

113. $(-x)^2y^4x^3$ x^5y^4

▶ **114.** $-x^2y^7y^3x^{-2}$ $-y^{10}$

115. $(b^{-8})^9$ $\frac{1}{b^{72}}$

116. $(y^7)^5$ y^{35}

117. $(r^{-3}s)^3$ $\frac{s^3}{r^9}$

118. $(m^5n^2)^{-3}$ $\frac{1}{m^{15}n^6}$

119. $(-d^2)^3(d^{-3})^3$ $-\frac{1}{d^3}$

▶ **120.** $(c^3)^2(c^4)^{-2}$ $\frac{1}{c^2}$

121. $\left(\frac{a^{-3}}{b^{-2}}\right)^{-2}$ $\frac{a^6}{b^4}$

▶ **122.** $\left(\frac{k^{-3}}{k^{-4}}\right)^{-1}$ $\frac{1}{k}$

123. $\frac{a^8}{a^3}$ a^5

124. $\frac{c^7}{c^2}$ c^5

125. $\frac{c^{12}c^5}{c^{10}}$ c^7

▶ **126.** $\frac{a^{33}}{a^2a^3}$ a^{28}

127. $\frac{1}{a^{-4}}$ a^4

▶ **128.** $\frac{3}{b^{-5}}$ $3b^5$

129. $\dfrac{(3x^2)^{-2}}{x^3x^{-4}x^0}$ $\dfrac{1}{9x^3}$

130. $\dfrac{y^{-3}y^{-4}y^0}{(2y^{-2})^3}$ $\dfrac{1}{8y}$

131. $\left(\dfrac{3a^{-2}b^2}{17a^2b^2}\right)^0$ 1

132. $\dfrac{a^0 + b^0}{2(a + b)^0}$ 1

133. $\left(-\dfrac{2a^3b^2}{3a^{-3}b^2}\right)^{-3}$ $-\dfrac{27}{8a^{18}}$

134. $\left(\dfrac{3x^5y^2}{6x^5y^{-2}}\right)^{-4}$ $\dfrac{16}{y^{16}}$

135. $(2x^{-4}y^3)^3(3x^2y^{-2})^{-2}$ $\dfrac{8y^{13}}{9x^{16}}$

136. $\left(\dfrac{1}{2}a^2b^{-3}\right)^{-2}(2ab^2)^2$ $\dfrac{16b^{10}}{a^2}$

137. $\dfrac{(3x^2y^{-4})^{-2}}{(2x^3y^2)^{-3}}$ $\dfrac{8x^5y^{14}}{9}$

138. $\dfrac{(-2m^{-3}n^2)^2}{(3m^2n^3)^{-2}}$ $\dfrac{36n^{10}}{m^2}$

Use a calculator to find each value.

139. 1.23^6 3.462825992

140. 0.0537^4 0.000008316

141. -6.25^3 -244.140625

142. $(-25.1)^5$ $-9{,}962{,}506.263$

Use a calculator to verify that each statement is true by showing that the values on either side of the equation are equal.

143. $(3.68)^0 = 1$

144. $(2.1)^4(2.1)^3 = (2.1)^7$

145. $(7.2)^2(2.7)^2 = [(7.2)(2.7)]^2$

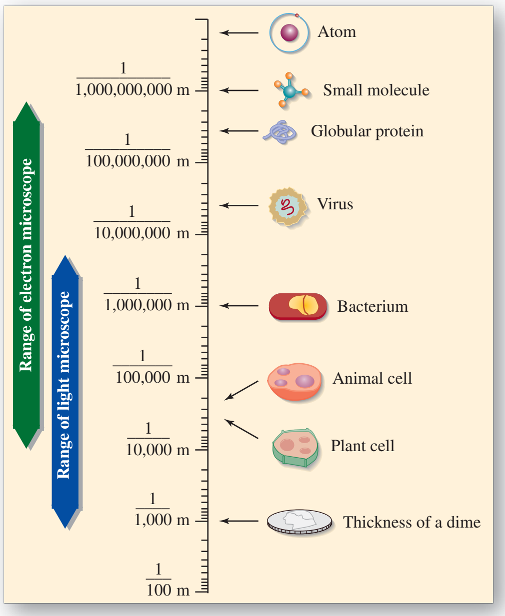
146. $\left(\dfrac{5.4}{2.7}\right)^{-4} = \left(\dfrac{2.7}{5.4}\right)^4$

147. $(3.2)^2(3.2)^{-2} = 1$

148. $(7.23)^{-3} = \dfrac{1}{(7.23)^3}$

APPLICATIONS

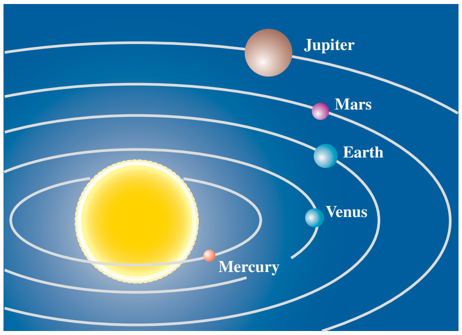
▶ **149.** MICROSCOPES The illustration shows the relative sizes of some chemical and biological structures, expressed as fractions of a meter (m). Express each fraction shown as a power of 10, from the largest to the smallest.
$10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}$



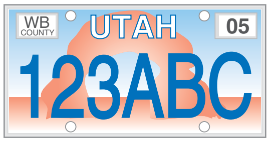
▶ **150.** ASTRONOMY The distance d, in miles, of the nth planet from the sun is given by the formula

$$d = 9{,}275{,}200[3(2^{n-2}) + 4]$$

From the illustration below, determine n for Earth and Mars. Then find the distance of Earth and the distance of Mars from the sun.
92,752,000 mi, 148,403,200 mi

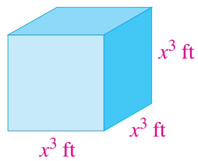


▶ **151.** LICENSE PLATES The number of different license plates of the form three digits followed by three letters is $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26$. Write this expression using exponents. Then evaluate it.
$10^3 \cdot 26^3$; 17,576,000



▶ **152.** PHYSICS Albert Einstein's work in relativity resulted in the observation that the total energy E of a body is equal to its total mass m times the square of the speed of light c. This relationship is given by the formula $E = mc^2$. Identify the base and exponent on the right-hand side of the equation. c; 2

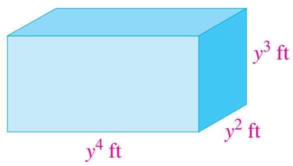
▶ **153.** GEOMETRY A cube is shown on the right.



a. Find the area of its base. x^6 ft^2

b. Find its volume. x^9 ft^3

▶ **154.** GEOMETRY A rectangular solid is shown on the right.



a. Find the area of its base. y^6 ft^2

b. Find its volume. y^9 ft^3

WRITING

155. Explain how an exponential expression with a negative exponent can be expressed as an equivalent expression with a positive exponent. Give an example.

156. In the definition of x^{-n}, x cannot be 0. Why not?

▶ **157.** Explain the error in the following solution.

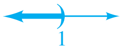
$$\cancel{-8ab^{-3} = \frac{a}{8b^3}}$$

158. Is a positive number greater than 1 raised to a negative power greater than or less than 1? Explain your answer.

REVIEW

Solve each inequality. Give the result in interval notation and graph it.

159. $a + 5 < 6$

$(-\infty, 1)$ 

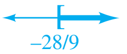
160. $-9x + 5 \geq 15$

$\left[-\infty, -\frac{10}{9}\right]$ 

161. $6(t - 2) \leq 4(t + 7)$

$(-\infty, 20]$ 

▶ **162.** $\frac{1}{4}p - \frac{1}{3} \leq p + 2$

$\left[-\frac{28}{9}, \infty\right)$ 

Objectives

1 Write numbers in scientific notation.

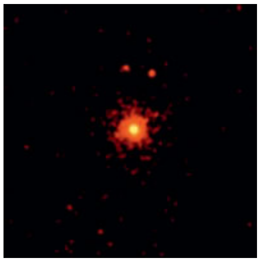
2 Convert from scientific notation to standard notation.

3 Perform computations with scientific notation.

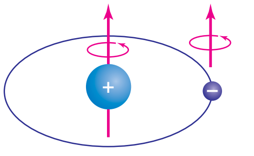
SECTION 5.2
Scientific Notation

Very large and small numbers occur in science and other disciplines. For example, the star nearest to Earth (excluding the sun) is Proxima Centauri, about 24,793,000,000,000 miles away, and the mass of a hydrogen atom is approximately 0.0000000000000000000000001673 gram.

These numbers, written in **standard** or **decimal notation,** are difficult to read and cumbersome to work with in computations because they contain many zeros. In this section, we will discuss a notation that enables us to express such numbers in a more manageable form.



© NASA/Science Source



1 Write numbers in scientific notation.

Scientific notation provides a compact way of writing very large and very small numbers.

Scientific Notation

A positive number is written in **scientific notation** when it is written in the form $N \times 10^n$, where $1 \leq N < 10$ and n represents an integer.

Some examples of numbers written in scientific notation are

$$3.67 \times 10^6 \qquad 2.24 \times 10^{-4} \qquad 9.875 \times 10^{22}$$

Every positive number written in scientific notation is the product of a decimal number between 1 (including 1) and 10 and an integer power of 10.

$$\underbrace{\square \cdot \square}_{\text{A decimal that is at least 1 but less than 10}} \times 10^{\overset{\text{An integer exponent}}{\square}}$$

EXAMPLE 1

Write each number in scientific notation:

a. 24,793,000,000,000 **b.** 0.0000000000000000000000001673

Strategy We will write each number as the product of a number between 1 and 10 and an integer power of 10.

WHY Numbers written in scientific notation have the form $N \times 10^n$ where $1 \le N < 10$ and n is an integer.

Solution

a. The number 2.4793 is between 1 and 10. To get 24,793,000,000,000, the decimal point in 2.4793 must be moved 13 places to the *right*.

$$\underbrace{24{,}793{,}000{,}000{,}000.}_{\text{13 places}}$$

We can move the decimal point 13 places to the right by multiplying 2.4793 by 10^{13}.

$$24{,}793{,}000{,}000{,}000 = 2.4793 \times 10^{13}$$

b. The number 1.673 is between 1 and 10. To get 0.0000000000000000000000001673, the decimal point in 1.673 must be moved 24 places to the *left*.

$$\underbrace{0.0000000000000000000000001673}_{\text{24 places}}$$

We can move the decimal point 24 places to the left by multiplying 1.673 by 10^{-24}.

$$0.0000000000000000000000001673 = 1.673 \times 10^{-24}$$

Self Check 1

Write each italicized number in scientific notation.

a. In 1998, the country earning the most money from tourism was the United States: *$74,240,000,000.* 7.424×10^{10}

b. DNA molecules contain and transmit the information that allows cells to reproduce. They are only *0.000000002* meter wide. 2×10^{-9}

Now Try **Problems 10 and 16**

Teaching Example 1 Write each number in scientific notation:
a. 0.0000004192
b. 50,100,000,000,000
Answers:
a. 4.192×10^{-7} **b.** 5.01×10^{13}

Numbers such as 47.2×10^3 and 0.063×10^{-2} appear to be written in scientific notation, because they are the product of a number and a power of 10. However, they are not. Their first factors (47.2 and 0.063) are not between 1 and 10.

EXAMPLE 2

Write each number in scientific notation:

a. 47.2×10^3 **b.** 0.063×10^{-2}

Strategy In each case, we will write the first factor in scientific notation and then multiply the powers of 10 using the product rule of exponents.

WHY Neither of the original numbers are written in scientific notation because the first factor is not between 1 and 10.

Solution

Since the first factors are not between 1 and 10, neither number is in scientific notation. However, we can change them to scientific notation as follows:

a.
$$\begin{aligned} \mathbf{47.2} \times 10^3 &= (\mathbf{4.72 \times 10^1}) \times 10^3 && \text{Write 47.2 in scientific notation.} \\ &= 4.72 \times (10^1 \times 10^3) && \text{Group the powers of 10 together.} \\ &= 4.72 \times 10^4 && \text{Apply the product rule for exponents: } 10^1 \times 10^3 = 10^{1+3} = 10^4. \end{aligned}$$

b.
$$\begin{aligned} \mathbf{0.063} \times 10^{-2} &= (\mathbf{6.3 \times 10^{-2}}) \times 10^{-2} && \text{Write 0.063 in scientific notation.} \\ &= 6.3 \times (10^{-2} \times 10^{-2}) \\ &= 6.3 \times 10^{-4} \end{aligned}$$

Self Check 2

Write each number in scientific notation:

a. 27.3×10^2

b. 0.0025×10^{-3}

Now Try **Problems 18 and 22**

Self Check 2 Answers
a. 2.73×10^3 **b.** 2.5×10^{-6}

Teaching Example 2 Write each number in scientific notation:
a. 0.0047×10^7 **b.** 52.1×10^{-3}
Answers:
a. 4.7×10^4 **b.** 5.21×10^{-2}

THINK IT THROUGH *The American Educational System*

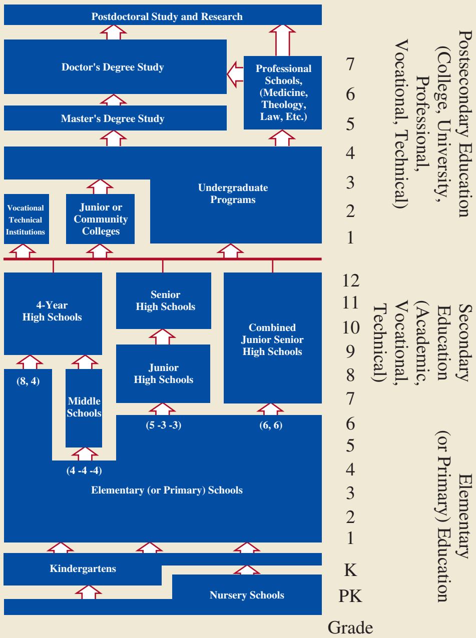
Between 2001 and 2013, the number of high school graduates is projected to increase nationally by 11 percent. Increases are expected in each region of the country, especially the West.

National Center for Education Statistics, Projections to 2013

The figure on the right shows the three-level structure of the American educational system as illustrated in the *Digest for Education Statistics*, 2002. Write each italicized number in the following excerpt from the digest using scientific notation.

For 2002, enrollment in U.S. elementary and secondary schools was estimated to be *53.6 million* and enrollment in U.S. postsecondary schools was estimated to be *15.6 million.* Total spending on all three levels of education was estimated to be *$745 billion.*

5.36×10^7, 1.56×10^7, 7.45×10^{11}



2 Convert from scientific notation to standard notation.

Each of the following numbers is written in both scientific and standard notation. In each case, the exponent gives the number of places that the decimal point moves, and the sign of the exponent indicates the direction that it moves:

$5.32 \times 10^4 = 53200.$
4 places to the right

$6.45 \times 10^7 = 64500000.$
7 places to the right

$2.37 \times 10^{-4} = 0.000237$
4 places to the left

$9.234 \times 10^{-2} = 0.09234$
2 places to the left

$4.89 \times 10^0 = 4.89$
No movement of the decimal point

Self Check 3

Convert each number to standard notation.

a. Russia is the largest country in land area, with over 6.5×10^6 square miles.

b. The average distance between molecules of air in a room is 3.937×10^{-7} inch.

Now Try **Problems 26 and 32**

EXAMPLE 3 Convert each number to standard notation:

a. 8.706×10^5 **b.** 1.1×10^{-3}

Strategy In each case, we need to identify the exponent on the power of 10 and consider its sign.

WHY The exponent gives the number of decimal places that we should move the decimal point. The sign of the exponent indicates whether it should be moved to the right or the left.

Solution

a. Since multiplication by 10^5 or 100,000 moves the decimal point 5 places to the right,

$$8.706 \times 10^5 = 870600. = 870{,}600$$

b. Since multiplication by 10^{-3} or 0.001 moves the decimal point 3 places to the left,

$$1.1 \times 10^{-3} = 0.0011 = 0.0011$$

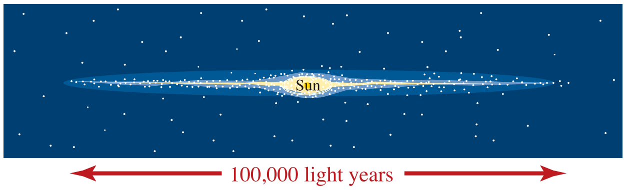
Self Check 3 Answers
a. 6,500,000 **b.** 0.0000003937

Teaching Example 3 Convert each number to standard notation:
a. 7.1364×10^{-5} **b.** 4.918×10^6
Answers:
a. 0.000071364 **b.** 4,918,000

3 Perform computations with scientific notation.

Scientific notation is useful when multiplying and dividing very large or very small numbers.

EXAMPLE 4 ***Astronomy*** The galaxy in which we live is called the Milky Way. This system of some 10^{11} stars, one of which is the sun, has a diameter of approximately 100,000 light years. (A light year is the distance light travels in a vacuum in one year: 9.46×10^{15} meters.) What is the diameter of the Milky Way in meters?



A cross-sectional representation of the Milky Way Galaxy

Strategy To find the diameter of the Milky Way, we will convert the number of light years, 100,000, to scientific notation and multiply it by the number of meters per light year, 9.46×10^{15}.

WHY When the numbers to be multiplied (100,000 and 9.46×10^{15}) are written in scientific notation, we can use the product rule for exponential expressions separately.

Solution
To find the diameter of the Milky Way in meters, we will multiply the diameter of the Milky Way, expressed in light years, by the number of meters in a light year. To perform the calculation, we write 100,000 in scientific notation as 1.0×10^5.

$1.0 \times 10^5 \cdot 9.46 \times 10^{15}$

$= (1.0 \cdot 9.46) \times (10^5 \cdot 10^{15})$ — Apply the commutative and associative properties of multiplication to group the first factors together and the powers of 10 together.

$= 9.46 \times 10^{5+15}$ — Perform the multiplication: $1.0 \cdot 9.46 = 9.46$. For the powers of 10, keep the base and add the exponents.

$= 9.46 \times 10^{20}$ — Perform the addition.

The Milky Way galaxy is about 9.46×10^{20} meters in diameter.

Self Check 4

ASTRONOMY A light year is 5.88×10^{12} miles. What is the diameter of the Milky Way in miles?

Now Try Problem 38

Self Check 4 Answer
5.88×10^{17} mi

Teaching Example 4 FARMING In 2007 the average yield of corn was 151.1 bushels per acre. If U.S. farmers planted 93,600,000 acres of corn in 2007, what is the total number of bushels of corn produced? (www.nass.usda.gov)
Answer:
about 1.414×10^{10} bushels

Self Check 5

WORLD OIL RESERVES/DEMAND There were 1,290,000,000,000 barrels of crude oil reserves in the ground at the start of 2006. At that time, world demand was 30,800,000,000 barrels per year, according to the Energy Information Administration. If annual demand as of 2006 remains the same and if no new oil discoveries are made, when will the world's oil supply run out? 42 years

Now Try **Problem 42**

Teaching Example 5 FARMING If the total number of bushels of soybeans produced in 2007 was 3,200,000,000 bushels for 62,800,000 acres planted, find the average number of bushels of soybeans produced per acre that year.
Answer:
about 52 bushels per acre

EXAMPLE 5 ***World Oil Reserves/Demand*** According to estimates in the *Oil and Gas Journal,* there were 1.03×10^{12} barrels of oil reserves in the ground at the start of 2003. At that time, world production was 2.89×10^{10} barrels per year. If annual production remains the same and if no new oil discoveries are made, when will the world's oil supply run out?

Strategy To find the number of years of crude oil that remains, we will convert the number of barrels in reserves to scientific notation and divide it by the annual number of barrels of demand, also written in scientific notation.

WHY When the numbers to be divided are written in scientific notation, we can use the quotient rule for exponents to simplify the computation.

Solution

If we divide the estimated number of barrels of oil in reserve, 1.03×10^{12}, by the number of barrels produced each year, 2.89×10^{10}, we can find the number of years of oil supply left.

$$\frac{1.03 \times 10^{12}}{2.89 \times 10^{10}} = \frac{1.03}{2.89} \times \frac{10^{12}}{10^{10}}$$
Divide the first factors and the second factors in the numerator and denominator separately.

$$\approx 0.36 \times 10^{12-10}$$
Perform the division: $\frac{1.03}{2.89} \approx 0.36$. For the powers of 10, keep the base and subtract the exponents.

$$\approx 0.36 \times 10^{2}$$
Perform the subtraction.

$$\approx 36$$
Write 0.36×10^2 in standard notation.

According to industry estimates, as of 2003, there were 36 years of oil reserves left. Under these conditions, the world's oil supply will run out in the year 2039.

Self Check 6

Use scientific notation to evaluate:

$$\frac{(320)(25{,}000)}{0.00004}$$

Now Try **Problem 44**

Self Check 6 Answer
$2 \times 10^{11} = 200{,}000{,}000{,}000$

Teaching Example 6 Use scientific notation to evaluate:
$$\frac{(5{,}200{,}000)(24{,}000)}{0.000008}$$
Answer:
1.56×10^{16}

EXAMPLE 6 Use scientific notation to evaluate:

$$\frac{(0.00000064)(24{,}000{,}000{,}000)}{(400{,}000{,}000)(0.0000000012)}$$

Strategy After writing each number in scientific notation, we will perform the arithmetic on the decimals and the exponential expressions separately.

WHY When the numbers to be multiplied and divided are written in scientific notation, we can use the product and quotient rules for exponents to simplify the computation.

Solution

After writing each number in scientific notation, we can do the arithmetic on the numbers and the exponential expressions separately.

$$\frac{(0.00000064)(24{,}000{,}000{,}000)}{(400{,}000{,}000)(0.0000000012)} = \frac{(6.4 \times 10^{-7})(2.4 \times 10^{10})}{(4 \times 10^{8})(1.2 \times 10^{-9})}$$
$$= \frac{(6.4)(2.4)}{(4)(1.2)} \times \frac{10^{-7}10^{10}}{10^{8}10^{-9}}$$
$$= \frac{15.36}{4.8} \times 10^{-7+10-8-(-9)}$$
$$= 3.2 \times 10^{4}$$

The result is 3.2×10^4. In standard notation, this is 32,000.

Using Your CALCULATOR Using Scientific Notation

Scientific calculators and graphing calculators often give answers in scientific notation. For example, if we use a calculator to find 301.2^8, the display will read

6.77391496 19	*On a scientific calculator*
301.2^8 6.77391496 1E19	*On a graphing calculator*

In either case, the answer is given in scientific notation, and we interpret it as

$$6.77391496 \times 10^{19}$$

Numbers can also be entered into a calculator in scientific notation. For example, to enter 24,000,000,000 (which is 2.4×10^{10} in scientific notation), we enter these numbers and press these keys:

2.4 $\boxed{\text{EXP}}$ 10 *On some scientific calculators*

2.4 $\boxed{\text{EE}}$ 10 *On a graphing calculator and on most scientific calculators*

To use a scientific calculator to evaluate

$$\frac{(24{,}000{,}000{,}000)(0.00000006495)}{0.00000004824}$$

we must enter each number in scientific notation, because each number has too many digits to be entered directly. In scientific notation, the three numbers are

$$2.4 \times 10^{10} \qquad 6.495 \times 10^{-8} \qquad 4.824 \times 10^{-8}$$

Using a scientific calculator, we enter these numbers and press these keys:

2.4 $\boxed{\text{EXP}}$ 10 $\boxed{\times}$ 6.495 $\boxed{\text{EXP}}$ 8 $\boxed{+/-}$ $\boxed{\div}$ 4.824 $\boxed{\text{EXP}}$ 8 $\boxed{+/-}$ $\boxed{=}$

The display will read $\boxed{3.231343284 \quad ^{10}}$. In standard notation, the answer is 32,313,432,840.

The keystrokes are similar on a graphing calculator.

ANSWERS TO SELF CHECKS

1. a. 7.424×10^{10} **b.** 2×10^{-9} **2. a.** 2.73×10^3 **b.** 2.5×10^{-6} **3. a.** 6,500,000 **b.** 0.0000003937 **4.** 5.88×10^{17} mi **5.** 42 years **6.** $2 \times 10^{11} = 200{,}000{,}000{,}000$

SECTION 5.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. 7.4×10^6 is written in scientific notation. 7,400,000 is written in standard notation.

2. 10^{-3}, 10^0, 10^1, and 10^4 are powers of 10.

Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

Fill in the blanks.

3. A positive number is written in scientific notation when it is written in the form $N \times 10^n$, where $1 \le N < 10$, and n is an integer.

▶ **4.** Fill in the blank with > or <:

The number 5.3×10^2 __>__ the number 5.3×10^{-2}.

5. To change 6.31×10^{-4} to standard notation, we move the decimal point four places to the __left__.

▶ **6.** To change 9.7×10^3 to standard notation, we move the decimal point three places to the __right__.

NOTATION

7. Explain why the number 60.22×10^{22} is not written in scientific notation. 60.22 is not between 1 and 10.

8. Explain why the number 0.6022×10^{24} is not written in scientific notation. 0.6022 is not between 1 and 10.

GUIDED PRACTICE

Write each number in scientific notation. See Example 1.

9. 3,900
3.9×10^3

10. 1,700
1.7×10^3

11. 0.0078
7.8×10^{-3}

12. 0.068
6.8×10^{-2}

13. 173,000,000,000,000
1.73×10^{14}

▶ **14.** 89,800,000,000
8.98×10^{10}

15. 0.0000096
9.6×10^{-6}

16. 0.000000046
4.6×10^{-8}

Write each number in scientific notation. See Example 2.

17. 323×10^5
3.23×10^7

▶ **18.** 689×10^9
6.89×10^{11}

19. $6{,}000 \times 10^{-7}$
6.0×10^{-4}

▶ **20.** 765×10^{-5}
7.65×10^{-3}

21. 0.0527×10^5
5.27×10^3

22. 0.0298×10^3
2.98×10^1

23. 0.0317×10^{-2}
3.17×10^{-4}

24. 0.0012×10^{-3}
1.2×10^{-6}

Write each number in standard notation. See Example 3.

25. 2.7×10^2 270

26. 7.2×10^3 7,200

▶ **27.** 3.23×10^{-3} 0.00323

▶ **28.** 6.48×10^{-2} 0.0648

29. 7.96×10^5 796,000

30. 9.67×10^6 9,670,000

31. 3.7×10^{-4} 0.00037

32. 4.12×10^{-5} 0.0000412

33. 5.23×10^0 5.23

34. 8.67×10^0 8.67

35. 23.65×10^6 23,650,000

▶ **36.** 75.6×10^{-5} 0.000756

Give all answers in scientific notation. Use a calculator to check your results. See Examples 4–6.

37. $(7.9 \times 10^5)(2.3 \times 10^6)$ 1.817×10^{12}

38. $(6.1 \times 10^8)(3.9 \times 10^5)$ 2.379×10^{14}

39. $(9.1 \times 10^{-5})(5.5 \times 10^{12})$ 5.005×10^8

▶ **40.** $(8.4 \times 10^{-13})(4.8 \times 10^9)$ 4.032×10^{-3}

41. $\dfrac{4.2 \times 10^{-12}}{8.4 \times 10^{-5}}$ 5×10^{-8}

▶ **42.** $\dfrac{1.21 \times 10^{-15}}{1.1 \times 10^2}$ 1.1×10^{-17}

43. $\dfrac{(3.9 \times 10^{-9})(9.5 \times 10^{-4})}{1.95 \times 10^{-2}}$ 1.9×10^{-10}

44. $\dfrac{(4.9 \times 10^{60})(2.7 \times 10^{30})}{6.3 \times 10^{40}}$ 2.1×10^{50}

TRY IT YOURSELF

Write each numeral in scientific notation and perform the operations. Give all answers in scientific notation and in standard form. Use a calculator to check your results.

45. (89,000,000,000)(4,500,000,000)
4.005×10^{20}; 400,500,000,000,000,000,000

46. (0.000000061)(3,500,000,000)
2.135×10^2; 213.5

47. $\dfrac{0.00000129}{0.0003}$
4.3×10^{-3}; 0.0043

▶ **48.** $\dfrac{4{,}400{,}000{,}000{,}000}{0.0002}$
2.2×10^{16}; 22,000,000,000,000,000

49. $\dfrac{(220{,}000)(0.000009)}{0.00033}$
6×10^3; 6,000

▶ **50.** $\dfrac{(640{,}000)(2{,}700{,}000)}{120{,}000}$
1.44×10^7; 14,400,000

51. $\dfrac{(0.00024)(96{,}000{,}000)}{640{,}000{,}000}$
3.6×10^{-5}; 0.000036

52. $\dfrac{(0.0000013)(0.00009)}{0.00039}$
3×10^{-7}; 0.0000003

APPLICATIONS

53. FIVE-CARD POKER The odds against being dealt the hand shown on the right are about 2.6×10^6 to 1. Express the odds using standard notation. 2,600,000 to 1

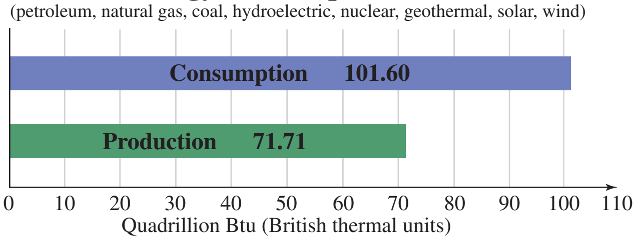
▶ **54.** ENERGY See the illustration. Express each of the following using scientific notation. (1 quadrillion is 10^{15}.)

a. 2007 U.S. energy consumption 1.016×10^{17} Btu

b. 2007 U.S. energy production 7.171×10^{16} Btu

c. The difference in 2007 consumption and production 2.989×10^{16} Btu

2007 U.S. Energy Consumption and Production
(petroleum, natural gas, coal, hydroelectric, nuclear, geothermal, solar, wind)



Source: Energy Information Administration, United States Department of Energy

55. THE YEAR 2000 Express in scientific notation each of the dollar amounts that appeared in the following excerpt from the *Federal Computer Week* website (February 16, 1998).

> *President Clinton's fiscal 1999 budget proposal of \$1.7 trillion includes expenditures of about \$3.9 billion to ensure that federal computers can accept dates after Dec. 31, 1999. Clinton has proposed spending \$275 million at the Defense Department and \$312 million at the Treasury Department to fix the year 2000 problem.*

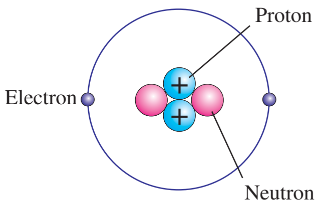
$\$1.7 \times 10^{12}$, $\$3.9 \times 10^{9}$, $\$2.75 \times 10^{8}$, $\$3.12 \times 10^{8}$

▶ **56.** STAR TREK In the science fiction series *Star Trek,* crew members talk of their spacecraft, the *U.S.S. Enterprise,* traveling at various warp speeds. To convert a warp speed, W, to an equivalent velocity in miles per second, v, we can use the equation

$$v = W^3c$$

where c is the speed of light, 1.86×10^5 miles per second. Find the velocity of a spacecraft traveling at warp 2. 1.488×10^6 mi/sec or 1,488,000 mi/sec

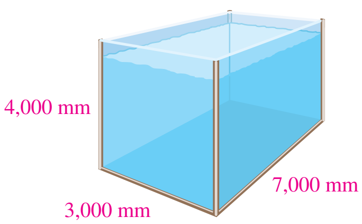
57. ATOMS A simple model of a helium atom is shown. If a proton has a mass of 1.7×10^{-24} grams, and if the mass of an electron is only about $\frac{1}{2{,}000}$ that of a proton, find the mass of an electron. 8.5×10^{-28} g



▶ **58.** OCEANS The mass of Earth's oceans is only about $\frac{1}{4{,}400}$ that of Earth. If the mass of Earth is 6.578×10^{21} tons, find the mass of the oceans. 1.495×10^{18} tons

59. LIGHT YEARS Light travels about 300,000,000 meters per second. A **light year** is the distance that light can travel in one year. Estimate the number of meters in one light year. about 9.5×10^{15} m

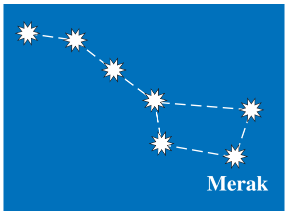
▶ **60.** AQUARIUMS Express the volume of the fish tank shown below in scientific notation. 8.4×10^{10} mm^3



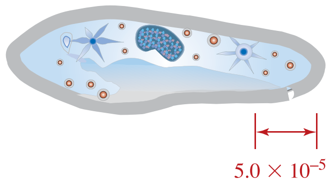
61. THE BIG DIPPER One of the stars in the Big Dipper is named Merak. It is approximately 4.65×10^{14} miles from Earth.

a. If light travels about 1.86×10^5 miles/sec, how many seconds does it take light emitted from Merak to reach Earth? $\left(\textit{Hint:}\ \text{Use the formula } t = \frac{d}{r}.\right)$
2.5×10^9 sec = 2,500,000,000 sec

b. Convert your result from part a to years.
about 79 years



▶ **62.** BIOLOGY A paramecium is a single-celled organism that propels itself with hair-like projections called *cilia.* Use the scale in the illustration below to estimate the length of the paramecium. Express the result in scientific and in standard notation.
2.5×10^{-4}*m*; 0.00025 m



63. HALE-BOPP On March 23, 1997, Comet Hale-Bopp made its closest approach to Earth, coming within 1.3 **astronomical units.** One astronomical unit (AU) is the distance from Earth to the sun—about 9.3×10^7 miles. Express this distance in miles, using scientific notation.
1.209×10^8 mi

▶ **64.** DIAMONDS The approximate number of atoms of carbon in a $\frac{1}{2}$–carat diamond can be found by computing

$$\frac{6.0 \times 10^{23}}{1.2 \times 10^2}$$

Express the number of carbon atoms in scientific and in standard notation.
5×10^{21}; 5,000,000,000,000,000,000,000

65. ATOMS A hydrogen atom is so small that a single drop of water contains more than a million million billion hydrogen atoms. Express this number in scientific notation.
1.0×10^{21}

66. ASTRONOMY The American Physical Society recently honored first-year graduate student Gwen Bell for coming up with what it considers the most accurate estimate of the mass of the Milky Way. In pounds, her estimate is a 3 with 42 zeros after it. Express this number in scientific notation. 3.0×10^{42}

WRITING

67. Explain how to change a number from standard notation to scientific notation.

68. Explain how to change a number from scientific notation to standard notation.

69. Explain why 9.99×10^n represents a number less than 1 but greater than 0 if n is a negative integer.

70. Explain the advantages of writing very large and very small numbers in scientific notation.

REVIEW

Solve each compound inequality. Give the result in interval notation and graph the solution set.

71. $4x \geq -x + 5$ and $6 \geq 4x - 3$

$\left[1, \frac{9}{4}\right]$

72. $15 > 2x - 7 > 9$

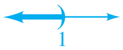
$(8, 11)$

73. $3x + 2 < 8$ or $2x - 3 > 11$

$(-\infty, 2) \cup (7, \infty)$

74. $-4(x + 2) \geq 12$ or $3x + 8 < 11$

$(-\infty, 1)$



Objectives

1. Define and classify polynomials.
2. Evaluate polynomial functions.
3. Graph polynomial functions.
4. Simplify polynomials by combining like terms.
5. Add polynomials.
6. Subtract polynomials.

SECTION 5.3 Polynomials and Polynomial Functions

In arithmetic, we add, subtract, multiply, divide, and find powers of numbers. In algebra, we perform these operations on algebraic expressions called *polynomials*.

1 Define and classify polynomials.

Recall from Chapter 1 that a **term** is a number or a product of a number and a variable (or variables) raised to a power. Some examples are

$$17, \qquad 9x, \qquad \frac{15}{16}y^2, \qquad \text{and} \qquad -2.4x^4y^5$$

If a term contains only a number, such as 17, it is called a **constant term,** or simply a **constant.**

The **numerical coefficient,** or simply the **coefficient,** is the numerical factor of a term. For example, the coefficient of $9x$ is 9 and the coefficient of $-2.4x^4y^5$ is -2.4. The coefficient of a constant term is that constant.

Polynomials

A **polynomial** is a single term or the sum of terms whose variables have whole-number exponents. No variable appears in a denominator.

The following expressions are polynomials in x:

$$-6x, \quad 3x^2 + 2x, \quad \frac{3}{2}x^5 - \frac{7}{3}x^4 - \frac{8}{3}x^3, \quad \text{and} \quad 19x^{20} + \sqrt{3}x^{14} + 4.5x^{11} - x^2$$

Caution! The following expressions are not polynomials:

$$\frac{2x}{x^2 + 1}, \qquad x^{1/2} - 8, \qquad \text{and} \qquad x^{-3} + 2x + 24$$

The first expression is a quotient and has a variable in the denominator. The last two have exponents that are not whole numbers.

If any terms of a polynomial contain more than one variable, we say that the polynomial is in more than one variable. Some examples are

$$3xy, \quad 5x^2y^2 + 2xy - 3y, \quad \text{and} \quad u^2v^2w^2 + uv + 1$$

Polynomials can be classified according to the number of terms they have. A polynomial with one term is called a **monomial,** a polynomial with two terms is called a **binomial,** and a polynomial with three terms is called a **trinomial.**

Monomials	Binomials	Trinomials
$2x^3$	$2x + 5$	$2x^2 + 4x + 3$
a^2b	$-17x^4 - \frac{3}{5}x$	$3mn^3 - m^2n^3 + 7n$
$3x^3y^5z^2$	$32x^{13}y^5z^3 + 47x^3yz$	$-12x^5y^2 + 13x^4y^3 - 7x^3y^3$

Polynomials and their terms can be classified according to the exponents on their variables.

Degree of a Term of a Polynomial

The **degree of a term** of a polynomial in one variable is the exponent on the variable. The degree of a term in several variables is the sum of the exponents on those variables. If the term is a nonzero constant, its degree is 0. The constant 0 has no defined degree.

EXAMPLE 1 Find the degree. **a.** $3x^4$ **b.** $-4x^2y^3$ **c.** 3

Strategy We will find the sum of the exponents on the variables for each term.

WHY The sum of the exponents on the variables of a term gives the degree of the term.

Solution

a. $3x^4$ is a monomial of degree 4, because the exponent on the variable is 4.

b. $-4x^2y^3$ is a monomial of degree 5, because the sum of the exponents on the variables is $2 + 3 = 5$.

c. 3 is a monomial of degree 0, because $3 = 3x^0$.

Self Check 1

Find the degree.

a. $-12a^2$ 2

b. $8a^3b^2$ 5

c. $\frac{1}{2}x^3y^2z^{12}$ 17

Now Try **Problems 26 and 30**

Teaching Example 1 Find the degree.

a. $27a^9$ **b.** $-25x^5y$ **c.** $\frac{1}{2}x^3y^2z^4$

Answers:

a. 9 **b.** 6 **c.** 9

We determine the degree of a polynomial by considering the degrees of each of its terms.

Degree of a Polynomial

The **degree of a polynomial** is the same as the degree of the term in the polynomial with largest degree.

EXAMPLE 2 State whether each polynomial is a monomial, binomial, or trinomial, and find the degree.

a. $3x^5 + 4x^2 + 7$ **b.** $7x^2y^8 - 3x^2y^2$ **c.** $3x + 2y - xy$

Strategy We will count the number of terms in the polynomial and determine the degree of each term.

WHY The number of terms determines the type of polynomial. The highest degree of any term of the polynomial determines its degree.

Self Check 2

State whether each polynomial is a monomial, binomial, or trinomial, and find the degree.

a. $x^2 - x + 1$

b. $-12x^7y^2 + 3x^9y^3$

Now Try **Problems 20 and 22**

Self Check 2 Answers

a. trinomial, degree 2

b. binomial, degree 12

Teaching Example 2 State whether each polynomial is a monomial, binomial, or trinomial, and find the degree.
a. $7x^4 - 3x^2 + 2$
b. $5xy^3 - 8x^2yz^9$
c. $-6x^2y^3$
Answers:
a. trinomial, degree 4
b. binomial, degree 12
c. monomial, degree 5

Solution

a. The polynomial has three terms, so it is a trinomial. The terms of $3x^5 + 4x^2 + 7$ have degree 5, 2, and 0, respectively. This trinomial is of degree 5, because the largest degree of the three terms is 5.

b. The polynomial has two terms, so it is a binomial. The terms have degree 10 and 4 respectively. This binomial has degree 10, because the largest degree of the two terms is 10.

c. $3x + 2y - xy$ is a trinomial of degree 2. (Recall that $xy = x^1y^1$.)

If the terms of a polynomial in one variable are written so that the exponents on the variable decrease as we move from left to right, we say that the terms are written with their exponents in *descending order.* If the terms are written so that the exponents on the variable increase as we move from left to right, we say that the terms are written with their exponents in *ascending order.*

$-5x^4 + 2x^3 + 7x^2 + 3x - 1$ This polynomial is written in descending powers of x.

$-1 + 3x + 7x^2 + 2x^3 - 5x^4$ The same polynomial is now written in ascending powers of x.

2 Evaluate polynomial functions.

In Chapter 2, we saw that linear functions are defined by equations of the form $f(x) = mx + b$. Some examples of linear functions are

$$f(x) = 3x + 1 \qquad g(x) = -\frac{1}{2}x - 1 \qquad h(x) = 5x$$

In each case, the right-hand side of the equation is a polynomial. For this reason, linear functions are members of a larger class of functions known as **polynomial functions.**

Polynomial Functions

A **polynomial function** is a function whose equation is defined by a polynomial in one variable.

Another example of a polynomial function is $f(x) = -x^2 + 6x - 8$. This is a second-degree polynomial function, called a **quadratic function.** Quadratic functions are of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$.

An example of a third-degree polynomial function is $f(x) = x^3 - 3x^2 - 9x + 2$. Third-degree polynomial functions, also called **cubic functions,** are of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Polynomial functions can be used to model many real-life situations. If we are given a polynomial function model, we can learn more about the situation by evaluating the function at specific values. To *evaluate a polynomial function* at a specific value, we replace the variable in the defining equation with that value, called the **input.** Then we simplify the resulting expression to find the **output.**

EXAMPLE 3 *Rocketry*

If a toy rocket is shot straight up with an initial velocity of 128 feet per second, its height, in feet, t seconds after liftoff is given by the function

$$h(t) = -16t^2 + 128t$$

Find the height of the rocket at:

a. 0 second **b.** 3 seconds **c.** 7.9 seconds

Strategy We will find $h(0)$, $h(3)$, and $h(7.9)$.

WHY The notation $h(0)$ represents the height of the rocket at 0 second, $h(3)$ represents the height 3 seconds after being launched, and $h(7.9)$ represents the height of the rocket 7.9 seconds after being launched.

Solution

a. To find the height of the rocket at 0 second, we substitute 0 for t and evaluate the right-hand side.

$$\begin{aligned} h(t) &= -16t^2 + 128t && \text{This is the given function.} \\ h(0) &= -16(0)^2 + 128(0) && \text{The input is 0.} \\ &= 0 && \text{The output is 0.} \end{aligned}$$

At 0 second, the rocket's height is 0. It is on the ground waiting to be launched.

b. To find the height at 3 seconds, we substitute 3 for t and evaluate the right-hand side.

$$\begin{aligned} h(t) &= -16t^2 + 128t && \text{This is the given function.} \\ h(3) &= -16(3)^2 + 128(3) && \text{The input is 3.} \\ &= -16(9) + 384 \\ &= -144 + 384 \\ &= 240 && \text{The output is 240.} \end{aligned}$$

At 3 seconds after liftoff, the height of the rocket is 240 feet.

c. To find the height at 7.9 seconds, we substitute 7.9 for t and evaluate the right-hand side.

$$\begin{aligned} h(t) &= -16t^2 + 128t && \text{This is the given function.} \\ h(7.9) &= -16(7.9)^2 + 128(7.9) && \text{The input is 7.9.} \\ &= -16(62.41) + 1{,}011.2 \\ &= -998.56 + 1{,}011.2 \\ &= 12.64 && \text{The output is 12.64.} \end{aligned}$$

At 7.9 seconds, the height is 12.64 feet. The rocket has almost fallen back to Earth.

Self Check 3

ROCKETRY In Example 3, find the height of the rocket 4 seconds after it is shot upward. 256 ft

Now Try **Problem 32**

Teaching Example 3 ROCKETRY In Example 3, find the height of the rocket 2 seconds after it is shot upward.
Answer:
192 ft

EXAMPLE 4 *Packaging*

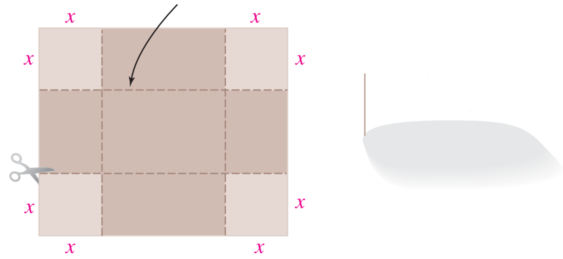
To make boxes, a manufacturer cuts equal-sized squares from each corner of a 10 in. × 12 in. piece of cardboard and then folds up the sides. See the figure on the next page. The polynomial function $f(x) = 4x^3 - 44x^2 + 120x$ gives the volume (in cubic inches) of the resulting box when a square with sides x inches long is cut from each corner. Find the volume of a box if 3-inch squares are cut out.

Self Check 4

PACKAGING In Example 4, find the volumeof the resulting box if 2-inch squares are cut from each corner of the cardboard. 96 in.^{3}

Now Try **Problem 34**

Teaching Example 4 PACKAGING In Example 4, find the volume of the resulting box if 1-inch squares are cut from each corner of the cardboard.
Answer:
80 in.^{3}



Strategy We will find $f(3)$.

WHY The notation $f(3)$ represents the volume of the box when a square with sides of 3 inches is cut from each corner.

Solution

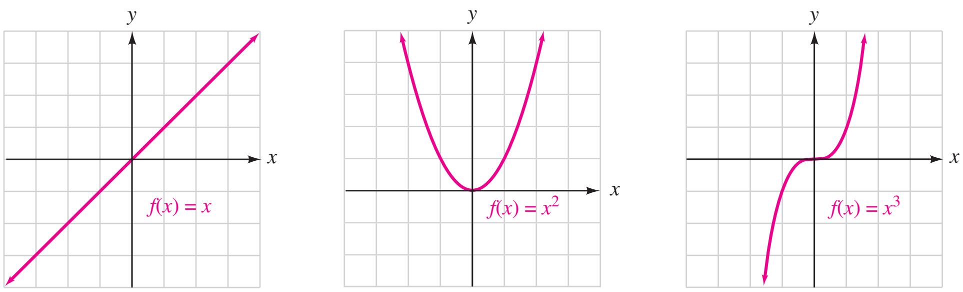
To find the volume of the box, we evaluate the function for $x = 3$.

$$f(x) = 4x^3 - 44x^2 + 120x \quad \text{This is the given function.}$$
$$f(3) = 4(3)^3 - 44(3)^2 + 120(3) \quad \text{Substitute 3 for x.}$$
$$= 4(27) - 44(9) + 120(3) \quad \text{Evaluate the right-hand side.}$$
$$= 108 - 396 + 360$$
$$= 72$$

If 3-inch squares are cut out, the box will have a volume of 72 in.^{3}.

3 Graph polynomial functions.

We have previously graphed three basic polynomial functions. The figure below shows the graph of a linear function $f(x) = x$, the graph of the squaring function $f(x) = x^2$, and the graph of the cubing function $f(x) = x^3$. From the graphs in the figure, it is easy to determine the domain and range of each of these functions.



The identity function
The domain is $(-\infty, \infty)$.
The range is $(-\infty, \infty)$.
(a)

The squaring function
The domain is $(-\infty, \infty)$.
The range is $[0, \infty)$.
(b)

The cubing function
The domain is $(-\infty, \infty)$.
The range is $(-\infty, \infty)$.
(c)

When graphing a linear function, we need to plot only two points, because the graph is a straight line. The graphs of polynomial functions of degree greater than 1 are smooth, continuous curves. To graph them, we must plot more points.

In Example 3, we saw that the polynomial function $h(t) = -16t^2 + 128t$ gives the height of the rocket t seconds after it is shot upward. Since the height of the rocket depends on time, we say that the height is a function of time. To graph this function, we can make a table of values, plot the points, and join them with a smooth curve.

EXAMPLE 5

Graph $h(t) = -16t^2 + 128t$ and find its domain and range.

Strategy We will graph the function by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

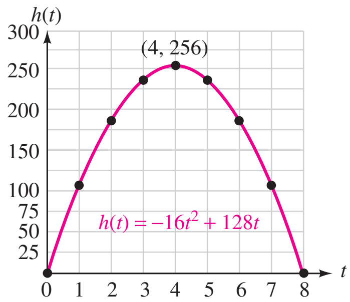
From Example 3, we have seen that

At 0 second, the rocket's height is 0: $h(0) = 0$.

At 3 seconds, the rocket's height is 240: $h(3) = 240$.

To graph the function, we select other values of t, evaluate the function at those values, and write the ordered pairs in the table shown in the figure below. Then we plot the pairs and join the resulting points to get the parabola shown in the figure. From the graph, we can see that 4 seconds into the flight, the rocket attains a maximum height of 256 feet.

$h(t) = -16t^2 + 128t$		
t	$h(t)$	$(t, h(t))$
0	0	(0, 0)
1	112	(1, 112)
2	192	(2, 192)
3	240	(3, 240)
4	256	(4, 256)
5	240	(5, 240)
6	192	(6, 192)
7	112	(7, 112)
8	0	(8, 0)

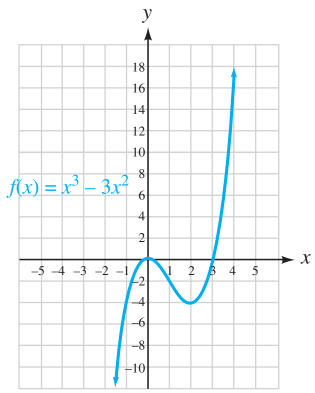


The parabola shown in the figure describes the height of the rocket in relation to time. It does not show the path of the rocket. The rocket goes straight up and then falls straight down.

From the graph, we see that the domain of the function is the interval [0, 8] and the range is the interval [0, 256].

Self Check 5

Graph $f(x) = x^3 - 3x^2$ and find its domain and range.



Now Try **Problem 35**

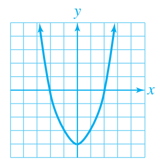
Self Check 5 Answer

D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

Teaching Example 5 Graph $f(x) = x^2 - 4$ and find its domain and range.

Answer:

D: $(-\infty, \infty)$, R: $[-4, \infty)$



EXAMPLE 6

Graph: $f(x) = x^3 - 3x^2 - 9x + 2$

Strategy We will graph the function by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

To graph this cubic function, we begin by evaluating it for $x = -3$.

$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 2 \\ f(-3) &= (-3)^3 - 3(-3)^2 - 9(-3) + 2 \\ &= -27 - 3(9) - 9(-3) + 2 \\ &= -27 - 27 + 27 + 2 \\ &= -25 \end{aligned}$$

Self Check 6

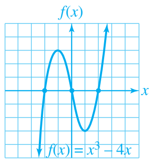
What are the domain and the range of the function graphed in the figure in Example 6?

Now Try **Problem 37**

Self Check 6 Answer

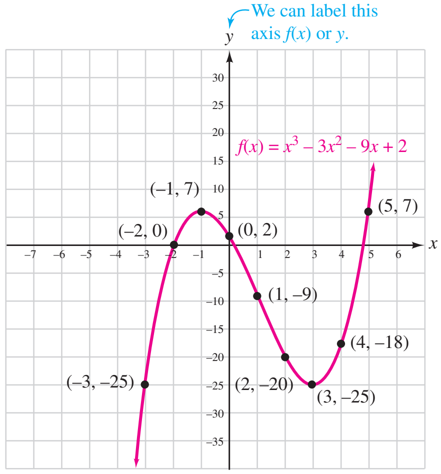
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

Teaching Example 6 Graph: $f(x) = x^3 - 4x$
Answer:



In the following table, we enter the ordered pair $(-3, -25)$. We continue the evaluation process for $x = -2, -1, 0, 1, 2, 3, 4,$ and 5, and list the results in the table. After plotting the ordered pairs, we draw a smooth curve through the points to get the graph of function f.

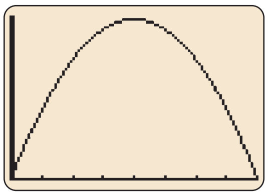
$f(x) = x^3 - 3x^2 - 9x + 2$		
x	$f(x)$	$(x, f(x))$
-3	-25	$(-3, -25)$
-2	0	$(-2, 0)$
-1	7	$(-1, 7)$
0	2	$(0, 2)$
1	-9	$(1, -9)$
2	-20	$(2, -20)$
3	-25	$(3, -25)$
4	-18	$(4, -18)$
5	7	$(5, 7)$

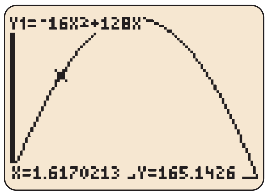


Using Your CALCULATOR Graphing Polynomial Functions

We can graph polynomial functions with a graphing calculator. For example, to graph the function from Example 5, $h(t) = -16t^2 + 128t$, we can rewrite it as $f(x) = -16x^2 + 128x$ and use window settings of $[0, 8]$ for x and $[0, 260]$ for y to get the parabola shown in figure (a).

We can trace to estimate the height of the rocket for any number of seconds into the flight. Figure (b) shows that the height of the rocket 1.6 seconds into the flight is approximately 165 feet.


(a)



(b)

4 Simplify polynomials by combining like terms.

Recall that **like terms** have the same variables with the same exponents:

Like terms	*Unlike terms*	
$-7x$ and $15x$	$-7x$ and $15a$	Different variables.
$4y^3$ and $16y^3$	$4y^3$ and $16y^2$	Different exponents on the same variable.
$\frac{1}{2}xy^2$ and $-\frac{1}{3}xy^2$	$\frac{1}{2}xy^2$ and $-\frac{1}{3}x^2y$	Different exponents on different variables.

Also recall that to **combine like terms,** we combine their coefficients and keep the same variables with the same exponents. For example,

$$4y + 5y = (4 + 5)y \qquad 8x^2 - x^2 = (8 - 1)x^2$$
$$= 9y \qquad\qquad = 7x^2$$

Polynomials with like terms can be simplified by combining like terms.

EXAMPLE 7 Simplify each polynomial by combining like terms.

a. $4x^4 + 81x^4$ **b.** $17x^2y^2 + 2x^2y - 6x^2y^2$

c. $r - 3r^2 - 4r^2 + 8r^2$ **d.** $\frac{3}{5}ab + \frac{4}{3}a - 7 + \frac{1}{2}ab - \frac{1}{6}a + 4$

Strategy We will use the distributive property in reverse to add (or subtract) the coefficients of the like terms. We will keep the same variables raised to the same powers.

WHY To *combine like terms* means to add or subtract the like terms in an expression.

Solution

a. $4x^4 + 81x^4 = 85x^4$ Think: $(4 + 81)x^4 = 85x^4$.

b. The first and third terms are like terms.

$$17x^2y^2 + 2x^2y - 6x^2y^2 = 11x^2y^2 + 2x^2y$$ Think: $(17 - 6)x^2y^2 = 11x^2y^2$.

c. The last three terms are like terms.

$$r - 3r^2 - 4r^2 + 8r^2 = r + r^2$$ Think: $(-3 - 4 + 8)r^2 = 1r^2 = r^2$.

$$= r^2 + r$$ Write the result in descending powers of r.

d. The first and fourth terms are like terms, the second and fifth terms are like terms, and the third and sixth terms are like terms.

$$\frac{3}{5}ab + \frac{4}{3}a - 7 + \frac{1}{2}ab - \frac{1}{6}a + 4$$

$$= \left(\frac{3}{5} + \frac{1}{2}\right)ab + \left(\frac{4}{3} - \frac{1}{6}\right)a - 7 + 4$$ Combine like terms.

$$= \left(\frac{6}{10} + \frac{5}{10}\right)ab + \left(\frac{8}{6} - \frac{1}{6}\right)a - 7 + 4$$ To add and subtract the fractions, build equivalent fractions: $\frac{3}{5} \cdot \frac{2}{2} = \frac{6}{10}$, $\frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10}$, and $\frac{4}{3} \cdot \frac{2}{2} = \frac{8}{6}$.

$$= \frac{11}{10}ab + \frac{7}{6}a - 3$$ Do the additions and the subtraction.

Caution! Do not try to clear this expression of fractions by multiplying it by the LCD 30. That strategy works only when we multiply *both sides of an equation* by the LCD.

$$\cancel{30\left(\frac{3}{5}ab + \frac{4}{3}a - 7 + \frac{1}{2}ab - \frac{1}{6}a + 4\right)}$$

Self Check 7

Simplify each polynomial by combining like terms.

a. $6m^4 + 3m^4$

b. $17s^3t + 3s^2t - 6s^3t$

c. $x - 19x^2 + 22x^2 - x^2$

d. $\frac{7}{8}rs + \frac{7}{9}r + 1 - \frac{3}{4}rs + \frac{4}{3}r - 2$

Now Try **Problems 42 and 46**

Self Check 7 Answers

a. $9m^4$ **b.** $11s^3t + 3s^2t$

c. $2x^2 + x$

d. $\frac{1}{8}rs + \frac{19}{9}r - 1$

Teaching Example 7 Simplify each polynomial by combining like terms.

a. $15a^4 + 2a^4$

b. $6a^2b^3 + 2ab^2 - 4a^2b^3 + ab^2$

c. $x - 3x^2 + 5x + 7x^2$

d. $\frac{1}{3}x^2 + \frac{1}{5}x^2 - 2 + 7$

Answers:

a. $17a^4$ **b.** $2a^2b^3 + 3ab^2$

c. $6x + 4x^2$ **d.** $\frac{8}{15}x^2 + 5$

5 Add polynomials.

When adding polynomials horizontally, each polynomial is usually enclosed within parentheses. For example, $(3x^2 - 2x + 4) + (2x^2 + 4x - 3)$.

Adding Polynomials

To add polynomials, remove the parentheses and combine their like terms.

Self Check 8

Add: $(2a^2 - 3a + 5) + (5a^2 + 4a - 2)$

Now Try **Problem 49**

Self Check 8 Answer

$7a^2 + a + 3$

Teaching Example 8 Add: $(4x^2 + 3x - 6) + (-x^2 + 6x + 12)$

Answer:

$3x^2 + 9x + 6$

EXAMPLE 8 Add: $(3x^2 - 2x + 4) + (2x^2 + 4x - 3)$

Strategy We will drop the parentheses and combine like terms.

WHY To add polynomials means to combine their like terms.

Solution

$(3x^2 - 2x + 4) + (2x^2 + 4x - 3)$ — We are to add two trinomials.

$= 3x^2 - 2x + 4 + 2x^2 + 4x - 3$ — Remove the parentheses.

$= 3x^2 + 2x^2 - 2x + 4x + 4 - 3$ — Use the commutative and associative properties of addition to rearrange terms so that like terms are together.

$= 5x^2 + 2x + 1$ — Combine like terms.

Self Check 9

Add: $(-6a^2b^3 - 5a^3b^2) + (3a^2b^3 + 2a^3b^2)$

Now Try **Problems 51 and 54**

Self Check 9 Answer

$-3a^2b^3 - 3a^3b^2$

Teaching Example 9 Add: $(-7x^2y + 2y^5) + (9x^2y - 3y^5)$

Answer:

$2x^2y - y^5$

EXAMPLE 9 Add: $(-5x^3y^2 - 4x^2y^3) + (2x^3y^2 + 5x^2y)$

Strategy We will drop the parentheses and combine like terms.

WHY To add polynomials means to combine their like terms.

Solution

$(-5x^3y^2 - 4x^2y^3) + (2x^3y^2 + 5x^2y^3)$

$= -5x^3y^2 + 2x^3y^2 - 4x^2y^3 + 5x^2y^3$ — Remove parentheses and rearrange terms.

$= -3x^3y^2 + x^2y^3$ — Combine like terms.

The additions in Examples 8 and 9 can be done by aligning the terms vertically and combining like terms column by column.

$$\begin{array}{rr} & 3x^2 - 2x + 4 \\ + & 2x^2 + 4x - 3 \\ \hline & 5x^2 + 2x + 1 \end{array} \qquad \begin{array}{rr} & -5x^3y^2 - 4x^2y^3 \\ + & 2x^3y^2 + 5x^2y^3 \\ \hline & -3x^3y^2 + x^2y^3 \end{array}$$

6 Subtract polynomials.

Recall that to subtract two real numbers, we add the opposite of the number that is being subtracted. We will use a similar definition to subtract polynomials. To write the **opposite (or negative) of a polynomial,** we change the sign of each of its terms.

- The opposite of $4x^3$ is $-4x^3$.
- The opposite of $y + 4$ is $-(y + 4) = -y - 4$.
- The opposite of $-a^2 + 5a - 16$ is $-(-a^2 + 5a - 16) = a^2 - 5a + 16$.
- The opposite of $7c^2d^2 - cd^2 + 5d - c$ is $-(7c^2d^2 - cd^2 + 5d - c) = -7c^2d^2 + cd^2 - 5d + c$.

Subtracting Polynomials

To subtract two polynomials, drop the parentheses, change the sign of the terms of the polynomial being subtracted, and combine like terms.

EXAMPLE 10 Subtract:

a. $8x^2 - 3x^2$ **b.** $3x^2y - (-9x^2y)$ **c.** $-5x^5y^3z^2 - 3x^5y^3z^2$

Strategy In each case, add the first polynomial to the opposite of the second polynomial.

WHY This is the definition of subtracting polynomials.

Solution

a. $8x^2 - 3x^2 = 8x^2 + (-3x^2)$ Add the opposite of $3x^2$, which is $-3x^2$.

$= 5x^2$ Combine like terms.

b. $3x^2y - (-9x^2y) = 3x^2y + 9x^2y$ The opposite of $-9x^2y$ is $9x^2y$.

$= 12x^2y$

c. $-5x^5y^3z^2 - 3x^5y^3z^2 = -5x^5y^3z^2 + (-3x^5y^3z^2)$

$= -8x^5y^3z^2$

Self Check 10

Subtract:

a. $-2a^2b^3 - 5a^2b^3$ $-7a^2b^3$

b. $-2a^2b^3 - (-5a^2b^3)$ $3a^2b^3$

Now Try **Problems 57 and 60**

Teaching Example 10 Subtract:
a. $15a^3 - 4a^3$
b. $4x^2y - (-2x^2y)$
Answers:
a. $11a^3$ **b.** $6x^2y$

When subtracting polynomials, we can add the opposite of the second polynomial by simply removing the parentheses, *changing the sign of every term of the second polynomial,* and then combining like terms.

EXAMPLE 11 Subtract:

a. $(8x^3 + 2x^2) - (2x^3 - 3x^2 - 1)$ **b.** $(3rt^2 + 4r^2t^2) - (8rt^2 - 4r^2t^2 + r^3t^2)$

Strategy In each case, add the first polynomial to the opposite of the second polynomial.

WHY We can remove a $-$ sign preceding parentheses by dropping the $-$ sign and the parentheses and changing the sign of every term within the parentheses.

Solution

a. $(8x^3 + 2x^2) - (2x^3 - 3x^2 - 1)$

$= 8x^3 + 2x^2 - 2x^3 + 3x^2 + 1$ Remove the parentheses. Change the sign of every term in the second polynomial.

$= 8x^3 - 2x^3 + 2x^2 + 3x^2 + 1$ Rearrange terms.

$= 6x^3 + 5x^2 + 1$ Combine like terms.

b. $(3rt^2 + 4r^2t^2) - (8rt^2 - 4r^2t^2 + r^3t^2)$

$= 3rt^2 + 4r^2t^2 - 8rt^2 + 4r^2t^2 - r^3t^2$ Remove the parentheses. Change the sign of every term in the second polynomial.

$= 3rt^2 - 8rt^2 + 4r^2t^2 + 4r^2t^2 - r^3t^2$ Rearrange terms.

$= -5rt^2 + 8r^2t^2 - r^3t^2$ Combine like terms.

Self Check 11

Subtract: $(6a^2b^3 - 2a^2b^2) - (-2a^2b^3 + a^2b^2)$.

Now Try **Problem 62**

Self Check 11 Answer
$8a^2b^3 - 3a^2b^2$

Teaching Example 11 Subtract:
$(5x^2y - 4xy + 3y^2) - (-2x^2y + xy - 5y^2)$
Answer:
$7x^2y - 5xy + 8y^2$

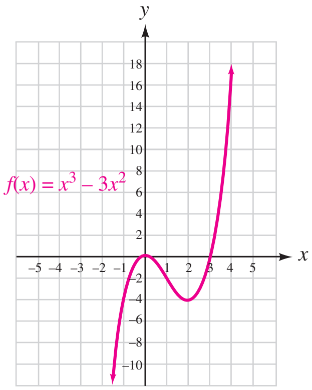
To subtract polynomials in vertical form, we add the opposite (or the negative) of the polynomial that is being subtracted.

$$\begin{array}{rr} & 8x^3y + 2x^2y \\ - & \underline{2x^3y - 3x^2y} \end{array} \Rightarrow \begin{array}{rr} & 8x^3y + 2x^2y \\ + & \underline{-2x^3y + 3x^2y} \\ & 6x^3y + 5x^2y \end{array}$$

This is the opposite of $2x^3y - 3x^2y$.

ANSWERS TO SELF CHECKS

1. a. 2 **b.** 5 **c.** 17 **2. a.** trinomial, degree 2 **b.** binomial, degree 12 **3.** 256 ft **4.** 96 in.^{3}
5. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



6. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$
7. a. $9m^4$ **b.** $11s^3t + 3s^2t$
c. $2x^2 + x$
d. $\frac{1}{8}rs + \frac{19}{9}r - 1$
8. $7a^2 + a + 3$
9. $-3a^2b^3 - 3a^3b^2$
10. a. $-7a^2b^3$ **b.** $3a^2b^3$
11. $8a^2b^3 - 3a^2b^2$

SECTION 5.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. A polynomial is the sum of one or more algebraic terms whose variables have whole-number exponents.

2. A monomial is a polynomial with one term.

3. A binomial is a polynomial with two terms.

4. A trinomial is a polynomial with three terms.

5. The degree of a monomial in one variable is the exponent on the variable.

6. A second-degree polynomial function is also called a quadratic function.

7. A third-degree polynomial function is also called a cubic function.

8. The coefficient of the term $-15x^2y^3$ is -15. The degree of the term is 5.

9. Terms having the same variables with the same exponents are called like terms.

10. The opposite (or negative) of $x^2 + x - 3$ is $-x^2 - x + 3$.

CONCEPTS

Write each polynomial with the exponents on x in descending order.

11. $3x - 2x^4 + 7 - 5x^2$ $-2x^4 - 5x^2 + 3x + 7$

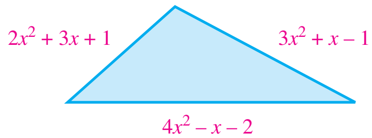
12. $a^2x - ax^3 + 7a^3x^5 - 5a^3x^2$ $7a^3x^5 - ax^3 - 5a^3x^2 + a^2x$

Write each polynomial with the exponents on y in ascending order.

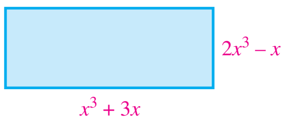
▶ **13.** $4y^2 - 2y^5 + 7y - 5y^3$ $7y + 4y^2 - 5y^3 - 2y^5$

▶ **14.** $x^3y^2 + x^2y^3 - 2x^3y + x^7y^6 - 3x^6$
$-3x^6 - 2x^3y + x^3y^2 + x^2y^3 + x^7y^6$

15. Write a polynomial that represents the perimeter of the triangle. $9x^2 + 3x - 2$



16. Write a polynomial that represents the perimeter of the rectangle. $6x^3 + 4x$



NOTATION

Complete each solution.

17. If $f(x) = 2x^2 + x + 2$, find $f(-1)$.

$$\begin{aligned} f(x) &= 2x^2 + x + 2 \\ f(-1) &= 2(\boxed{-1})^2 + (\boxed{-1}) + 2 \\ &= 2(\boxed{1}) + (-1) + 2 \\ &= \boxed{3} \end{aligned}$$

18. If $h(t) = -t^3 - t^2 + 2t + 1$, find $h(3)$.

$$\begin{aligned} h(t) &= -t^3 - t^2 + 2t + 1 \\ h(3) &= -(\boxed{3})^3 - (\boxed{3})^2 + 2(3) + 1 \\ &= \boxed{-27} - 9 + 6 + 1 \\ &= \boxed{-29} \end{aligned}$$

▶ Selected exercises available online at
www.webassign.net/brookscole

GUIDED PRACTICE

Classify each polynomial as a monomial, a binomial, a trinomial, or none of these. Then determine its degree. See Examples 1–2.

19. $3x^2$
monomial, 2

20. $2y^3 + 4y^2$
binomial, 3

21. $3x^2y - 2x + 3y$
trinomial, 3

22. $a^2 + ab + b^2$
trinomial, 2

23. $x^2 - y^2$
binomial, 2

▶ **24.** $\frac{17}{2}x^3 + 3x^2 - x - 4$
none of these, 3

25. 5
monomial, 0

26. $4x^3y^5$
monomial, 8

27. $9x^2y^4 - x - y^{10} + 1$
none of these, 10

28. x^{17}
monomial, 17

▶ **29.** $4x^9 + 3x^2y^4$
binomial, 9

30. -12
monomial, 0

Evaluate the polynomial function $f(x) = 2x^2 - 4$ at each value of x. See Examples 3–4.

31. $x = 0$ -4

32. $x = 3$ 14

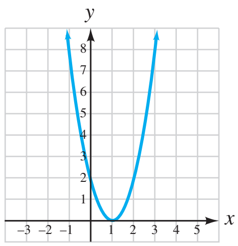
33. $x = -1$ -2

▶ **34.** $x = -4$ 28

Complete each table, graph the polynomial function, and find its domain and range. See Examples 5–6.

▶ **35.** $f(x) = 2x^2 - 4x + 2$ D: $(-\infty, \infty)$, R: $[0, \infty)$

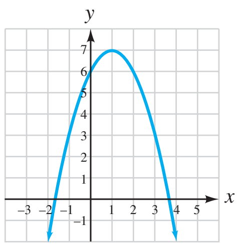
x	$f(x)$
-1	8
0	2
1	0
2	2
3	8



$f(x) = 2x^2 - 4x + 2$

▶ **36.** $f(x) = -x^2 + 2x + 6$ D: $(-\infty, \infty)$, R: $(-\infty, 7]$

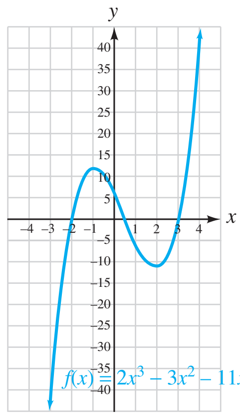
x	$f(x)$
-2	-2
-1	3
0	6
1	7
2	6
3	3
4	-2



$f(x) = -x^2 + 2x + 6$

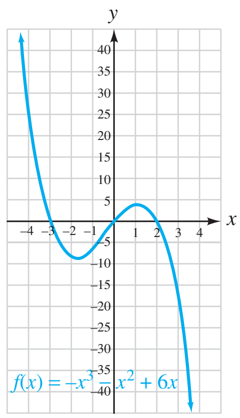
37. $f(x) = 2x^3 - 3x^2 - 11x + 6$ D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

x	$f(x)$
-3	-42
-2	0
-1	12
0	6
1	-6
2	-12
3	0
4	42



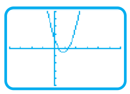
38. $f(x) = -x^3 - x^2 + 6x$ D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

x	$f(x)$
-4	24
-3	0
-2	-8
-1	-6
0	0
1	4
2	0
3	-18

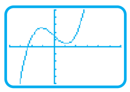


Use a graphing calculator to graph each polynomial function. Use window settings of $[-4, 6]$ for x and $[-5, 5]$ for y.

39. $f(x) = 2.75x^2 - 4.7x + 1.5$



40. $f(x) = 0.37x^3 - 1.4x + 1.5$



Simplify each polynomial by combining like terms. See Example 7.

41. $3x + 7x$
$10x$

42. $-8x + 3y + 5x$
$-3x + 3y$

43. $5x^2 + 6x - 3x^2 - 2x$
$2x^2 + 4x$

▶ **44.** $3mn + 5mn - 2mn$
$6mn$

45. $3r^2t^3 - 8r^2t^3 + 2r^2t^3$
$-3r^2t^3$

46. $9u^2v - 10u^2v - 2uv$
$-u^2v - 2uv$

47. $\frac{2}{3}x^2y^3 + \frac{1}{4}x^2y^2 - \frac{1}{3}x^2y^3$ $\frac{1}{3}x^2y^3 + \frac{1}{4}x^2y^2$

▶ **48.** $\frac{3}{5}x^6y^4z - \frac{1}{4}x^6y^4z - \frac{1}{5}x^6y^4z$ $\frac{3}{20}x^6y^4z$

Perform each addition. See Examples 8–9.

49. $(3x^2 + 2x + 1) + (-2x^2 - 7x + 5)$
$x^2 - 5x + 6$

50. $(-2a^2 - 5a - 7) + (-3a^2 + 7a + 1)$
$-5a^2 + 2a - 6$

51. $(2a^2 + 4a - 7) + (3a^2 - a - 2)$
$5a^2 + 3a - 9$

52. $(7y^3 + 4y^2 + y + 3) + (-8y^3 - y + 3)$
$-y^3 + 4y^2 + 6$

53. $$\begin{array}{r} 2x^2 + 3x + 2 \\ \underline{3x^2 - 2x - 3} \\ 5x^2 + x - 1 \end{array}$$

54. $$\begin{array}{r} 3a^2 - 5a - 21 \\ \underline{3a^2 + 4a + 12} \\ 6a^2 - a - 9 \end{array}$$

55. $$\begin{array}{r} x^2 + 5x + 6 \\ 4x^2 - 3x - 1 \\ \underline{2x^2 + 4x + 5} \\ 7x^2 + 6x + 10 \end{array}$$

56. $$\begin{array}{r} 3b^2 - 4b - 5 \\ -2b^2 + 4b + 3 \\ \underline{-3b^2 - 5b - 2} \\ -2b^2 - 5b - 4 \end{array}$$

Perform each subtraction. See Examples 10–11.

57. $6a^2 - 4a^2$
$2a^2$

58. $7t^2 - (-2t^2)$
$9t^2$

59. $5a^2b - (-8a^2b)$
$13a^2b$

60. $-4x^2yz^3 - 3x^2yz^3$
$-7x^2yz^3$

61. $(-a^2 + 2a + 3) - (4a^2 - 2a - 1)$
$-5a^2 + 4a + 4$

62. $(x^2 - 3x + 8) - (3x^2 + x + 3)$
$-2x^2 - 4x + 5$

63. $(6x^3 + 3x - 2) - (2x^3 + 3x^2 + 5)$
$4x^3 - 3x^2 + 3x - 7$

64. $(-8p^3 - 2p - 4) - (2p^3 + p^2 - p)$
$-10p^3 - p^2 - p - 4$

65. $$\begin{array}{rr} & 3x^2 - 4x + 17 \\ - & \underline{2x^2 + 4x - 5} \\ & x^2 - 8x + 22 \end{array}$$

66. $$\begin{array}{rr} & -2y^2 - 4y + 3 \\ - & \underline{3y^2 + 10y - 5} \\ & -5y^2 - 14y + 8 \end{array}$$

67. $$\begin{array}{rr} & -5y^3 + 4y^2 - 11y + 3 \\ - & \underline{-2y^3 - 14y^2 + 17y - 32} \\ & -3y^3 + 18y^2 - 28y + 35 \end{array}$$

68. $$\begin{array}{rr} & 17x^4 - 3x^2 - 65x - 12 \\ - & \underline{23x^4 + 14x^2 + 3x - 23} \\ & -6x^4 - 17x^2 - 68x + 11 \end{array}$$

TRY IT YOURSELF

Perform each operation.

69. $(3pq + p - q) + (-pq - p - q)$
$2pq - 2q$

70. $(-2mn + 2m - n) - (-2mn - 2m + n)$
$4m - 2n$

71. $(-2x^2y^3 + 6xy + 5y^2) - (-4x^2y^3 - 7xy + 2y^2)$
$2x^2y^3 + 13xy + 3y^2$

72. $(3ax^3 - 2ax^2 + 3a^3) + (4ax^3 + 3ax^2 - 2a^3)$
$7ax^3 + ax^2 + a^3$

73. $(3x^2 + 4x - 3) + (2x^2 - 3x - 1) - (x^2 + x + 7)$
$4x^2 - 11$

74. $(-2x^2 + 6x + 5) - (-4x^2 - 7x + 2) - (4x^2 + 10x + 5)$
$-2x^2 + 3x - 2$

75. $$\begin{array}{rr} & 3x^3 - 2x^2 + 4x - 3 \\ & -2x^3 + 3x^2 + 3x - 2 \\ + & \underline{5x^3 - 7x^2 + 7x - 12} \\ & 6x^3 - 6x^2 + 14x - 17 \end{array}$$

76. $$\begin{array}{rr} & 7a^3 + 3a + 7 \\ & -2a^3 + 4a^2 - 13 \\ + & \underline{3a^3 - 3a^2 + 4a + 5} \\ & 8a^3 + a^2 + 7a - 1 \end{array}$$

77. $$\begin{array}{rr} & 4x^2 + 0x \\ - & \underline{4x^2 - 2x} \\ & 2x \end{array}$$

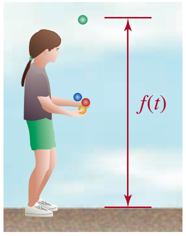
78. $$\begin{array}{rr} & -2a^3 + 0a^2 \\ - & \underline{-2a^3 + 3a^2} \\ & -3a^2 \end{array}$$

79. Subtract $(-x^2 - 2x + 1)$ from the sum of $(4x^2 + 5x - 3)$ and $(7x^2 + 2x - 10)$.
$12x^2 + 9x - 14$

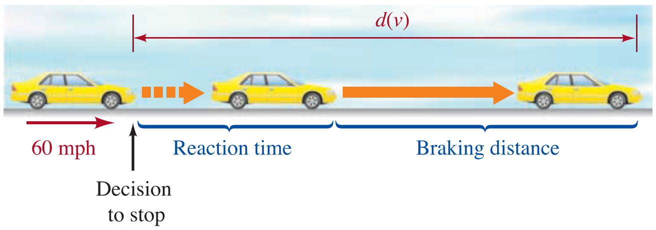
80. Subtract $(a^2b - 2ab + b)$ from the sum of $(-6a^2b + 4ab - a)$ and $(8a^2b + 2ab - 10b)$.
$a^2b + 8ab - a - 11b$

APPLICATIONS

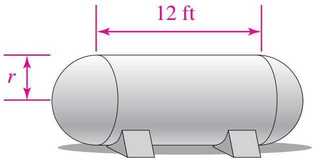
81. JUGGLING A juggler tosses one ball straight upward while continuing to juggle three others. The height $f(t)$, in feet, of the ball is given by the polynomial function $f(t) = -16t^2 + 32t + 4$, where t is the time in seconds since the ball was thrown. Find the height of the ball 1 second after it is tossed upward. 20 ft



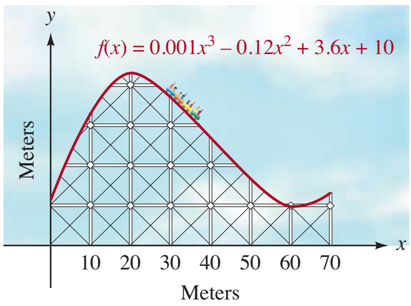
82. STOPPING DISTANCES The number of feet that a car travels before stopping depends on the driver's reaction time and the braking distance. (See the illustration.) For one driver, the stopping distance $d(v)$, in feet, is given by the polynomial function $d(v) = 0.04v^2 + 0.9v$, where v is the velocity of the car in mph. Find the stopping distance at 60 mph. 198 ft



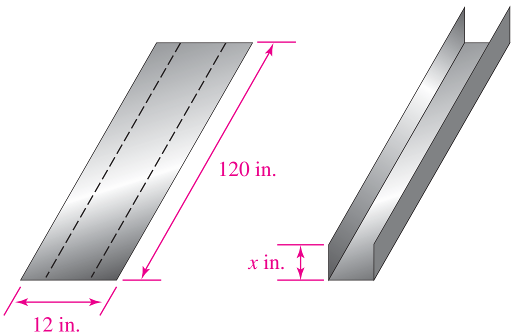
83. STORAGE TANKS The volume $V(r)$ of the gasoline storage tank, in cubic feet, is given by the polynomial function $V(r) = 4.2r^3 + 37.7r^2$, where r is the radius in feet of the cylindrical part of the tank. What is the capacity of the tank shown if its radius is 4 feet? 872 ft^3



84. ROLLER COASTERS The polynomial function $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$ models the path of a portion of the track of a roller coaster, as shown below. Find the height of the track for $x = 0, 20, 40,$ and 60. 10 m, 42 m, 26 m, 10 m



85. RAIN GUTTERS A rectangular sheet of metal will be used to make a rain gutter by bending up its sides, as shown below. If the ends are covered, the capacity $f(x)$ of the gutter is a polynomial function of x: $f(x) = -240x^2 + 1{,}440x$. Find the capacity of the gutter if x is 3 inches. 2,160 in.^{3}



86. CUSTOMER SERVICE A software company has found that on Mondays, the polynomial function $C(t) = -0.0625t^4 + t^3 - 6t^2 + 16t$ approximates the number of callers to its hotline at any one time. Here, t represents the time, in hours, since the hotline opened at 8:00 A.M. How many service technicians should be on duty on Mondays at noon if the company doesn't want any callers to the hotline waiting to be helped by a technician? 16

87. REAL ESTATE A computer analysis of two properties on the market generated functions to predict the value, in dollars, of each property after x years.

Rental home: $R(x) = 1{,}100x + 125{,}000$

Duplex: $D(x) = 1{,}400x + 150{,}000$

a. Find one polynomial function V that will give the combined value of the two properties after x years.
$V(x) = 2{,}500x + 275{,}000$

b. Use your answer to part a to find the combined value of the two properties after 20 years.
\$325,000

88. BUSINESS EXPENSES A company purchased two cars for its sales force to use. The following functions give the respective values of the vehicles after x years.

Toyota Camry LE: $T(x) = -2{,}100x + 16{,}600$

Ford Explorer Sport: $F(x) = -2{,}700x + 19{,}200$

a. Find one polynomial function V that will give the value of both cars after x years.
$V(x) = -4{,}800x + 35{,}800$

b. Use your answer to part a to find the combined value of the two cars after 3 years.
\$21,400

89. MATHEMATICS In calculus, an important polynomial function is

$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

Refer to the polynomial on the right side of the equation.

a. How many terms does the polynomial have? 4

b. Are the terms written in descending or ascending powers of x?
ascending

c. What is the coefficient of the fourth term?
$-\frac{1}{720}$

d. What is the degree of the third term?
fourth

e. What is the degree of the polynomial?
sixth

90. CALCULUS See Exercise 89. Another polynomial function used in advanced mathematics is

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

a. Find $f(0)$.
1

b. Find $f(1)$.
$\frac{65}{24} = 2\frac{17}{24}$

WRITING

91. Explain why the terms x^2y and xy^2 are not like terms.

92. The family of polynomial functions contains linear, quadratic, and cubic functions. Explain.

93. Explain what is wrong in the following solution.

$$\cancel{\begin{aligned}(2x - 3) - (3x + 4) &= 2x - 3 - 3x - 4\\ &= -x - 7\end{aligned}}$$

▶ **94.** Explain why $f(x) = \frac{1}{x + 1}$ is not a polynomial function.

95. Use the word *descending* in a sentence in which the context is not mathematical. Do the same for the word *ascending*.

96. Look up the meaning of the prefix *poly* in a dictionary. Why do you think the name *polynomial* was given to expressions such as $x^3 - x^2 + 2x + 15$?

REVIEW

Solve each inequality. Write the solution set in interval notation.

97. $|x| \le 5$ $[-5, 5]$

▶ **98.** $|x| > 7$ $(-\infty, -7) \cup (7, \infty)$

99. $|x - 4| < 5$ $(-1, 9)$

100. $|2x + 1| \ge 7$ $(-\infty, -4] \cup [3, \infty)$

Objectives

1. Multiply monomials.
2. Multiply a polynomial by a monomial.
3. Multiply two binomials.
4. Multiply any two polynomials.
5. Multiply three polynomials.
6. Find special products.
7. Use multiplication to simplify expressions.
8. Solve problems using multiplication of polynomials.

SECTION 5.4 Multiplying Polynomials

In this section, we will show how to multiply polynomials. These procedures involve the application of several algebraic concepts introduced in earlier chapters, such as the commutative and associative properties of multiplication, the rules for exponents, and the distributive property.

1 Multiply monomials.

We begin by considering the simplest case of polynomial multiplication, multiplying two monomials.

Multiplying Monomials

To multiply two monomials, multiply the numerical factors (the coefficients) and then multiply the variable factors.

Self Check 1

Multiply:

a. $(-2a^3)(4a^2)$ $-8a^5$

b. $(-5b^3)(-3a)(a^2b)$ $15a^3b^4$

Now Try **Problems 26 and 30**

Teaching Example 1 Multiply:

a. $(-5x^4)(3x^6)$

b. $(-3x^2)(2x^3y)(-x^3y)$

Answers:

a. $-15x^{10}$ **b.** $6x^8y^2$

EXAMPLE 1 Multiply:

a. $(3x^2)(6x^3)$ **b.** $(-8x)(2y)(xy)$ **c.** $(2a^3b)(-7b^2c)(-12ac^4)$

Strategy We will multiply the numerical factors and then multiply the variable factors.

WHY The commutative and associative properties of multiplication enable us to reorder and regroup the factors.

Solution

We can use the commutative and associative properties of multiplication to rearrange and regroup the factors.

a.
$$\begin{aligned}(3x^2)(6x^3) &= 3 \cdot x^2 \cdot 6 \cdot x^3\\ &= (3 \cdot 6)(x^2 \cdot x^3)\\ &= 18x^5\end{aligned}$$

To simplify $x^2 \cdot x^3$, keep the base and add the exponents.

b. $(-8x)(2y)(xy) = -8 \cdot x \cdot 2 \cdot y \cdot x \cdot y$
$= (-8 \cdot 2) \cdot x \cdot x \cdot y \cdot y$
$= -16x^2y^2$

c. $(2a^3b)(-7b^2c)(-12ac^4) = 2 \cdot a^3 \cdot b \cdot (-7) \cdot b^2 \cdot c \cdot (-12) \cdot a \cdot c^4$
$= 2(-7)(-12) \cdot a^3 \cdot a \cdot b \cdot b^2 \cdot c \cdot c^4$
$= 168a^4b^3c^5$

2 Multiply a polynomial by a monomial.

To multiply a polynomial by a monomial, we use the distributive property.

Multiplying Polynomials by Monomials

To multiply a monomial and a polynomial, multiply each term of the polynomial by the monomial.

EXAMPLE 2 Multiply:

a. $3x^2(6xy + 3y^2)$ **b.** $5x^3y^2(xy^3 - 2x^2y)$ **c.** $-2ab^2(3bz - 2az + 4z^3)$

Strategy We will multiply each term of the polynomial by the monomial.

WHY We use the distributive property (and extensions of it) to multiply a monomial and a polynomial.

Solution

We can use the distributive property to remove parentheses.

a. $3x^2(6xy + 3y^2) = 3x^2(6xy) + 3x^2(3y^2)$ Distribute the multiplication by $3x^2$.
$= 18x^3y + 9x^2y^2$ Perform the multiplications.

Since $18x^3y$ and $9x^2y^2$ are not like terms, we cannot add them.

b. $5x^3y^2(xy^3 - 2x^2y) = 5x^3y^2(xy^3) - 5x^3y^2(2x^2y)$ Distribute $5x^3y^2$.
$= 5x^4y^5 - 10x^5y^3$

c. $-2ab^2(3bz - 2az + 4z^3)$
$= -2ab^2 \cdot 3bz - (-2ab^2) \cdot 2az + (-2ab^2) \cdot 4z^3$
$= -6ab^3z + 4a^2b^2z - 8ab^2z^3$

Self Check 2

Multiply: $-2a^2(a^2 - a + 3)$

Now Try Problems 36 and 40

Self Check 2 Answer
$-2a^4 + 2a^3 - 6a^2$

Teaching Example 2 Multiply: $-5x^2y(3xy^4 - 2x^3y^4 - 7)$
Answer:
$-15x^3y^5 + 10x^5y^5 + 35x^2y$

3 Multiply two binomials.

When we multiply two binomials, the distributive property requires that each term of one binomial be multiplied by each term of the other binomial. This fact can be emphasized by drawing arrows to show the indicated products. For example, to multiply $3x + 2$ and $x + 4$, we can write

First terms Last terms

$(3x + 2)(x + 4) = 3x(x) + 3x(4) + 2(x) + 2(4)$
$= 3x^2 + 12x + 2x + 8$
$= 3x^2 + 14x + 8$ Combine like terms: $12x + 2x = 14x$.

Inner terms

Outer terms

We note that

- the product of the **F**irst terms is $3x \cdot x = 3x^2$,
- the product of the **O**uter terms is $3x \cdot 4 = 12x$,
- the product of the **I**nner terms is $2 \cdot x = 2x$, and
- the product of the **L**ast terms is $2 \cdot 4 = 8$.

The procedure is called the **FOIL** method of multiplying two binomials. Foil is an acronym for **F**irst terms, **O**uter terms, **I**nner terms, and **L**ast terms. Of course, the resulting terms of the product must be combined, if possible.

It is easy to multiply binomials by sight using the FOIL method. We find the product of the first terms, then find the products of the outer terms and the inner terms and add them (when possible), and then find the product of the last terms.

Self Check 3

Multiply:

a. $(3a + 4b)(2a - b)$

b. $(a^2b - 3)(a^2b - 1)$

c. $(6c^4 - d)(3c + d)$

Now Try **Problems 42 and 46**

Self Check 3 Answers

a. $6a^2 + 5ab - 4b^2$

b. $a^4b^2 - 4a^2b + 3$

c. $18c^5 + 6c^4d - 3cd - d^2$

Teaching Example 3 Multiply:

a. $(x - 3y)(4x + 5y)$

b. $(2x^2 + 5)(7x^2 + 1)$

c. $(6x - 2y)(x^2 - 5y)$

Answers:

a. $4x^2 - 7xy - 15y^2$

b. $14x^4 + 37x^2 + 5$

c. $6x^3 - 30xy - 2x^2y + 10y^2$

EXAMPLE 3 Multiply:

a. $(2x - 3)(3x + 2)$ **b.** $(3x^2 + 1)(3x^2 + 4)$ **c.** $(4xy - 5)(2x^2 - 3y)$

Strategy We will use the FOIL method to multiply the binomials.

WHY In each case we are to find the product of two binomials, and the FOIL method is a shortcut for multiplying two binomials.

Solution

a. $(2x - 3)(3x + 2) = 6x^2 - 5x - 6$

The product of the first terms is $2x \cdot 3x = 6x^2$. The middle term in the result comes from combining the outer and inner products of $4x$ and $-9x$:

$$4x + (-9x) = -5x$$

The product of the last terms is $-3 \cdot 2 = -6$.

b. $(3x^2 + 1)(3x^2 + 4) = 9x^4 + 15x^2 + 4$

The product of the first terms is $3x^2 \cdot 3x^2 = 9x^4$. The middle term in the result comes from combining the products $12x^2$ and $3x^2$:

$$12x^2 + 3x^2 = 15x^2$$

The product of the last terms is $1 \cdot 4 = 4$.

c. $(4xy - 5)(2x^2 - 3y) = 8x^3y - 12xy^2 - 10x^2 + 15y$

The product of the first terms is $4xy \cdot 2x^2 = 8x^3y$. The product of the outer terms is $4xy(-3y) = -12xy^2$. The product of the inner terms is $-5(2x^2) = -10x^2$. The product of the last terms is $-5(-3y) = 15y$. Since the terms of $8x^3y - 12xy^2 - 10x^2 + 15y$ are unlike, we cannot simplify this result.

4 Multiply any two polynomials.

To multiply a polynomial by a polynomial, we use the distributive property repeatedly.

EXAMPLE 4 Multiply: $(2a + b)(3a^2 - 4ab - b^2)$

Strategy We will multiply each term of the trinomial $3a^2 - 4ab + b^2$ by each term of the binomial $2a + b$.

WHY To multiply two polynomials, we must multiply each term of one polynomial by each term of the other polynomial.

Solution

$$(2a + b)(3a^2 - 4ab - b^2)$$

$$\begin{aligned} &= 2a(3a^2) + 2a(-4ab) + 2a(-b^2) + b(3a^2) + b(-4ab) + b(-b^2) \\ &= 6a^3 - 8a^2b - 2ab^2 + 3a^2b - 4ab^2 - b^3 && \text{Multiply the monomials.} \\ &= 6a^3 - 5a^2b - 6ab^2 - b^3 && \text{Combine like terms.} \end{aligned}$$

Self Check 4

Multiply:
$(2a + b)(3a^2 - ab + 7b^2)$

Now Try **Problem 52**

Self Check 4 Answer
$6a^3 + a^2b + 13ab^2 + 7b^3$

Teaching Example 4 Multiply:
$(6x + 2y)(7x^2 - xy + 3y^2)$
Answer:
$42x^3 + 8x^2y + 16xy^2 + 6y^3$

The results of Example 4 suggest the following rule.

Multiplying Polynomials

To multiply two polynomials, multiply each term of one polynomial by each term of the other polynomial, and then combine like terms.

It is often convenient to multiply polynomials using vertical form.

EXAMPLE 5 Use vertical form to multiply:

a. $(3x + 2)(4x + 9)$ **b.** $(3a^2 - 4ab + b^2)(2a - b)$

Strategy We will write one polynomial underneath the other and multiply each term of the upper polynomial by each term of the lower polynomial, lining up like terms. Then we will combine like terms column by column.

WHY *Vertical form* means to use an approach similar to that used in arithmetic to multiply two whole numbers.

Solution

a.

$$\begin{array}{rrrl} & 3x & + 2 & \\ & 4x & + 9 & \\ \hline & 27x & + 18 & \leftarrow \text{This is the result of multiplying } 3x + 2 \text{ by } 9. \\ 12x^2 & + 8x & & \leftarrow \text{This is the result of multiplying } 3x + 2 \text{ by } 4x. \\ \hline 12x^2 & + 35x & + 18 & \text{Combine like terms, column by column.} \end{array}$$

b.

$$\begin{array}{rrrrl} & 3a^2 & - 4ab & + b^2 & \\ & & 2a & - b & \\ \hline & - 3a^2b & + 4ab^2 & - b^3 & \leftarrow \text{Multiply } 3a^2 - 4ab + b^2 \text{ by } -b. \\ 6a^3 & - 8a^2b & + 2ab^2 & & \leftarrow \text{Multiply } 3a^2 - 4ab + b^2 \text{ by } 2a. \\ \hline 6a^3 & - 11a^2b & + 6ab^2 & - b^3 & \text{Combine like terms, column by column.} \end{array}$$

Self Check 5

Multiply:
$$\begin{array}{r} 3x^2 + 2x - 5 \\ 2x + 1 \\ \hline \end{array}$$
$6x^3 + 7x^2 - 8x - 5$

Now Try **Problem 56**

Teaching Example 5 Multiply:
$$\begin{array}{r} 4a^2 - 2a + 3 \\ 3a - 1 \\ \hline \end{array}$$
Answer:
$12a^3 - 10a^2 + 11a - 3$

Self Check 6

Find the product of $2a^2 + 6a - 12$ and $3a^2 + 9a - 9$.

Now Try **Problem 60**

Self Check 6 Answer
$6a^4 + 36a^3 - 162a + 108$

Teaching Example 6 Find the product of $3x^2 - 2x + 5$ and $2x^2 + 7x - 4$.
Answer:
$6x^4 + 17x^3 - 16x^2 + 43x - 20$

EXAMPLE 6 Find the product of $-2y^3 - 6y^2 + 12$ and $5y^2 - 10y - 10$.

Strategy We will write one polynomial underneath the other and multiply each term of the upper polynomial by each term of the lower polynomial, lining up like terms. Then we will combine like terms column by column.

WHY *Vertical form* means to use an approach similar to that used in arithmetic to multiply two whole numbers.

Solution

For lengthy multiplications like this, we can use vertical form. We begin by multiplying $-2y^3 - 6y^2 + 12$ by -10; then we multiply $-2y^3 - 6y^2 + 12$ by $-10y$; and finally we multiply $-2y^3 - 6y^2 + 12$ by $5y^2$. Then we combine like terms, column by column.

$$\begin{array}{rrrrrrl}
 & & -2y^3 & -\ 6y^2 & & +\ 12 & \\
 & & & 5y^2 & -\ 10y & -\ 10 & \\
\hline
 & & 20y^3 & +\ 60y^2 & & -\ 120 & \text{There is no } y\text{-term; leave a space.} \\
 & 20y^4 & +\ 60y^3 & & -\ 120y & & \text{There is no } y^2\text{-term; leave a space.} \\
-10y^5 & -\ 30y^4 & & +\ 60y^2 & & & \text{There is no } y^3\text{-term; leave a space.} \\
\hline
-10y^5 & -\ 10y^4 & +\ 80y^3 & +\ 120y^2 & -\ 120y & -\ 120 &
\end{array}$$

5 Multiply three polynomials.

When finding the product of three polynomials, we begin by multiplying *any* two of them, and then we multiply that result by the third polynomial.

Self Check 7

Multiply: $-2r(r - 2s)(5r - 4s)$

Now Try **Problem 62**

Self Check 7 Answer
$-10r^3 + 28r^2s - 16rs^2$

Teaching Example 7 Multiply:
$-3x(2x + 5y)(4x - y)$
Answer:
$-24x^3 - 54x^2y + 15xy^2$

EXAMPLE 7 Multiply: $3cd(c + 2d)(3c - d)$

Strategy We will find the product of $c + 2d$ and $3c - d$ and then multiply that result by $3cd$.

WHY It is wise to perform the most difficult multiplication first. (In this case, this is the product of two binomials). Then carry out the simpler multiplication last.

Solution

First, we find the product of the two binomials. Then we multiply that result by $3cd$.

$$\begin{aligned}
3cd\mathbf{(c + 2d)(3c - d)} &= 3cd\mathbf{(3c^2 - cd + 6cd - 2d^2)} && \text{Use the FOIL method to find } (c + 2d)(3c - d). \\
&= 3cd(3c^2 + 5cd - 2d^2) && \text{Combine like terms: } -cd + 6cd = 5cd. \\
&= 9c^3d + 15c^2d^2 - 6cd^3 && \text{Distribute the multiplication by } 3cd.
\end{aligned}$$

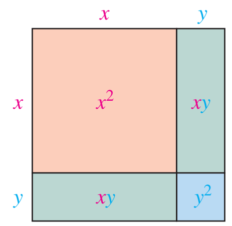
6 Find special products.

We often must find the square of a binomial. To do so, we can use the FOIL method. For example, to find $(x + y)^2$ and $(x - y)^2$, we proceed as follows.

$$\begin{aligned}
(x + y)^2 &= (x + y)(x + y) \\
&= x^2 + xy + xy + y^2 \\
&= x^2 + 2xy + y^2
\end{aligned}
\qquad\qquad
\begin{aligned}
(x - y)^2 &= (x - y)(x - y) \\
&= x^2 - xy - xy + y^2 \\
&= x^2 - 2xy + y^2
\end{aligned}$$

In each case, we see that the square of the binomial is the square of its first term, twice the product of its two terms, and the square of its last term.

The figure below shows how $(x + y)^2$ can be found geometrically.



The area of the largest square is the product of its length and width: $(x + y)(x + y) = (x + y)^2$.

The area of the largest square is also the sum of its four parts: $x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$.

Thus, $(x + y)^2 = x^2 + 2xy + y^2$.

Another common binomial product is the product of the sum and difference of the same two terms. An example of such a product is $(x + y)(x - y)$. To find this product, we use the FOIL method.

$$\begin{aligned}(x + y)(x - y) &= x^2 - xy + xy + y^2 \\ &= x^2 - y^2 \qquad \text{Combine like terms: } -xy + xy = 0.\end{aligned}$$

We see that the product of the sum and the difference of the same two terms is the square of the first term minus the square of the second term.

The square of a binomial and the product of the sum and the difference of the same two terms are called **special products.** Because special products occur so often, it is useful to learn their forms.

Special Product Formulas

$(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$ — The square of a sum

$(x - y)^2 = (x - y)(x - y) = x^2 - 2xy + y^2$ — The square of a difference

$(x + y)(x - y) = x^2 - y^2$ — The product of the sum and difference of two terms

Caution! The squares $(x + y)^2$ and $(x - y)^2$ have trinomials for their products. Don't forget to write the middle terms in these products. Remember that

$$(x + y)^2 \neq x^2 + y^2 \qquad \text{and} \qquad (x - y)^2 \neq x^2 - y^2$$

Missing $2xy$ — Missing $-2xy$ — Should be $+$ symbol

Also remember that the product $(x + y)(x - y)$ is the binomial $x^2 - y^2$. And since $(x + y)(x - y) = (x - y)(x + y)$ by the commutative property of multiplication,

$$(x - y)(x + y) = x^2 - y^2$$

EXAMPLE 8 Multiply:

a. $(5c + 3d)^2$ **b.** $\left(\frac{1}{2}a^4 - b^2\right)^2$ **c.** $(0.2m^3 + 2.5n)(0.2m^3 - 2.5n)$

Strategy We will find the product of each pair of binomials, and we will use the appropriate special-product rule.

WHY This approach is faster than using the FOIL method.

Self Check 8

Multiply:

a. $(8r + 2s)^2$

b. $\left(\frac{1}{3}a^3 - b^6\right)^2$

c. $(0.4x + 1.2y^4)(0.4x - 1.2y^4)$

Now Try **Problems 66, 70, and 76**

Self Check 8 Answers
a. $64r^2 + 32rs + 4s^2$
b. $\frac{1}{9}a^6 - \frac{2}{3}a^3b^6 + b^{12}$
c. $0.16x^2 - 1.44y^8$

Teaching Example 8 Multiply:
a. $(5x - 2y)^2$
b. $(3x + 4y)^2$
c. $\left(\frac{1}{2}x + \frac{1}{3}y\right)\left(\frac{1}{2}x - \frac{1}{3}y\right)$
Answers:
a. $25x^2 - 20xy + 4y^2$
b. $9x^2 + 24xy + 16y^2$
c. $\frac{1}{4}x^2 - \frac{1}{9}y^2$

Solution

a. To find $(5c + 3d)^2$ using a special product formula, we begin by noting that the first term of the binomial is $5c$ and the last term is $3d$.

$$(5c + 3d)^2 = \underset{\text{The square of the first term}}{(5c)^2} + \underset{\text{Twice the product of the two terms}}{2(5c)(3d)} + \underset{\text{The square of the last term}}{(3d)^2}$$
$$= 25c^2 + 30cd + 9d^2$$

b. To find $\left(\frac{1}{2}a^4 - b^2\right)^2$ using a special product formula, we begin by noting that the first term of the binomial is $\frac{1}{2}a^4$ and the last term is $-b^2$.

$$\left(\frac{1}{2}a^4 - b^2\right)^2 = \underset{\text{The square of the first term}}{\left(\frac{1}{2}a^4\right)^2} + \underset{\text{Twice the product of the two terms}}{2\left(\frac{1}{2}a^4\right)(-b^2)} + \underset{\text{The square of the last term}}{(-b^2)^2}$$
$$= \frac{1}{4}a^8 - a^4b^2 + b^4$$

c. $(0.2m^3 + 2.5n)(0.2m^3 - 2.5n)$ is the product of the sum and the difference of the same two terms: $0.2m^3$ and $2.5n$. Using a special product formula, we proceed as follows.

$$(0.2m^3 + 2.5n)(0.2m^3 - 2.5n) = \underset{\text{The square of the first term}}{(0.2m^3)^2} - \underset{\text{The square of the last term}}{(2.5n)^2}$$
$$= 0.04m^6 - 6.25n^2$$

7 Use multiplication to simplify expressions.

The procedures discussed in this section are often useful when we simplify algebraic expressions that involve the multiplication of polynomials.

Self Check 9
Simplify:
$(y - 7)(y + 7) - (4y + 3)^2$
Now Try **Problem 80**
Self Check 9 Answer
$-15y^2 - 24y - 58$

Teaching Example 9 Simplify:
$(x - 5)(x + 2) - (2x - 3)^2$
Answer:
$-3x^2 + 9x - 19$

EXAMPLE 9 Simplify: $(5x - 4)^2 - (x - 7)(x + 1)$

Strategy We will use a special product formula to find $(5x - 4)^2$ and the FOIL method to find $(x - 7)(x + 1)$.

WHY To simplify expressions, we need to follow the order of operations rule. We find the powers and perform the multiplication before performing the subtraction.

Solution

$(5x - 4)^2 - (x - 7)(x + 1)$

$= 25x^2 - 40x + 16 - (x^2 - 6x - 7)$ $\quad$ $(5x-4)^2 = (5x)^2 + 2(5x)(-4) + (-4)^2$.

$= 25x^2 - 40x + 16 - x^2 + 6x + 7$ $\quad$ To subtract $(x^2 - 6x - 7)$, remove the parentheses and change the sign of each term within the parentheses.

$= 24x^2 - 34x + 23$ $\quad$ Combine like terms.

8 Solve problems using multiplication of polynomials.

Profit, revenue, and cost are terms used in the business world. The profit p earned on the sale of one or more items is given by the formula

$$\text{Profit} = \text{revenue} - \text{cost}$$

If a salesperson has 12 vacuum cleaners and sells them for \$225 each, the revenue will be $r = 12 \cdot \$225 = \$2{,}700$. This illustrates the following formula for finding the revenue r:

$$r = \boxed{\text{number of items sold } x} \cdot \boxed{\text{selling price of each item } p} = xp = px$$

EXAMPLE 10 ***Selling Vacuum Cleaners*** Over the years, a saleswoman has found that the number of vacuum cleaners she can sell depends on price. The lower the price, the more she can sell. She has determined that the number of vacuums x that she can sell at a price p is related by the equation $x = -\frac{2}{25}p + 28$.

a. Find a formula for the revenue r.

b. How much revenue will be taken in if the vacuums are priced at \$250?

Strategy We will write polynomials that represent the selling price and the number of items sold and find their product.

WHY The revenue is the product of the price and the number of items sold.

Solution

a. To find a formula for revenue, we substitute $-\frac{2}{25}p + 28$ for x in the formula $r = px$ and multiply.

$$r = px \quad \text{This is the formula for revenue.}$$

$$r = p\left(-\frac{2}{25}p + 28\right) \quad \text{Substitute } -\frac{2}{25}p + 28 \text{ for } x.$$

$$r = -\frac{2}{25}p^2 + 28p \quad \text{Multiply the polynomials.}$$

b. To find how much revenue will be taken in if the vacuums are priced at \$250, we substitute 250 for p in the formula for revenue.

$$r = -\frac{2}{25}p^2 + 28p \quad \text{This is the formula for revenue.}$$

$$r = -\frac{2}{25}(250)^2 + 28(250) \quad \text{Substitute 250 for } p.$$

$$= -5{,}000 + 7{,}000$$

$$= 2{,}000$$

The revenue will be \$2,000.

Self Check 10

SCRAPBOOKS An 8.5-inch by 11-inch scrapbook page has a border of uniform width surrounding a rectangular-shaped birth announcement. Write a polynomial that represents the area of the birth announcement.

Now Try **Problem 126**

Self Check 10 Answer
$4x^2 - 39x + 93.5$

Teaching Example 10 GIFT BOXES
The corners of an 8-inch by 8-inch piece of cardboard are creased, folded inward, and glued to make a gift box. Write a polynomial that gives the volume of the resulting box.
Answer:
$x(8 - 2x)(8 - 2x)$ in.^{3} = $(4x^3 - 32x^2 + 64x)$ in.^{3}

ANSWERS TO SELF CHECKS

1. a. $-8a^5$ **b.** $15a^3b^4$ **2.** $-2a^4 + 2a^3 - 6a^2$ **3. a.** $6a^2 + 5ab - 4b^2$ **b.** $a^4b^2 - 4a^2b + 3$ **c.** $18c^5 + 6c^4d - 3cd - d^2$ **4.** $6a^3 + a^2b + 13ab^2 + 7b^3$ **5.** $6x^3 + 7x^2 - 8x - 5$ **6.** $6a^4 + 36a^3 - 162a + 108$ **7.** $-10r^3 + 28r^2s - 16rs^2$ **8. a.** $64r^2 + 32rs + 4s^2$ **b.** $\frac{1}{9}a^6 - \frac{2}{3}a^3b^6 + b^{12}$ **c.** $0.16x^2 - 1.44y^8$ **9.** $-15y^2 - 24y - 58$ **10.** $4x^2 - 39x + 93.5$

SECTION 5.4 STUDY SET

VOCABULARY

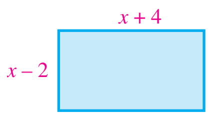
Fill in the blanks.

1. The expression $(x + 4)(x - 5)$ is the product of two binomials.
2. The expression $(x + 4)^2$ is the square of a binomial.
3. The polynomial $3x^2 - 3x + 2$ contains three terms.
4. To find $-2x(3x^2 - 2)$, we use the distributive property.

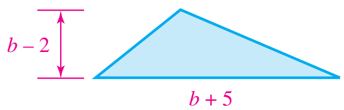
CONCEPTS

Fill in the blanks.

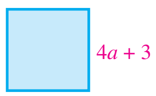
5. To multiply a monomial by a monomial, we multiply the numerical factors and then multiply the variable factors.
▶ 6. To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial.
7. To multiply a polynomial by a polynomial, we multiply each term of one polynomial by each term of the other polynomial.
▶ 8. FOIL is an acronym for First terms, Outer terms, Inner terms, and Last terms.
9. $(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$
10. $(x - y)^2 = (x - y)(x - y) = x^2 - 2xy + y^2$
11. $(x + y)(x - y) = x^2 - y^2$
12. The square of a binomial is the square of its first term, twice the product of its two terms, and the square of its last term.
13. The product of the sum and difference of the same two terms is the square of the first term minus the square of the second term.
14. Write a polynomial that represents the area of the rectangle. $x^2 + 2x - 8$



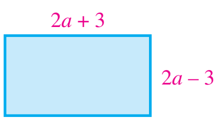
15. Write a polynomial that represents the area of the triangle. $\frac{1}{2}b^2 + \frac{3}{2}b - 5$



16. Write a polynomial that represents the area of the square. $16a^2 + 24a + 9$



17. Write a polynomial that represents the area of the rectangle. $4a^2 - 9$



Consider $(2x + 4)(4x - 3)$. *Give the*

18. First terms
$2x, 4x$
19. Outer terms
$2x, -3$
20. Inner terms
$4, 4x$
21. Last terms
$4, -3$

Perform each operation.

▶ 22. $(4b - 1) + (2b - 1)$
$6b - 2$
▶ 23. $(4b - 1)(2b - 1)$
$8b^2 - 6b + 1$
▶ 24. $(4b - 1) - (2b - 1)$
$2b$

GUIDED PRACTICE

Find each product. See Example 1.

25. $(2a^2)(-3ab)$
$-6a^3b$
▶ 26. $(-3x^2y)(3xy)$
$-9x^3y^2$
27. $(-3ab^2c)(5ac^2)$
$-15a^2b^2c^3$
28. $(-2m^2n)(-4mn^3)$
$8m^3n^4$
29. $(4a^2b)(-5a^3b^2)(6a^4)$
$-120a^9b^3$
30. $(2x^2y^3)(4xy^5)(-5y^6)$
$-40x^3y^{14}$
31. $(-5xx^2)(-3xy)^4$
$-405x^7y^4$
32. $(-2a^2ab^2)^3(-3ab^2b^2)$
$24a^{10}b^{10}$

Find each product. See Example 2.

33. $3(x + 2)$
$3x + 6$
▶ 34. $-5(a + b)$
$-5a - 5b$
35. $3x(x^2 + 3x)$
$3x^3 + 9x^2$
▶ 36. $-2x(3x^2 - 2)$
$-6x^3 + 4x$
37. $-2x(3x^2 - 3x + 2)$
$-6x^3 + 6x^2 - 4x$
38. $3a(4a^2 + 3a - 4)$
$12a^3 + 9a^2 - 12a$
39. $7rst(r^2 + s^2 - t^2)$
$7r^3st + 7rs^3t - 7rst^3$
▶ 40. $3x^2yz(x^2 - 2y + 3z^2)$
$3x^4yz - 6x^2y^2z + 9x^2yz^3$

Find each product. See Example 3.

41. $(x + 2)(x + 3)$
$x^2 + 5x + 6$
42. $(y - 3)(y + 4)$
$y^2 + y - 12$
43. $(3t - 2)(2t + 3)$
$6t^2 + 5t - 6$
▶ 44. $(p + 3)(3p - 4)$
$3p^2 + 5p - 12$
45. $(3y - z)(2y - z)$
$6y^2 - 5yz + z^2$
46. $(2m - n)(3m - n)$
$6m^2 - 5mn + n^2$
47. $(b^3 - 1)(b + 1)$
$b^4 + b^3 - b - 1$
48. $(c^3 + 1)(1 - c)$
$-c^4 + c^3 - c + 1$

Find each product. See Example 4.

49. $(x - y)(x^2 + xy + y^2)$
$x^3 - y^3$
▶ 50. $(x + y)(x^2 - xy + y^2)$
$x^3 + y^3$
51. $(3y + 1)(2y^2 + 3y + 2)$
$6y^3 + 11y^2 + 9y + 2$
52. $(a + 2)(3a^2 + 4a - 2)$
$3a^3 + 10a^2 + 6a - 4$

Find each product. See Example 5.

53. $\begin{array}{r} 2a + 3 \\ \underline{2a - 1} \end{array}$
$4a^2 + 4a - 3$

54. $\begin{array}{r} 3b - 5 \\ \underline{4b + 5} \end{array}$
$12b^2 - 5b - 25$

55. $\begin{array}{r} 4x^2 + 2x - 1 \\ \underline{3x - 2} \end{array}$
$12x^3 - 2x^2 - 7x + 2$

▶ **56.** $\begin{array}{r} 3x^2 - 4x + 3 \\ \underline{2x + 1} \end{array}$
$6x^3 - 5x^2 + 2x + 3$

Find each product. See Example 6.

57. $\begin{array}{r} x + y + z \\ \underline{x - y + z} \end{array}$
$x^2 + 2xz - y^2 + z^2$

58. $\begin{array}{r} x + y - z \\ \underline{x + y + z} \end{array}$
$x^2 + 2xy + y^2 - z^2$

59. $\begin{array}{r} 2a + b + z \\ \underline{a - b - z} \end{array}$
$2a^2 - ab - az - b^2 - 2bz - z^2$

▶ **60.** $\begin{array}{r} 3a + 2b - z \\ \underline{a - 2b + 2z} \end{array}$
$3a^2 - 4ab + 5az - 4b^2 + 6bz - 2z^2$

Find each product. See Example 7.

61. $6p^2(3p - 4)(p + 3)$
$18p^4 + 30p^3 - 72p^2$

▶ **62.** $4a(2a + 3)(3a - 2)$
$24a^3 + 20a^2 - 24a$

63. $-5mn(2m + 3n)(m - 3n)$
$-10m^3n + 15m^2n^2 + 45mn^3$

64. $4pq(p + 4q)(p - 4q)$
$4p^3q - 64pq^3$

Find each product. See Example 8.

65. $(x + 2)^2$
$x^2 + 4x + 4$

66. $(x - 3)^2$
$x^2 - 6x + 9$

67. $(a - 4)^2$
$a^2 - 8a + 16$

68. $(y + 5)^2$
$y^2 + 10y + 25$

69. $(2a + b)^2$
$4a^2 + 4ab + b^2$

▶ **70.** $(a - 2b)^2$
$a^2 - 4ab + 4b^2$

71. $(5r^2 + 6)^2$
$25r^4 + 60r^2 + 36$

72. $(6p^2 - 3)^2$
$36p^4 - 36p^2 + 9$

73. $(x + 2)(x - 2)$
$x^2 - 4$

74. $(z + 3)(z - 3)$
$z^2 - 9$

75. $(y^3 + 2)(y^3 - 2)$
$y^6 - 4$

▶ **76.** $(y^4 + 3)(y^4 - 3)$
$y^8 - 9$

Simplify each expression. See Example 9.

77. $3x(2x + 4) - 3x^2$
$3x^2 + 12x$

78. $2y - 3y(y^2 + 4)$
$-3y^3 - 10y$

79. $(x + 3)(x - 3) + (2x - 1)(x + 2)$
$3x^2 + 3x - 11$

▶ **80.** $(3y + 1)^2 + (2y - 4)^2$
$13y^2 - 10y + 17$

TRY IT YOURSELF

Perform each multiplication.

81. $\left(\frac{1}{2}b + 8\right)(4b + 6)$
$2b^2 + 35b + 48$

82. $\left(\frac{2}{3}x + 1\right)(15x - 9)$
$10x^2 + 9x - 9$

83. $(0.4t - 3)(0.5t - 3)$
$0.2t^2 - 2.7t + 9$

84. $(0.7d - 2)(0.1d + 3)$
$0.07d^2 + 1.9d - 6$

85. $(3tu - 1)(-2tu + 3)$
$-6t^2u^2 + 11tu - 3$

86. $(-5st + 1)(10st - 7)$
$-50s^2t^2 + 45st - 7$

87. $(9b^3 - c)(3b^2 - c)$
$27b^5 - 9b^3c - 3b^2c + c^2$

88. $(h^5 - k)(4h^3 - k)$
$4h^8 - h^5k - 4h^3k + k^2$

89. $(11m^2 + 3n^3)(5m + 2n^2)$
$55m^3 + 22m^2n^2 + 15mn^3 + 6n^5$

90. $(50m^4 - 3n^4)(2m + 2n^3)$
$100m^5 + 100m^4n^3 - 6mn^4 - 6n^7$

91. $(3m - y)(4my)(2m - y)$
$24m^3y - 20m^2y^2 + 4my^3$

92. $-3a^2b^3(2b)(3a + b)$
$-18a^3b^4 - 6a^2b^5$

93. $a^2b^2(-3b^2)(6a + 2b)$
$-18a^3b^4 - 6a^2b^5$

94. $(2h - z)(-3hz)(3h - z)$
$-18h^3z + 15h^2z^2 - 3hz^3$

95. $(9ab - 4)^2$
$81a^2b^2 - 72ab + 16$

▶ **96.** $(2yz^2 + 5)^2$
$4y^2z^4 + 20yz^2 + 25$

97. $\left(\frac{1}{4}b + 2\right)^2$
$\frac{1}{16}b^2 + b + 4$

98. $\left(\frac{2}{3}y - 7\right)^2$
$\frac{4}{9}y^2 - \frac{28}{3}y + 49$

99. $(4k - 1.3)^2$
$16k^2 - 10.4k + 1.69$

100. $(0.5k + 6)^2$
$0.25k^2 + 6k + 36$

101. $(xy - 6)(xy + 6)$
$x^2y^2 - 36$

102. $(a^4b - c)(a^4b + c)$
$a^8b^2 - c^2$

103. $\left(\frac{1}{2}x - 16\right)\left(\frac{1}{2}x + 16\right)$
$\frac{1}{4}x^2 - 256$

104. $\left(\frac{3}{4}h^2 - \frac{2}{3}\right)\left(\frac{3}{4}h^2 + \frac{2}{3}\right)$
$\frac{9}{16}h^4 - \frac{4}{9}$

105. $(2.4 + y)(2.4 - y)$
$5.76 - y^2$

106. $(3.5t + 4.1u)(3.5t - 4.1u)$
$12.25t^2 - 16.81u^2$

107. $(2a - b)(4a^2 + 2ab + b^2)$
$8a^3 - b^3$

108. $(x - 3y)(x^2 + 3xy + 9y^2)$
$x^3 - 27y^3$

109. $(a + b)(a - b)(a - 3b)$
$a^3 - 3a^2b - ab^2 + 3b^3$

▶ **110.** $(x - y)(x + 2y)(x - 2y)$
$x^3 - x^2y - 4xy^2 + 4y^3$

111. $(a + b + c)(2a - b - 2c)$
$2a^2 + ab - b^2 - 3bc - 2c^2$

112. $(x + 2y + 3z)^2$
$x^2 + 4xy + 6xz + 4y^2 + 12yz + 9z^2$

113. $(r + s)^2(r - s)^2$
$r^4 - 2r^2s^2 + s^4$

▶ **114.** $r(r + s)(r - s)^2$
$r^4 - r^3s - r^2s^2 + rs^3$

Simplify each expression.

115. $3pq - p(p - q)$
$-p^2 + 4pq$

116. $-4rs(r - 2) + 4rs$
$-4r^2s + 12rs$

117. $(2b + 3)(b - 1) - (b + 2)(3b - 1)$
$-b^2 - 4b - 1$

118. $(3x - 4)^2 - (2x + 3)^2$
$5x^2 - 36x + 7$

Use a calculator to help find each product.

119. $(3.21x - 7.85)(2.87x + 4.59)$
$9.2127x^2 - 7.7956x - 36.0315$

120. $(7.44y + 56.7)(-2.1y - 67.3)$
$-15.624y^2 - 619.782y - 3{,}815.91$

121. $(-17.3y + 4.35)^2$
$299.29y^2 - 150.51y + 18.9225$

122. $(-0.31x + 29.3)(-0.31x - 29.3)$
$0.0961x^2 - 858.49$

APPLICATIONS

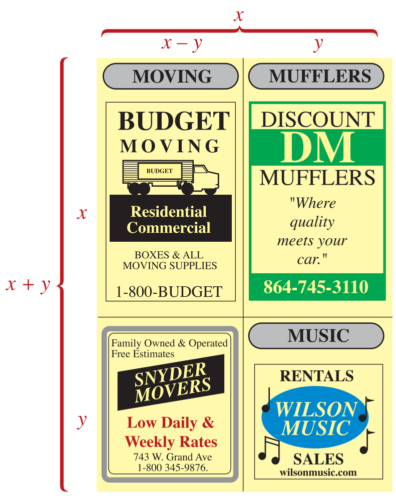
123. THE YELLOW PAGES Refer to the illustration below.

a. Describe the area occupied by the ads for movers by using a product of two binomials.
$(x + y)(x - y)$

b. Describe the area occupied by the ad for Budget Moving by using a product. Then perform the multiplication.
$x(x - y)$; $x^2 - xy$

c. Describe the area occupied by the ad for Snyder Movers by using a product. Then perform the multiplication.
$y(x - y)$; $xy - y^2$

d. Explain why your answer to part a is equal to the sum of your answers to parts b and c. What special product does this exercise illustrate?
They represent the same area. $(x + y)(x - y) = x^2 - y^2$

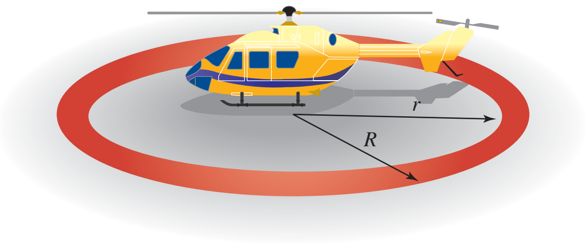


124. HELICOPTER PADS To determine the amount of fluorescent paint needed to paint the circular ring on the landing pad design shown in the illustration, painters must find its area. The area of the ring is given by the expression $\pi(R + r)(R - r)$.

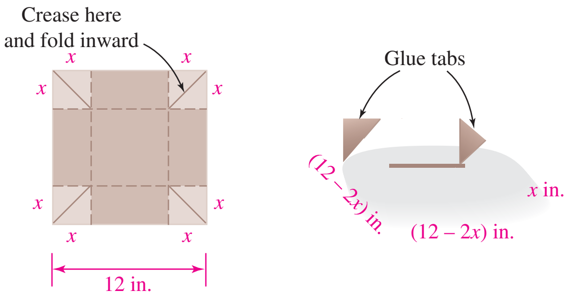
a. Find the product $\pi(R + r)(R - r)$.
$\pi R^2 - \pi r^2$

b. If $R = 25$ feet and $r = 20$ feet, find the area to be painted. Round to the nearest tenth.
706.9ft^2

c. If a quart of fluorescent paint covers 65 ft^2, how many quarts will be needed to paint the ring?
11 qt



125. GIFT BOXES The corners of a 12-in.-by-12-in. piece of cardboard are creased, folded inward, and glued to make a gift box. Write a polynomial that gives the volume of the resulting box.
$x(12 - 2x)(12 - 2x)$ in.^{3} = $(144x - 48x^2 + 4x^3)$ in.^{3}



126. REVENUE A salesperson has found that the number x of televisions she can sell at a certain price p is related by the equation $x = -\frac{1}{5}p + 90$

a. Find the number of TVs she will sell if the price is \$375.
15

b. Write a formula for the revenue when x TVs are sold.
$r = -\frac{1}{5}p^2 + 90p$

c. Find the revenue generated by TV sales if they are priced at \$400 each.
\$4,000

WRITING

127. Explain how to use the FOIL method.

128. Explain how you would multiply two trinomials.

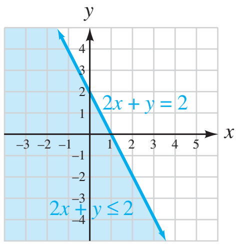
129. On a test, when asked to find $(x - y)^2$, a student answered $x^2 - y^2$. What error did the student make?

130. Explain how the distributive property is used to find the following product: $2x^3(x^2 - 5x + 1)$.

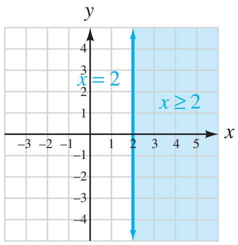
REVIEW

Graph each inequality or system of inequalities.

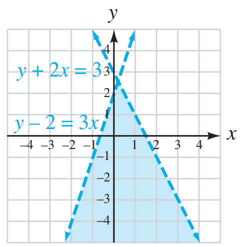
131. $2x + y \le 2$



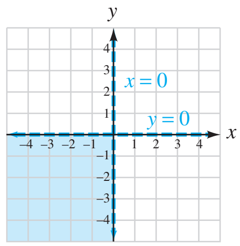
132. $x \ge 2$



133. $\begin{cases} y - 2 < 3x \\ y + 2x < 3 \end{cases}$



134. $\begin{cases} y < 0 \\ x < 0 \end{cases}$



SECTION 5.5

The Greatest Common Factor and Factoring by Grouping

Objectives

1. Find the prime-factored form of a natural number.
2. Find the greatest common factor of a list of terms.
3. Factor out the greatest common factor.
4. Factor by grouping.
5. Use factoring to solve formulas for a specified variable.

We have discussed how to multiply polynomials. We will now reverse the operation of multiplication and show how to find the factors of a known product. The process of finding the individual factors of a known product is called **factoring.**

1 Find the prime-factored form of a natural number.

If one number a divides a second number b, then a is called a **factor** of b. For example, because 3 divides 24, it is a factor of 24. Each number in the following list is a factor of 24, because each number divides 24.

1, 2, 3, 4, 6, 8, 12, and 24

To factor a natural number means to write it as a product of other natural numbers. If each factor is a prime number, the natural number is said to be written in **prime-factored form.** Example 1 shows how to find the prime-factored forms of 60, 84, and 180, respectively.

EXAMPLE 1 Find the prime factorization of each number:

a. 60 **b.** 84 **c.** 180

Strategy We will use a series of steps to express each number as a product of only prime numbers.

WHY To *prime factor* a number means to write it as a product of prime numbers.

Solution

a.
$$\begin{aligned} 60 &= 6 \cdot 10 \\ &= 2 \cdot 3 \cdot 2 \cdot 5 \\ &= 2^2 \cdot 3 \cdot 5 \end{aligned}$$

b.
$$\begin{aligned} 84 &= 4 \cdot 21 \\ &= 2 \cdot 2 \cdot 3 \cdot 7 \\ &= 2^2 \cdot 3 \cdot 7 \end{aligned}$$

c.
$$\begin{aligned} 180 &= 10 \cdot 18 \\ &= 2 \cdot 5 \cdot 3 \cdot 6 \\ &= 2 \cdot 5 \cdot 3 \cdot 3 \cdot 2 \\ &= 2^2 \cdot 3^2 \cdot 5 \end{aligned}$$

Self Check 1

Find the prime factorization of 120. $2^3 \cdot 3 \cdot 5$

Now Try **Problem 18**

Teaching Example 1 Find the prime factorization of 280.
Answer:
$2^3 \cdot 5 \cdot 7$

The largest natural number that divides 60, 84, and 180 is called the **greatest common factor (GCF)** of the numbers. Because 60, 84, and 180 all have two factors of 2 and one factor of 3, the GCF of these three numbers is $2^2 \cdot 3 = 12$. We note that

$$\frac{60}{12} = 5, \quad \frac{84}{12} = 7, \quad \text{and} \quad \frac{180}{12} = 15$$

There is no natural number greater than 12 that divides 60, 84, and 180.

Algebraic monomials can also have greatest common factors.

2 Find the greatest common factor of a list of terms.

To find the GCF of several monomials, we follow these steps.

Steps for Finding the GCF

1. Find the prime-factored form of each monomial.
2. Identify the prime factors that are common to each monomial.
3. Find the product of the factors found in step 2, with each factor raised to the smallest power that occurs in any one monomial.

Self Check 2

Find the GCF of $24x^2y^3$, $3x^3y$, and $18x^2y^2$. $3x^2y$

Now Try **Problem 30**

Teaching Example 2 Find the GCF of $15x^3y^5$, $25x^2y^4$, and $30x^4y^9$.
Answer:
$5x^2y^4$

EXAMPLE 2 Find the GCF of $6a^2b^3c$, $9a^3b^2c$, and $18a^4c^3$.

Strategy We will prime factor each coefficient of each term in the list. Then we will identify the numerical and variable factors common to each term and find their product.

WHY The product of the common factors is the GCF of the terms in the list.

Solution

We begin by factoring each monomial.

$$6a^2b^3c = \mathbf{3} \cdot 2 \cdot \mathbf{a} \cdot \mathbf{a} \cdot b \cdot b \cdot b \cdot \mathbf{c} \qquad \text{This can be written as } 2 \cdot 3 \cdot a^2 \cdot b^3 \cdot c.$$
$$9a^3b^2c = \mathbf{3} \cdot 3 \cdot \mathbf{a} \cdot \mathbf{a} \cdot a \cdot b \cdot b \cdot \mathbf{c} \qquad \text{This can be written as } 3^2 \cdot a^3 \cdot b^2 \cdot c.$$
$$18a^4c^3 = 2 \cdot \mathbf{3} \cdot 3 \cdot \mathbf{a} \cdot \mathbf{a} \cdot a \cdot a \cdot \mathbf{c} \cdot c \cdot c \qquad \text{This can be written as } 2 \cdot 3^2 \cdot a^4 \cdot c^3.$$

Since each monomial has one factor of 3, two factors of a, and one factor of c in common, the GCF is

$$3^1 \cdot a^2 \cdot c^1 = 3a^2c$$

3 Factor out the greatest common factor.

We have seen that the distributive property provides a method for multiplying a polynomial by a monomial. For example,

$$\begin{aligned} \mathbf{2x^3y^3}(3x^2 - 4y^3) &= \mathbf{2x^3y^3} \cdot 3x^2 - \mathbf{2x^3y^3} \cdot 4y^3 \\ &= 6x^5y^3 - 8x^3y^6 \end{aligned}$$

If the product of a multiplication is $6x^5y^3 - 8x^3y^6$, we can use the distributive property in reverse to find the individual factors.

$$\begin{aligned} 6x^5y^3 - 8x^3y^6 &= \mathbf{2x^3y^3} \cdot 3x^2 - \mathbf{2x^3y^3} \cdot 4y^3 \\ &= \mathbf{2x^3y^3}(3x^2 - 4y^3) \end{aligned}$$

Since $2x^3y^3$ is the GCF of the terms of $6x^5y^3 - 8x^3y^6$, this process is called **factoring out the greatest common factor.**

Self Check 3

Factor: $9x^4y^2 - 12x^3y^3$

Now Try **Problem 34**

Self Check 3 Answer
$3x^3y^2(3x - 4y)$

EXAMPLE 3 Factor: $25a^3b + 15ab^3$

Strategy We will determine the GCF of the terms and write each term of the polynomial as the product of the GCF and other factors.

WHY We can then use the distributive property in reverse to factor out the GCF.

Solution

We begin by factoring each monomial:

$$25a^3b = 5 \cdot 5 \cdot a \cdot a \cdot a \cdot b$$
$$15ab^3 = 5 \cdot 3 \cdot a \cdot b \cdot b \cdot b$$

Since each term has one factor of 5, one factor of a, and one factor of b in common, and there are no other common factors, $5ab$ is the GCF of the two terms. We can use the distributive property to factor it out.

$$\begin{aligned} 25a^3b + 15ab^3 &= 5ab \cdot 5a^2 + 5ab \cdot 3b^2 \\ &= 5ab(5a^2 + 3b^2) \end{aligned}$$

Teaching Example 3 Factor: $6x^2y^5 - 21xy^4$
Answer:
$3xy^4(2xy - 7)$

EXAMPLE 4 Factor: $3xy^2z^3 + 6xyz^3 - 3xz^2$

Strategy We will determine the GCF of the terms and write each term of the polynomial as the product of the GCF and other factors.

WHY We can then use the distributive property in reverse to factor out the GCF.

Solution

We begin by factoring each monomial:

$$3xy^2z^3 = 3 \cdot x \cdot y \cdot y \cdot z \cdot z \cdot z$$
$$6xyz^3 = 3 \cdot 2 \cdot x \cdot y \cdot z \cdot z \cdot z$$
$$-3xz^2 = -1 \cdot 3 \cdot x \cdot z \cdot z$$

Since each term has one factor of 3, one factor of x, and two factors of z in common, and because there are no other common factors, $3xz^2$ is the GCF of the three terms. We can use the distributive property to factor it out.

$$\begin{aligned} 3xy^2z^3 + 6xyz^3 - 3xz^2 &= 3xz^2 \cdot y^2z + 3xz^2 \cdot 2yz - 3xz^2 \cdot 1 \\ &= 3xz^2(y^2z + 2yz - 1) \end{aligned}$$

Self Check 4

Factor: $2a^4b^2 + 6a^3b^2 - 2a^2b$

Now Try **Problem 40**

Self Check 4 Answer
$2a^2b(a^2b + 3ab - 1)$

Teaching Example 4 Factor: $10x^4y^3 - 15xy^4 + 5xy^2$
Answer:
$5xy^2(2x^3y - 3y^2 + 1)$

Caution! In Example 4, when the $3xz^2$ is factored out from the third term, remember to write the -1.

A polynomial that cannot be factored is called a **prime polynomial** or an **irreducible polynomial.**

EXAMPLE 5 Factor, if possible: $3x^2 + 4y + 7$

Strategy We will try to determine the GCF of each term.

WHY If the polynomial cannot be factored, it is called a prime polynomial.

Solution

We factor each monomial:

$$3x^2 = 3 \cdot x \cdot x \qquad 4y = 2 \cdot 2 \cdot y \qquad 7 = 7$$

Since there are no common factors other than 1, this polynomial cannot be factored. It is a prime polynomial.

Self Check 5

Factor, if possible:
$6a^3 + 7b^2 + 5$

Now Try **Problem 44**

Self Check 5 Answer
a prime polynomial

Teaching Example 5 Factor, if possible:
$3x^3 + 5y^2 - 2$
Answer:
a prime polynomial

Self Check 6

Factor out the opposite of the GCF from $-8a^2b^2 - 12ab^3$.

Now Try **Problem 54**

Self Check 6 Answer
$-4ab^2(2a + 3b)$

Teaching Example 6 Factor out the opposite of the GCF from $-20x^2y^3 + 10x^3y^4$
Answer:
$-10x^2y^3(2 - xy)$

EXAMPLE 6 Factor out the opposite of the GCF from $-6u^2v^3 + 8u^3v^2$.

Strategy We will determine the GCF of the terms of the polynomial. Then we will write each term of the polynomial as the product of the opposite of the GCF and other factors.

WHY We can then use the distributive property in reverse to factor out the opposite of the GCF.

Solution

Because the GCF of the two terms is $2u^2v^2$, the opposite of the GCF is $-2u^2v^2$. To factor out $-2u^2v^2$, we proceed as follows:

$$\begin{aligned} -6u^2v^3 + 8u^3v^2 &= -2u^2v^2 \cdot 3v + 2u^2v^2 \cdot 4u \\ &= \mathbf{-2u^2v^2} \cdot 3v - (\mathbf{-2u^2v^2})4u \\ &= \mathbf{-2u^2v^2}(3v - 4u) \end{aligned}$$

Self Check 7

Factor:
$x(a + b - c) - y(a + b - c)$

Now Try **Problem 56**

Self Check 7 Answer
$(a + b - c)(x - y)$

Teaching Example 7 Factor:
$x(w + z) - y(w + z)$
Answer:
$(w + z)(x - y)$

EXAMPLE 7 Factor: $a(x - y + z) - b(x - y + z) + 3(x - y + z)$

Strategy We will determine the terms of the expression and find their GCF.

WHY We can then use the distributive property in reverse to factor out the GCF.

Solution

We can factor out the GCF of the three terms, which is $(x - y + z)$.

$$\begin{aligned} &a(x - y + z) - b(x - y + z) + 3(x - y + z) \\ &\quad = \mathbf{(x - y + z)}a - \mathbf{(x - y + z)}b + \mathbf{(x - y + z)}3 \\ &\quad = \mathbf{(x - y + z)}(a - b + 3) \end{aligned}$$

4 Factor by grouping.

Suppose that we wish to factor

$$ac + ad + bc + bd$$

Although there is no factor common to all four terms, there is a common factor of a in the first two terms and a common factor of b in the last two terms. We can factor out these common factors to get

$$ac + ad + bc + bd = a\mathbf{(c + d)} + b\mathbf{(c + d)}$$

We can now factor out the common factor of $c + d$ on the right-hand side:

$$ac + ad + bc + bd = \mathbf{(c + d)}(a + b)$$

The grouping in this type of problem is not always unique. For example, if we write the expression $ac + ad + bc + bd$ in the form

$$ac + bc + ad + bd$$

and factor c from the first two terms and d from the last two terms, we obtain

$$\begin{aligned} ac + bc + ad + bd &= c\mathbf{(a + b)} + d\mathbf{(a + b)} \\ &= \mathbf{(a + b)}(c + d) \end{aligned}$$

This is equivalent to $(c + d)(a + b)$.

The method used in the previous examples is called **factoring by grouping.**

Factoring by Grouping

1. Group the terms of the polynomial so that each group has a common factor.
2. Factor out the common factor from each group.
3. Factor out the resulting common factor. If there is no common factor, regroup the terms of the polynomial and repeat steps 2 and 3.

EXAMPLE 8 Factor: $3ax^2 + 3bx^2 + a + 5bx + 5ax + b$

Strategy We will group the terms of the polynomial so that each group has a common factor. Then we will factor out the common factor from each group.

WHY If there is a resulting common factor, we can factor it out. If not, we will regroup the terms of the original polynomial and try the strategy again.

Solution

Although there is no factor common to all six terms, $3x^2$ can be factored out of the first two terms, and $5x$ can be factored out of the fourth and fifth terms to get

$$3ax^2 + 3bx^2 + a + 5bx + 5ax + b = 3x^2(a + b) + a + 5x(b + a) + b$$

This result can be written in the form

$$3ax^2 + 3bx^2 + a + 5bx + 5ax + b = 3x^2(a + b) + 5x(a + b) + 1(a + b)$$

Since $a + b$ is common to all three terms, it can be factored out to get

$$3ax^2 + 3bx^2 + a + 5bx + 5ax + b = (a + b)(3x^2 + 5x + 1)$$

Self Check 8

Factor: $2x^3 + x^2 + x + 2x^2y + xy + y$

Now Try Problems 62 and 66

Self Check 8 Answer
$(x + y)(2x^2 + x + 1)$

Teaching Example 8 Factor: $ax + 2bx + ay + 2by + az + 2bz$
Answer:
$(a + 2b)(x + y + z)$

A polynomial is **factored completely** when no factor can be factored further. To factor an expression completely, it is often necessary to factor more than once, as the following example illustrates.

EXAMPLE 9 Factor: $3x^3y - 4x^2y^2 - 6x^2y + 8xy^2$

Strategy Since all four terms have a common factor of xy, we will factor it out first. Then we will factor the resulting polynomial by grouping.

WHY Factoring out the GCF first makes the factoring process easier.

Solution

We begin by factoring out the common factor xy.

$$3x^3y - 4x^2y^2 - 6x^2y + 8xy^2 = xy(3x^2 - 4xy - 6x + 8y)$$

We can now factor $3x^2 - 4xy - 6x + 8y$ by grouping:

$$\begin{aligned} &3x^3y - 4x^2y^2 - 6x^2y + 8xy^2 \\ &\quad = xy(3x^2 - 4xy - 6x + 8y) \\ &\quad = xy[x(3x - 4y) - 2(3x - 4y)] && \text{Factor } x \text{ from } 3x^2 - 4xy \text{ and } -2 \text{ from } -6x + 8y. \\ &\quad = xy(3x - 4y)(x - 2) && \text{Factor out } 3x - 4y. \end{aligned}$$

Because no more factoring can be done, the factorization is complete.

Self Check 9

Factor: $3a^3b + 3a^2b - 2a^2b^2 - 2ab^2$

Now Try Problem 70

Self Check 9 Answer
$ab(3a - 2b)(a + 1)$

Teaching Example 9 Factor: $2ax^2 - 4bx^2 + 2axy - 4bxy$
Answer:
$2x(x + y)(a - 2b)$

Success Tip Whenever you factor an expression, always factor it completely. Each factor of a completely factored expression will be prime.

5 Use factoring to solve formulas for a specified variable.

Factoring is often required to solve a **literal equation** (an equation containing more than one variable) for one of its variables.

Self Check 10

Solve $A = p + prt$ for p.

Now Try **Problem 72**

Self Check 10 Answer

$p = \dfrac{A}{1 + rt}$

Teaching Example 10 Solve: $y - y_1 = mx - mx_1$ for m

Answer:

$m = \dfrac{y - y_1}{x - x_1}$

EXAMPLE 10 ***Electronics*** The formula $r_1r_2 = rr_2 + rr_1$ is used in electronics to relate the combined resistance r of two resistors wired in parallel. The variable r_1 represents the resistance of the first resistor, and the variable r_2 represents the resistance of the second. Solve for r_2.

Strategy To isolate r_2 on one side of the equation, we will get all the terms involving r_2 on the left side and all the terms not involving r_2 on the right side.

WHY To *solve a formula for a specified variable* means to isolate that variable on one side of the equation, with all other variables and constants on the opposite side.

Solution

We want to isolate this variable on one side of the equation.

$$r_1r_2 = rr_2 + rr_1$$

$$r_1r_2 - rr_2 = rr_1 \quad \text{Subtract } rr_2 \text{ from both sides.}$$

$$r_2(r_1 - r) = rr_1 \quad \text{Factor out } r_2 \text{ on the left-hand side.}$$

$$r_2 = \frac{rr_1}{r_1 - r} \quad \text{Divide both sides by } r_1 - r.$$

ANSWERS TO SELF CHECKS

1. $2^3 \cdot 3 \cdot 5$ **2.** $3x^2y$ **3.** $3x^3y^2(3x - 4y)$ **4.** $2a^2b(a^2b + 3ab - 1)$ **5.** a prime polynomial **6.** $-4ab^2(2a + 3b)$ **7.** $(a + b - c)(x - y)$ **8.** $(x + y)(2x^2 + x + 1)$ **9.** $ab(3a - 2b)(a + 1)$ **10.** $p = \frac{A}{1 + rt}$

SECTION 5.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. When we write $2x + 4$ as $2(x + 2)$, we say that we have factored $2x + 4$.
2. When we write 100 as $2^2 \cdot 5^2$, we say that we have written 100 in prime-factored form.
3. Because 5 divides 20, we say that 5 is a factor of 20.
4. The polynomial $2x^2y^3 - 4xy^2 + 6xy$ has three terms.
5. The abbreviation GCF stands for greatest common factor.
6. If a polynomial cannot be factored, it is called a prime polynomial or an irreducible polynomial.

Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

7. The prime factorizations of three monomials are shown here. Find their GCF. $6xy^2$

$$2 \cdot 2 \cdot 3 \cdot x \cdot x \cdot y \cdot y \cdot y$$
$$2 \cdot 3 \cdot 3 \cdot x \cdot y \cdot y \cdot y \cdot y$$
$$2 \cdot 3 \cdot 3 \cdot 7 \cdot x \cdot x \cdot x \cdot y \cdot y$$

8. **a.** What property is illustrated below?

$$4a^2b(2ab^3 - 3a^2b^4) = 4a^2b \cdot 2ab^3 - 4a^2b \cdot 3a^2b^4$$

the distributive property

b. Explain how we use the distributive property in reverse to factor $8a^3b^4 - 12a^4b^5$.

9. Explain why each factorization of $30t^2 - 20t^3$ is not complete.

a. $5t^2(6 - 4t)$
The terms within the parentheses have a common factor 2.

b. $10t(3t - 2t^2)$
The terms within the parentheses have a common factor t.

▶ **10. a.** Factor $-5y^3 - 10y^2 + 15y$ by factoring out the positive GCF.
$5y(-y^2 - 2y + 3)$

b. Factor $-5y^3 - 10y^2 + 15y$ by factoring out the opposite GCF.
$-5y(y^2 + 2y - 3)$

NOTATION

Complete each factorization.

11. $3a - 12 = 3(a - \boxed{4})$

▶ **12.** $8z^3 + 4z^2 + 2z = 2z(4z^2 + 2z + \boxed{1})$

13. $x^3 - x^2 + 2x - 2 = \boxed{x^2}(x - 1) + \boxed{2}(x - 1)$
$= (x - 1)(x^2 + 2)$

▶ **14.** $-24a^3b^2 + 12ab^2 = -12ab^2(2a^2 \boxed{-} 1)$

GUIDED PRACTICE

Find the prime factorization of each number. See Example 1.

15. 6 $2 \cdot 3$

16. 10 $2 \cdot 5$

17. 135 $3^3 \cdot 5$

▶ **18.** 98 $2 \cdot 7^2$

19. 128 2^7

20. 357 $3 \cdot 7 \cdot 17$

21. 325 $5^2 \cdot 13$

22. 288 $2^5 \cdot 3^2$

Find the GCF of each set of monomials. See Example 2.

23. 36, 48 12

24. 45, 75 15

25. 42, 36, 98 2

26. 16, 40, 60 4

27. $4a^2b, 8a^3c$ $4a^2$

▶ **28.** $6x^3y^2z, 9xyz^2$ $3xyz$

29. $18x^4y^3z^2, 12xy^2z^3$
$6xy^2z^2$

30. $6x^2y^3, 24xy^3, 40x^2y^2z^3$
$2xy^2$

Factor each polynomial. See Example 3.

31. $2x + 8$
$2(x + 4)$

32. $3y - 9$
$3(y - 3)$

33. $2x^2 - 6x$
$2x(x - 3)$

34. $3y^3 + 3y^2$
$3y^2(y + 1)$

35. $15x^2y - 10x^2y^2$
$5x^2y(3 - 2y)$

▶ **36.** $13ab^2c^3 - 26a^3b^2c$
$13ab^2c(c^2 - 2a^2)$

37. $28a^3b - 21ab^3$
$7ab(4a^2 - 3b^2)$

38. $12p^4q + 8p^2q^3$
$4p^2q(3p^2 + 2q^2)$

Factor each polynomial. See Example 4.

39. $45x^{10}y^3 - 63x^7y^7 + 9x^7y^3$ $9x^7y^3(5x^3 - 7y^4 + 1)$

40. $48u^6v^6 - u^4v^3 - 3u^6v^3$ $u^4v^3(48u^2v^3 - 1 - 3u^2)$

41. $12a^6b^5 + 8a^4b^2 - 6a^3b^4$ $2a^3b^2(6a^3b^3 + 4a - 3b^2)$

▶ **42.** $6p^5q^4 - 9p^4q^2 + 6p^2q^4$ $3p^2q^2(2p^3q^2 - 3p^2 + 2q^2)$

Factor each polynomial, if possible. See Example 5.

43. $5xy + 12ab^2$ prime

44. $11m^3n^2 - 12x^2y$ prime

45. $14r^2s^3 + 15t^6$ prime

46. $3r^2s^3 + 6rs^4 - 7$ prime

Factor out the opposite of the greatest common factor. See Example 6.

47. $-3a - 6$
$-3(a + 2)$

48. $-6b + 12$
$-6(b - 2)$

49. $-3x^2 - x$
$-x(3x + 1)$

50. $-4a^3 + a^2$
$-a^2(4a - 1)$

51. $-6x^2 - 3xy$
$-3x(2x + y)$

52. $-15y^3 + 25y^2$
$-5y^2(3y - 5)$

53. $-18a^2b - 12ab^2$
$-6ab(3a + 2b)$

▶ **54.** $-21t^5 + 28t^3$
$-7t^3(3t^2 - 4)$

Factor each expression. See Example 7.

55. $4(x + y) + t(x + y)$
$(x + y)(4 + t)$

56. $5(a - b) - t(a - b)$
$(a - b)(5 - t)$

57. $3(m + n + p) + x(m + n + p)$
$(m + n + p)(3 + x)$

▶ **58.** $x(x - y - z) + y(x - y - z)$
$(x - y - z)(x + y)$

Factor each expression. See Example 8.

59. $ax + bx + ay + by$
$(x + y)(a + b)$

60. $ar - br + as - bs$
$(r + s)(a - b)$

61. $x^2 + yx + 2x + 2y$
$(x + 2)(x + y)$

62. $2c + 2d - cd - d^2$
$(c + d)(2 - d)$

63. $7u + v^2 - 7v - uv$
$(v - u)(v - 7)$

64. $ax + bx - a - b$
$(a + b)(x - 1)$

▶ **65.** $x^2 + xy + xz + xy + y^2 + zy$
$(x + y)(x + y + z)$

66. $ab - b^2 - bc + ac - bc - c^2$
$(b + c)(a - b - c)$

Factor each expression. See Example 9.

67. $mpx + mqx + npx + nqx$
$x(m + n)(p + q)$

68. $abd - abe + acd - ace$
$a(b + c)(d - e)$

69. $x^2y + xy^2 + 2xyz + xy^2 + y^3 + 2y^2z$
$y(x + y)(x + y + 2z)$

▶ **70.** $a^3 - 2a^2b + a^2c - a^2b + 2ab^2 - abc$
$a(a - b)(a - 2b + c)$

Solve for the indicated variable. See Example 10.

71. $r_1r_2 = rr_2 + rr_1$ for r_1
$r_1 = \frac{rr_2}{r_2 - r}$

72. $r_1r_2 = rr_2 + rr_1$ for r
$r = \frac{r_1r_2}{r_2 + r_1}$

73. $d_1d_2 = fd_2 + fd_1$ for f
$f = \frac{d_1d_2}{d_2 + d_1}$

▶ **74.** $d_1d_2 = fd_2 + fd_1$ for d_1
$d_1 = \frac{fd_2}{d_2 - f}$

75. $b^2x^2 + a^2y^2 = a^2b^2$ for a^2 $a^2 = \frac{b^2x^2}{b^2 - y^2}$

76. $b^2x^2 + a^2y^2 = a^2b^2$ for b^2 $b^2 = \frac{a^2y^2}{a^2 - x^2}$

77. $S(1 - r) = a - lr$ for r $r = \frac{S - a}{S - l}$

78. $Sn = (n - 2)180°$ for n $n = \frac{360°}{180° - S}$

TRY IT YOURSELF

Factor each expression completely, including −1 if necessary.

79. $7x^2 + 14x$
$7x(x + 2)$

80. $27z^3 + 12z^2 + 3z$
$3z(9z^2 + 4z + 1)$

81. $25t^6 - 10t^3 + 5t^2$
$5t^2(5t^4 - 2t + 1)$

82. $5ax^2 + 5ay^2$
$5a(x^2 + y^2)$

83. $-63u^3v^6z^9 + 28u^2v^7z^2 - 21u^3v^3z^4$
$-7u^2v^3z^2(9uv^3z^7 - 4v^4 + 3uz^2)$

84. $-56x^4y^3z^2 - 72x^3y^4z^5 + 80xy^2z^3$
$-8xy^2z^2(7x^3y + 9x^2y^2z^3 - 10z)$

85. $-a(x + y) + b(x + y)$
$-(x + y)(a - b)$

86. $-bx(a - b) - cx(a - b)$
$-x(a - b)(b + c)$

87. $(u + v)^2 - (u + v)$
$(u + v)(u + v - 1)$

88. $a(x - y) - (x - y)^2$
$(x - y)(a - x + y)$

89. $-6x^3y - 12x^2y^2 - 18xy^3$
$-6xy(x^2 + 2xy + 3y^2)$

90. $-10m^3n^2 + 5m^2n^3 - 15mn^4$
$-5mn^2(2m^2 - mn + 3n^2)$

91. $3c - cd + 3d - c^2$
$(3 - c)(c + d)$

92. $x^2y - ax - xy + a$
$(xy - a)(x - 1)$

93. $2n^4p - 2n^2 - n^3p^2 + np + 2mn^3p - 2mn$
$n(2n - p + 2m)(n^2p - 1)$

94. $a^2c^3 + ac^2 + a^3c^2 - 2a^2bc^2 - 2bc^2 + c^3$
$c^2(a^2 + 1)(c + a - 2b)$

95. $(a - b)r - (a - b)s$
$(a - b)(r - s)$

96. $(x + y)u + (x + y)v$
$(x + y)(u + v)$

97. $x^2 + 4y - xy - 4x$
$(x - y)(x - 4)$

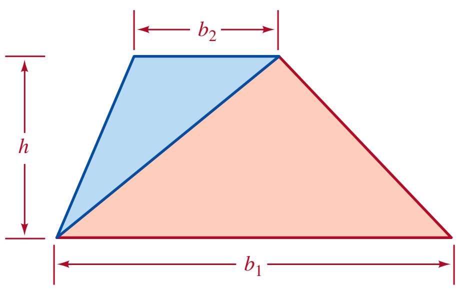
98. $a^2 - 4b + ab - 4a$
$(a + b)(a - 4)$

APPLICATIONS

99. GEOMETRIC FORMULAS

a. Write an expression that gives the area of the portion of the figure below that is shaded red. $\frac{1}{2}b_1h$

b. Do the same for the portion of the figure that is shaded blue. $\frac{1}{2}b_2h$



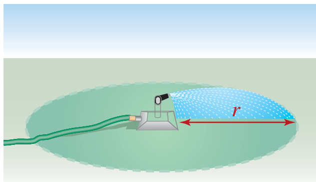
c. Add the results from parts a and b and then factor that expression. What important formula from geometry do you obtain?
$\frac{1}{2}h(b_1 + b_2)$; the formula for the area of a trapezoid

100. PACKAGING The amount of cardboard needed to make the cereal box shown below can be found by computing the area A, which is given by the formula

$$A = 2wh + 4wl + 2lh$$

where w is the width, h the height, and l the length. Solve the equation for the width. $w = \frac{A - 2lh}{2h + 4l}$

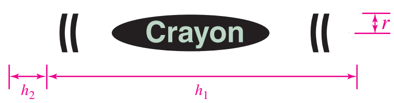
101. LANDSCAPING The combined area of the portions of the square lot that the sprinkler doesn't reach is given by $4r^2 - \pi r^2$, where r is the radius of the circular spray. Factor this expression. $r^2(4 - \pi)$



102. CRAYONS The amount of colored wax used to make the crayon shown below can be found by computing its volume using the formula

$$V = \pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2$$

Factor the expression on the right side of this equation. $\pi r^2(h_1 + \frac{1}{3}h_2)$



WRITING

103. One student commented, "Factoring undoes the distributive property." What do you think she meant?

104. Explain how to find the greatest common factor of two natural numbers.

105. Explain what is wrong with the following factorization.

$$5x^2 + x - 2 = 5(x^2 + x - 2)$$

106. What is a prime polynomial?

REVIEW

107. What figure results when the function $f(x) = 3x + 1$ is graphed? a line

108. What figure results when the function $g(x) = x^2$ is graphed? a parabola

109. Solve the inequality $-x > 3$. Express the solution set using interval notation. $(-\infty, -3)$

110. Evaluate: $2|-25-(-6)(3)|$ 14

111. If two different lines are parallel, what can be said about their slopes? They are the same.

112. Are $-3t^2$ and $12t^2$ like terms? If so, combine them. yes, $9t^2$

SECTION 5.6
The Difference of Two Squares; the Sum and Difference of Two Cubes

Objectives

1. Identify perfect squares.
2. Factor the difference of two squares.
3. Identify perfect cubes.
4. Factor the sum and difference of two cubes.

In this section, we will discuss rules of factoring that apply to polynomials that can be written as the difference of two squares or as the sum or difference of two cubes. To use these methods, we must be able to recognize such polynomials. We begin with a discussion that will help you recognize polynomials with terms that are *perfect squares*.

1 Identify perfect squares.

To factor the difference of two squares, it is helpful to know the first 20 integers that are **perfect squares.**

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400

Expressions such as $x^6y^4z^2$ are also perfect squares, because they can be written as the square of another quantity:

$$x^6y^4z^2 = (x^3y^2z)^2$$

2 Factor the difference of two squares.

In Section 5.4, we developed the special product formula

(1) $(x + y)(x - y) = x^2 - y^2$

The binomial $x^2 - y^2$ is called the **difference of two squares,** because x^2 represents the square of x, y^2 represents the square of y, and $x^2 - y^2$ represents the difference of these squares.

Equation 1 can be written in reverse order to give a rule for factoring the difference of two squares.

Factoring the Difference of Two Squares

$$x^2 - y^2 = (x + y)(x - y)$$

If we think of the difference of two squares as the square of a **F**irst quantity minus the square of a **L**ast quantity, we have the formula

$$F^2 - L^2 = (F + L)(F - L)$$

and we say: *To factor the square of a **First** quantity minus the square of a **Last** quantity, we multiply the **First** plus the **Last** by the **First** minus the **Last**.*

Self Check 1

Factor: $81p^2 - 25$

Now Try **Problem 18**

Self Check 1 Answer

$(9p + 5)(9p - 5)$

Teaching Example 1 Factor: $25x^2 - 36$

Answer:

$(5x + 6)(5x - 6)$

EXAMPLE 1 Factor: $49x^2 - 16$

Strategy The terms of this binomial do not have a common factor (other than 1). Then we will rewrite the binomial $49x^2 - 16$ as a difference of two squares: $(7x)^2 - (4)^2$.

WHY If the binomial is a difference of two squares, we can factor it using a special-product rule.

Solution

$$\begin{array}{ccccc} F^2 & - & L^2 & = & (F + L)(F - L) \\ \downarrow & & \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ (7x)^2 & - & 4^2 & = & (7x + 4)(7x - 4) \end{array}$$

We can verify this result using the FOIL method to perform the multiplication.

$$\begin{aligned} (7x + 4)(7x - 4) &= 49x^2 - 28x + 28x - 16 \\ &= 49x^2 - 16 \end{aligned}$$

Caution! Many expressions that represent the sum of two squares, such as $(7x)^2 + 4^2$, cannot be factored in the real number system. The binomial $49x^2 + 16$ is a prime binomial.

Self Check 2

Factor: $36r^4 - s^2$

Now Try **Problem 20**

Self Check 2 Answer

$(6r^2 + s)(6r^2 - s)$

Teaching Example 2 Factor: $49x^2 - 81y^4$

Answer:

$(7x + 9y^2)(7x - 9y^2)$

EXAMPLE 2 Factor: $64a^4 - 25b^2$

Strategy We note that the terms of this binomial do not have a common factor (other than 1). Then we will rewrite the binomial $64x^4 - 25b^2$ as a difference of two squares: $(8x^2)^2 - (5b)^2$.

WHY If the binomial is a difference of two squares, we can factor it using a special-product rule.

Solution

$$\begin{array}{ccccc} F^2 & - & L^2 & = & (F + L)(F - L) \\ \downarrow & & \downarrow & & \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ (8a^2)^2 & - & (5b)^2 & = & (8a^2 + 5b)(8a^2 - 5b) \end{array}$$

Verify by multiplication.

Self Check 3

Factor: $a^4 - 81$

Now Try **Problem 24**

Self Check 3 Answer

$(a^2 + 9)(a + 3)(a - 3)$

Teaching Example 3 Factor: $x^4 - 16$

Answer:

$(x^2 + 4)(x + 2)(x - 2)$

EXAMPLE 3 Factor: $x^4 - 1$

Strategy We note that the terms of this binomial do not have a common factor (other than 1). Then we will rewrite the binomial $x^4 - 1$ as a difference of two squares: $(x^2)^2 - (1)^2$.

WHY If the binomial is a difference of two squares, we can factor it using a special-product rule.

Solution

Because the binomial is the difference of the squares of x^2 and 1, it factors into the sum of x^2 and 1 and the difference of x^2 and 1.

$$x^4 - 1 = (x^2)^2 - (1)^2$$
$$= (x^2 + 1)(x^2 - 1)$$

The factor $x^2 + 1$ is the sum of two quantities and is prime. However, the factor $x^2 - 1$ is the difference of two squares and can be factored as $(x + 1)(x - 1)$. Thus,

$$x^4 - 1 = (x^2 + 1)(x^2 - 1)$$
$$= (x^2 + 1)(x + 1)(x - 1)$$

Success Tip When asked to factor a polynomial, we must be sure to factor it completely. After factoring a polynomial, always check to see whether any of the factors in the result can be factored further.

EXAMPLE 4

Factor: $(x + y)^4 - z^4$

Strategy We will use a substitution to factor this difference of two squares.

WHY For more complicated expressions, especially those involving a quantity within parentheses, a substitution often helps simplify the factoring process.

Solution

If we use the substitution $a = x + y$, we obtain

$$(x + y)^4 - z^4 = a^4 - z^4 \quad \text{Replace } x + y \text{ with } a.$$
$$= (a^2 + z^2)(a^2 - z^2) \quad \text{Factor the difference of two squares.}$$
$$= (a^2 + z^2)(a + z)(a - z) \quad \text{Factor } a^2 - z^2.$$

To find the factorization of $(x + y)^4 - z^4$, we substitute $x + y$ for each a in the expression $(a^2 + z^2)(a + z)(a - z)$.

$$(a^2 + z^2)(a + z)(a - z) = [(x + y)^2 + z^2](x + y + z)(x + y - z)$$

Thus, $(x + y)^4 - z^4 = [(x + y)^2 + z^2](x + y + z)(x + y - z)$.

If we square the binomial within the brackets, we have

$$(x + y)^4 - z^4 = [x^2 + 2xy + y^2 + z^2](x + y + z)(x + y - z)$$

Self Check 4

Factor: $(a - b)^4 - c^4$

Now Try Problem 30

Self Check 4 Answer
$[(a - b)^2 + c^2](a - b + c)(a - b - c)$

Teaching Example 4 Factor: $(x - y)^4 - z^4$
Answer:
$[(x - y)^2 + z^2](x - y + z)(x - y - z)$

When possible, we always factor out a common factor before factoring the difference of two squares. The factoring process is easier when all common factors are factored out first.

EXAMPLE 5

Factor: $2x^4y - 32y$

Strategy We will factor out the GCF of $2y$ and factor the remaining difference of two squares.

WHY The first step in factoring any polynomial is to factor out the GCF.

Solution

$$2x^4y - 32y = 2y(x^4 - 16) \quad \text{Factor out the GCF, which is } 2y.$$
$$= 2y(x^2 + 4)(x^2 - 4) \quad \text{Factor } x^4 - 16.$$
$$= 2y(x^2 + 4)(x + 2)(x - 2) \quad \text{Factor } x^2 - 4.$$

Self Check 5

Factor: $3a^4 - 3$

Now Try Problem 32

Self Check 5 Answer
$3(a^2 + 1)(a + 1)(a - 1)$

Teaching Example 5 Factor: $5x^4 - 405$
Answer:
$5(x^2 + 9)(x + 3)(x - 3)$

Self Check 6

Factor: $a^2 - b^2 + a + b$

Now Try **Problem 36**

Self Check 6 Answer
$(a + b)(a - b + 1)$

Teaching Example 6 Factor: $x^2 - 4 + x + 2$
Answer:
$(x + 2)(x - 2 + 1)$

EXAMPLE 6 Factor: $x^2 - y^2 + x - y$

Strategy The terms of each expression do not have a common factor (other than 1) and traditional factoring by grouping will not work. Instead, we will group the first two terms of the polynomial and the last two terms.

WHY Hopefully, those steps will produce equivalent expressions that can be factored.

Solution

If we group the first two terms and factor the difference of two squares, we have

$$x^2 - y^2 + x - y = (x + y)(x - y) + (x - y) \quad \text{Factor } x^2 - y^2. \text{ The terms of the resulting expression have a common factor, } x - y.$$

$$= (x - y)(x + y + 1) \quad \text{Factor out } x - y.$$

3 Identify perfect cubes.

The number 125 is called a perfect cube, because $5^3 = 125$. To factor the sum or difference of two cubes, it is helpful to know the first ten **perfect cubes:**

1, 8, 27, 64, 125, 216, 343, 512, 729, 1,000

Expressions such as $x^9y^6z^3$ are also perfect cubes, because they can be written as the cube of another quantity:

$$x^9y^6z^3 = (x^3y^2z)^3$$

4 Factor the sum and difference of two cubes.

To find rules for factoring the sum or difference of two cubes, we use the following product formulas:

(2) $(x + y)(x^2 - xy + y^2) = x^3 + y^3$

(3) $(x - y)(x^2 + xy + y^2) = x^3 - y^3$

To verify Equation 2, we multiply $x^2 - xy + y^2$ by $x + y$.

$$\begin{aligned}(x + y)(x^2 - xy + y^2) &= (x + y)x^2 - (x + y)xy + (x + y)y^2 \\ &= x \cdot x^2 + y \cdot x^2 - x \cdot xy - y \cdot xy + x \cdot y^2 + y \cdot y^2 \\ &= x^3 + x^2y - x^2y - xy^2 + xy^2 + y^3 \\ &= x^3 + y^3\end{aligned}$$

Equation 3 can also be verified by multiplication.

If we write Equations 2 and 3 in reverse order, we have the formulas for factoring the sum and difference of two cubes.

Factoring the Sum and Difference of Two Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

If we think of the sum of two cubes as the sum of the cube of a **F**irst quantity plus the cube of a **L**ast quantity, we have the formula

$$\text{F}^3 + \text{L}^3 = (\text{F} + \text{L})(\text{F}^2 - \text{FL} + \text{L}^2)$$

To factor the cube of a ***First*** *quantity plus the cube of a* ***Last*** *quantity, we multiply the sum of the* ***First*** *and* ***Last*** *by*

- *the* ***First*** *squared*
- *minus the* ***First*** *times the* ***Last***
- *plus the* ***Last*** *squared.*

The formula for the difference of two cubes is

$$\mathrm{F}^3 - \mathrm{L}^3 = (\mathrm{F} - \mathrm{L})(\mathrm{F}^2 + \mathrm{FL} + \mathrm{L}^2)$$

To factor the cube of a ***First*** *quantity minus the cube of a* ***Last*** *quantity, we multiply the difference of the* ***First*** *and* ***Last*** *by*

- *the* ***First*** *squared*
- *plus the* ***First*** *times the* ***Last***
- *plus the* ***Last*** *squared.*

EXAMPLE 7 Factor: $a^3 + 8$

Strategy The terms of each expression do not have a common factor (other than 1) and the expression has two terms. We will write the binomial in a form that shows it is the sum of two cubes.

WHY We can then use the rule for factoring the sum of two cubes.

Solution

Since $a^3 + 8$ can be written as $a^3 + 2^3$, we have the sum of two cubes, which factors as follows:

$$\begin{aligned} \mathrm{F}^3 + \mathrm{L}^3 &= (\mathrm{F} + \mathrm{L})(\mathrm{F}^2 - \mathrm{FL} + \mathrm{L}^2) \\ \downarrow \quad \downarrow \quad & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow\downarrow \quad \downarrow \\ a^3 + 2^3 &= (a + 2)(a^2 - a2 + 2^2) \\ &= (a + 2)(a^2 - 2a + 4) \end{aligned}$$

Thus, $a^3 + 8 = (a + 2)(a^2 - 2a + 4)$. Check by multiplication.

Self Check 7

Factor: $p^3 + 27$

Now Try **Problem 40**

Self Check 7 Answer

$(p + 3)(p^2 - 3p + 9)$

Teaching Example 7 Factor: $x^3 + 125$

Answer:

$(x + 5)(x^2 - 5x + 25)$

EXAMPLE 8 Factor: $27a^3 - 64b^3$

Strategy The terms of each expression do not have a common factor (other than 1) and the expression has two terms. We will write the binomial in a form that shows it is the difference of two cubes.

WHY We can then use the rule for factoring the difference of two cubes.

Solution

Since $27a^3 - 64b^3$ can be written as $(3a)^3 - (4b)^3$, we have the difference of two cubes, which factors as follows:

$$\begin{aligned} \mathrm{F}^3 - \mathrm{L}^3 &= (\mathrm{F} - \mathrm{L})(\mathrm{F}^2 + \mathrm{F}\ \mathrm{L} + \mathrm{L}^2) \\ \downarrow \quad \downarrow \quad & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ (3a)^3 - (4b)^3 &= (3a - 4b)[(3a)^2 + (3a)(4b) + (4b)^2] \\ &= (3a - 4b)(9a^2 + 12ab + 16b^2) \end{aligned}$$

Thus, $27a^3 - 64b^3 = (3a - 4b)(9a^2 + 12ab + 16b^2)$. Check by multiplication.

Self Check 8

Factor: $8p^3 - 125q^3$

Now Try **Problem 46**

Self Check 8 Answer

$(2p - 5q)(4p^2 + 10pq + 25q^2)$

Teaching Example 8 Factor: $1000x^3 - 27y^3$

Answer:

$(10x - 3y)(100x^2 + 30xy + 9y^2)$

Self Check 9

Factor: $(p + q)^3 - r^3$

Now Try Problem 48

Self Check 9 Answer

$(p + q - r)(p^2 + 2pq + q^2 + pr + qr + r^2)$

Teaching Example 9 Factor: $(x + y)^3 - b^3$

Answer:

$(x + y - b)(x^2 + 2xy + y^2 + bx + by + b^2)$

EXAMPLE 9

Factor: $a^3 - (c + d)^3$

Strategy The terms of each expression do not have a common factor (other than 1) and the expression has two terms. We will use the rule for factoring the difference of two cubes.

WHY The terms cubes a^3 and $(c + d)^3$ are perfect cubes.

Solution

$$a^3 - (c + d)^3 = [a - (c + d)][a^2 + a(c + d) + (c + d)^2]$$

Now we simplify the expressions inside both sets of brackets.

$$a^3 - (c + d)^3 = (a - c - d)(a^2 + ac + ad + c^2 + 2cd + d^2)$$

Self Check 10

Factor: $x^6 - 1$

Now Try Problem 50

Self Check 10 Answer

$(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$

Teaching Example 10 Factor: $a^6 - b^6$

Answer:

$(a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$

EXAMPLE 10

Factor: $x^6 - 64$

Strategy The terms of each expression do not have a common factor (other than 1) and the expression has two terms. The binomial is both the difference of two squares and the difference of two cubes. We will write it in a form that shows it as the difference of two squares to begin the factoring process.

WHY It is easier to factor it as the difference of two squares first.

Solution

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - 8^2 \\ &= \mathbf{(x^3 + 8)(x^3 - 8)} \end{aligned}$$

Each of these factors further, however, for one is the sum of two cubes and the other is the difference of two cubes:

$$x^6 - 64 = \mathbf{(x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4)}$$

Self Check 11

Factor: $3x^5 + 24x^2$

Now Try Problem 52

Self Check 11 Answer

$3x^2(x + 2)(x^2 - 2x + 4)$

Teaching Example 11 Factor: $5a^4 + 40a$

Answer:

$5a(a + 2)(a^2 - 2a + 4)$

EXAMPLE 11

Factor: $2a^5 + 128a^2$

Strategy We will factor out the GCF of $2a^2$ and factor the resulting sum of two cubes.

WHY The first step in factoring any polynomial is to factor out the GCF.

Solution

We first factor out the common monomial factor $2a^2$ to obtain

$$2a^5 + 128a^2 = 2a^2\mathbf{(a^3 + 64)}$$

Then we factor $a^3 + 64$ as the sum of two cubes to obtain

$$2a^5 + 128a^2 = 2a^2\mathbf{(a + 4)(a^2 - 4a + 16)}$$

ANSWERS TO SELF CHECKS

1. $(9p + 5)(9p - 5)$ **2.** $(6r^2 + s)(6r^2 - s)$ **3.** $(a^2 + 9)(a + 3)(a - 3)$
4. $[(a - b)^2 + c^2](a - b + c)(a - b - c)$ **5.** $3(a^2 + 1)(a + 1)(a - 1)$
6. $(a + b)(a - b + 1)$ **7.** $(p + 3)(p^2 - 3p + 9)$ **8.** $(2p - 5q)(4p^2 + 10pq + 25q^2)$
9. $(p + q - r)(p^2 + 2pq + q^2 + pr + qr + r^2)$
10. $(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)$ **11.** $3x^2(x + 2)(x^2 - 2x + 4)$

SECTION 5.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. When the polynomial $4x^2 - 25$ is rewritten as $(2x)^2 - (5)^2$, we see that it is the difference of two squares.

2. When the polynomial $8x^3 + 125$ is rewritten as $(2x)^3 + (5)^3$, we see that it is the sum of two cubes.

CONCEPTS

3. Write the first ten integers that are perfect squares.
1, 4, 9, 16, 25, 36, 49, 64, 81, 100

4. Write the first ten perfect cubes.
1, 8, 27, 64, 125, 216, 343, 512, 729, 1,000

5. Use multiplication to verify that the sum of two squares $x^2 + 25$ does not factor as $(x + 5)(x + 5)$.
$(x + 5)(x + 5) = x^2 + 10x + 25$

6. Use multiplication to verify that the difference of two squares $x^2 - 25$ factors as $(x + 5)(x - 5)$.
$(x + 5)(x - 5) = x^2 - 25$

Explain why each factorization is not complete.

7. $4g^2 - 16 = (2g + 4)(2g - 4)$
A common factor of 2 can be factored out of each binomial.

8. $1 - t^8 = (1 + t^4)(1 - t^4)$
$(1 - t^4)$ factors as the difference of two squares.

NOTATION

Complete each factorization.

9. $p^3 + q^3 = (p + q)(p^2 - pq + q^2)$

10. $p^3 - q^3 = (p - q)(p^2 + pq + q^2)$

11. $p^2 - q^2 = (p + q)(p - q)$

12. $p^2q + pq^2 = pq(p + q)$

13. $36y^2 - 49m^2 = (6y)^2 - (7m)^2 = (6y + 7m)(6y - 7m)$

14. $h^3 - 27k^3 = (h)^3 - (3k)^3 = (h - 3k)(h^2 + 3hk + 9k^2)$

GUIDED PRACTICE

Factor each polynomial. See Example 1.

15. $x^2 - 4$
$(x + 2)(x - 2)$

16. $y^2 - 9$
$(y + 3)(y - 3)$

17. $9y^2 - 64$
$(3y + 8)(3y - 8)$

18. $25a^2 - 49$
$(5a - 7)(5a + 7)$

Factor each polynomial. See Example 2.

19. $16x^4 - 81y^2$
$(4x^2 + 9y)(4x^2 - 9y)$

20. $144a^2 - b^4$
$(12a + b^2)(12a - b^2)$

21. $625a^2 - 169b^4$
$(25a + 13b^2)(25a - 13b^2)$

22. $64r^6 - 121s^2$
$(8r^3 + 11s)(8r^3 - 11s)$

Factor each polynomial. See Example 3.

23. $s^4 - 16$
$(s^2 + 4)(s + 2)(s - 2)$

24. $t^4 - 625$
$(t^2 + 25)(t + 5)(t - 5)$

25. $m^4 - 81$
$(m^2 + 9)(m + 3)(m - 3)$

26. $z^4 - 256$
$(z^2 + 16)(z + 4)(z - 4)$

Factor each polynomial. See Example 4.

27. $(x + y)^2 - z^2$
$(x + y + z)(x + y - z)$

28. $(a - b)^2 - c^2$
$(a - b + c)(a - b - c)$

29. $(m + n)^4 - p^4$
$[(m + n)^2 + p^2](m + n + p)(m + n - p)$

30. $(p + q)^4 - t^4$
$[(p + q)^2 + t^2](p + q + t)(p + q - t)$

Factor each polynomial. See Example 5.

31. $2x^3 - 32x$
$2x(x + 4)(x - 4)$

32. $3x^3 - 243x$
$3x(x + 9)(x - 9)$

33. $32a^4 - 162b^4$
$2(4a^2 + 9b^2)(2a + 3b)(2a - 3b)$

34. $256x^4y^4z^2 - z^{10}$
$z^2(16x^2y^2 + z^4)(4xy + z^2)(4xy - z^2)$

Factor each polynomial. See Example 6.

35. $a^2 - b^2 + a + b$
$(a + b)(a - b + 1)$

36. $x^2 - y^2 - x - y$
$(x + y)(x - y - 1)$

37. $a^2 - b^2 + 2a - 2b$
$(a - b)(a + b + 2)$

38. $m^2 - n^2 + 3m + 3n$
$(m + n)(m - n + 3)$

Factor each polynomial. See Example 7.

39. $t^3 + 27$
$(t + 3)(t^2 - 3t + 9)$

40. $p^3 + 64$
$(p + 4)(p^2 - 4p + 16)$

41. $r^3 + s^3$
$(r + s)(r^2 - rs + s^2)$

42. $x^3 + 8y^3$
$(x + 2y)(x^2 - 2xy + 4y^2)$

Factor each polynomial. See Example 8.

43. $r^3 - 125$
$(r - 5)(r^2 + 5r + 25)$

44. $s^3 - 1{,}000$
$(s - 10)(s^2 + 10s + 100)$

45. $8a^3 - 27b^3$
$(2a - 3b)(4a^2 + 6ab + 9b^2)$

46. $64a^3 - 125b^3$
$(4a - 5b)(16a^2 + 20ab + 25b^2)$

Factor each polynomial. See Examples 9–10.

47. $27 - (x + y)^3$
$(3 - x - y)(9 + 3x + 3y + x^2 + 2xy + y^2)$

48. $x^3 - (y + z)^3$
$(x - y - z)(x^2 + xy + xz + y^2 + 2yz + z^2)$

49. $t^6 - 1$
$(t + 1)(t^2 - t + 1)(t - 1)(t^2 + t + 1)$

50. $t^6 - 1{,}000{,}000$
$(t + 10)(t^2 - 10t + 100)(t - 10)(t^2 + 10t + 100)$

Selected exercises available online at
www.webassign.net/brookscole

Factor each polynomial. See Examples 11.

51. $5x^3 + 625$
$5(x + 5)(x^2 - 5x + 25)$

52. $2x^6 + 54x^3$
$2x^3(x + 3)(x^2 - 3x + 9)$

53. $2x^3 - 128$
$2(x - 4)(x^2 + 4x + 16)$

▶ **54.** $4x^5 - 256x^2$
$4x^2(x - 4)(x^2 + 4x + 16)$

TRY IT YOURSELF

Factor each polynomial. If a polynomial is prime, so indicate.

55. $5p^2 + 20$
$5(p^2 + 4)$

56. $5p^2 - 20$
$5(p + 2)(p - 2)$

57. $5p^3 + 20$
$5(p^3 + 4)$

58. $5p^3 + 40$
$5(p + 2)(p^2 - 2p + 4)$

59. $x^2 + 25$
prime

60. $4y^2 + 9z^4$
prime

▶ **61.** $81a^2 - 49b^2$
$(9a + 7b)(9a - 7b)$

62. $36x^4y^2 - 49z^4$
$(6x^2y + 7z^2)(6x^2y - 7z^2)$

63. $4a^2b^4c^6 - 9d^8$
$(2ab^2c^3 + 3d^4)(2ab^2c^3 - 3d^4)$

64. $a^2 - (b - c)^2$
$(a + b - c)(a - b + c)$

▶ **65.** $x^4 - y^4$
$(x^2 + y^2)(x + y)(x - y)$

▶ **66.** $(m + n)^2 - p^4$
$(m + n + p^2)(m + n - p^2)$

67. $225a^4 - 16b^8c^{12}$
$(15a^2 + 4b^4c^6)(15a^2 - 4b^4c^6)$

68. $2x^2 - 288$
$2(x + 12)(x - 12)$

▶ **69.** $8x^2 - 72$
$8(x + 3)(x - 3)$

70. $5x^3 + 125x$
$5x(x^2 + 25)$

▶ **71.** $6x^4 - 216x^2$
$6x^2(x + 6)(x - 6)$

72. $r^2s^2t^2 - t^2x^4y^2$
$t^2(rs + x^2y)(rs - x^2y)$

▶ **73.** $16a^4b^3c^4 - 64a^2bc^6$
$16a^2bc^4(ab + 2c)(ab - 2c)$

74. $2x + y + 4x^2 - y^2$
$(2x + y)(1 + 2x - y)$

75. $m - 2n + m^2 - 4n^2$
$(m - 2n)(1 + m + 2n)$

76. $t^3 - v^3$
$(t - v)(t^2 + tv + v^2)$

77. $27a^3 + b^3$
$(3a + b)(9a^2 - 3ab + b^2)$

78. $8x^6 + 125y^3$
$(2x^2 + 5y)(4x^4 - 10x^2y + 25y^2)$

79. $125x^3y^6 + 216z^9$
$(5xy^2 + 6z^3)(25x^2y^4 - 30xy^2z^3 + 36z^6)$

80. $1{,}000a^6 - 343b^3c^6$
$(10a^2 - 7bc^2)(100a^4 + 70a^2bc^2 + 49b^2c^4)$

81. $x^6 + y^6$
$(x^2 + y^2)(x^4 - x^2y^2 + y^4)$

▶ **82.** $x^9 + y^9$
$(x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$

83. $128u^2v^3 - 2t^3u^2$
$2u^2(4v - t)(16v^2 + 4tv + t^2)$

84. $56rs^2t^3 + 7rs^2v^6$
$7rs^2(2t + v^2)(4t^2 - 2tv^2 + v^4)$

85. $(a + b)x^3 + 27(a + b)$
$(a + b)(x + 3)(x^2 - 3x + 9)$

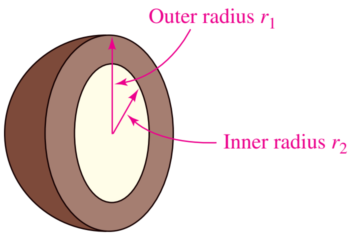
86. $(c - d)r^3 - (c - d)s^3$
$(c - d)(r - s)(r^2 + rs + s^2)$

APPLICATIONS

▶ **87.** CANDY To find the amount of chocolate used in the outer coating of the malted-milk ball shown in the next column, we can find the volume V of the chocolate shell using the formula

$$V = \frac{4}{3}\pi r_1^3 - \frac{4}{3}\pi r_2^3$$

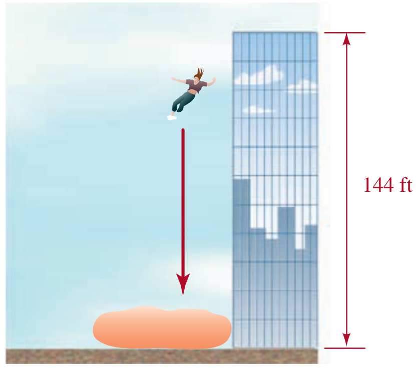
Factor the expression on the right side of the formula.
$\frac{4}{3}\pi(r_1 - r_2)(r_1{}^2 + r_1r_2 + r_2{}^2)$



▶ **88.** MOVIE STUNTS The function that gives the distance a stuntwoman is above the ground t seconds after she falls over the side of a 144-foot-tall building is

$$h(t) = 144 - 16t^2$$

Factor the right side of the equation. $16(3 - t)(3 + t)$



WRITING

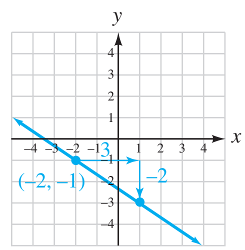
89. Describe the pattern used to factor the difference of two squares.

▶ **90.** Describe the patterns used to factor the sum and the difference of two cubes.

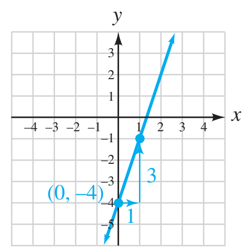
REVIEW

Graph the line with the given characteristics.

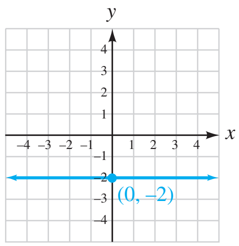
91. Passing through $(-2, -1)$, slope $= -\frac{2}{3}$



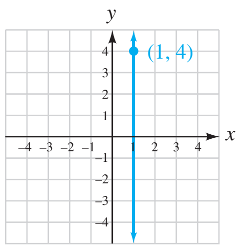
92. y-intercept $(0, -4)$, slope $= 3$



93. Horizontal; y-intercept $(0, -2)$

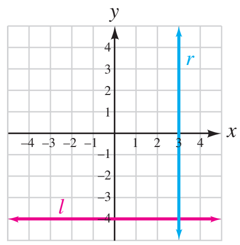


94. Parallel to the y-axis, passing through $(1, 4)$



95. Write the equation of line l shown in the illustration to the right. $y = -4$

96. Write the equation of line r shown in the illustration to the right. $x = 3$



SECTION 5.7 Factoring Trinomials

Objectives

1. Factor perfect-square trinomials.
2. Factor trinomials of the form $x^2 + bx + c$.
3. Factor trinomials of the form $ax^2 + bx + c$.
4. Test for factorability.
5. Use substitution to factor trinomials.
6. Use grouping to factor trinomials.

In this section, we will discuss several techniques for factoring trinomials. These techniques are based on the fact that the product of two binomials is often a trinomial. With that observation in mind, we begin the study of trinomial factoring by considering two special products.

1 Factor perfect-square trinomials.

Many trinomials can be factored by using the following special product formulas.

(1) $(x + y)(x + y) = x^2 + 2xy + y^2$

(2) $(x - y)(x - y) = x^2 - 2xy + y^2$

To factor $x^2 + 6x + 9$, we note that it can be written in the form $x^2 + 2(3)x + 3^2$. If $y = 3$, this form matches the right-hand side of Equation 1. Thus, $x^2 + 6x + 9$ factors as

$$\begin{aligned} x^2 + 6x + 9 &= x^2 + 2(3)x + 3^2 \\ &= (x + 3)(x + 3) \\ &= (x + 3)^2 \end{aligned}$$

Since $x^2 + 6x + 9$ is the square of $x + 3$, $x^2 + 6x + 9$ is called a **perfect square trinomial.** This result can be verified by multiplication:

$$\begin{aligned} (x + 3)(x + 3) &= x^2 + 3x + 3x + 9 \\ &= x^2 + 6x + 9 \end{aligned}$$

EXAMPLE 1 Factor: $x^2 - 4xz + 4z^2$

Strategy The terms of this trinomial do not have a common factor (other than 1). We will determine whether it is a perfect-square trinomial.

WHY If it is, we can factor it using a special-product rule.

Self Check 1

Factor: $b^2 - 10b + 25$ $(b - 5)^2$

Now Try **Problem 26**

Teaching Example 1 Factor: $x^2 - 14x + 49$
Answer: $(x - 7)^2$

Solution

To factor the perfect square trinomial $x^2 - 4xz + 4z^2$, we note that it can be written in the form $x^2 - 2x(2z) + (2z)^2$. If $y = 2z$, this form matches the right-hand side of Equation 2.

$$\begin{aligned} x^2 - 4xz + 4z^2 &= x^2 - 2x(2z) + (2z)^2 \\ &= (x - 2z)(x - 2z) \\ &= (x - 2z)^2 \end{aligned}$$

This result can be verified by multiplication.

We begin our discussion of *general trinomials* by considering trinomials with leading coefficients (the coefficient of the squared term) of 1.

2 Factor trinomials of the form $x^2 + bx + c$.

Since the product of two binomials is often a trinomial, we expect that many trinomials will factor as two binomials. For example, to factor $x^2 + 7x + 12$, we must find two binomials $x + a$ and $x + b$ such that

$$x^2 + 7x + 12 = (x + a)(x + b)$$

where $ab = 12$ and $ax + bx = 7x$.

To find the numbers a and b, we list the possible factorizations of 12 and find the one where the sum of the factors is 7.

This is the one to choose.

$12(1)$ $6(2)$ $4(3)$ $-12(-1)$ $-6(-2)$ $-4(-3)$

Thus, $a = 4$, $b = 3$, and

$$x^2 + 7x + 12 = (x + a)(x + b)$$

(3) $$x^2 + 7x + 12 = (x + 4)(x + 3)$$

This factorization can be verified by multiplying $x + 4$ and $x + 3$ and observing that the product is $x^2 + 7x + 12$.

Because of the commutative property of multiplication, the order of the factors in Equation 3 is not important.

To factor trinomials with lead coefficients of 1, we follow these steps.

Factoring Trinomials with Leading Coefficients of 1

1. Write the trinomial in descending powers of one variable.
2. List the factorizations of the third term of the trinomial.
3. Pick the factorization where the sum of the factors is equal to the coefficient of the middle term.

Self Check 2

Factor: $a^2 - 7a + 12$

Now Try **Problem 32**

Self Check 2 Answer
$(a - 4)(a - 3)$

EXAMPLE 2

Factor: $x^2 - 6x + 8$

Strategy The terms of this trinomial do not have a common factor (other than 1). We will assume that $x^2 - 6x + 8$ is a product of two binomials. We must find the terms of the binomials.

WHY The only option is to try to factor $x^2 - 6x + 8$ as a product of two binomials, once we first check for a common factor.

Solution

Since the trinomial is written in descending powers of x, we can move to step 2 and list the possible factorizations of the third term of the trinomial, which is 8.

This is the one to choose.

$$8(1) \qquad 4(2) \qquad -8(-1) \qquad -4(-2)$$

In the trinomial, the coefficient of the middle term is -6. The only factorization where the sum of the factors is -6 is $-4(-2)$. Thus, $a = -4$, $b = -2$, and

$$\begin{aligned} x^2 - 6x + 8 &= (x + a)(x + b) \\ &= (x - 4)(x - 2) \end{aligned}$$

We can verify this result by multiplication:

$$\begin{aligned} (x - 4)(x - 2) &= x^2 - 2x - 4x + 8 \quad \text{Use the FOIL method.} \\ &= x^2 - 6x + 8 \end{aligned}$$

Teaching Example 2 Factor: $x^2 - 16x + 15$
Answer:
$(x - 15)(x - 1)$

EXAMPLE 3 Factor: $-x + x^2 - 12$

Strategy We will write the terms of the trinomial in descending powers of x.

WHY It is easier to factor a trinomial if its terms are written in descending powers of the variable.

Solution

We begin by writing the trinomial in descending powers of x:

$$-x + x^2 - 12 = x^2 - x - 12$$

The possible factorizations of the third term are

This is the one to choose.

$$12(-1) \qquad 6(-2) \qquad 4(-3) \qquad 1(-12) \qquad 2(-6) \qquad 3(-4)$$

In the trinomial, the coefficient of the middle term is -1. The only factorization where the sum of the factors is -1 is $3(-4)$. Thus, $a = 3$, $b = -4$, and

$$\begin{aligned} -x + x^2 - 12 &= (x + a)(x + b) \\ &= (x + 3)(x - 4) \end{aligned}$$

Self Check 3

Factor: $-3a + a^2 - 10$

Now Try **Problem 34**

Self Check 3 Answer
$(a + 2)(a - 5)$

Teaching Example 3 Factor: $-8x + x^2 - 20$
Answer:
$(x - 10)(x + 2)$

EXAMPLE 4 Factor: $30x - 4xy - 2xy^2$

Strategy We will factor out the GCF of $-2x$ and factor the resulting trinomial.

WHY The first step in factoring any polynomial is to factor out the GCF. Factoring out the GCF first makes factoring easier.

Solution

We begin by writing the trinomial in descending powers of y:

$$30x - 4xy - 2xy^2 = -2xy^2 - 4xy + 30x$$

Each term in this trinomial has a common factor of $-2x$, which we will factor out.

$$30x - 4xy - 2xy^2 = -2x(y^2 + 2y - 15)$$

Self Check 4

Factor: $18a + 3ab - 3ab^2$

Now Try **Problem 40**

Self Check 4 Answer
$-3a(b + 2)(b - 3)$

Teaching Example 4 Factor: $-10ax - 28a + 2ax^2$
Answer:
$2a(x - 7)(x + 2)$

To factor $y^2 + 2y - 15$, we list the factors of -15 and find the pair whose sum is 2.

This is the one to choose.

$$15(-1) \qquad 5(-3) \qquad 1(-15) \qquad 3(-5)$$

The only factorization where the sum of the factors is 2 (the coefficient of the middle term of $y^2 + 2y - 15$) is $5(-3)$. Thus, $a = 5$, $b = -3$, and

$$\begin{aligned} 30x - 4xy - 2xy^2 &= -2x(\mathbf{y^2 + 2y - 15}) \\ &= -2x(\mathbf{y + 5})(\mathbf{y - 3}) \end{aligned}$$

Caution! In Example 4, be sure to include all factors in the final result. It is a common error to forget to write the $-2x$.

3 Factor trinomials of the form $ax^2 + bx + c$.

There are more combinations of factors to consider when factoring trinomials with lead coefficients other than 1. To factor $5x^2 + 7x + 2$, for example, we must find two binomials of the form $ax + b$ and $cx + d$ such that

$$5x^2 + 7x + 2 = (ax + b)(cx + d)$$

Since the first term of the trinomial $5x^2 + 7x + 2$ is $5x^2$, the first terms of the binomial factors must be $5x$ and x.

$$5x^2 + 7x + 2 = (\overbrace{5x + b)(x}^{5x^2} + d)$$

Since the product of the last terms must be 2, and the sum of the products of the outer and inner terms must be $7x$, we must find two numbers whose product is 2 that will give a middle term of $7x$.

$$5x^2 + 7x + 2 = (5x + \overbrace{b)(x + d}^{2})$$

$O + I = 7x$

Since $2(1)$ and $(-2)(-1)$ give a product of 2, there are four possible combinations to consider:

$$\begin{array}{ll} \mathbf{(5x + 2)(x + 1)} & (5x - 2)(x - 1) \\ (5x + 1)(x + 2) & (5x - 1)(x - 2) \end{array}$$

Of these possibilities, only the first one gives the correct middle term of $7x$. Thus,

(4) $5x^2 + 7x + 2 = (5x + 2)(x + 1)$

We can verify this result by multiplication:

$$\begin{aligned} (5x + 2)(x + 1) &= 5x^2 + 5x + 2x + 2 \\ &= 5x^2 + 7x + 2 \end{aligned}$$

4 Test for factorability.

If a trinomial has the form $ax^2 + bx + c$, with integer coefficients and $a \neq 0$, we can test to see whether it is factorable.

- If the value of $b^2 - 4ac$ is a perfect square, the trinomial can be factored using only integers.
- If the value of $b^2 - 4ac$ is not a perfect square, the trinomial cannot be factored using only integers.

For example, $5x^2 + 7x + 2$ is a trinomial in the form $ax^2 + bx + c$ with

$$a = 5, \qquad b = 7, \qquad \text{and} \qquad c = 2$$

For this trinomial, the value of $b^2 - 4ac$ is

$$\begin{aligned} b^2 - 4ac &= 7^2 - 4(5)(2) \\ &= 49 - 40 \\ &= 9 \end{aligned}$$

Since 9 is a perfect square, the trinomial is factorable. Its factorization is shown in Equation 4.

Test for Factorability

A trinomial of the form $ax^2 + bx + c$, with integer coefficients and $a \neq 0$, will factor into two binomials with integer coefficients if the value of $b^2 - 4ac$ is a perfect square. If $b^2 - 4ac = 0$, the factors will be the same.

EXAMPLE 5

Factor: $3p^2 - 4p - 4$

Strategy First, we see that the terms of the trinomial do not have a common factor (other than 1). Then we will use the test for factorability to see whether the trinomial is factorable.

WHY If it is, we can factor it as the product of two binomials by making educated guesses and then checking them using multiplication.

Solution

In the trinomial, $a = 3$, $b = -4$, and $c = -4$. To see whether it factors, we evaluate $b^2 - 4ac$.

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(3)(-4) \\ &= 16 + 48 \\ &= 64 \end{aligned}$$

Since 64 is a perfect square, the trinomial is factorable.

To factor the trinomial, we note that the first terms of the binomial factors must be $3p$ and p to give the first term of $3p^2$.

$3p^2$

$$3p^2 - 4p - 4 = (3p + ?)(p + ?)$$

The product of the last terms must be -4, and the sum of the products of the outer terms and the inner terms must be $-4p$.

-4

$$3p^2 - 4p - 4 = (3p + ?)(p + ?)$$

$\text{O} + \text{I} = -4p$

Self Check 5

Factor: $4q^2 - 9q - 9$

Now Try **Problem 44**

Self Check 5 Answer
$(4q + 3)(q - 3)$

Teaching Example 5 Factor: $10x^2 + x - 3$
Answer:
$(5x + 3)(2x - 1)$

Because $1(-4)$, $-1(4)$, and $-2(2)$ all give a product of -4, there are six possible combinations to consider:

$$(3p + 1)(p - 4) \qquad (3p - 4)(p + 1)$$
$$(3p - 1)(p + 4) \qquad (3p + 4)(p - 1)$$
$$(3p - 2)(p + 2) \qquad \mathbf{(3p + 2)(p - 2)}$$

Of these possibilities, only the last gives the required middle term of $-4p$. Thus,

$$3p^2 - 4p - 4 = (3p + 2)(p - 2)$$

Self Check 6

Factor, if possible: $5a^2 - 8a + 2$

Now Try **Problem 46**

Self Check 6 Answer
a prime polynomial

Teaching Example 6 Factor, if possible: $4x^2 + 6x + 9$
Answer:
a prime polynomial

EXAMPLE 6 Factor, if possible: $4t^2 - 3t - 5$

Strategy First, we see that the terms of the trinomial do not have a common factor (other than 1). Then we will use the test for factorability to see whether the trinomial is factorable.

WHY If it is not factorable, the polynomial is a prime polynomial.

Solution

In the trinomial, $a = 4$, $b = -3$, and $c = -5$. To see whether the trinomial is factorable, we evaluate $b^2 - 4ac$ by substituting the values of a, b, and c.

$$\begin{aligned} b^2 - 4ac &= (-3)^2 - 4(4)(-5) \\ &= 9 + 80 \\ &= 89 \end{aligned}$$

Since 89 is not a perfect square, the trinomial is not factorable using only integer coefficients.

It is not easy to give specific rules for factoring general trinomials. However, the following hints are helpful.

Factoring a General Trinomial

1. Write the trinomial in descending powers of one variable.
2. Factor out any greatest common factor (including -1, if that is necessary to make the coefficient of the first term positive).
3. Test the trinomial for factorability.
4. When the sign of the third term of the trinomial is $+$, the signs between the terms of each binomial factor are the same as the sign of the middle term of the trinomial.
   When the sign of the third term of the trinomial is $-$, the signs between the terms of the binomials are opposite.
5. Try various combinations of the factors of the first terms and the last terms until you find the one that works.
6. Check the factorization by multiplication.

Self Check 7

Factor: $-6x^2 - 15xy - 6y^2$

Now Try **Problem 52**

Self Check 7 Answer
$-3(x + 2y)(2x + y)$

EXAMPLE 7 Factor: $24y^2 + 10xy - 6x^2$

Strategy We will write the trinomial in descending powers of x and factor out the GCF of -2. Then we will factor the resulting trinomial.

WHY The first step in factoring any polynomial is to factor out the GCF. Factoring out the GCF first makes factoring easier.

Solution

$$24y^2 + 10xy - 6x^2 = -6x^2 + 10xy + 24y^2$$
$$= -2(3x^2 - 5xy - 12y^2)$$

In the trinomial $3x^2 - 5xy - 12y^2$, $a = 3$, $b = -5$, and $c = -12$.

$$b^2 - 4ac = (-5)^2 - 4(3)(-12)$$
$$= 25 + 144$$
$$= 169$$

Since 169 is a perfect square, the trinomial will factor.

The sign of the first term of $3x^2 - 5xy - 12y^2$ is positive and the sign of the third term is negative. Therefore, the signs between the binomial factors will be opposite. Because the first term is $3x^2$, the first terms of the binomial factors must be $3x$ and x.

$$-2(3x^2 - 5xy - 12y^2) = -2(\overbrace{3x \qquad)(x}^{3x^2} \qquad)$$

The product of the last terms must be $-12y^2$, and the sum of the product of the outer terms and the product of the inner terms must be $-5xy$.

$$24y^2 + 10xy - 6x^2 = -2(3x \quad \overbrace{?y)(x \quad ?y}^{-12y^2})$$

$O + I = -5xy$

Since $1(-12)$, $2(-6)$, $3(-4)$, $12(-1)$, $6(-2)$, and $4(-3)$ all give a product of -12, there are 12 possible combinations to consider.

	$(3x + 1y)(x - 12y)$	$\mathbf{(3x - 12y)(x + 1y)}$
	$(3x + 2y)(x - 6y)$	$\mathbf{(3x - 6y)(x + 2y)}$
	$\mathbf{(3x + 3y)(x - 4y)}$	$(3x - 4y)(x + 3y)$
	$\mathbf{(3x + 12y)(x - 1y)}$	$(3x - 1y)(x + 12y)$
	$\mathbf{(3x + 6y)(x - 2y)}$	$(3x - 2y)(x + 6y)$
This is the one to choose. →	$\mathbf{(3x + 4y)(x - 3y)}$	$\mathbf{(3x - 3y)(x + 4y)}$

The combinations marked in color cannot work, because one of the binomial factors has a common factor. This implies that $3x^2 - 5xy - 12y^2$ would have a common factor, which it doesn't.

After mentally trying the remaining combinations, we find that only $(3x + 4y)(x - 3y)$ gives the proper middle term of $-5xy$.

$$24y^2 + 10xy - 6x^2 = -2(3x^2 - 5xy - 12y^2)$$
$$= -2(3x + 4y)(x - 3y)$$

Verify this result by multiplication.

Teaching Example 7 Factor: $-18x^2 + 21xy + 15y^2$
Answer: $-3(2x + y)(3x - 5y)$

EXAMPLE 8

Factor: $6y + 13x^2y + 6x^4y$

Strategy We will write the trinomial in descending powers of x and factor out the GCF of y. Then we will factor the resulting trinomial.

WHY The first step in factoring any polynomial is to factor out the GCF. Factoring out the GCF first makes factoring easier.

Self Check 8

Factor: $4b + 11a^2b + 6a^4b$

Now Try **Problem 54**

Self Check 8 Answer
$b(2a^2 + 1)(3a^2 + 4)$

Teaching Example 8 Factor: $10x^4y + 11x^2y - 6y$
Answer: $y(5x^2 - 2)(2x^2 + 3)$

Solution

$$6y + 13x^2y + 6x^4y = 6x^4y + 13x^2y + 6y$$
$$= y(6x^4 + 13x^2 + 6)$$

A test for factorability will show that $6x^4 + 13x^2 + 6$ will factor.

Since the coefficients of the first and last terms of $6x^4 + 13x^2 + 6$ are positive, the signs between the terms in each binomial will be +.

Since the first term of the trinomial is $6x^4$, the first terms of the binomial factors must be either $2x^2$ and $3x^2$ or x^2 and $6x^2$.

Since the product of the last terms of the binomial factors must be 6, we must find two numbers whose product is 6 that will lead to a middle term of $7x^2$. After trying some combinations, we find the one that works.

$$6y + 13x^2y + 6x^4y = y(6x^4 + 13x^2 + 6)$$
$$= y(2x^2 + 3)(3x^2 + 2)$$

Verify this result by multiplication.

Self Check 9

Factor: $a^2 + 4a + 4 - b^2$

Now Try **Problem 58**

Self Check 9 Answer $(a + 2 + b)(a + 2 - b)$

Teaching Example 9 Factor: $x^2 + 14x + 49 - y^2$
Answer: $(x + 7 + y)(x + 7 - y)$

EXAMPLE 9 Factor: $x^2 + 6x + 9 - z^2$

Strategy The terms of each expression do not have a common factor (other than 1) and traditional factoring by grouping will not work. Instead, we will group the first three terms of the polynomial.

WHY Hopefully, those steps will produce equivalent expressions that can be factored.

Solution

We group the first three terms together and factor the trinomial to get

$$x^2 + 6x + 9 - z^2 = (x + 3)(x + 3) - z^2$$
$$= (x + 3)^2 - z^2$$

We can now factor the difference of two squares to get

$$x^2 + 6x + 9 - z^2 = (x + 3 + z)(x + 3 - z)$$

5 Use substitution to factor trinomials.

For more complicated expressions, a substitution sometimes helps to simplify the factoring process.

Self Check 10

Factor: $(a + b)^2 - 3(a + b) - 10$

Now Try **Problem 62**

Self Check 10 Answer $(a + b + 2)(a + b - 5)$

Teaching Example 10 Factor: $(x + y)^2 - 2(x + y) - 8$
Answer: $(x + y - 4)(x + y + 2)$

EXAMPLE 10 Factor: $(x + y)^2 + 7(x + y) + 12$

Strategy We will use a substitution where we will replace each expression $x + y$ with the variable z and factor the resulting trinomial.

WHY The resulting trinomial will be easier to factor because it will be in only one variable, z.

Solution

We rewrite the trinomial $(x + y)^2 + 7(x + y) + 12$ as $z^2 + 7z + 12$, where $z = x + y$. The trinomial $z^2 + 7z + 12$ factors as $(z + 4)(z + 3)$.

To find the factorization of $(x + y)^2 + 7(x + y) + 12$, we substitute $x + y$ for z in the expression $(z + 4)(z + 3)$ to obtain

$$z^2 + 7z + 12 = (z + 4)(z + 3)$$
$$(x + y)^2 + 7(x + y) + 12 = (x + y + 4)(x + y + 3) \quad \text{Replace } z \text{ with } x + y.$$

6 Use grouping to factor trinomials.

The method of factoring by grouping can be used to help factor trinomials of the form $ax^2 + bx + c$. For example, to factor the trinomial $6x^2 + 7x - 3$, we proceed as follows:

1. First find the product ac: $6(-3) = -18$. This number is called the **key number.**
2. Find two factors of the key number -18 whose sum is $b = 7$:

$$9(-2) = -18 \qquad \text{and} \qquad 9 + (-2) = 7$$

3. Use the factors 9 and -2 as coefficients of two terms to be placed between $6x^2$ and -3:

$$6x^2 \mathbf{+ 7x} - 3 = 6x^2 \mathbf{+ 9x - 2x} - 3 \qquad \text{Express } 7x \text{ as } 9x - 2x.$$

4. Factor by grouping:

$$\begin{aligned} 6x^2 + 9x - 2x - 3 &= 3x\mathbf{(2x + 3)} - 1\mathbf{(2x + 3)} && \text{From } 6x^2 + 9x, \text{ factor out } 3x. \\ & && \text{From } -2x - 3, \text{ factor out } -1. \\ &= \mathbf{(2x + 3)}(3x - 1) && \text{Factor out } 2x + 3. \end{aligned}$$

We can verify this factorization by multiplication.

Factoring by grouping is especially useful when the lead coefficient, a, and the constant term, c, have many factors.

Factoring Trinomials by Grouping

To factor a trinomial by grouping:

1. Factor out any GCF (including -1 if that is necessary to make $a > 0$ in a trinomial of the form $ax^2 + bx + c$).
2. Identify a, b, and c, and find the key number ac.
3. Find two numbers whose product is the key number and whose sum is b.
4. Enter the two numbers as coefficients of x between the first and last terms and factor the polynomial by grouping.

The product of these numbers must be ac.

$$ax^2 + \boxed{}\,x + \boxed{}\,x + c$$

The sum of these numbers must be b.

5. Check by multiplying.

EXAMPLE 11 Factor: $10x^2 + 17x - 6$

Strategy We will express the middle term of the trinomial as the sum of two terms.

WHY We want to produce an equivalent four-termed polynomial that can be factored by grouping.

Solution

Since $a = 10$ and $c = -6$ in the trinomial, $ac = -60$. We now find two factors of -60 whose sum is 17. Two such factors are 20 and -3. We use these factors as coefficients of two terms to be placed between $10x^2$ and -6:

$$10x^2 \mathbf{+ 17x} - 6 = 10x^2 \mathbf{+ 20x - 3x} - 6 \qquad \text{Express } 17x \text{ as } 20x - 3x.$$

Self Check 11

Factor: $15a^2 + 17a - 4$

Now Try Problem 68

Self Check 11 Answer
$(3a + 4)(5a - 1)$

Teaching Example 11 Factor: $15x^2 - 11x - 12$
Answer:
$(5x + 3)(3x - 4)$

Finally, we factor by grouping.

$= 10x(x + 2) - 3(x + 2)$ From $10x^2 + 20x$, factor out $10x$. From $-3x - 6$, factor out -3.

$= (x + 2)(10x - 3)$ Factor out $x + 2$.

ANSWERS TO SELF CHECKS

1. $(b - 5)^2$ **2.** $(a - 4)(a - 3)$ **3.** $(a + 2)(a - 5)$ **4.** $-3a(b + 2)(b - 3)$ **5.** $(4q + 3)(q - 3)$ **6.** a prime polynomial **7.** $-3(x + 2y)(2x + y)$ **8.** $b(2a^2 + 1)(3a^2 + 4)$ **9.** $(a + 2 + b)(a + 2 - b)$ **10.** $(a + b + 2)(a + b - 5)$ **11.** $(3a + 4)(5a - 1)$

SECTION 5.7 STUDY SET

VOCABULARY

Fill in the blanks.

1. A polynomial with three terms, such as $3x^2 - 2x + 4$, is called a trinomial.

2. Since $y^2 + 2y + 1$ is the square of $y + 1$, we call $y^2 + 2y + 1$ a perfect-square trinomial.

3. For $a^2 - a - 6$, the leading coefficient (the coefficient of the a^2 term) is 1.

▶ **4.** The trinomial $4a^2 - 5a - 6$ is written in descending powers of a.

CONCEPTS

Consider $3x^2 - x + 16$. Find the sign of the

5. First term positive

6. Middle term negative

7. Last term positive

▶ **8.** Use the substitution $x = a + b$ to rewrite the trinomial $6(a + b)^2 - 17(a + b) - 3$. $6x^2 - 17x - 3$

NOTATION

Find each product.

9. $(x + y)(x + y) = x^2 + \boxed{2xy + y^2}$

10. $(x - y)(x - y) = x^2 - \boxed{2xy + y^2}$

11. $(x + y)(x - y) = \boxed{x^2 - y^2}$

12. $(a + b)(a + b) = \boxed{a^2 + 2ab} + b^2$

13. The trinomial $4m^2 - 4m + 1$ is written in $ax^2 + bx + c$ form. Identify a, b, and c. 4, −4, 1

▶ **14.** Consider the trinomial $15s^2 + 4s - 4$. Is $b^2 - 4ac$ a perfect square? yes

Complete each factorization.

15. $x^2 + 5x + 6 = (x + 3)\boxed{(x + 2)}$

▶ **16.** $x^2 - 6x + 8 = (x - 4)\boxed{(x - 2)}$

17. $x^2 + 2x - 15 = (x + 5)\boxed{(x - 3)}$

18. $x^2 - 3x - 18 = (x - 6)\boxed{(x + 3)}$

19. $2a^2 + 9a + 4 = \boxed{(2a + 1)}(a + 4)$

▶ **20.** $6p^2 - 5p - 4 = \boxed{(3p - 4)}(2p + 1)$

GUIDED PRACTICE

Use a special product formula to factor each perfect-square trinomial. See Example 1.

21. $x^2 + 2x + 1$ $(x + 1)^2$

22. $y^2 - 2y + 1$ $(y - 1)^2$

23. $a^2 - 18a + 81$ $(a - 9)^2$

24. $b^2 + 12b + 36$ $(b + 6)^2$

25. $4y^2 + 4yz + z^2$ $(2y + z)^2$

▶ **26.** $9x^2 + 6xy + y^2$ $(3x + y)^2$

27. $9a^2 - 12ab + 4b^2$ $(3a - 2b)^2$

28. $4a^2 - 12ab + 9b^2$ $(2a - 3b)^2$

Factor each trinomial. See Example 2.

29. $x^2 + x - 20$ $(x + 5)(x - 4)$

30. $x^2 + 10x + 21$ $(x + 3)(x + 7)$

31. $p^2 - 17p + 72$ $(p - 9)(p - 8)$

▶ **32.** $q^2 - 8q - 33$ $(q - 11)(q + 3)$

Factor each trinomial. See Example 3.

▶ **33.** $-a + a^2 - 12$ $(a + 3)(a - 4)$

34. $-16 + b^2 - 6b$ $(b + 2)(b - 8)$

35. $-42 + p + p^2$ $(p + 7)(p - 6)$

36. $27 + m^2 - 12m$ $(m - 9)(m - 3)$

Factor each trinomial. See Example 4.

37. $3x + 3x^3 - 10x^2$ $x(3x - 1)(x - 3)$

▶ **38.** $-3t^2 + 3t^3 + t$ $t(3t^2 - 3t + 1)$

39. $15a - 3ab^2 - 12ab$ $-3a(b + 5)(b - 1)$

40. $-30m^2 + 5m^3 + 40m$ $5m(m - 4)(m - 2)$

Test each trinomial for factorability and factor it, if possible. See Examples 5–6.

41. $x^2 - 5x + 6$ $(x - 3)(x - 2)$

42. $y^2 + 7y + 6$ $(y + 1)(y + 6)$

43. $6y^2 + 7y + 2$ $(3y + 2)(2y + 1)$

▶ **44.** $6x^2 - 11x + 3$ $(3x - 1)(2x - 3)$

45. $b^2 + 8b + 18$ prime

46. $x^2 + 4x - 28$ prime

47. $x^2 - x + 30$ prime

▶ **48.** $5x^2 + 4x + 1$ prime

Factor each trinomial. See Example 7.

49. $-16x^2 + 21x^4 - 10x^3$ $x^2(7x - 8)(3x + 2)$

50. $-50x^2 + 16x^3 + 36x$ $2x(x - 2)(8x - 9)$

51. $-18y^2 - 8x^2 - 30xy$ $-2(4x + 3y)(x + 3y)$

▶ **52.** $7axy - 6ax^2 - 2ay^2$ $-a(2x - y)(3x - 2y)$

Factor each trinomial. See Example 8.

53. $-2p^2 - 2pq + 4q^2$ $-2(p + 2q)(p - q)$

54. $-6m^2 + 3mn + 3n^2$ $-3(2m + n)(m - n)$

55. $b^2x^2 - 12bx^2 + 35x^2$ $x^2(b - 7)(b - 5)$

▶ **56.** $c^3x^2 + 11c^3x - 42c^3$ $c^3(x + 14)(x - 3)$

Factor each trinomial. See Example 9.

▶ **57.** $x^2 + 4x + 4 - y^2$ $(x + 2 + y)(x + 2 - y)$

58. $x^2 - 6x + 9 - 4y^2$ $(x - 3 + 2y)(x - 3 - 2y)$

59. $x^2 + 2x + 1 - 9z^2$ $(x + 1 + 3z)(x + 1 - 3z)$

60. $x^2 + 10x + 25 - 16z^2$ $(x + 5 + 4z)(x + 5 - 4z)$

Factor each trinomial. See Example 10.

61. $(x + a)^2 + 2(x + a) + 1$ $(x + a + 1)^2$

▶ **62.** $(a + b)^2 - 2(a + b) + 1$ $(a + b - 1)^2$

63. $3(a + b)^2 - 14(a + b) - 24$ $(3a + 3b + 4)(a + b - 6)$

▶ **64.** $2(x - y)^2 + (x - y) - 10$ $(2x - 2y + 5)(x - y - 2)$

Use grouping to factor each trinomial. See Example 11.

65. $a^2 + 4a - 45$ $(a + 9)(a - 5)$

▶ **66.** $c^2 - 7c + 12$ $(c - 4)(c - 3)$

67. $6z^2 + 17z + 12$ $(2z + 3)(3z + 4)$

68. $8x^2 - 10x + 3$ $(4x - 3)(2x - 1)$

TRY IT YOURSELF

Factor out all common monomials first (including −1 if the leading coefficient is negative). If a trinomial is prime, so indicate.

69. $3x^2 + 12x - 63$ $3(x + 7)(x - 3)$

70. $2y^2 + 4y - 48$ $2(y + 6)(y - 4)$

71. $x^2 - 7x + 10$ $(x - 2)(x - 5)$

72. $-a^2 + 4a + 32$ $-(a - 8)(a + 4)$

▶ **73.** $-x^2 - 2x + 15$ $-(x + 5)(x - 3)$

74. $-3x^2 + 15x - 18$ $-3(x - 3)(x - 2)$

75. $-2y^2 - 16y + 40$ $-2(y + 10)(y - 2)$

76. $a^2 + 5a - 50$ $(a + 10)(a - 5)$

77. $x^2 - 4xy - 21y^2$ $(x + 3y)(x - 7y)$

78. $a^2 + 4ab - 5b^2$ $(a + 5b)(a - b)$

79. $8a^2 + 6a - 9$ $(4a - 3)(2a + 3)$

80. $b^2 + 9b - 36$ $(b + 12)(b - 3)$

81. $15b^2 + 4b - 4$ $(5b - 2)(3b + 2)$

82. $6x^2 - 5xy - 4y^2$ $(3x - 4y)(2x + y)$

83. $18y^2 - 3yz - 10z^2$ $(6y - 5z)(3y + 2z)$

84. $4a^2 + 20a + 3$ prime

85. $a^2 - 3ab - 4b^2$ $(a + b)(a - 4b)$

86. $b^2 + 2bc - 80c^2$ $(b + 10c)(b - 8c)$

87. $-3a^2 + ab + 2b^2$ $-(3a + 2b)(a - b)$

▶ **88.** $-2x^2 + 3xy + 5y^2$ $-(2x - 5y)(x + y)$

89. $5a^2 + 45b^2 - 30ab$ $5(a - 3b)^2$

90. $-4x^2 - 9 + 12x$ $-(2x - 3)^2$

91. $x^4 + 8x^2 + 15$ $(x^2 + 5)(x^2 + 3)$

92. $x^4 + 11x^2 + 24$ $(x^2 + 8)(x^2 + 3)$

93. $y^4 - 13y^2 + 30$ $(y^2 - 10)(y^2 - 3)$

94. $y^4 - 13y^2 + 42$ $(y^2 - 7)(y^2 - 6)$

95. $a^4 - 29a^2 + 100$ $(a + 5)(a - 5)(a + 2)(a - 2)$

96. $b^4 - 17b^2 + 16$ $(b + 1)(b - 1)(b + 4)(b - 4)$

97. $2a^2 - 33a + 16$ $(a - 16)(2a - 1)$

98. $3b^2 + 2b - 21$ $(b + 3)(3b - 7)$

99. $2u^2 + 5u + 3$ $(2u + 3)(u + 1)$

100. $6y^2 + 5y - 6$ $(2y + 3)(3y - 2)$

101. $20r^2 - 7rs - 6s^2$ $(5r + 2s)(4r - 3s)$

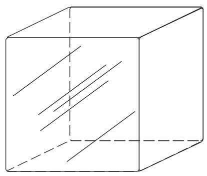
102. $6s^2 + st - 12t^2$ $(2s + 3t)(3s - 4t)$

103. $c^2 - 4a^2 + 4ab - b^2$ $(c + 2a - b)(c - 2a + b)$

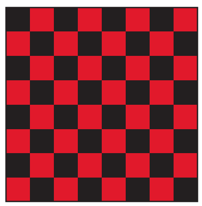
104. $4c^2 - a^2 - 6ab - 9b^2$ $(2c + a + 3b)(2c - a - 3b)$

APPLICATIONS

▶ **105.** ICE The surface area of the cubical block of ice shown on the right is $6x^2 + 36x + 54$. Find the length of an edge of the block. $x + 3$



▶ **106.** CHECKERS The area of the square checkerboard in the illustration is $25x^2 - 40x + 16$. Find the length of a side. $5x - 4$



WRITING

107. Explain how you would factor −1 from a trinomial.

108. Explain how you would test the polynomial $ax^2 + bx + c$ for factorability.

REVIEW

109. If $f(x) = |2x - 1|$, find $f(-2)$. 5

110. If $g(x) = 2x^2 - 1$, find $g(-2)$. 7

111. Solve: $-3 = -\frac{9}{8}s$ $\frac{8}{3}$

112. Solve: $2x + 3 = \frac{2}{3}x - 1$ -3

113. Simplify: $3p^2 - 6(5p^2 + p) + p^2$ $-26p^2 - 6p$

114. Solve: $\begin{cases} 2(2x + 3y) = 5 \\ 8x = 3(1 + 3y) \end{cases}$ $\left(\frac{3}{4}, \frac{1}{3}\right)$

Objectives

1 Factor random polynomials.

SECTION 5.8 Summary of Factoring Techniques

Factoring some polynomials involves several steps in which two or more factoring techniques must be used. In this section, we will discuss a general factoring strategy—a step-by-step plan to follow when factoring any polynomial.

1 Factor random polynomials.

In this section we will discuss ways to approach a randomly chosen factoring problem. For example, suppose we wish to factor the trinomial

$$x^2y^2z^3 + 7xy^2z^3 + 6y^2z^3$$

We begin by attempting to identify the problem type. The first possibility to look for is factoring out a common factor. Because the trinomial has a common factor y^2z^3, we factor it out:

$$x^2y^2z^3 + 7xy^2z^3 + 6y^2z^3 = y^2z^3(x^2 + 7x + 6)$$

We note that $x^2 + 7x + 6$ is a trinomial that can be factored as $(x + 6)(x + 1)$. Thus,

$$\begin{aligned} x^2y^2z^3 + 7xy^2z^3 + 6y^2z^3 &= y^2z^3(x^2 + 7x + 6) \\ &= y^2z^3(x + 6)(x + 1) \end{aligned}$$

To identify the type of factoring problem, we follow these steps.

Steps for Factoring a Polynomial

1. Is there a common factor? If so, factor out the GCF, or the opposite of the GCF, so that the leading coefficient is positive.
2. How many terms does the polynomial have?
   If it has *two terms,* look for the following problem types:
   - **a.** The difference of two squares
   - **b.** The sum of two cubes
   - **c.** The difference of two cubes

   If it has *three terms,* look for the following problem types:
   - **a.** A perfect-square trinomial
   - **b.** If the trinomial is not a perfect square, use the trial-and-check method or the grouping method

   If it has *four* or more terms, try to factor by grouping.
3. Can any factors be factored further? If so, factor them completely.
4. Does the factorization check? Check by multiplying.

For more complicated expressions, a substitution sometimes helps to simplify the factoring process.

EXAMPLE 1 Factor: $48a^4c^3 - 3b^4c^3$

Strategy We will answer the four questions listed in the *Steps for Factoring a Polynomial.*

WHY The answers will help us determine which factoring techniques to use.

Solution

We begin by factoring out the common factor $3c^3$:

$$48a^4c^3 - 3b^4c^3 = 3c^3(16a^4 - b^4)$$

Since the expression $16a^4 - b^4$ has two terms, we check to see whether it is the difference of two squares, which it is. As the difference of two squares, it factors as $(4a^2 + b^2)(4a^2 - b^2)$.

$$\begin{aligned} 48a^4c^3 - 3b^4c^3 &= 3c^3(16a^4 - b^4) \\ &= 3c^3(4a^2 + b^2)(4a^2 - b^2) \end{aligned}$$

The binomial $4a^2 + b^2$ is the sum of two squares and is prime. However, $4a^2 - b^2$ is the difference of two squares and factors as $(2a + b)(2a - b)$.

$$\begin{aligned} 48a^4c^3 - 3b^4c^3 &= 3c^3(16a^4 - b^4) \\ &= 3c^3(4a^2 + b^2)(4a^2 - b^2) \\ &= 3c^3(4a^2 + b^2)(2a + b)(2a - b) \end{aligned}$$

Since each of the individual factors is prime, the factorization is complete.

Self Check 1

Factor: $3p^4r^3 - 3q^4r^3$

Now Try **Problem 37**

Self Check 1 Answer

$3r^3(p^2 + q^2)(p + q)(p - q)$

Teaching Example 1 Factor: $32x^2y^4 - 2x^2z^4$

Answer:

$2x^2(4y^2 + z^2)(2y + z)(2y - z)$

EXAMPLE 2 Factor: $x^5y + x^2y^4 - x^3y^3 - y^6$

Strategy We will answer the four questions listed in the *Steps for Factoring a Polynomial.*

WHY The answers will help us determine which factoring techniques to use.

Solution

We begin by factoring out the common factor y:

$$x^5y + x^2y^4 - x^3y^3 - y^6 = y(x^5 + x^2y^3 - x^3y^2 - y^5)$$

Because the expression $x^5 + x^2y^3 - x^3y^2 - y^5$ has four terms, we try factoring by grouping to obtain

$$\begin{aligned} &x^5y + x^2y^4 - x^3y^3 - y^6 \\ &\quad = y(x^5 + x^2y^3 - x^3y^2 - y^5) && \text{Factor out } y. \\ &\quad = y[x^2(x^3 + y^3) - y^2(x^3 + y^3)] && \text{Factor by grouping.} \\ &\quad = y(x^3 + y^3)(x^2 - y^2) && \text{Factor out } x^3 + y^3. \end{aligned}$$

Finally, we factor $x^3 + y^3$ (the sum of two cubes) and $x^2 - y^2$ (the difference of two squares) to obtain

$$x^5y + x^2y^4 - x^3y^3 - y^6 = y(x + y)(x^2 - xy + y^2)(x + y)(x - y)$$

Because each of the individual factors is prime, the factorization is complete.

Self Check 2

Factor:

$a^5p - a^3b^2p + a^2b^3p - b^5p$

Now Try **Problem 46**

Self Check 2 Answer

$p(a + b)(a^2 - ab + b^2)(a + b)(a - b)$

Teaching Example 2 Factor: $ax^5 - ax^3z^2 + 8ax^2 - 8az^2$

Answer:

$a(x + 2)(x^2 - 2x + 4)(x + z)(x - z)$

EXAMPLE 3 Factor: $x^3 + 5x^2 + 6x + x^2y + 5xy + 6y$

Strategy We will answer the four questions listed in the *Steps for Factoring a Polynomial.*

WHY The answers will help us determine which factoring techniques to use.

Self Check 3

Factor:

$a^3 - 5a^2 + 6a + a^2b - 5ab + 6b$

Now Try **Problem 47**

Self Check 3 Answer

$(a - 2)(a - 3)(a + b)$

Teaching Example 3 Factor: $x^3 + x^2y + 3x^2 + 3xy - 10x - 10y$
Answer: $(x - 2)(x + 5)(x + y)$

Solution

There are no common factors. Since there are more than three terms, we try factoring by grouping. We can factor x from the first three terms and y from the last three terms.

$$\begin{aligned} &x^3 + 5x^2 + 6x + x^2y + 5xy + 6y \\ &= x(x^2 + 5x + 6) + y(x^2 + 5x + 6) \\ &= (x^2 + 5x + 6)(x + y) && \text{Factor out } x^2 + 5x + 6. \\ &= (x + 3)(x + 2)(x + y) && \text{Factor } x^2 + 5x + 6. \end{aligned}$$

Self Check 4

Factor: $a^4 - a^3 - 2a^2 + a - 2$

Now Try **Problem 48**

Self Check 4 Answer
$(a - 2)(a^3 + a^2 + 1)$

Teaching Example 4 Factor: $x^4 + 6x^3 + 9x^2 + x + 3$
Answer: $(x + 3)(x^3 + 3x^2 + 1)$

EXAMPLE 4 Factor: $x^4 + 2x^3 + x^2 + x + 1$

Strategy We will answer the four questions listed in the *Steps for Factoring a Polynomial.*

WHY The answers will help us determine which factoring techniques to use.

Solution

There are no common factors. Since there are more than three terms, we try factoring by grouping. We can factor x^2 from the first three terms.

$$\begin{aligned} x^4 + 2x^3 + x^2 + x + 1 &= x^2(x^2 + 2x + 1) + (x + 1) \\ &= x^2(x + 1)(x + 1) + (x + 1) && \text{Factor } x^2 + 2x + 1. \\ &= (x + 1)[x^2(x + 1) + 1] && \text{Factor out } x + 1. \\ &= (x + 1)(x^3 + x^2 + 1) \end{aligned}$$

ANSWERS TO SELF CHECKS

1. $3r^3(p^2 + q^2)(p + q)(p - q)$ **2.** $p(a + b)(a^2 - ab + b^2)(a + b)(a - b)$
3. $(a - 2)(a - 3)(a + b)$ **4.** $(a - 2)(a^3 + a^2 + 1)$

SECTION 5.8 STUDY SET

VOCABULARY

Fill in the blanks.

1. The process of finding the individual factors of a known product is called factoring.
2. $x^3 + y^3$ is called a sum of two cubes.
3. $x^3 - y^3$ is called a difference of two cubes.
4. $x^2 - y^2$ is called a difference of two squares.

CONCEPTS

Fill in the blanks.

5. In any factoring problem, always factor out any common factors first.
6. When factoring, if an expression has two terms, check to see whether the problem type is the difference of two squares, the sum of two cubes, or the difference of two cubes.
7. When factoring, if an expression has three terms, try to factor it as a trinomial.
8. When factoring, if an expression has four or more terms, try factoring it by grouping.
9. Explain how to verify that $y^2z^3(x + 6)(x + 1)$ is the factored form of $x^2y^2z^3 + 7xy^2z^3 + 6y^2z^3$.
Multiply the factors of $y^2z^3(x + 6)(x + 1)$ to see if the product is $x^2y^2z^3 + 7xy^2z^3 + 6y^2z^3$.
10. Why is the polynomial $x + 6$ classified as prime?
It cannot be factored.

NOTATION

Complete each factorization.

11. $18a^3b + 3a^2b^2 - 6ab^3 = 3ab(6a^2 + ab - 2b^2)$
$= 3ab(3a + 2b)(2a - b)$

Selected exercises available online at **www.webassign.net/brookscole**

12. $2x^4 - 1{,}250 = 2(\boxed{x^4 - 625})$

$= 2(\boxed{x^2 + 25})(x^2 - 25)$

$= 2(x^2 + 25)(x + 5)(\boxed{x - 5})$

TRY IT YOURSELF

In the following list of random problems, factor each polynomial, if possible. See Examples 1–4.

13. $x^2 + 16 + 8x$ $(x + 4)^2$

14. $20 + 11x - 3x^2$ $-(3x + 4)(x - 5)$

15. $8x^3y^3 - 27$ $(2xy - 3)(4x^2y^2 + 6xy + 9)$

16. $3x^2y + 6xy^2 - 12xy$ $3xy(x + 2y - 4)$

17. $xy - ty + xs - ts$ $(x - t)(y + s)$

18. $bc + b + cd + d$ $(b + d)(c + 1)$

19. $25x^2 - 16y^2$ $(5x + 4y)(5x - 4y)$

20. $27x^9 - y^3$ $(3x^3 - y)(9x^6 + 3x^3y + y^2)$

21. $12x^2 + 52x + 35$ $(6x + 5)(2x + 7)$

22. $12x^2 + 14x - 6$ $2(3x - 1)(2x + 3)$

23. $6x^2 - 14x + 8$ $2(3x - 4)(x - 1)$

24. $12x^2 - 12$ $12(x + 1)(x - 1)$

25. $4x^2y^2 + 4xy^2 + y^2$ $y^2(2x + 1)^2$

26. $100z^2 - 81t^2$ $(10z + 9t)(10z - 9t)$

27. $x^3 + (a^2y)^3$ $(x + a^2y)(x^2 - a^2xy + a^4y^2)$

28. $4x^2y^2z^2 - 26x^2y^2z^3$ $2x^2y^2z^2(2 - 13z)$

29. $2x^3 - 54$ $2(x - 3)(x^2 + 3x + 9)$

30. $4(xy)^3 + 256$ $4(xy + 4)(x^2y^2 - 4xy + 16)$

31. $ae + bf + af + be$ $(a + b)(f + e)$

32. $a^2x^2 + b^2y^2 + b^2x^2 + a^2y^2$ $(x^2 + y^2)(a^2 + b^2)$

33. $2(x + y)^2 + (x + y) - 3$ $(2x + 2y + 3)(x + y - 1)$

34. $(x - y)^3 + 125$ $(x - y + 5)[(x - y)^2 - 5(x - y) + 25]$

35. $625x^4 - 256y^4$ $(25x^2 + 16y^2)(5x + 4y)(5x - 4y)$

36. $2(a - b)^2 + 5(a - b) + 3$ $(2a - 2b + 3)(a - b + 1)$

37. $36x^4 - 36$ $36(x^2 + 1)(x + 1)(x - 1)$

38. $6x^2 - 63 - 13x$ $(2x - 9)(3x + 7)$

39. $a^4 - 13a^2 + 36$ $(a + 3)(a - 3)(a + 2)(a - 2)$

40. $x^4 - 17x^2 + 16$ $(x + 1)(x - 1)(x + 4)(x - 4)$

41. $x^2 + 6x + 9 - y^2$ $(x + 3 + y)(x + 3 - y)$

42. $x^2 + 10x + 25 - y^8$ $(x + 5 + y^4)(x + 5 - y^4)$

43. $4x^2 + 4x + 1 - 4y^2$ $(2x + 1 + 2y)(2x + 1 - 2y)$

44. $9x^2 - 6x + 1 - 25y^2$ $(3x - 1 - 5y)(3x - 1 + 5y)$

45. $x^2 - y^2 - 2y - 1$ $(x + y + 1)(x - y - 1)$

46. $a^2c - b^2c + 4bc - 4c$ $c(a + b - 2)(a - b + 2)$

47. $p^2x + 6px + 8x + p^2y + 6py + 8y$ $(p + 2)(p + 4)(x + y)$

48. $p^4 + 3p^3 + 2p^2 + p + 2$ $(p + 2)(p^3 + p^2 + 1)$

WRITING

49. What is your strategy for factoring a polynomial?

50. For the factorization below, explain why the polynomial is not factored completely.

$$48a^4c^3 - 3b^4c^3 = 3c^3(16a^4 - b^4)$$

REVIEW

51. Determine whether the graphs of $x + y = 2$ and $y = x + 5$ are parallel or perpendicular. perpendicular

52. When expressed as a decimal, is $\frac{7}{8}$ a terminating or a repeating decimal? terminating

53. Evaluate: $\begin{vmatrix} 1 & 15 \\ 15 & 0 \end{vmatrix}$ -225

54. If a triangle has exactly two sides with equal measures, what type of triangle is it? isosceles

SECTION 5.9
Solving Equations by Factoring

Objectives

1. Solve quadratic equations.
2. Solve higher-degree polynomial equations.
3. Use quadratic equations to solve problems.

Equations that involve *first-degree* polynomials, such as $3x + 6 = 0$, are called linear equations. Equations such as $3x^2 + 6x = 0$ that involve *second-degree* polynomials are called quadratic equations. The techniques that we have used to solve linear equations cannot be used to solve quadratic equations. However, we can solve many quadratic equations using factoring.

1 Solve quadratic equations.

An equation such as $3x^2 + 4x - 7 = 0$ or $-5y^2 + 3y + 8 = 0$ is called a **quadratic** or **second-degree** equation.

Quadratic Equations

A **quadratic equation** is any equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b, and c represent real numbers and $a \neq 0$.

Many quadratic equations can be solved by factoring and then by using the **zero-factor property.**

Zero-Factor Property

If a and b represent real numbers, then

If $ab = 0$, then $a = 0$ or $b = 0$.

The zero-factor property states that *if the product of two or more numbers is 0, then at least one of the numbers must be 0.*

To solve the quadratic equation $x^2 + 5x + 6 = 0$, we factor its left-hand side to obtain

$$(x + 3)(x + 2) = 0$$

Since the product of $x + 3$ and $x + 2$ is 0, at least one of the factors must be 0. Thus, we can set each factor equal to 0 and solve each resulting linear equation for x:

$$x + 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = -3 \quad \Big| \quad x = -2$$

To check these solutions, we substitute -3 and -2 for x in the equation and verify that each number satisfies the equation.

Check:

$$\begin{aligned} x^2 + 5x + 6 &= 0 \\ (-3)^2 + 5(-3) + 6 &\stackrel{?}{=} 0 \\ 9 - 15 + 6 &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x^2 + 5x + 6 &= 0 \\ (-2)^2 + 5(-2) + 6 &\stackrel{?}{=} 0 \\ 4 - 10 + 6 &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

Both -3 and -2 are solutions, because both satisfy the equation.

Self Check 1

Solve: $4p^2 - 12p = 0$ 0, 3

Now Try **Problem 18**

Teaching Example 1 Solve: $5x^2 - 25x = 0$

Answer:

0, 5

EXAMPLE 1

Solve: $3x^2 + 6x = 0$

Strategy We note that 0 is on the right side of the equation. We will factor the left side and use the zero-factor property.

WHY To use the zero-factor property, we need one side of the equation to be factored completely and the other side to be 0.

Solution

To solve the equation, we factor the left-hand side, set each factor equal to 0, and solve each resulting equation for x.

$3x^2 + 6x = 0$	The equation to solve.
$3x(x + 2) = 0$	Factor out the common factor of 3x.
$3x = 0$ or $x + 2 = 0$	By the zero-factor property, at least one of the factors must be equal to zero.
$x = 0$ | $x = -2$	Solve each linear equation.

Verify that both solutions, 0 and -2, check.

Caution! In Example 1, do not attempt to solve the equation by dividing both sides by $3x$, or you will lose the solution 0.

EXAMPLE 2 Solve: $x^2 - 16 = 0$

Strategy We note that 0 is on the right side of the equation. We will factor the left side and use the zero-factor property.

WHY To use the zero-factor property, we need one side of the equation to be factored completely and the other side to be 0.

Solution

To solve the equation, we factor the difference of two squares on the left-hand side, set each factor equal to 0, and solve each resulting equation.

$$x^2 - 16 = 0$$
$$(x + 4)(x - 4) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = -4 \quad \Big| \quad x = 4$$

Verify that both solutions, -4 and 4, check.

Self Check 2

Solve: $a^2 - 81 = 0$ 9, -9

Now Try **Problem 20**

Teaching Example 2 Solve: $x^2 - 64 = 0$
Answer:
8, -8

The following steps can be used to solve a quadratic equation by factoring.

Solving a Quadratic Equation by the Factoring Method

1. Write the equation in $ax^2 + bx + c = 0$ form (called *quadratic* form).
2. Factor the polynomial.
3. Use the zero-factor property to set each factor equal to zero.
4. Solve each resulting equation.
5. Check the proposed solutions in the original equation.

Many equations that do not appear to be quadratic can be put into quadratic form and then solved by factoring.

EXAMPLE 3 Solve: $x = \frac{6}{5} - \frac{6}{5}x^2$

Strategy We will multiply both sides of the equation by 5 to clear it of fractions and use factoring to solve the resulting quadratic equation.

WHY It is easier to factor a polynomial that contains integer coefficients than one that contains fractions.

Solution

We must write the equation in quadratic form. To clear the equation of fractions, we multiply both sides by 5.

$$x = \frac{6}{5} - \frac{6}{5}x^2$$
$$5x = 6 - 6x^2 \qquad \text{Multiply both sides by 5.}$$

Self Check 3

Solve: $x = \frac{6}{7}x^2 - \frac{3}{7}$ $\frac{3}{2}, -\frac{1}{3}$

Now Try **Problem 25**

Teaching Example 3 Solve: $x = \frac{15}{14} - \frac{4}{7}x^2$
Answer:
$\frac{3}{4}, \frac{-5}{2}$

To use factoring to solve this quadratic equation, one side of the equation must be 0. Since it is easier to factor a second-degree polynomial if the coefficient of the squared term is positive, we add $6x^2$ to both sides and subtract 6 from both sides to obtain

$$6x^2 + 5x - 6 = 0$$

$$(3x - 2)(2x + 3) = 0 \qquad \text{Factor the trinomial.}$$

$$3x - 2 = 0 \quad \text{or} \quad 2x + 3 = 0 \qquad \text{Set each factor equal to 0 and solve for } x.$$

$$3x = 2 \quad \Big| \quad 2x = -3$$

$$x = \frac{2}{3} \quad \Big| \quad x = -\frac{3}{2}$$

Verify that both solutions, $\frac{2}{3}$ and $-\frac{3}{2}$, check.

Caution! To solve a quadratic equation by factoring, be sure to set the quadratic polynomial equal to 0 before factoring and applying the zero-factor property. Do not make the following error:

$$6x^2 + 5x = 6$$

$$x(6x + 5) = 6 \qquad \text{If the product of two numbers is 6, neither number need be 6. For example, } 2 \cdot 3 = 6.$$

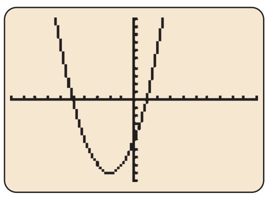
$$x = 6 \quad \text{or} \quad 6x + 5 = 6$$

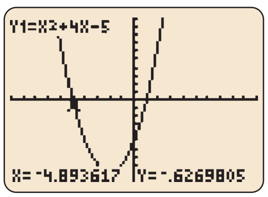
$$\Big| \quad x = \frac{1}{6}$$

Neither solution checks.

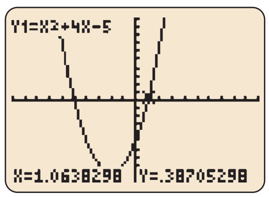
Using Your CALCULATOR Solving Quadratic Equations

To solve a quadratic equation such as $x^2 + 4x - 5 = \mathbf{0}$ with a graphing calculator, we can use standard window settings of $[-10, 10]$ for x and $[-10, 10]$ for y and graph the quadratic function $\mathbf{y} = x^2 + 4x - 5$, as shown in figure (a). We can then trace to find the x-coordinates of the x-intercepts of the parabola. See figures (b) and (c). For better results, we can zoom in. Since these are the numbers x that make $y = 0$, they are the solutions of the equation.


(a)

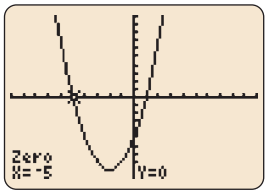


(b)

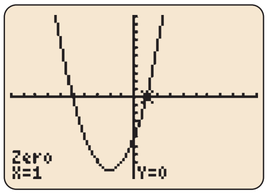


(c)

We can also find the x-intercepts of the graph of $y = x^2 + 4x - 5$ by using the ZERO feature found on most graphing calculators. Figures (a) and (b) on the next page show how this feature locates the x-intercept and displays its coordinates. (Consult your owner's manual for the specific instructions about how to use this feature.) From the displays, we can conclude that -5 and 1 are solutions of $x^2 + 4x - 5 = 0$. Verify this by checking them in the original equation.



(a)



(b)

2 Solve higher-degree polynomial equations.

We can solve many polynomial equations with degree greater than 2 by factoring and applying an extension of the zero-factor property.

EXAMPLE 4 Solve: $6x^3 - x^2 = 2x$

Strategy To solve this equation, we will get 0 on the right side, factor the polynomial on the left side, and use the zero-factor property.

WHY To use the zero-factor property, we need one side of the equation to be factored completely and the other side to be 0.

Solution

First, we subtract $2x$ from both sides so that the right-hand side of the equation is 0.

$$6x^3 - x^2 - 2x = 0$$

Then we factor x from the third-degree polynomial on the left-hand side and proceed as follows:

$$\begin{aligned} 6x^3 - x^2 - 2x &= 0 \\ x(6x^2 - x - 2) &= 0 \quad \text{Factor out } x. \\ x(3x - 2)(2x + 1) &= 0 \quad \text{Factor } 6x^2 - x - 2. \end{aligned}$$

$$x = 0 \quad \text{or} \quad 3x - 2 = 0 \quad \text{or} \quad 2x + 1 = 0 \qquad \text{Set each of the three factors equal to 0.}$$

$$x = \frac{2}{3} \qquad\qquad x = -\frac{1}{2} \qquad \text{Solve each equation.}$$

Verify that the three solutions, $0, \frac{2}{3}$, and $-\frac{1}{2}$, check.

Self Check 4

Solve: $5x^3 + 13x^2 = 6x$

Now Try **Problem 32**

Self Check 4 Answer

$0, \frac{2}{5}, -3$

Teaching Example 4 Solve: $x^3 = 2x^2 + 15x$

Answer:

$0, 5, -3$

EXAMPLE 5 Solve: $x^4 + 4 - 5x^2 = 0$

Strategy To solve this equation, we note that 0 is on the right side, write the polynomial in descending powers of x, factor the polynomial on the left side, and use the zero-factor property.

WHY To use the zero-factor property, we need one side of the equation to be factored completely and the other side to be 0.

Solution

First, we write the powers of x in descending order. Then we factor the trinomial on the left-hand side and proceed as follows:

$$\begin{aligned} x^4 - 5x^2 + 4 &= 0 \\ (x^2 - 1)(x^2 - 4) &= 0 \\ (x + 1)(x - 1)(x + 2)(x - 2) &= 0 \quad \text{Factor } x^2 - 1 \text{ and } x^2 - 4. \end{aligned}$$

$$x + 1 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = -1 \qquad\quad x = 1 \qquad\quad x = -2 \qquad\quad x = 2$$

Verify that each solution checks.

Self Check 5

Solve: $a^4 + 36 - 13a^2 = 0$

Now Try **Problem 34**

Self Check 5 Answer

$2, -2, 3, -3$

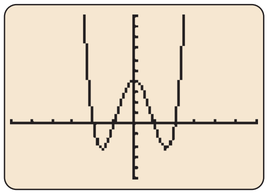
Teaching Example 5 Solve: $x^4 = 10x^2 - 9$

Answer:

$3, -3, 1, -1$

Using Your CALCULATOR **Solving Equations**

To solve the equation $x^4 - 5x^2 + 4 = 0$ with a graphing calculator, we can use window settings of $[-6, 6]$ for x and $[-5, 10]$ for y and graph the polynomial function $y = x^4 - 5x^2 + 4$ as shown in the figure. We can then read the values of x that make $y = 0$. They are $x = -2, -1, 1,$ and 2. If the x-coordinates of the x-intercepts were not obvious, we could approximate their values by using TRACE and ZOOM or by using the ZERO feature.



3 Use quadratic equations to solve problems.

Self Check 6

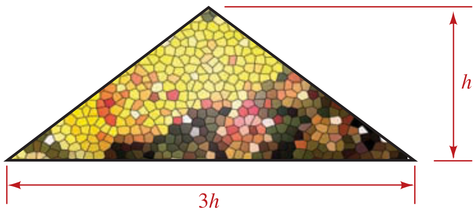
LANDSCAPING A rectangular garden is 3 feet longer than it is wide. If the total area of the garden is 40 square feet, find the dimensions of the garden.

Now Try Problem 63

Self Check 6 Answer
5 ft by 8 ft

Teaching Example 6 LANDSCAPING
A rectangular garden is 1 meter longer than twice its width. If the area is 36 square meters, find the dimensions of the garden.
Answer:
4 m by 9 m

EXAMPLE 6 ***Stained Glass*** The triangular stained glass window shown in the figure below is to be installed in a chapel. The length of the base of the window is 3 times its height. The area of the window is 96 square feet. Find its base and height.



Analyze We are to find the length of the base and the height of the window. The formula that gives the area of a triangle is $A = \frac{1}{2}bh$, where b is the length of the base and h the height.

Form We can let $h =$ the positive number that represents the height of the window. Then $3h =$ the length of the base. To form an equation in terms of h, we can substitute $3h$ for b and 96 for A in the formula for the area of a triangle.

$$A = \frac{1}{2}bh$$

$$96 = \frac{1}{2}(3h)h$$

Solve To solve this equation, we must write it in quadratic form.

$$96 = \frac{1}{2}(3h)h$$

$$192 = 3h^2 \quad (3h)h = 3h^2\text{. To clear the equation of the fraction, multiply both sides by 2.}$$

$$64 = h^2 \quad \text{Divide both sides by 3.}$$

$$0 = h^2 - 64 \quad \text{To obtain 0 on the left-hand side, subtract 64 from both sides.}$$

$$0 = (h + 8)(h - 8) \quad \text{Factor the difference of two squares.}$$

$$h + 8 = 0 \quad \text{or} \quad h - 8 = 0$$
$$h = -8 \quad | \quad h = 8$$

State Since the height of a triangle cannot be negative, we must discard the negative solution. Thus, the height of the window is 8 feet, and the length of its base is 3(8), or 24 feet.

Check The area of a triangle with a base of 24 feet and a height of 8 feet is 96 square feet:

$$A = \frac{1}{2}bh = \frac{1}{2}(24)(8) = 12(8) = 96$$

The result checks.

EXAMPLE 7 ***Ballistics*** If the initial velocity of an object thrown straight up into the air is 176 feet per second, when will the object strike the ground?

Analyze The height, in feet, of an object thrown straight up into the air with an initial velocity of v feet per second is given by the formula

$$h = -16t^2 + vt$$

The height h is in feet, and t represents the number of seconds since the object was released. When the object hits the ground, its height will be 0.

Form In the formula, we set h equal to 0 and set v equal to 176.

$$h = -16t^2 + vt$$
$$0 = -16t^2 + 176t$$

Solve To solve this equation, we will use the factoring method.

$$0 = -16t^2 + 176t$$
$$0 = -16t(t - 11) \quad \text{Factor out } -16t.$$
$$-16t = 0 \quad \text{or} \quad t - 11 = 0 \quad \text{Set each factor equal to 0.}$$
$$t = 0 \quad | \quad t = 11$$

State When t is 0, the object's height above the ground is 0 feet, because it has not been released. When t is 11, the height is again 0 feet, and the object has returned to the ground. The solution is 11 seconds.

Check Verify that h is 0 when t is 11.

Self Check 7

BALLISTICS If the initial velocity of an object thrown straight up into the air is 144 feet per second, when will the object strike the ground? 9 sec

Now Try **Problem 69**

Teaching Example 7 BALLISTICS If the initial velocity of an object thrown straight up into the air is 128 feet per second, when will the object strike the ground?
Answer:
8 sec

ANSWERS TO SELF CHECKS

1. 0, 3 **2.** 9, −9 **3.** $\frac{3}{2}, -\frac{1}{3}$ **4.** $0, \frac{2}{5}, -3$ **5.** 2, −2, 3, −3 **6.** 5 ft by 8 ft **7.** 9 sec

SECTION 5.9 STUDY SET

VOCABULARY

Fill in the blanks.

1. A quadratic equation is any equation that can be written in the form $ax^2 + bx + c = 0$ where $a \neq 0$.

2. To solve an equation means to find all the values of the variable that make the equation true.

Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

3. If the product of two numbers is 0, what must be true about at least one of the numbers? At least one is 0.

▶ **4.** Use a check to determine whether -5 and 4 are solutions of $a^2 - 9a + 20 = 0$. 4 is; -5 is not.

Determine whether each equation is a quadratic equation.

5. $w^2 + 7w + 12 = 0$ yes

6. $6t + 11 = 0$ no

7. $x(x + 3) = -2$ yes

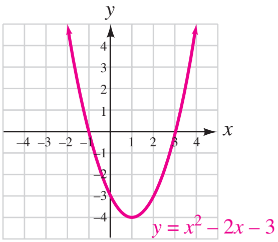
8. $k^3 - 4k^2 + k - 15 = 0$ no

9. Find the error.

$$x(x+2) = 8$$
$$x = 8 \quad \text{or} \quad x + 2 = 8$$
$$x = 8 \quad | \quad x = 6$$

If the product of two numbers is 8, neither number need be 8.

10. Use the graph to the right to solve the quadratic equation $x^2 - 2x - 3 = 0$. 3, -1



NOTATION

Complete each solution.

11. Solve: $y^2 - 3y - 54 = 0$

$$(y - 9)(\boxed{y + 6}) = 0$$
$$\boxed{y - 9} = 0 \quad \text{or} \quad y + 6 = 0$$
$$y = 9 \quad | \quad y = \boxed{-6}$$

▶ **12.** Solve: $x^2 - x = 12$

$$x^2 - x - 12 = 0$$
$$(x - 4)(\boxed{x + 3}) = 0$$
$$x - 4 = 0 \quad \text{or} \quad \boxed{x + 3} = 0$$
$$x = 4 \quad | \quad x = \boxed{-3}$$

GUIDED PRACTICE

Solve each equation. See Examples 1–2.

13. $4x^2 + 8x = 0$ 0, -2

14. $x^2 - 9 = 0$ 3, -3

▶ **15.** $y^2 - 16 = 0$ 4, -4

16. $y^2 - 25 = 0$ 5, -5

17. $x^2 + x = 0$ 0, -1

18. $x^2 - 3x = 0$ 0, 3

19. $5y^2 - 25y = 0$ 0, 5

▶ **20.** $y^2 - 36 = 0$ 6, -6

Solve each equation. See Example 3

21. $z^2 + 8z + 15 = 0$ $-3, -5$

22. $w^2 + 7w + 12 = 0$ $-3, -4$

23. $2y^2 - 5y + 2 = 0$ $\frac{1}{2}, 2$

▶ **24.** $2x^2 - 3x + 1 = 0$ $\frac{1}{2}, 1$

25. $\dfrac{3a^2}{2} = \dfrac{1}{2} - a$ $\frac{1}{3}, -1$

26. $x^2 = \dfrac{1}{2}(x + 1)$ $1, -\frac{1}{2}$

27. $x^2 + 1 = \dfrac{5}{2}x$ $2, \frac{1}{2}$

28. $\dfrac{3}{5}(x^2 - 4) = -\dfrac{9}{5}x$ $1, -4$

Solve each equation. See Example 4.

29. $x^3 + x^2 = 0$ $0, 0, -1$

30. $2x^4 + 8x^3 = 0$ $0, 0, 0, -4$

31. $x^3 - 4x^2 = 21x$ $0, 7, -3$

▶ **32.** $x^3 + 8x^2 - 9x = 0$ $1, 0, -9$

Solve each equation. See Example 5.

33. $z^4 - 13z^2 + 36 = 0$ $3, -3, 2, -2$

34. $y^4 + 9 - 10y^2 = 0$ $3, -3, 1, -1$

35. $x^4 - 20x^2 + 64 = 0$ $2, -2, 4, -4$

▶ **36.** $x^4 - 25x^2 + 144 = 0$ $3, -3, 4, -4$

TRY IT YOURSELF

Solve each equation.

37. $x^2 + 6x + 8 = 0$ $-2, -4$

38. $x^2 + 9x + 20 = 0$ $-4, -5$

39. $3m^2 + 10m + 3 = 0$ $-\frac{1}{3}, -3$

40. $2r^2 + 5r + 3 = 0$ $-\frac{3}{2}, -1$

41. $2x^2 - x - 1 = 0$ $1, -\frac{1}{2}$

42. $2x^2 - 3x - 5 = 0$ $\frac{5}{2}, -1$

43. $y^3 - 49y = 0$ $0, 7, -7$

44. $2z^3 - 200z = 0$ $0, 10, -10$

Write each equation in quadratic form and solve it by factoring.

45. $x(x - 6) + 9 = 0$ 3, 3

46. $x^2 + 8(x + 2) = 0$ $-4, -4$

47. $8a^2 = 3 - 10a$ $\frac{1}{4}, -\frac{3}{2}$

▶ **48.** $5z^2 = 6 - 13z$ $\frac{2}{5}, -3$

49. $b(6b - 7) = 10$ $2, -\frac{5}{6}$

▶ **50.** $2y(4y + 3) = 9$ $\frac{3}{4}, -\frac{3}{2}$

51. $x\left(3x + \dfrac{22}{5}\right) = 1$ $\frac{1}{5}, -\frac{5}{3}$

52. $x\left(\dfrac{x}{11} - \dfrac{1}{7}\right) = \dfrac{6}{77}$ $-\frac{3}{7}, 2$

53. $3a(a^2 + 5a) = -18a$ $0, -2, -3$

54. $7t^3 = 2t\left(t + \dfrac{5}{2}\right)$ $0, 1, -\frac{5}{7}$

▶ **55.** $\dfrac{x^2(6x + 37)}{35} = x$ $0, \frac{5}{6}, -7$

56. $x^2 = -\dfrac{4x^3(3x + 5)}{3}$ $0, 0, -\frac{1}{6}, -\frac{3}{2}$

57. INTEGER PROBLEM The product of two consecutive even integers is 288. Find the integers. (*Hint:* Let x = the first even integer. Then represent the second even integer in terms of x.) 16, 18 or $-18, -16$

▶ **58.** INTEGER PROBLEM The product of two consecutive odd integers is 143. Find the integers. (*Hint:* Let x = the first odd integer. Then represent the second odd integer in terms of x.) 11, 13, or $-13, -11$

Use a graphing calculator to find the solutions of each equation, if one exists. If an answer is not exact, give the answer to the nearest hundredth.

59. $2x^2 - 7x + 4 = 0$
2.78, 0.72

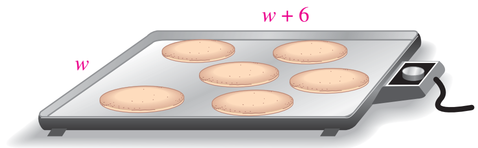
60. $x^2 - 4x + 7 = 0$
no solution

61. $-3x^3 - 2x^2 + 5 = 0$
1

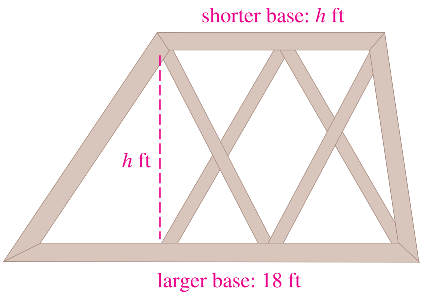
▶ **62.** $-2x^3 - 3x - 5 = 0$
-1

APPLICATIONS

63. COOKING The electric griddle shown below has a cooking surface of 160 square inches. Find the length and the width of the griddle. 10 in., 16 in.



▶ **64.** STRUCTURAL ENGINEERING The formula for the area of a trapezoid is $A = \frac{h(B + b)}{2}$. The area of the trapezoidal truss in the illustration is 44 square feet. Find the height of the truss shown below if the shorter base is the same as the height. 4 ft



65. WATER FOUNTAINS Building codes require that the 6-foot by 8-foot fountain pool be surrounded by a uniform-width concrete walkway with a surface area of 120 square feet. How wide should the walkway be? 3 ft

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Landscape Architect


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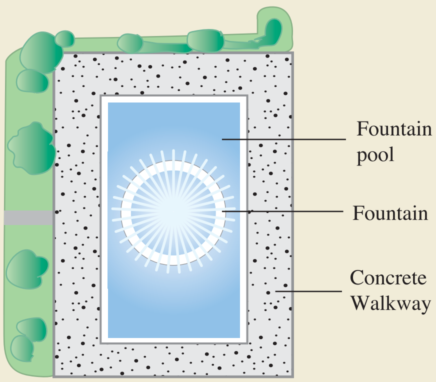
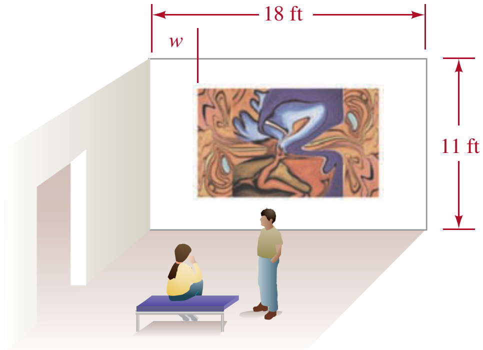
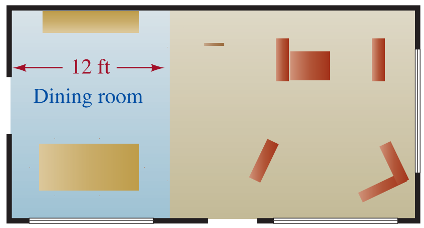


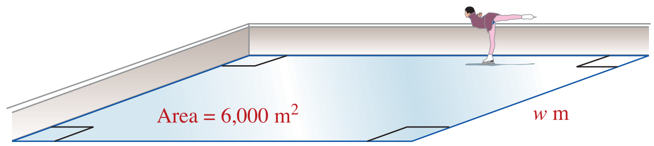
▶ **66.** FINE ARTS An artist intends to paint a 60-square-foot mural on the large wall shown below. Find the dimensions of the mural if the artist leaves a border of uniform width around it. 5 ft by 12 ft



67. ARCHITECTURE The rectangular room shown below is twice as long as it is wide. It is divided into two rectangular parts by a partition, positioned as shown. If the larger part of the room contains 560 square feet, find the dimensions of the entire room. 20 ft by 40 ft



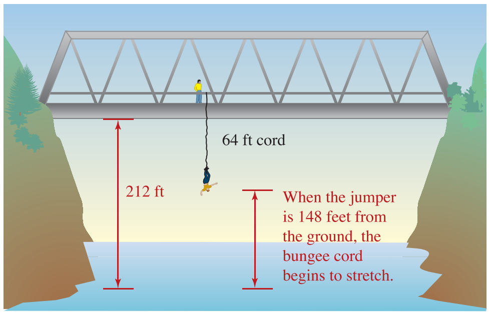
▶ **68.** WINTER RECREATION The length of the rectangular ice-skating rink shown below is 20 meters greater than twice its width. Find the width. 50 m



69. BALLISTICS The muzzle velocity of a cannon is 480 feet per second. If a cannonball is fired vertically, at what times will it be at a height of 3,344 feet? 11 sec and 19 sec

70. SLINGSHOTS A slingshot can provide an initial velocity of 128 feet per second. At what times will a stone, shot vertically upward, be 192 feet above the ground? 2 sec and 6 sec

71. BUNGEE JUMPING The formula $h = -16t^2 + 212$ gives the distance a bungee jumper is from the ground for the free-fall portion of a jump, t seconds after leaping off a bridge, as shown below. We can find the number of seconds it takes the jumper to reach the point in the fall where the 64-foot bungee cord starts to stretch by substituting 148 for h and solving for t. Find t. 2 sec



72. BASEBALL In 2001, pitcher Pedro Martinez of the Boston Red Sox threw a fastball that was clocked at 97 mph. This is a velocity of approximately 144 feet per second. If he could throw the baseball vertically into the air with this velocity, how long would it take for the ball to fall to the ground? 9 sec

73. FORENSIC MEDICINE The kinetic energy E of a moving object is given by $E = \frac{1}{2}mv^2$, where m is the mass of the object (in kilograms) and v is the object's velocity (in meters per second). Kinetic energy is measured in joules. By measuring the damage done to a victim who has been struck by a 3-kilogram club, a police pathologist finds that the energy at impact was 54 joules. Find the velocity of the club at impact. 6 m/sec

74. TRAFFIC ACCIDENTS Investigators at a traffic accident used the function $d(v) = 0.04v^2 + 0.8v$, where v is the velocity of the car (in mph) and $d(v)$ is the stopping distance of the car (in feet), to reconstruct the events leading up to a collision. From physical evidence, it was concluded that it took one car 32 feet to stop. At what velocity was the car traveling prior to the accident? 20 mph

75. BREAK-EVEN POINT The cost for a guitar maker to hand-craft x guitars is given by the function $C(x) = \frac{1}{8}x^2 - x + 6$. The revenue taken in with the sale of x guitars is given by the function $R(x) = \frac{1}{4}x^2$. Find the number of guitars that must be sold so that the cost equals the revenue. 4

76. REVENUE Over the years, the manager of a crafts store has found that the number of scented candles x she can sell in a month depends on the price p according to the formula $x = 200 - 10p$. At what price should she sell the candles if she needs to bring in \$750 in revenue a month from their sale? (*Hint:* Revenue = price · number sold = px.) \$5 or \$15

WRITING

77. Explain the zero-factor property.

78. In the work shown below, explain why the student has not solved for x.

$$\text{Solve: } \begin{aligned} x^2 + x - 6 &= 0 \\ x^2 + x &= 6 \\ x &= 6 - x^2 \end{aligned}$$

79. Find the error.

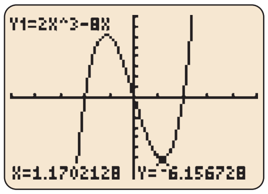
$$\begin{aligned} x^2 - x &= 0 \\ \frac{x^2}{x} - \frac{x}{x} &= \frac{0}{x} \\ x - 1 &= 0 \\ x &= 1 \end{aligned}$$

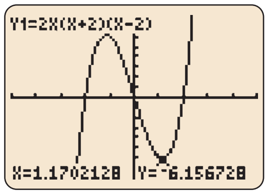
80. Explain what is wrong with the following solution.

$$\text{Solve: } \begin{aligned} x^2 - x &= 6 \\ x(x - 1) &= 6 \end{aligned}$$

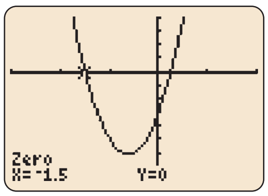
$$x = 6 \quad \text{or} \quad x - 1 = 6$$
$$x = 7$$

81. The graphs of the two polynomial functions, $f(x) = 2x^3 - 8x$ and $f(x) = 2x(x + 2)(x - 2)$, in the illustrations below, appear to be the same. After examining the defining equations, explain why we know that they indeed are identical graphs.





82. Explain why the x-coordinate of the x-intercept of the graph of $y = 8x^2 + 10x - 3$ (indicated as a *zero* in the illustration below) is a solution of $8x^2 + 10x - 3 = 0$.



REVIEW

83. ALUMINUM FOIL Find the number of square feet of aluminum foil on a roll if it has dimensions of $8\frac{1}{3}$ yards × 12 inches. 25 ft^2

84. HOCKEY A hockey puck is a vulcanized rubber disk 2.5 cm (1 in.) thick and 7.6 cm (3 in.) in diameter. Find the volume of a puck in cubic centimeters and cubic inches. Round to the nearest tenth. 113.4 cm^3, 7.1 in.^{3}

STUDY SKILL CHECKLIST

Preparing for the Chapter 5 Test

Rules of exponents, scientific notation, polynomials, and different factoring methods were discussed in this chapter. In Section 8, an overall factoring strategy was presented on page 448. This strategy is helpful when factoring randomly chosen polynomials. As you study the material for the test on this chapter, review the following checklist, including factoring and solving quadratic equations:

☐ Review the rules of exponents summarized on page 384. Remember to always write your final answers with positive exponents. When simplifying an exponential expression with negative exponents, write the reciprocal and change the sign of the exponent.

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25} \qquad 7x^{-3} = \frac{7}{1} \cdot \frac{1}{x^3} = \frac{7}{x^3}$$

☐ The square of a binomial is a *trinomial.* A common error when squaring a binomial is to forget the middle term of the product.

$$\begin{aligned}(7x-3)^2 &= (7x-3)(7x-3)\\ &= 49x^2 - 21x - 21x + 9\\ &= 49x^2 - 42x + 9\end{aligned}$$

☐ Always factor out the Greatest Common Factor first. If this step is not done, it is easy to miss factorizations that need to be done to factor the expression completely.

Factor: $2x^5 - 32x$

$$\begin{aligned}2x^5 - 32x &= 2x(x^4 - 16) && \text{Factor out the GCF } 2x.\\ &= 2x[(x^2)^2 - 4^2] && x^4 - 16 \text{ is the difference of two squares.}\\ &= 2x(x^2+4)(x^2-4) && x^2 - 4 \text{ doesn't factor further; } x^2 - 4 \text{ is the difference of two squares.}\\ &= 2x(x^2+4)(x+2)(x-2)\end{aligned}$$

☐ Although $x^2 - 9x$ and $x^2 - 9$ look very similar, they have entirely different factorizations. Be sure to look closely at each term and remember the *Steps for Factoring a Polynomial* found on page 448 of this text:

$x^2 - 9x = x(x-9)$ Factor out the GCF x.

$x^2 - 9 = (x+3)(x-3)$ There are no common factors. It is the difference of two squares and factors as the product of two binomials.

☐ To factor the sum or difference of two cubes, write the two terms as the base to the third power that is equivalent to the original polynomial. Then follow the rule for factoring the sum or difference of two cubes:

Factor: $x^3 + 8y^3$

$x^3 + 8y^3 = x^3 + (2y)^3$ This polynomial is the sum of two cubes. Write each term as a base to the third power that is equivalent to the original polynomial.

$$\begin{aligned}&= (x+2y)[x^2 - x(2y) + (2y)^2]\\ &= (x+2y)(x^2 - 2xy + 4y^2)\end{aligned}$$

☐ Although there is a factorization for the sum of two *cubes,* in general the sum of two squares is a prime polynomial.

$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^2 + y^2$ This is a prime polynomial

☐ To solve a quadratic equation by factoring, write the equation in $ax^2 + bx + c = 0$ form, factor the polynomial completely, use the zero-factor property to set each factor equal to zero, and solve the resulting linear equations.

Solve: $x(x+5) = 14$

$x(x+5) = 14$ This is the equation to solve.

$x^2 + 5x = 14$ Distribute the multiplication by x.

$x^2 + 5x - 14 = 0$ Subtract 14 from both sides to get 0 on the right side of the equation.

$(x+7)(x-2) = 0$ Factor $x^2 + 5x - 14$.

$x + 7 = 0$ or $x - 2 = 0$ Set each factor equal to 0.

$x = -7$ | $x = 2$ Solve each linear equation.

CHAPTER 5 SUMMARY AND REVIEW

SECTION 5.1 Exponents

DEFINITIONS AND CONCEPTS	EXAMPLES
An **exponent** indicates repeated multiplication. It tells how many times the **base** is to be used as a factor. If n is a natural number, $x^n = \overbrace{x \cdot x \cdot x \cdot x \cdot \cdots \cdot x}^{n \text{ factors of } x}$ where x is the **base** and n the **exponent.**	Identify the base and the exponent in each expression. $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ 2 is the base, 6 is the exponent $p^5 = p \cdot p \cdot p \cdot p \cdot p$ p is the base, 5 is the exponent $(-ab)^3 = (-ab)(-ab)(-ab)$ $-ab$ is the base, 3 is the exponent $7c^4 = 7 \cdot c \cdot c \cdot c \cdot c$ c is the base, 4 is the exponent
Rules for Exponents: If m and n represent integers and there are no divisions by 0, then ***Product rule:*** $x^m x^n = x^{m+n}$ ***Quotient rule:*** $\frac{x^m}{x^n} = x^{m-n}$ ***Power rule:*** $(x^m)^n = x^{mn}$ ***Power of a product:*** $(xy)^m = x^m y^m$ ***Power of a quotient:*** $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$ ***Zero exponent:*** $x^0 = 1$ ***Negative exponent:*** $x^{-n} = \frac{1}{x^n}$ ***Exponent of 1:*** $x^1 = x$ ***Negative exponents appearing in fractions:*** $\frac{1}{x^{-n}} = x^n$ $\frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$ $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$	Simplify each expression: $x^2 \cdot x^3 = x^{2+3} = x^5$ $\frac{m^9}{m^3} = m^{9-3} = m^6$ $(r^4)^5 = r^{4 \cdot 5} = r^{20}$ $(2y)^5 = 2^5 y^5 = 32y^5$ $\left(\frac{a}{3}\right)^4 = \frac{a^4}{3^4} = \frac{a^4}{81}$ $5^0 = 1$ $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ $7^1 = 7$ $\frac{1}{10^{-2}} = 10^2 = 100$ $\frac{4^{-2}}{m^{-3}} = \frac{m^3}{4^2} = \frac{m^3}{16}$ $\left(\frac{7}{a}\right)^{-2} = \left(\frac{a}{7}\right)^2 = \frac{a^2}{7^2} = \frac{a^2}{49}$

REVIEW EXERCISES

Evaluate each expression.

1. 3^5 243

2. -2^5 -32

3. $(-4)^3$ -64

4. $\left(\frac{2}{3}\right)^2$ $\frac{4}{9}$

Simplify each expression. Write answers using positive exponents.

5. $x^4 \cdot x^2$ x^6

6. $m^{-3}n^{-4}m^6n^{-1}$ $\frac{m^3}{n^5}$

7. $\frac{(4m^6)^3}{m^2}$ $64m^{16}$

8. $(-t^2)^2(t^3)^3$ t^{13}

9. $(3x^2y^3)^2$ $9x^4y^6$

10. $\left(\frac{x^4}{b}\right)^4$ $\frac{x^{16}}{b^4}$

11. $-10x^0$ -10

12. $\frac{1}{5h^{-12}}$ $\frac{h^{12}}{5}$

13. $\left(\frac{2a}{b}\right)^{-1}$ $\frac{b}{2a}$

14. $\frac{x}{5x^{-4}}$ $\frac{x^5}{5}$

15. $(3x^{-3})^{-2}$ $\frac{x^6}{9}$

16. $\frac{2x^{-4}x^3}{9}$ $\frac{2}{9x}$

17. $-\left(\frac{c^{-3}}{c^{-5}}\right)^5$ $-c^{10}$

18. $\left(\frac{4}{5}\right)^{-2}$ $\frac{25}{16}$

19. $\frac{3^{-3}}{4^{-2}}$ $\frac{16}{27}$

20. $\frac{1}{4^{-3}}$ 64

21. $\left(\frac{s^7}{s^{-8}}\right)^3\left(\frac{s^{-5}}{s^2}\right)^{-4}$ s^{73}

22. $\left(\frac{-2a^4bc^{-6}}{a^{-3}b^2c^{-5}}\right)^{-3}$ $-\frac{b^3c^3}{8a^{21}}$

SECTION 5.2 Scientific Notation

DEFINITIONS AND CONCEPTS	EXAMPLES
Scientific notation is a compact way of writing large and small numbers. A positive number is written in **scientific notation** when it is written in the form $N \times 10^n$, where $1 \le N < 10$ and n is an integer.	Write each number in scientific notation. $98{,}100 = 9.81 \times 10^4$ (4 decimal places) and $0.0043 = 4.3 \times 10^{-3}$ (3 decimal places) Write each number in standard notation. $6.42 \times 10^5 = 642{,}000$ (5 decimal places) and $7.7 \times 10^{-6} = 0.0000077$ (6 decimal places)
To **multiply or divide with very small or very large numbers,** write the numbers in scientific notation and perform the arithmetic on the decimals and the powers of 10 separately	Use scientific notation to perform the calculation: $\frac{(5{,}400{,}000{,}000)(0.0066)}{(0.000000022)(64{,}800)} = \frac{(5.4 \times 10^9)(6.6 \times 10^{-3})}{(2.2 \times 10^{-8})(6.48 \times 10^4)}$ $= \frac{(5.4)(6.6)}{(2.2)(6.48)} \times \frac{10^9 10^{-3}}{10^{-8}10^4}$ $= 2.5 \times 10^{9+(-3)-(-8)-4}$ $= 2.5 \times 10^{10}$

REVIEW EXERCISES

Write each number in scientific notation.

23. 19,300,000,000 1.93×10^{10}

24. 0.00000002735 2.735×10^{-8}

Write each number in standard notation.

25. 7.277×10^7 72,770,000

26. 8.3×10^{-9} 0.0000000083

Write each number in scientific notation and perform the operations. Give the answers in scientific notation.

27. SPEED OF LIGHT Light travels at about 300,000 kilometers per second. If the average distance from the sun to the planet Mars is approximately 228,000,000 kilometers, how long does it take light from the sun to reach Mars? 7.6×10^2 sec

28. PROTONS If the mass of 1 proton is 0.00000000000000000000000167248 gram, find the mass of 1 million protons. 1.67248×10^{-18} g

Write each number in scientific notation and perform the operations. Give the answers in scientific notation.

29. $\frac{(616{,}000{,}000)(0.000009)}{(0.00066)}$ 8.4×10^6

30. $\frac{0.0000000495}{(33{,}000)(800{,}000{,}000)}$ 1.875×10^{-21}

SECTION 5.3 Polynomials and Polynomial Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
A **polynomial** is a single term or the sum of terms in which all variables have whole-number exponents. No variable appears in a denominator.	Examples of polynomials: $6, \quad -2mn^3, \quad r^2 - 14s^4, \quad 3x^2 + x - 8$ Not polynomials: $x^3 - x^{-2}, \quad 6ab^4 - \frac{1}{b} + 5a$
A **monomial** is a polynomial with one term. A **binomial** is a polynomial with two terms. A **trinomial** is a polynomial with three terms.	Monomials: $17m \quad -5x^2y$ Binomials: $3y + 2 \quad 3p^4 - 2q^3$ Trinomials: $3t^2 + 5t - 6 \quad 8r^4s^3 - 6r^3s^2 + 4r^2s$
The **degree of a term** of a polynomial in one variable is the value of the exponent on the variable. The **degree of a term** that has more than one variable is the sum of the exponents on the variables. The **degree of a nonzero constant** is 0.	***Term*** / ***Degree of the term***: $9d^3$ — 3; $-1.6r^7s^4$ — $7 + 4 = 11$; 19 — 0
The **degree of a polynomial** is equal to the highest degree of any term of the polynomial.	***Polynomial*** / ***Degree of the polynomial***: $8a^4 - 3a^2 + 5a - 9$ — 4; $4x^4y + 11x^3y^2 - x^2y^6$ — $2 + 6 = 8$
To **evaluate a polynomial function,** we replace the variable in the defining equation with its value, called the **input.** Then we simplify to find the **output.** **Polynomial functions** can be graphed on the rectangular coordinate system.	Evaluate $f(x) = x^3 - 3x^2 - 9x + 2$ for $x = 2$. $f(x) = x^3 - 3x^2 - 9x + 2$ $f(2) = (2)^3 - 3(2)^2 - 9(2) + 2$ Substitute 2 for x. $= 8 - 3(4) - 18 + 2$ $= -20$ The polynomial function $f(x) = x^3 - 3x^2 - 9x + 2$ is graphed in Example 6 on page 401.
To **simplify a polynomial,** combine like terms.	Simplify: $8a^5 - 2a^4 + 6a^5 + a^2 = 14a^5 - 2a^4 + a^2$
To **add polynomials,** drop the parentheses and combine like terms (terms having the same variables with the same exponents).	Add: $(3x^2 - 2x + 7) + (2x^2 - 17) = 3x^2 - 2x + 7 + 2x^2 - 17$ $= 5x^2 - 2x - 10$
To **subtract polynomials,** change the signs of the terms of the polynomial being subtracted, drop the parentheses, and combine like terms.	Subtract: $(3x^2 - 2x + 7) - (2x^2 - 17) = 3x^2 - 2x + 7 - 2x^2 + 17$ $= x^2 - 2x + 24$

REVIEW EXERCISES

Determine whether each expression is a polynomial.

31. $\dfrac{2x^2}{x + 1}$ no

32. $-5x^3 + x^2 - 5x - 4$ yes

33. $2.8y^{15} - y^{10} + y^8 - \dfrac{3}{2}y^6$ yes

34. $x^{-3} + x^{-2} - x^{-1} - 1$ no

Classify each polynomial as a monomial, binomial, trinomial, or none of these. Then determine the degree of the polynomial.

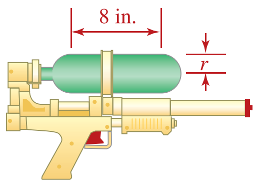
35. $x^2 - 8$ binomial, 2

36. $-15a^3b$ monomial, 4

37. $x^4 + x^3 - x^2 + x - 4$ none of these, 4

38. $9x^2y + 13x^3y^2 + 8x^4y^4$ trinomial, 8

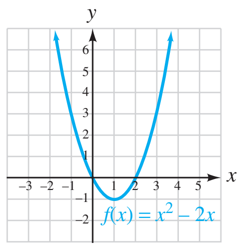
39. SQUIRT GUNS The volume of the reservoir on top of the squirt gun is given by the polynomial function $V(r) = 4.19r^3 + 25.13r^2$, where r is the radius in inches. Find $V(2)$ to the nearest cubic inch. 134 in.^{3}



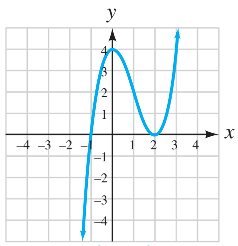
40. Evaluate $f(x) = x^3 - 5x^2 - 2x - 5$ for $x = 3$. -29

Graph each polynomial function.

41. $f(x) = x^2 - 2x$



42. $f(x) = x^3 - 3x^2 + 4$



$f(x) = x^3 - 3x^2 + 4$

Simplify each polynomial.

43. $9t^3 - 5t^2 - 5t + 3t^2 - 7t^3$ $2t^3 - 2t^2 - 5t$

44. $\dfrac{5}{4}ab^2c - \dfrac{5}{3}abc - \dfrac{1}{2}ab^2c + \dfrac{5}{6}abc$ $\frac{3}{4}ab^2c - \frac{5}{6}abc$

Perform each operation.

45. $(2x^2y^3 - 5x^2y + 9y) + (x^2y^3 - 3x^2y - y)$
$3x^2y^3 - 8x^2y + 8y$

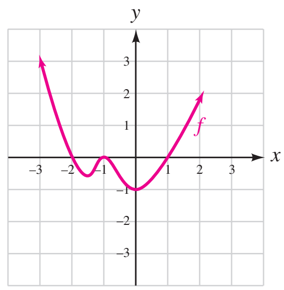
46. $(8m^2 - 7m) - (-11m^2 + 6m + 9)$
$19m^2 - 13m - 9$

47. $\left(\dfrac{2}{3}s^6 - \dfrac{1}{6}s^4\right) + \left(-\dfrac{1}{6}s^6 - \dfrac{3}{2}s^4\right)$ $\frac{1}{2}s^6 - \frac{5}{3}s^4$

48.
$$\begin{array}{r} -10k^4 - 4k^3 + 5k^2 - k + 1 \\ -\ \underline{(-16k^4 + 2k^3 - 4k^2 - k + 3)} \\ 6k^4 - 6k^3 + 9k^2 - 2 \end{array}$$

49. Subtract $6c^2d^2 + 4c^2d - 5cd^2$ from the sum of $-c^2d^2 + 5c^2d - 10cd^2$ and $11c^2d^2 - c^2d + 9cd^2$.
$4c^2d^2 + 4cd^2$

50. Use the graph of function f to find each of the following.



a. $f(0)$
-1

b. The values of x for which $f(x) = 0$
$-2, -1, 1$

c. Write the domain and range of f in interval notation.
D: $(-\infty, \infty)$; R: $[-1, \infty)$

SECTION 5.4 Multiplying Polynomials

DEFINITIONS AND CONCEPTS	EXAMPLES
To **multiply two monomials,** multiply their numerical factors and multiply their variable factors.	Multiply: $(3x^2y)(4x^3y^2) = (3 \cdot 4)(x^2 \cdot x^3)(y \cdot y^2)$ *Group the coefficients together and the variables with like bases together.* $= 12x^5y^3$

To **multiply a polynomial by a monomial,** multiply each term of the polynomial by the monomial.

Multiply:

$$\begin{aligned} \mathbf{5m^2n^3}(2m - 3n^2) &= \mathbf{5m^2n^3}(2m) - \mathbf{5m^2n^3}(3n^2) && \text{Distribute.} \\ &= 10m^3n^3 - 15m^2n^5 && \text{Multiply the monomials.} \end{aligned}$$

To **multiply two binomials,** multiply each term of one binomial by each term of the other binomial.

The **FOIL method** can be used to multiply two binomials.

F: First **O:** Outer **I:** Inner **L:** Last

Multiply:

$$\begin{aligned} (7x + 4)(3x + 5) &= \overset{\text{F}}{7x \cdot 3x} + \overset{\text{O}}{7x \cdot 5} + \overset{\text{I}}{4 \cdot 3x} + \overset{\text{L}}{4 \cdot 5} \\ &= 21x^2 + 35x + 12x + 20 && \text{Multiply the monomials.} \\ &= 21x^2 + 47x + 20 && \text{Combine like terms.} \end{aligned}$$

To **multiply two polynomials,** multiply each term of one polynomial by each term of the other polynomial.

Multiply:

$$\begin{aligned} &(3x + 2)(x^2 - 3x + 6) \\ &= 3x(x^2) + (3x)(-3x) + 3x(6) + 2(x^2) + 2(-3x) + 2(6) \\ &= 3x^3 - 9x^2 + 18x + 2x^2 - 6x + 12 && \text{Multiply the monomials.} \\ &= 3x^3 - 7x^2 + 12x + 12 && \text{Combine like terms.} \end{aligned}$$

When finding the **product of three polynomials,** begin by multiplying any two of them, and then multiply that result by the third polynomial.

Multiply:

$$\begin{aligned} -4x^3\mathbf{(x - 5)(x - 1)} &= -4x^3\mathbf{(x^2 - x - 5x + 5)} && \text{Multiply the binomials.} \\ &= -4x^3(x^2 - 6x + 5) \\ &= -4x^5 + 24x^4 - 20x^3 && \text{Distribute.} \end{aligned}$$

Special products: It is worthwhile to learn these forms.

The square of a binomial sum:

$(x + y)^2 = x^2 + 2xy + y^2$

Multiply: $(\mathbf{2x} + \mathbf{3})^2 = (\mathbf{2x})^2 + 2(\mathbf{2x})(\mathbf{3}) + \mathbf{3}^2$

$= 4x^2 + 12x + 9$

The square of a binomial difference:

$(x - y)^2 = x^2 - 2xy + y^2$

Multiply: $(\mathbf{3x} - \mathbf{y})^2 = (\mathbf{3x})^2 - 2(\mathbf{3x})(\mathbf{y}) + \mathbf{y}^2$

$= 9x^2 - 6xy + y^2$

The sum and difference of two terms:

$(x + y)(x - y) = x^2 - y^2$

Multiply: $(\mathbf{5p} + \mathbf{3q})(\mathbf{5p} - \mathbf{3q}) = (\mathbf{5p})^2 - (\mathbf{3q})^2$

$= 25p^2 - 9q^2$

REVIEW EXERCISES

Find each product.

51. $(8a^2)\left(-\frac{1}{2}a\right)$
$-4a^3$

52. $(-3xy^2z)(-2xz^3)(xz)$
$6x^3y^2z^5$

53. $2xy^2(x^3y - 4xy^5)$
$2x^4y^3 - 8x^2y^7$

54. $-a^2b(-a^2 - 2ab + b^2)$
$a^4b + 2a^3b^2 - a^2b^3$

55. $(3x^2 + 2)(2x - 4)$
$6x^3 - 12x^2 + 4x - 8$

56. $(5at - 6)^2$
$25a^2t^2 - 60at + 36$

57. $(7c^4d^3 - d)(7c^4d^3 + d)$
$49c^8d^6 - d^2$

58. $(5x^2 - 4x)(3x^2 - 2x + 10)$
$15x^4 - 22x^3 + 58x^2 - 40x$

59. $(r + s)(r - s)(r - 3s)$
$r^3 - 3r^2s - rs^2 + 3s^3$

60. $\left(3c - \frac{3}{4}\right)^2$
$9c^2 - \frac{9}{2}c + \frac{9}{16}$

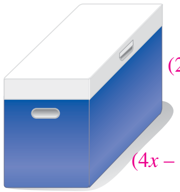
61. $[5 - (a - b)]^2$
$25 - 10a + 10b + a^2 - 2ab + b^2$

62. $(2x - y - 2z)(x + y + z)$
$2x^2 + xy - y^2 - 3yz - 2z^2$

63. Simplify: $(4a + 1)^2 + (5a - 2)^2$
$41a^2 - 12a + 5$

64. If $f(x) = x^2 + 5x - 1$, find: $f(b - 3)$
$b^2 - b - 7$

65. ICE CHESTS Write a polynomial that represents:

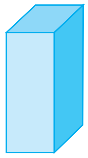


a. the perimeter of the base of the ice chest
$(12x - 2)$ in.

b. the area of the base of the ice chest
$(8x^2 - 2x)$ in.^{2}

c. the volume of the ice chest
$(16x^3 + 12x^2 - 4x)$ in.^{3}

66. GEOMETRY The length, width, and height of the rectangular solid shown here are consecutive integers.



a. Write a polynomial function that gives the volume of the solid.
$f(x) = x^3 + 3x^2 + 2x$

b. What is the volume of the solid if the shortest dimension is 5 inches?
210 in.^{3}

SECTION 5.5 The Greatest Common Factor and Factoring by Grouping

DEFINITIONS AND CONCEPTS	EXAMPLES
Factoring is multiplication reversed. To **factor a polynomial** means to express it as a product of two (or more) polynomials.	Multiplication: Given the factors, we find a polynomial. → $4x(x + 9) = 4x^2 + 36x$ ← Factoring: Given a polynomial, we find the factors.
To find the **greatest common factor, GCF,** of a list of terms **1.** Write each coefficient as a product of prime factors. **2.** Identify the numerical and variable factors common to each term. **3.** Multiply the common numerical and variable factors identified in Step 2 to obtain the GCF. If there are no common factors, the GCF is 1.	Find the GCF of $14a^4$, $35a^3$, and $56a^2$. $\left.\begin{aligned} 14a^4 &= 2 \cdot 7 \cdot a \cdot a \cdot a \cdot a \\ 35a^3 &= 5 \cdot 7 \cdot a \cdot a \cdot a \\ 56a^2 &= 2 \cdot 2 \cdot 2 \cdot 7 \cdot a \cdot a \end{aligned}\right\} \text{GCF} = 7 \cdot a \cdot a = 7a^2$
When we **factor a polynomial,** we write *a sum of terms as a product of factors.* The first step of factoring a polynomial is to see whether the terms of the polynomial have a common factor. If they do, **factor out the GCF.**	Factor: $14a^4 + 35a^3 - 56a^2 = \mathbf{7a^2}(2a^2 + 5a - 8)$ Factor out the GCF, $7a^2$. Use multiplication to check the factorization: $7a^2(2a^2 + 5a - 8) = 14a^4 + 35a^3 - 56a^2$ This is the original polynomial.
A polynomial that cannot be factored is a **prime polynomial.**	Since $2x + 13$ cannot be factored using integer coefficients, it is a prime binomial.

If a polynomial has four terms, try **factoring by grouping.** **1.** Group the terms of the polynomial so that the first two terms have a common factor and the last two terms have a common factor. **2.** Factor out the common factor from each group. **3.** Factor out the resulting common binomial factor. If there is no common binomial factor, regroup the terms of the polynomial and repeat steps 2 and 3.	Factor: $ax - 2x + 3a - 6$ $ax - 2x + 3a - 6 = ax - 2x + 3a - 6$ Group the terms. $= x(a - 2) + 3(a - 2)$ Factor x from $ax - 2x$. Factor 3 from $3a - 6$. $= (a - 2)(x + 3)$ Factor out the GCF, $a - 2$.

REVIEW EXERCISES

Find the GCF of each list.

67. 42, 36, 54 6

68. $6x^2y^5, 15xy^3$ $3xy^3$

Factor, if possible.

69. $4x^4 + 8$
$4(x^4 + 2)$

70. $\frac{3x^3}{5} - \frac{6x^2}{5} + \frac{x}{5}$
$\frac{x}{5}(3x^2 - 6x + 1)$

71. $6x - 11$
prime

72. $7a^4b^2 + 49a^3b$
$7a^3b(ab + 7)$

73. $5x^2(x + y) - 15x^3(x + y)$
$5x^2(x + y)(1 - 3x)$

74. $27x^3y^3z^3 + 81x^4y^5z^2 - 90x^2y^3z^7$
$9x^2y^3z^2(3xz + 9x^2y^2 - 10z^5)$

Factor −1 from each binomial.

75. $-x - 9$
$-(x + 9)$

76. $4r - 7$
$-(-4r + 7)$ or $-(7 - 4r)$

Factor out the opposite of the greatest common factor.

77. $-7b^3 + 14c$
$-7(b^3 - 2c)$

78. $-49a^3b^2(a - b)^4 + 63a^2b^4(a - b)^3$
$-7a^2b^2(a - b)^3(7a^2 - 7ab - 9b^2)$

Factor by grouping.

79. $xy + 2y + 4x + 8$
$(x + 2)(y + 4)$

80. $r^2y - ar - ry + a + r - 1$
$(ry - a + 1)(r - 1)$

81. $t^3 - 9 + t - 9t^2$
$(t^2 + 1)(t - 9)$

82. $1 - x - 3z + 3xz$
$(1 - x)(1 - 3z)$

83. Solve $m_1m_2 = mm_2 + mm_1$ for m_1.
$m_1 = \frac{mm_2}{m_2 - m}$

84. GEOMETRY The formula for the surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$. Write the formula with the right side in factored form.
$A = 2\pi r(r + h)$

SECTION 5.6 The Difference of Two Squares; the Sum and Difference of Two Cubes

DEFINITIONS AND CONCEPTS	EXAMPLES
The **difference of two squares:** To factor the square of a First quantity minus the square of a Last quantity, multiply the First plus the Last by the First minus the Last. $F^2 - L^2 = (F + L)(F - L)$	Factor: $x^2y^2 - 100$ $x^2y^2 - 100 = (xy)^2 - 10^2$ This is a difference of two squares. $= (xy + 10)(xy - 10)$

In general, the **sum of two squares** (with no common factor other than 1) cannot be factored using real numbers.	$x^2 + 100$ and $36y^2 + 49z^4$ are prime polynomials.
The **sum of two cubes:** To factor the cube of a First quantity plus the cube of a Last quantity, multiply the First plus the Last by the First squared, minus the First times the Last, plus the Last squared. $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$	Factor: $y^3 + 27z^6$ $y^3 + 27z^6 = y^3 + (3z^2)^3$ This is a sum of two cubes. $= (y + 3z^2)[y^2 - y \cdot 3z^2 + (3z^2)^2]$ $= (y + 3z^2)(y^2 - 3yz^2 + 9z^4)$
The **difference of two cubes:** To factor the cube of a First quantity minus the cube of a Last quantity, multiply the First minus the Last by the First squared, plus the First times the Last, plus the Last squared. $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$	Factor: $125s^3 - 64$ $125s^3 - 64 = (5s)^3 - 4^3$ This is a difference of two cubes. $= (5s - 4)[(5s)^2 + 5s \cdot 4 + 4^2]$ $= (5s - 4)(25s^2 + 20s + 16)$

REVIEW EXERCISES

Factor, if possible.

85. $z^2 - 16$ $(z + 4)(z - 4)$

86. $x^2y^4 - 64z^6$ $(xy^2 + 8z^3)(xy^2 - 8z^3)$

87. $a^2b^2 + c^2$ prime

88. $c^2 - (a + b)^2$ $(c + a + b)(c - a - b)$

89. $10m^6 - 160m^2$ $10m^2(m^2 + 4)(m + 2)(m - 2)$

90. $m^2 - n^2 + m + n$ $(m + n)(m - n + 1)$

91. $32a^4c - 162b^4c$ $2c(4a^2 + 9b^2)(2a + 3b)(2a - 3b)$

92. $k^2 + 2k + 1 - 9m^2$ $(k + 1 + 3m)(k + 1 - 3m)$

93. $t^3 + 64$ $(t + 4)(t^2 - 4t + 16)$

94. $8a^3 - 125b^9$ $(2a - 5b^3)(4a^2 + 10ab^3 + 25b^6)$

95. $4d^7 + 4d^4$ $4d^4(d + 1)(d^2 - d + 1)$

96. $(b + c)^3 + 27$ $(b + c + 3)(b^2 + 2bc + c^2 - 3b - 3c + 9)$

SECTION 5.7 Factoring Trinomials

DEFINITIONS AND CONCEPTS	EXAMPLES
Trinomials that are squares of a binomial are called **perfect-square trinomials.** We can factor perfect-square trinomials by applying the special-product rules in reverse. $x^2 + 2xy + y^2 = (x + y)^2$ $x^2 - 2xy + y^2 = (x - y)^2$	Factor: $x^2 + 22x + 121$ and $s^2 - 18st + 81t^2$ Match each trinomial to a special-product form in the left column. $x^2 + 22x + 121 = x^2 + 2 \cdot x \cdot 11 + 11^2 = (x + 11)^2$ $s^2 - 18st + 81t^2 = s^2 - 2 \cdot s \cdot 9t + (9t)^2 = (s - 9t)^2$
Many trinomials factor as the product of two binomials. To **factor a trinomial** of the form $x^2 + bx + c$, whose **leading coefficient is 1,** find two integers whose product is c and whose sum is b. $x^2 + bx + c = (x\ \square)(x\ \square)$ The product of these numbers must be c and their sum must be b. Use the FOIL method to check the factorization.	Factor: $p^2 + 14p + 45$ Here $c = 45$ and $b = 14$. We must find two integers whose product is 45 and whose sum is 14. Since $5 \cdot 9 = 45$ and $5 + 9 = 14$, two such numbers are **5** and **9,** and we have $p^2 + 14p + 45 = (p + 5)(p + 9)$ ***Check:*** $(p + 5)(p + 9) = p^2 + 9p + 5p + 45 = p^2 + 14p + 45$

We can use the **trial-and-check method** to factor trinomials with **leading coefficients other than 1.** Write the trinomial as the product of two binomials and determine four integers. The product of these numbers must be a. $ax^2 + bx + c = (\boxed{}x\,\boxed{})(\boxed{}x\,\boxed{})$ The product of these numbers must be c. Use the FOIL method to check.	Factor: $2x^2 - 5x - 12$ Since the first term is $2x^2$, the first terms of the binomial factors must be $2x$ and x. $(2x\,\boxed{})(x\,\boxed{})$ Because $2x \cdot x$ will give $2x^2$. The second terms of the binomials must be two integers whose product is -12. There are six such pairs: $1(-12), 2(-6), \mathbf{3(-4)}, 4(-3), 6(-2),$ and $12(-1)$ The pair in blue gives the correct middle term, $-5x$, when we use the FOIL method to check: Outer: $-8x$ $(2x + \mathbf{3})(x - \mathbf{4})$ $-8x + 3x = -5x$ Inner: $3x$ Thus, $2x^2 - 5x - 12 = (2x + 3)(x - 4)$.
To factor $ax^2 + bx + c$ by **grouping**, write it as an equivalent four-term polynomial: $ax^2 + \boxed{}x + \boxed{}x + c$ The product of these numbers must be ac, and their sum must be b. Then factor the four-term polynomial by grouping. Use the FOIL method to check.	Factor by grouping: $2x^2 - 5x - 12$ We must find two integers whose product is $ac = 2(-12) = -24$ and whose sum is $b = -5$. Two such numbers are -8 and 3. They serve as the coefficients of $-8x$ and $3x$, the two terms that we use to represent the middle term, $-5x$, of the trinomial. $2x^2 \mathbf{- 5x} - 12 = 2x^2 \mathbf{- 8x + 3x} - 12$ Express $-5x$ as $-8x + 3x$. $= 2x(x - 4) + 3(x - 4)$ $= (x - 4)(2x + 3)$ Factor out $(x - 4)$.
The GCF should always be factored out first. A trinomial is **factored completely** when no factor can be factored further.	Factor: $10x^4 - 50x^3 + 60x^2$ $10x^4 - 50x^3 + 60x^2 = \mathbf{10x^2}(x^2 - 5x + 6)$ Factor out the GCF, $10x^2$. $= 10x^2(x - 3)(x - 2)$ Factor the trinomial.

REVIEW EXERCISES

Factor, if possible.

97. $x^2 + 10x + 25$
$(x + 5)^2$

98. $49a^6 + 84a^3b^2 + 36b^4$
$(7a^3 + 6b^2)^2$

99. $y^2 - 21y + 20$
$(y - 20)(y - 1)$

100. $z^2 + 30 - 11z$
$(z - 5)(z - 6)$

101. $28 - x^2 - 3x$
$-(x + 7)(x - 4)$

102. $a^2 - 24b^2 - 5ab$
$(a - 8b)(a + 3b)$

103. $4a^2 - 5a + 1$
$(4a - 1)(a - 1)$

104. $3b^2 + 2b + 1$
prime

105. $y^8 + y^7 - 2y^6$
$y^6(y + 2)(y - 1)$

106. $27r^2st + 90rst - 72st$
$9st(3r - 2)(r + 4)$

107. $6t^2(r + s) + 13t(r + s) - 15(r + s)$
$(r + s)(6t - 5)(t + 3)$

108. $v^4 - 13v^2 + 42$
$(v^2 - 7)(v^2 - 6)$

109. $w^8 - w^4 - 90$
$(w^4 - 10)(w^4 + 9)$

110. Use a substitution to factor $(s + t)^2 - 2(s + t) + 1$.
$(s + t - 1)^2$

SECTION 5.8 Summary of Factoring Techniques

DEFINITIONS AND CONCEPTS

To factor a random polynomial, use this **factoring strategy:**

1. Factor out all common factors.
2. How many terms does the polynomial have? If it has *two terms,* check to see whether it is
   a. The difference of two squares
   b. The sum of two cubes
   c. The difference of two cubes

   If it has *three terms,* try to factor it as
   a. A perfect-square trinomial or
   b. A general trinomial

   If it has *four or more terms,* try factoring it by grouping.
3. Continue until each individual factor is prime.
4. Check the results by multiplying.

EXAMPLES

Factor: $-4t^2x^5 + 4t^2x^3y^2 - 4t^2x^2y^3 + 4t^2y^5$

We first factor out the opposite of the GCF, $-4t^2$.

$$-4t^2x^5 + 4t^2x^3y^2 - 4t^2x^2y^3 + 4t^2y^5$$
$$= -4t^2(x^5 - x^3y^2 + x^2y^3 - y^5)$$

Since the expression within parentheses has four terms, we try to factor it by grouping.

$$= -4t^2[x^3(x^2 - y^2) + y^3(x^2 - y^2)]$$
$$= -4t^2(x^2 - y^2)(x^3 + y^3)$$

Finally, we factor the difference of two squares and the sum of two cubes.

$$= -4t^2(x + y)(x - y)(x + y)(x^2 - xy + y^2)$$

Since each factor is prime, the factoring is complete. Check the result by multiplication.

REVIEW EXERCISES

Factor, if possible.

111. $4q^2rs + 4qrst - 120rst^2$ $4rs(q - 5t)(q + 6t)$

112. $2(m + n)^2 + (m + n) - 3$ $(2m + 2n + 3)(m + n - 1)$

113. $z^2 - 4 + zx - 2x$ $(z - 2)(z + x + 2)$

114. $m^4 + 16n^2$ prime

115. $x^2 + 4x + 4 - 4p^4$ $(x + 2 + 2p^2)(x + 2 - 2p^2)$

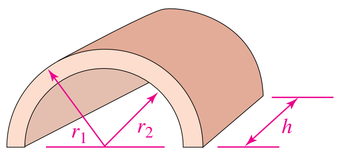
116. $y^2 + 3y + 2 + 2x + xy$ $(y + 2)(y + 1 + x)$

117. $4a^3b^3c^2 + 256c^2$ $4c^2(ab + 4)(a^2b^2 - 4ab + 16)$

118. $-13a^2 + 36 + a^4$ $(a + 3)(a - 3)(a + 2)(a - 2)$

119. $4x^4 + 12x^3 + 9x^2 + 2x + 3$ $(2x + 3)(2x^3 + 3x^2 + 1)$

120. SPANISH ROOF TILES The amount of clay used to make a roof tile is given by



$$V = \frac{\pi}{2}r_1^2h - \frac{\pi}{2}r_2^2h$$

Factor the right side of the formula. $\frac{\pi}{2}h(r_1 + r_2)(r_1 - r_2)$

SECTION 5.9 Solving Equations by Factoring

DEFINITIONS AND CONCEPTS

A **quadratic equation** is an equation that can be written in the **standard form** $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

The **zero-factor property:** If a and b are real numbers, then

If $ab = 0$, then $a = 0$ or $b = 0$.

EXAMPLES

Examples of quadratic equations are:

$8x^2 + 64x = 0$, $25y^2 - 16 = 0$, and $5m^2 - 13m = 6$

If $(x + 1)(x - 9) = 0$, then $x + 1 = 0$ or $x - 9 = 0$.

To **solve a quadratic equation by factoring:**

1. Write the equation in standard form $ax^2 + bx + c = 0$.
2. Factor the polynomial.
3. Use the zero-factor property to set each factor equal to zero.
4. Solve each resulting linear equation.
5. Check each solution.

Solve: $6x^2 = 5x + 6$

$6x^2 - 5x - 6 = 0$ Write the equation in standard form.

$(2x - 3)(3x + 2) = 0$ Factor the trinomial.

$2x - 3 = 0$ or $3x + 2 = 0$ Set each factor equal to 0.

$x = \frac{3}{2}$ | $x = -\frac{2}{3}$ Solve each equation.

The solutions are $\frac{3}{2}$ and $-\frac{2}{3}$. Check both in the original equation.

Some **polynomial equations of higher degree** can be solved by factoring. We use the extension of the zero-factor property: When the product of two or more real numbers is 0, at least one of them is 0.

Solve: $x^3 - 2x^2 - 8x = 0$

$x(x^2 - 2x - 8) = 0$ Factor out the GCF, x.

$x(x - 4)(x + 2) = 0$ Factor the trinomial.

$x = 0$ or $x - 4 = 0$ or $x + 2 = 0$ Set each factor equal to 0.

$x = 4$ | $x = -2$ Solve each equation.

The solutions are 0, 4, and -2. Check each in the original equation.

REVIEW EXERCISES

Solve each equation by factoring.

121. $4x^2 - 3x = 0$ $0, \frac{3}{4}$

122. $x^2 = 36$ $6, -6$

123. $12x^2 = 5 - 4x$ $\frac{1}{2}, -\frac{5}{6}$

124. $-d^4 + 10d^2 - 9 = 0$ $3, -3, 1, -1$

125. $t^2(15t - 2) = 8t$ $0, -\frac{2}{3}, \frac{4}{5}$

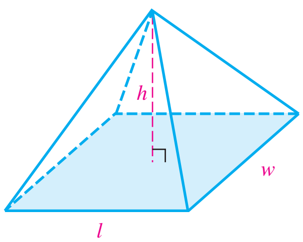
126. $u^3 = \frac{1}{3}u(19u + 14)$ $-\frac{2}{3}, 7, 0$

127. $(y + 7)^2 + 8 = -2(y + 7) + 7$ a repeated solution of -8

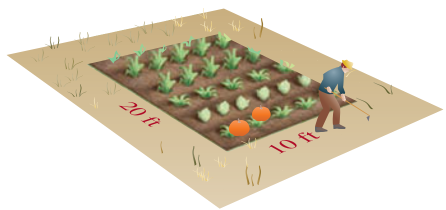
128. $x^3 + 7x^2 - x - 7 = 0$ $-7, -1, 1$

129. Let $f(x) = x^2 - 4x + 2$. For what value(s) of x is $f(x) = -1$? 1, 3

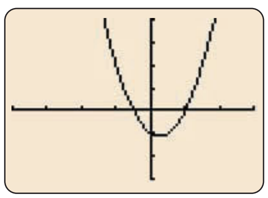
130. PYRAMIDS The volume of a pyramid is given by the formula $V = \frac{1}{3}lwh$, where l is the length and w is the width of its rectangular base and h is its height. The volume of the pyramid is 210 cubic meters. Find the dimensions of its rectangular base if one edge of the base is 3 meters longer than the other, and the height of the pyramid is 9 meters. 7 m by 10 m



131. GARDENING A family wants to increase the size of their vegetable garden by 400 square feet by clearing the weeds off of a uniform width border around the existing garden. How wide should the border be if the rectangular-shaped garden is currently 10 feet wide and 20 feet long? 5 ft



132. Use the graph of $y = 2x^2 - x - 1$, shown in the illustration, to estimate the solutions of $2x^2 - x - 1 = 0$. $-\frac{1}{2}, 1$



CHAPTER 5 TEST

1. Fill in the blanks.

a. Since $x^2 + 16x + 64$ is the square of $x + 8$, it is called a _perfect_-square trinomial.

b. When we factor a polynomial, we write a sum of terms as a _product_ of factors.

c. The statement $x^2 - x - 12 = (x - 4)(x + 3)$ shows that the trinomial $x^2 - x - 12$ factors as the product of two _binomials_.

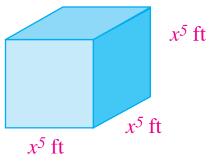
d. A _quadratic_ equation is any equation that can be written in the form $ax^2 + bx + c = 0$, where $a \neq 0$.

e. $x^2 - y^2$ is called a _difference_ of two squares and $x^3 + y^3$ is called a sum of two _cubes_.

f. The _greatest_ _common_ factor of $12x^3$ and $8x^5$ is $4x^3$.

2. a. Write an algebraic expression that represents the area of the base of the cube. x^{10} ft^2

b. Write an algebraic expression that represents the volume of the cube. x^{15} ft^3



Simplify each expression. Write answers using positive exponents. Assume that no denominators are zero.

3. $9^{-1}a^{-5}m^3(a^5m^{-4})^2$ $\frac{a^5}{9m^5}$

4. $\left(\frac{-2x^2y^3}{5}\right)^3$ $-\frac{8x^6y^9}{125}$

5. $\frac{(-4b^2)^3(b^{-3})^3}{b^5}$ $-\frac{64}{b^8}$

6. $\left(\frac{3m^2n^3}{m^4n^{-2}}\right)^{-2}$ $\frac{m^4}{9n^{10}}$

7. Use scientific notation to find the quotient:

$$\frac{3{,}190{,}000{,}000{,}000}{0.00022(5{,}000{,}000{,}000)}$$

Express the answer in scientific notation and standard notation. $2.9 \times 10^6 = 2{,}900{,}000$

8. SPEED OF LIGHT Light travels 1.86×10^5 miles per second. How far does it travel in a minute? Express the answer in scientific notation. 1.116×10^7 mi

Find the coefficient of each term and the degree of the polynomial.

9. a. $3x^3 - 4x^5 - 3x^2 - \frac{5}{3}$ $3, -4, -3, -\frac{5}{3}; 5$

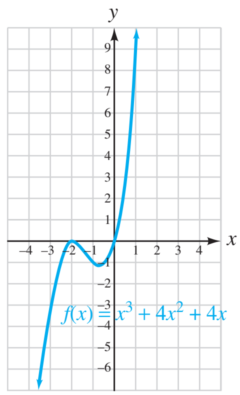
b. $8x^5y^3 - x^8y^2 + x^9y^4 - 6x^2y^5 + 4$ $8, -1, 1, -6, 4; 13$

10. BOATING The height (in feet) of a warning flare from the surface of the ocean t seconds after being shot into the air is given by the polynomial function $h(t) = -16t^2 + 80t + 10$. What is the height of the flare 2.5 seconds after being fired? 110 ft

11. a. Graph the function: $f(x) = x^3 + 4x^2 + 4x$

b. Write the domain and the range of the function in interval notation. D: $(-\infty, \infty)$; R: $(-\infty, \infty)$

c. From the graph, determine the solutions of $x^3 + 4x^2 + 4x = 0$. $-2, 0$

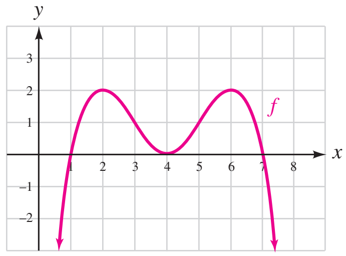


12. Use the graph of function f to find each of the following.

a. $f(4)$ 0

b. The values of x for which $f(x) = 2$. 2, 6

c. The domain and range of f. D: $(-\infty, \infty)$, R: $(-\infty, 2]$



Perform the operations.

13. $(-2x^2y^3 + 6xy + 5y^2) - (-4x^2y^3 - 7xy + 2y^2)$
$2x^2y^3 + 13xy + 3y^2$

14. $\left(\frac{1}{2}x^5 + \frac{1}{3}x^2\right) + \left(\frac{3}{2}x^5 - \frac{1}{5}x^2\right)$ $2x^5 + \frac{2}{15}x^2$

15. $(a^2yz^4)(2ay^5z)(-6ay^6z^7)$ $-12a^4y^{12}z^{12}$

16. $-5a^2b(3ab^3 - 2ab^4)$ $-15a^3b^4 + 10a^3b^5$

17. $(3y + 1)(2y^2 + 3y + 2)$ $6y^3 + 11y^2 + 9y + 2$

18. $(0.6d - 2)(0.1d + 3)$ $0.06d^2 + 1.6d - 6$

19. $[6 + (m - n)]^2$ $36 + 12m - 12n + m^2 - 2mn + n^2$

20. $2s(4s + 5t)(4s - 5t)$ $32s^3 - 50st^2$

21. Simplify: $(4t - 3)^2 - (t + 1)(t - 4)$ $15t^2 - 21t + 13$

22. If $f(x) = x^2 - 3x + 6$, find: $f(c + 1)$ $c^2 - c + 4$

Factor, if possible.

23. $12a^3b^2c - 3a^2b^2c^2 + 6abc^3$ $3abc(4a^2b - abc + 2c^2)$

24. $hk + bz + hz + bk$ $(k + z)(h + b)$

25. $x^2 - x - 30$ $(x - 6)(x + 5)$

26. $y^4 - 81$ $(y^2 + 9)(y + 3)(y - 3)$

27. $25m^2 - 60mn + 36n^2$ $(5m - 6n)^2$

28. $s^4 - 13s^2 + 36$ $(s + 3)(s - 3)(s + 2)(s - 2)$

29. $-21x^4 + 10x^3 + 16x^2$ $-x^2(7x - 8)(3x + 2)$

30. $16b^2 + 25$ prime

31. $5x^3 + 625$ $5(x + 5)(x^2 - 5x + 25)$

32. $64a^3 - 125b^6$ $(4a - 5b^2)(16a^2 + 20ab^2 + 25b^4)$

33. $(x - y)^2 + 3(x - y) - 10$ $(x - y + 5)(x - y - 2)$

34. $6b^2 + bc - 2c^2$ $(3b + 2c)(2b - c)$

35. $a^2 - b^2 + a + b$ $(a + b)(a - b + 1)$

36. Solve for v: $v_1v_3 - v_3v = v_1v$ $v = \frac{v_1v_3}{v_3 + v_1}$

Solve each equation.

37. $5m^2 = 25m$ $0, 5$

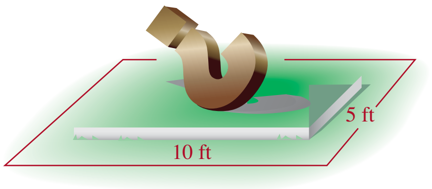
38. $\frac{2}{5}x^2 + \frac{3}{5}x - 7 = 0$ $-5, \frac{7}{2}$

39. $(3x + 1)^2 + 2x = x^2 + 2(x + 5)$ $\frac{3}{4}, -\frac{3}{2}$

40. $x^3 + 8x^2 - 9x = 0$ $1, 0, -9$

41. $x^3 - 16x = 16 - x^2$ $-1, -4, 4$

42. STATUES A bronze statue is mounted on a 5 foot by 10 foot rectangular concrete slab. The slab is surrounded by a flowerbed border of uniform width that has a total area of 54 square feet. Find the width of the border. 1.5 ft



CHAPTERS 1–5 CUMULATIVE REVIEW

Determine whether each of the following statements is true or false.

1. Every integer is a real number. [Section 1.2] true

2. Rational numbers are nonterminating, nonrepeating decimals. [Section 1.2] false

3. An irrational number can be expressed as a fraction with an integer numerator and a nonzero integer denominator. [Section 1.2] false

4. 0 is a whole number but not a natural number. [Section 1.2] true

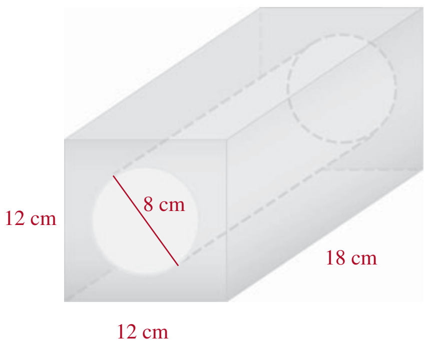
5. All natural numbers are rational numbers. [Section 1.2] true

6. Evaluate: $-3|(3^2 \cdot 5 - 2^3 \cdot 6)^2|$ [Section 1.3] -27

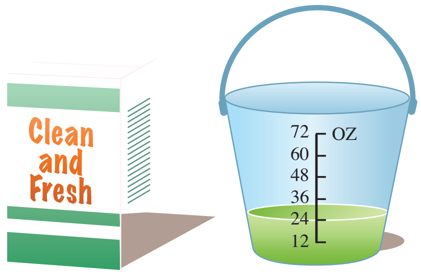
7. Solve: $\dfrac{x+2}{5} - 4x = \dfrac{8}{5} - \dfrac{x+9}{2}$ [Section 1.5] 1

8. Solve $F = \dfrac{9}{5}C + 32$ for C. [Section 1.6] $C = \frac{5}{9}(F - 32)$

9. MACHINING Find the volume that is left when a hole is drilled through the metal block shown below. Round to the nearest hundredth. [Section 1.7] 1,687.22 cm^3



10. HOUSEKEEPING To clean a kitchen floor, a maid mixed a concentrated cleaner with water to get the 15% solution shown below. If this mixture was too harsh on her hands, how much water would she have to add to the bucket to make it a 10% solution? [Section 1.8] 12 oz

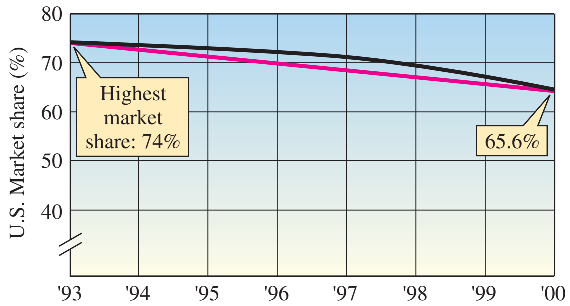


Write an equation of the line with the given properties. Express the answer in slope–intercept form.

11. Perpendicular to a line with slope $m = -\frac{1}{8}$, passing through $(-2, 5)$ [Section 2.4] $y = 8x + 21$

12. Passing through $(-5, 4)$ and $(8, -6)$ [Section 2.4] $y = -\frac{10}{13}x + \frac{2}{13}$

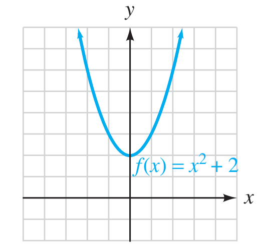
13. AUTOMOBILE SALES The following graph shows the decline of the market share for the Big Three automakers (General Motors, Ford, and Chrysler) from 1993 to 2000. The straight red line can be used to model the decline. Use the line to determine the average annual rate of change in the Big Three's market share during these years. [Section 2.3] -1.2% per year



14. If $h(x) = x^4 - x^3 - x^2 - x - 1$, find $h(1)$. [Section 2.5] -3

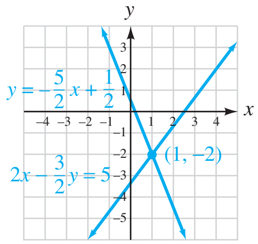
15. COPIER COSTS A lease agreement for a copier charges the user a \$95 monthly service fee and $2\frac{1}{2}$¢ per copy. Write a linear function that gives the cost per month to lease the copier, where x represents the total number of copies made during the month.
[Section 2.5] $f(x) = 0.025x + 95$

16. Graph $f(x) = x^2 + 2$. Then specify the domain and range.
[Section 2.6]
D: the set of real numbers, R: the set of all real numbers greater than or equal to 2



17. Solve the system by graphing. [Section 3.1]

$$\begin{cases} y = -\dfrac{5}{2}x + \dfrac{1}{2} \\ 2x - \dfrac{3}{2}y = 5 \end{cases}$$



18. Solve: $\begin{cases} 2(2x + 3y) = 5 \\ 8x = 3(1 + 3y) \end{cases}$ [Section 3.2] $\left(\frac{3}{4}, \frac{1}{3}\right)$

19. Solve: $\begin{cases} 3x + 2y - z = -8 \\ 2x - y + 7z = 10 \\ 2x + 2y - 3z = -10 \end{cases}$ [Section 3.4] $(-2, 0, 2)$

20. Evaluate: $\begin{vmatrix} 2 & -3 & 4 \\ -1 & 2 & 4 \\ 3 & -3 & 1 \end{vmatrix}$ [Section 3.7] -23

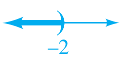
Solve each inequality. Give the solution set in interval notation and then graph it.

21. $-9(t - 3) + 2t \leq 8(4 - t)$ [Section 4.1]
$(-\infty, 5]$

22. $-6 \leq \frac{1}{3}h + 1 < 0$ [Section 4.2]
$[-21, -3)$

23. $|m + 5| \geq 7$ [Section 4.3]
$(-\infty, -12] \cup [2, \infty)$

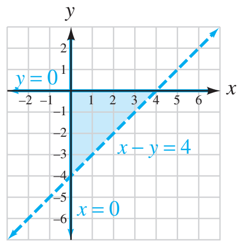
24. $4.5x - 1 < -10$ or $6 - 2x \geq 12$ [Section 4.2]
$(-\infty, -2)$



25. GROCERY SHOPPING Let x = the number of items in the grocery bag shown on the right. Which statement below mathematically describes the number of items this type of bag can hold?
[Section 4.2] iii

i. $x > 10$ or $x < 15$
ii. $x < 10$ and $x > 15$
iii. $10 \leq x \leq 15$
iv. $x > 10$ and $x < 15$

26. Graph the solution set:
$$\begin{cases} x - y < 4 \\ y \leq 0 \\ x \geq 0 \end{cases}$$
[Section 4.5]



27. Simplify $\left(\dfrac{-3a^4b^2}{-9a^5b^{-2}}\right)^{-2}$. Write the answer without using negative exponents. [Section 5.1] $\frac{9a^2}{b^8}$

28. Write 9.0895×10^{-8} in standard notation.
[Section 5.2] 0.000000090895

29. Find the product: $(2a - b)(4a^2 + 2ab + b^2)$
[Section 5.4] $8a^3 - b^3$

30. Simplify: $(3k - 1)^2 + (2k - 4)^2$
[Section 5.4] $13k^2 - 10k + 17$

Factor each polynomial completely.

31. $x^2 + 4y - xy - 4x$ [Section 5.5] $(x - y)(x - 4)$

32. $6s^4 - 216s^2$ [Section 5.6] $6s^2(s + 6)(s - 6)$

33. $8x^6 + 125y^3$ [Section 5.6] $(2x^2 + 5y)(4x^4 - 10x^2y + 25y^2)$

34. $-3a^2 + ab + 2b^2$ [Section 5.7] $-(3a + 2b)(a - b)$

35. Solve: $x^2 = \frac{1}{2}(x + 1)$ [Section 5.9] $1, -\frac{1}{2}$

36. Solve $m_1m_2 = m_2g + m_1g$ for m_1.
[Section 5.9] $m_1 = \frac{m_2g}{m_2 - g}$

6

Rational Expressions and Equations

Reggie Casagrande/Getty Images

from Campus to Careers

Webmaster

If you use the Internet, then you have seen first-hand what webmasters do. They design and maintain websites for individuals and companies on the World Wide Web. The job of webmaster requires excellent computer and technical skills. A background in business, art, and design is also helpful. Since webmasters are often called on to be troubleshooters when technical difficulties arise, those considering entering the field are encouraged to study math to strengthen their problem-solving abilities.

In **Problem 77** of **Study Set 6.9,** you will use the language of variation to describe various aspects of Internet usage and website design.

JOB TITLE: Webmaster

EDUCATION: Many webmasters have college degrees. However, some have only a year or two of college training.

JOB OUTLOOK: Excellent—job opportunities are expected to increase between 18% to 26% through the year 2014.

ANNUAL EARNINGS: Average salary $68,300

FOR MORE INFORMATION: http://www.bls.gov/k12/computers05.htm

Objectives

1. Define rational expressions.
2. Evaluate rational functions.
3. Graph rational functions.
4. Find the domain of a rational function.
5. Simplify rational expressions.
6. Simplify rational expressions that have factors that are opposites.

SECTION 6.1

Rational Functions and Simplifying Rational Expressions

Linear and polynomial functions can be used to model many real-world situations. In this section, we introduce another family of functions known as *rational functions*. Rational functions get their name from the fact that their defining equation contains a *ratio* (fraction) of two polynomials.

1 Define rational expressions.

Fractions that are the quotient of two integers are *rational numbers*. Fractions that are the quotient of two polynomials are called *rational expressions*.

Rational Expressions

A **rational expression** is an expression of the form $\frac{A}{B}$, where A and B are polynomials and B does not equal 0.

Some examples of rational expressions are

$$\frac{3x}{x-7}, \quad \frac{8yz^4}{6y^2z^2}, \quad \frac{5m+n}{8m+16}, \quad \text{and} \quad \frac{6a^2-13a+6}{3a^2+a-2}$$

Caution! Since division by 0 is undefined, the value of a polynomial in the denominator of a rational expression cannot be 0. For example, x cannot be 7 in the rational expression $\frac{3x}{x-7}$, because the value of the denominator would be 0. In $\frac{5m+n}{8m+16}$, m cannot be -2, because the value of the denominator would be 0.

2 Evaluate rational functions.

Rational expressions in one variable often define functions. For example, if the cost of subscribing to an online research network is \$6 per month plus \$1.50 per hour of access time, the average (mean) hourly cost of the service is the total monthly cost, divided by the number of hours of access time used that month:

$$\frac{C}{n} = \frac{1.50n + 6}{n}$$

C is the total monthly cost, and n is the number of hours the service is used that month.

The right side of this equation is a rational expression: the quotient of the binomial $1.50n + 6$ and the monomial n.

The rational function that gives the average hourly cost of using the network for n hours per month can be written

$$f(n) = \frac{1.50n + 6}{n}$$

We are assuming that at least one access call will be made each month, so the function is defined for $n > 0$.

Rational Functions

A **rational function** is a function whose equation is defined by a rational expression in one variable, where the value of the polynomial in the denominator is never zero.

EXAMPLE 1 Use the function $f(n) = \dfrac{1.50n + 6}{n}$ to find the average hourly cost when the network described earlier is used for:

a. 1 hour **b.** 9 hours

Strategy We will find $f(1)$ and $f(9)$.

WHY The notation $f(1)$ represents the average hourly cost for 1 hour of use and $f(9)$ represents average hourly cost for 9 hours of use.

Solution

a. To find the average hourly cost for 1 hour of time, we find $f(1)$:

$$f(\mathbf{1}) = \frac{1.50(\mathbf{1}) + 6}{\mathbf{1}} = 7.5 \quad \text{Input 1 for } n \text{ and simplify.}$$

The average hourly cost for 1 hour of access time is \$7.50.

b. To find the average hourly cost for 9 hours of time, we find $f(9)$:

$$f(\mathbf{9}) = \frac{1.50(\mathbf{9}) + 6}{\mathbf{9}} = 2.166666666\ldots \quad \text{Input 9 for } n \text{ and simplify.}$$

The average hourly cost for 9 hours of access time is approximately \$2.17.

Self Check 1

Find the average hourly cost when the network is used for:

a. 3 hours \$3.50

b. 100 hours \$1.56

Now Try **Problem 91**

Teaching Example 1 Use the function $f(n) = \frac{1.50n + 6}{n}$ to find the average hourly cost when the network is used for

a. 5 hours **b.** 10 hours

Answers:

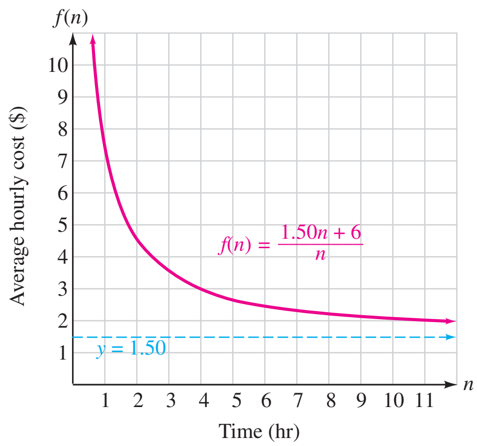
a. \$2.70 **b.** \$2.10

3 Graph rational functions.

To graph the rational function $f(n) = \frac{1.50n + 6}{n}$, we substitute values for n (the inputs) in the function, compute the corresponding values of $f(n)$ (the outputs), and express the results as ordered pairs. From the evaluations in Example 1 and its Self Check, we know four such ordered pairs are: (1, 7.50), (3, 3.50), (9, 2.17), and (100, 1.56). Those pairs and others are listed in the table below. We then plot the points and draw a smooth curve through them to get the graph.

$f(n) = \dfrac{1.50n + 6}{n}$

n	$f(n)$	
1	7.50	→ (1, 7.50)
2	4.50	→ (2, 4.50)
3	3.50	→ (3, 3.50)
4	3.00	→ (4, 3.00)
5	2.70	→ (5, 2.70)
6	2.50	→ (6, 2.50)
7	2.36	→ (7, 2.36)
8	2.25	→ (8, 2.25)
9	2.17	→ (9, 2.17)
10	2.10	→ (10, 2.10)
100	1.56	→ (100, 1.56)



As the access time increases, the graph approaches the line $y = 1.50$, which indicates that the average hourly cost approaches \$1.50 as the hours of use increase.

From the graph, we can see that the average hourly cost decreases as the number of hours of access time increases. Since the cost of each extra hour of access time is \$1.50, the average hourly cost can approach \$1.50 but never drop below it. Thus, the graph of the function approaches the line $y = 1.5$ as n increases. When a graph approaches a line, we call the line an **asymptote.** The line $y = 1.5$ is a **horizontal asymptote** of the graph.

As n gets smaller and approaches 0, the graph approaches the y-axis. The y-axis is a **vertical asymptote** of the graph.

4 Find the domain of a rational function.

Since division by 0 is undefined, any values that make the denominator 0 in a rational function must be excluded from the domain of the function.

Self Check 2

Find the domain of:

$f(x) = \frac{x^2 + 1}{x^2 - 49}$

Now Try **Problem 25**

Self Check 2 Answer

The domain is the set of all real numbers except -7 and 7: $(-\infty, -7) \cup (-7, 7) \cup (7, \infty)$.

Teaching Example 2 Find the domain of: $f(x) = \frac{4x - 8}{x^2 - x - 12}$

Answer:

The set of all real numbers except 4 and -3. $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$ $\{x \mid x$ is a real number and $x \neq -3, x \neq 4\}$

EXAMPLE 2

Find the domain of: $f(x) = \frac{3x + 2}{x^2 + x - 6}$

Strategy We will set $x^2 + x - 6$ equal to 0 and solve for x.

WHY We don't need to examine the numerator of the rational expression; it can be any value, including 0. The domain of the function includes all real numbers, except those that make the *denominator equal to 0.*

Solution

$$x^2 + x - 6 = 0 \qquad \text{Set the denominator equal to 0.}$$
$$(x + 3)(x - 2) = 0 \qquad \text{Factor the trinomial.}$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Set each factor equal to 0.}$$
$$x = -3 \quad \Big| \quad x = 2 \qquad \text{Solve each linear equation.}$$

Thus, the domain of the function is the set of all real numbers except -3 and 2. Using set-builder notation we can describe the domain as $\{x \mid x$ is a real number and $x \neq -3, x \neq 2\}$. In interval notation, the domain is $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

The Language of Algebra Another way that Example 2 could be phrased is: *State the restrictions on the variable.* For $\frac{3x + 2}{x^2 + x - 6}$, we can write $x \neq -3$ and $x \neq 2$.

Using Your CALCULATOR **Finding the Domain and Range of a Rational Function**

We can find the domain and range of the function in Example 2 by looking at its graph. If we use window settings of $[-10, 10]$ for x and $[-10, 10]$ for y and graph the function

$$f(x) = \frac{3x + 2}{x^2 + x - 6}$$

we will obtain the graph shown on the next page.

From the figure, we can see that

- As x approaches -3 from the left, the values of y decrease, and the graph approaches the vertical line $x = -3$. As x approaches -3 from the right, the values of y increase, and the graph approaches the vertical line $x = -3$.

From the figure, we can also see that

- As x approaches 2 from the left, the values of y decrease, and the graph approaches the vertical line $x = 2$. As x approaches 2 from the right, the values of y increase, and the graph approaches the vertical line $x = 2$.

The lines $x = -3$ and $x = 2$ are vertical asymptotes. These asymptotes seem to appear in the figure. This is because graphing calculators draw graphs by

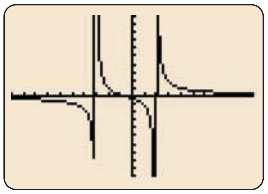
connecting dots whose x-coordinates are close together. Often when two such points straddle a vertical asymptote and their y-coordinates are far apart, the calculator draws a line between them, producing what appears to be a vertical asymptote. If instead of connected mode, you set your calculator to dot mode by pressing MODE, pressing ▼ five times, pressing ▶ once, and then pressing ENTER, the vertical lines will not appear.

From the figure, we can also see that

- As x increases to the right of 2, the values of y decrease and approach the line $y = 0$.
- As x decreases to the left of -3, the values of y increase and approach the line $y = 0$.

The line $y = 0$ (the x-axis) is a horizontal asymptote. Graphing calculators do not draw lines that appear to be horizontal asymptotes.

From the graph, we can see that every real number x, except -3 and 2, gives a value of y. This observation confirms that the domain of the function is $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$. We can also see that y can be any value. Thus, the range is $(-\infty, \infty)$.



THINK IT THROUGH *Learning and Remembering*

"Most students express a wish to be more efficient in their studies. Knowing how your brain takes in and processes information, and then working with this system, will greatly improve your efficiency."

From the Study Skills Package, University of Waterloo Counseling Services

The graph below is called the *curve of forgetting.* It shows how quickly a typical student forgets the new information presented in a one-hour lecture if he or she does not review the material later. Use the graph to estimate each of the following.

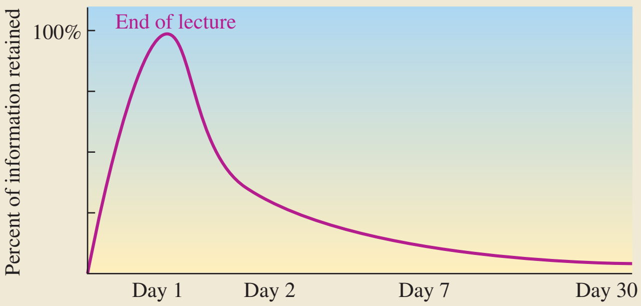
1. What percent of the information is retained by Day 2? about 25%

2. What percent of the information is forgotten by Day 7? about 88%

3. What percent of the information is retained by Day 30? about 10%

4. Do you think the curve of forgetting has an asymptote? Explain why or why not. yes

For ways to improve retention, visit the website www.adm.uwaterloo.ca/infocs/study_skills/curve.html



Source: The University of Waterloo, Canada, Counseling Service

5 Simplify rational expressions.

When working with rational expressions, we will use some familiar rules from arithmetic.

Properties of Fractions

If a, b, c, d, and k represent real numbers, and if there are no divisions by 0, then

1. $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$
2. $\frac{a}{1} = a$ and $\frac{a}{a} = 1$
3. $\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b}$
4. $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$

Property 3 is true because any number times 1 is that number.

$$\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k} = \frac{a}{b} \cdot 1 = \frac{a}{b} \quad \text{where} \quad b \neq 0 \quad \text{and} \quad k \neq 0$$

To streamline this process, we can replace $\frac{k}{k}$ in $\frac{ak}{bk}$ with the equivalent fraction $\frac{1}{1}$.

$$\frac{ak}{bk} = \frac{a\overset{1}{\cancel{k}}}{b\underset{1}{\cancel{k}}} = \frac{a}{b} \qquad \frac{k}{k} = \frac{1}{1} = 1$$

We say that we have simplified $\frac{ak}{bk}$ by *removing a factor equal to 1.*

The Language of Algebra Property 3 is known as the *fundamental property of fractions.* Stated in another way, it enables us to divide out factors that are common to the numerator and denominator of a fraction.

To **simplify a rational expression** means to write it so that the numerator and denominator have no common factors other than 1.

Simplifying Rational Expressions

1. Factor the numerator and denominator completely to determine their common factors.
2. Remove factors equal to 1 by replacing each pair of factors common to the numerator and denominator with the equivalent fraction $\frac{1}{1}$.
3. Multiply the remaining factors in the numerator and in the denominator.

Self Check 3

Simplify: $\frac{12a^4b^2}{20ab^4}$ $\frac{3a^3}{5b^2}$

Now Try **Problem 33**

Teaching Example 3 Simplify: $\frac{10kr^3}{25k^2r^2}$

Answer:
$\frac{2r}{5k}$

EXAMPLE 3 Simplify: $\frac{8yz^4}{6y^2z^2}$

Strategy We will begin by writing the numerator and denominator in factored form. Then we will remove any factors common to the numerator and denominator.

WHY The rational expression is simplified when the numerator and denominator have no common factors other than 1.

Solution

$$\frac{8yz^4}{6y^2z^2} = \frac{2 \cdot 2 \cdot 2 \cdot y \cdot z \cdot z \cdot z \cdot z}{2 \cdot 3 \cdot y \cdot y \cdot z \cdot z}$$ Factor $8yz^4$ and $6y^2z^2$ completely.

$$= \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot \overset{1}{\cancel{y}} \cdot \overset{1}{\cancel{z}} \cdot \overset{1}{\cancel{z}} \cdot z \cdot z}{\underset{1}{\cancel{2}} \cdot 3 \cdot \underset{1}{\cancel{y}} \cdot y \cdot \underset{1}{\cancel{z}} \cdot \underset{1}{\cancel{z}}}$$ Replace $\frac{2}{2}$, $\frac{y}{y}$, and $\frac{z}{z}$ with $\frac{1}{1}$. This removes the factor $\frac{2 \cdot y \cdot z \cdot z}{2 \cdot y \cdot z \cdot z} = 1$.

$$= \frac{4z^2}{3y}$$

We say that $\frac{8yz^4}{6y^2z^2}$ simplifies to $\frac{4z^2}{3y}$.

An alternate approach is to use rules for exponents to simplify the rational expression.

$$\frac{8yz^4}{6y^2z^2} = \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot y^{1-2}z^{4-2}}{\underset{1}{\cancel{2}} \cdot 3} = \frac{4y^{-1}z^2}{3} = \frac{4z^2}{3y}$$ To divide exponential expressions with the same base, keep the base and subtract the exponents.

To simplify rational expressions, we often make use of the factoring methods discussed in Chapter 5.

EXAMPLE 4

Simplify: **a.** $\frac{6x^3}{3x^4 - 9x^3}$ **b.** $\frac{x^2 - 16}{2x^2 + 8x}$

Strategy We will begin by factoring the numerator and denominator. Then we will remove any factors common to the numerator and denominator.

WHY We need to make sure that the numerator and denominator have no common factors other than 1. If that is the case, then the rational expression is simplified.

Solution

a. $$\frac{6x^3}{3x^4 - 9x^3} = \frac{2 \cdot 3 \cdot x^3}{3x^3(x - 3)}$$ Factor the numerator. In the denominator, factor out the GCF, $3x^3$.

$$= \frac{2 \cdot \overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{x^3}}}{\underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{x^3}} \cdot (x - 3)}$$ Remove the factors common to the numerator and denominator.

$$= \frac{2}{x - 3}$$

b. In the numerator, we factor the difference of two squares. In the denominator, we factor out the GCF, $2x$.

$$\frac{x^2 - 16}{2x^2 + 8x} = \frac{\overset{1}{\cancel{(x + 4)}}(x - 4)}{2x\underset{1}{\cancel{(x + 4)}}}$$ Remove the binomial factor $x + 4$ that is common to the numerator and denominator.

$$= \frac{x - 4}{2x}$$ This rational expression does not simplify further.

Self Check 4

Simplify:

a. $\frac{28x^4}{7x^5 - 14x^4}$ $\frac{4}{x - 2}$

b. $\frac{x^2 - 9}{5x^2 - 15x}$ $\frac{x + 3}{5x}$

Now Try **Problems 35 and 39**

Teaching Example 4 Simplify:
a. $\frac{7a^3}{5a^4 - 7a^3}$ **b.** $\frac{x^2 - 4x - 12}{x^2 - 36}$
Answers:
a. $\frac{7}{5a - 7}$ **b.** $\frac{x + 2}{x + 6}$

When simplifying rational expressions, we can only remove *factors* common to the entire numerator and denominator. It is incorrect to remove *terms* common to the numerator and denominator.

$$\frac{\overset{1}{\cancel{x}} - 4}{2\underset{1}{\cancel{x}}} \qquad \frac{a^2 - 3a + \overset{1}{\cancel{2}}}{a + \underset{1}{\cancel{2}}} \qquad \frac{\overset{1}{\cancel{y^2}} - 36}{\underset{1}{\cancel{y^2}} - y - 7}$$

x is a term of $x - 4$. 2 is a term of $a^2 - 3a + 2$ and a term of $a + 2$. y^2 is a term of $y^2 - 36$ and a term of $y^2 - y - 7$.

Using Your CALCULATOR Checking an Algebraic Simplification

After simplifying an expression, we can use a scientific calculator to check the answer. One way to check whether $\frac{x^2 - 16}{2x^2 + 8x} = \frac{x - 4}{2x}$ is correct in Example 4b is to evaluate $\frac{x^2 - 16}{2x^2 + 8x}$ and $\frac{x - 4}{2x}$ for a value of x (say, **2**). The expressions should give identical results.

For $\dfrac{x^2 - 16}{2x^2 + 8x}$:

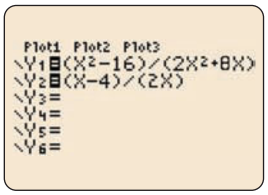
$\boxed{(}\ \mathbf{2}\ \boxed{x^2}\ \boxed{-}\ 16\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ 2\ \boxed{\times}\ \mathbf{2}\ \boxed{x^2}\ \boxed{+}\ 8\ \boxed{\times}\ \mathbf{2}\ \boxed{)}\ \boxed{=}$ $\quad \boxed{-0.5}$

For $\dfrac{x - 4}{2x}$:

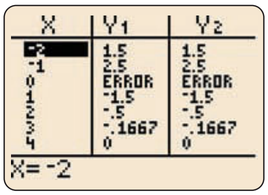
$\boxed{(}\ \mathbf{2}\ \boxed{-}\ 4\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ 2\ \boxed{\times}\ \mathbf{2}\ \boxed{)}\ \boxed{=}$ $\quad \boxed{-0.5}$

The results of the evaluations are indeed the same. Evaluate the expressions for several other values of x. If the results differ for any given value, the original expression was not simplified correctly.

We can also use a graphing calculator to show that the simplification in Example 4b is correct. We enter the functions $f(x) = \frac{x^2 - 16}{2x^2 + 8x}$ and $g(x) = \frac{x - 4}{2x}$ as Y_1 and Y_2, respectively. See figure (a). Then select the TABLE feature. Reading across the table, the values of Y_1 and Y_2 should be the same for each value of x as shown in figure (b). Note for $x = -4$ the Y_1 value says error while the Y_2 value is 1. This happens as a result of removing the common factor $x + 4$ in the simplification.

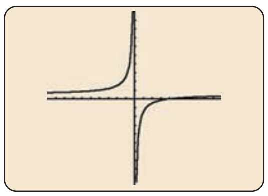


(a)

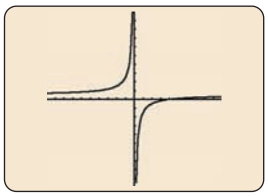


(b)

A third method to informally check the simplification is to compare the graphs of $f(x) = \frac{x^2 - 16}{2x^2 + 8x}$, shown in figure (c), and $g(x) = \frac{x - 4}{2x}$, shown in figure (d). Since the graphs appear to be the same, we can conclude that the simplification is probably correct.



(c)



(d)

EXAMPLE 5 Simplify:

a. $\dfrac{x^2 - 10x + 25}{8x - 40}$ **b.** $\dfrac{6a^2 - 13a + 6}{3a^2 + a - 2}$ **c.** $\dfrac{y^3 - 8}{y^3 - 2y^2 + 3y - 6}$

Strategy We will begin by factoring the numerator and denominator completely. Then we will remove any factors common to the numerator and denominator.

WHY We need to make sure that the numerator and denominator have no common factors other than 1. If that is the case, then the rational expression is simplified.

Solution

a. We factor the perfect-square trinomial in the numerator. In the denominator, we factor out the GCF, 8. Then we remove the common factor, $x - 5$.

$$\frac{x^2 - 10x + 25}{8x - 40} = \frac{\overset{1}{\cancel{(x - 5)}}(x - 5)}{8\underset{1}{\cancel{(x - 5)}}}$$ Remove a factor equal to 1: $\frac{x-5}{x-5} = 1$.

$$= \frac{x - 5}{8}$$

b. We factor the trinomials in the numerator and the denominator and then remove the common factor, $3a - 2$.

$$\frac{6a^2 - 13a + 6}{3a^2 + a - 2} = \frac{\overset{1}{\cancel{(3a - 2)}}(2a - 3)}{\underset{1}{\cancel{(3a - 2)}}(a + 1)}$$ Remove a factor equal to 1: $\frac{3a-2}{3a-2} = 1$.

$$= \frac{2a - 3}{a + 1}$$ This expression does not simplify further.

c. $$\frac{y^3 - 8}{y^3 - 2y^2 + 3y - 6} = \frac{(y - 2)(y^2 + 2y + 4)}{y^2(y - 2) + 3(y - 2)}$$ In the numerator, factor the sum of two cubes. In the denominator, begin the process of factoring by grouping.

$$= \frac{(y - 2)(y^2 + 2y + 4)}{(y - 2)(y^2 + 3)}$$ In the denominator, complete the factoring by grouping.

$$= \frac{\overset{1}{\cancel{(y - 2)}}(y^2 + 2y + 4)}{\underset{1}{\cancel{(y - 2)}}(y^2 + 3)}$$ Remove the factor common to the numerator and denominator: $\frac{y-2}{y-2} = 1$.

$$= \frac{y^2 + 2y + 4}{y^2 + 3}$$ This expression does not simplify further.

Self Check 5

Simplify:

a. $\dfrac{x^2 - 6x + 9}{6x - 18}$ $\dfrac{x-3}{6}$

b. $\dfrac{2b^2 + 7b - 15}{2b^2 + 13b + 15}$ $\dfrac{2b-3}{2b+3}$

c. $\dfrac{a^3 - 1}{a^3 - a^2 + 6a - 6}$ $\dfrac{a^2+a+1}{a^2+6}$

Now Try **Problems 43, 47, and 51**

Teaching Example 5 Simplify:

a. $\dfrac{x^2 - 6x + 9}{5x - 15}$ **b.** $\dfrac{3x^2 + 2x - 8}{3x^2 - x - 4}$

c. $\dfrac{8x^3 + 27}{2x^2 + x - 3}$

Answers:

a. $\dfrac{x - 3}{5}$ **b.** $\dfrac{x + 2}{x + 1}$ **c.** $\dfrac{4x^2 - 6x + 9}{x - 1}$

Sometimes we will encounter rational expressions that are already in simplified form. For example, to attempt to simplify

$$\frac{x^2 + xa + 2x + 2a}{x^2 + x - 6}$$

we factor the numerator and denominator:

$$\frac{x^2 + xa + 2x + 2a}{x^2 + x - 6} = \frac{x(x + a) + 2(x + a)}{(x - 2)(x + 3)}$$ In the numerator, begin the process of factoring by grouping. In the denominator, factor the trinomial.

$$= \frac{(x + a)(x + 2)}{(x - 2)(x + 3)}$$ In the numerator, complete the factoring by grouping.

Since there are no common factors in the numerator and denominator, the rational expression is in *lowest terms.* It cannot be simplified.

6 Simplify rational expressions that have factors that are opposites.

If the terms of two polynomials are the same, except that they are opposite in sign, the polynomials are *opposites.* For example, $b - a$ and $a - b$ are opposites.

To simplify $\frac{b-a}{a-b}$, the quotient of opposites, we factor -1 from the numerator and remove any factors common to both the numerator and the denominator:

$$\frac{b-a}{a-b} = \frac{-a+b}{a-b} \quad \text{Rewrite the numerator.}$$

$$= \frac{-\overset{1}{\cancel{(a-b)}}}{\underset{1}{\cancel{(a-b)}}} \quad \text{Factor out } -1 \text{ from each term in the numerator and remove the common factor } a-b.$$

$$= \frac{-1}{1}$$

$$= -1$$

In general, we have the following principle.

The Quotient of Opposites

The quotient of any nonzero polynomial and its opposite is -1.

Self Check 6

Simplify: $\dfrac{2a^2 - 3ab - 9b^2}{3b^2 - ab}$

Now Try **Problem 55**

Self Check 6 Answer

$-\dfrac{2a+3b}{b}$ or $\dfrac{-2a-3b}{b}$

Teaching Example 6 Simplify:
$\dfrac{x^2 - 13xy + 12y^2}{144y^2 - x^2}$
Answer:
$-\dfrac{x-y}{x+12y}$

EXAMPLE 6

Simplify: $\dfrac{3x^2 - 10xy - 8y^2}{4y^2 - xy}$

Strategy We will begin by factoring the numerator and denominator. Then we look for common factors, or factors that are opposites, and remove them.

WHY We need to make sure that the numerator and denominator have no common factors other than 1. If that is the case, then the rational expression is simplified.

Solution

We factor the numerator and denominator. Because $x - 4y$ and $4y - x$ are opposites, their quotient is -1.

$$\frac{3x^2 - 10xy - 8y^2}{4y^2 - xy} = \frac{(3x+2y)\overset{-1}{\cancel{(x-4y)}}}{y\underset{1}{\cancel{(4y-x)}}} \quad \text{Since } x-4y \text{ and } 4y-x \text{ are opposites, simplify by replacing } \tfrac{x-4y}{4y-x} \text{ with the equivalent fraction } \tfrac{-1}{1} = -1.$$

$$= \frac{-(3x+2y)}{y}$$

This result can also be written as $-\dfrac{3x+2y}{y}$ or $\dfrac{-3x-2y}{y}$.

Caution! A $-$ symbol preceding a fraction may be applied to the numerator or to the denominator, but not to both. For example,

$$-\frac{3x+2y}{y} \neq \frac{-3x-2y}{-y}$$

ANSWERS TO SELF CHECKS

1. a. \$3.50 **b.** \$1.56 **2.** The domain is the set of all real numbers except -7 and 7: $(-\infty, -7) \cup (-7, 7) \cup (7, \infty)$. **3.** $\frac{3a^3}{5b^2}$ **4. a.** $\frac{4}{x-2}$ **b.** $\frac{x+3}{5x}$ **5. a.** $\frac{x-3}{6}$ **b.** $\frac{2b-3}{2b+3}$ **c.** $\frac{a^2+a+1}{a^2+6}$ **6.** $-\frac{2a+3b}{b}$ or $\frac{-2a-3b}{b}$

SECTION 6.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. A quotient of two polynomials, such as $\frac{x^2+x}{x^2-3x}$, is called a _rational_ expression.
2. In the rational expression $\frac{(x+2)(3x-1)}{(x+2)(4x+2)}$, $x+2$ is a common _factor_ of the numerator and the denominator.
3. To _simplify_ a rational expression, we remove factors common to the numerator and denominator.
4. Because of the division by 0, the expression $\frac{8}{0}$ is _undefined_.
5. The binomials $x - 15$ and $15 - x$ are called _opposites_, because their terms are the same, except that they are opposite in sign.
6. The graph of the function shown in Problem 7 below approaches the positive x-axis. When a graph approaches a line, we call the line an _asymptote_.

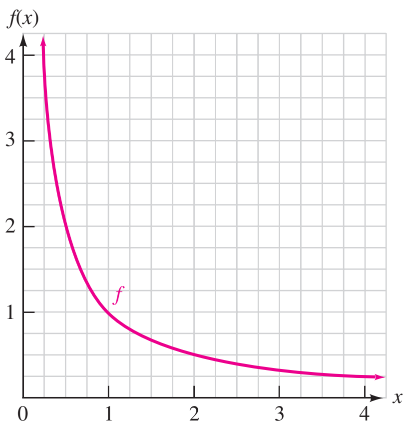
CONCEPTS

7. The graph of rational function f for $x > 0$ is shown in the illustration. Find each of the following.

   a. $f(1)$ 1 **b.** $f(2)$ 0.5

   c. The value(s) of x for which $f(x) = 4$ 0.25

   d. The domain and range of f D: $(0, \infty)$, R: $(0, \infty)$



8. Fill in the blanks to show that $\frac{x-y}{y-x} = -1$ by factoring out -1 from each term in the numerator.

$$\frac{x-y}{y-x} = \frac{-y+x}{y-x} = \frac{-1\overset{1}{\cancel{(y-x)}}}{\underset{1}{\cancel{(y-x)}}} = -1$$

9. Simplify each expression.

   a. $\frac{3 \cdot 5 \cdot x \cdot y \cdot y}{5 \cdot 7 \cdot x \cdot x \cdot x \cdot y}$ $\frac{3y}{7x^2}$ **b.** $\frac{(x+8)(x-3)}{(x+2)(x+8)}$ $\frac{x-3}{x+2}$

   c. $\frac{a^3(a-9)}{(9-a)(9+a)}$ $-\frac{a^3}{a+9}$

10. Simplify each rational expression, if possible.

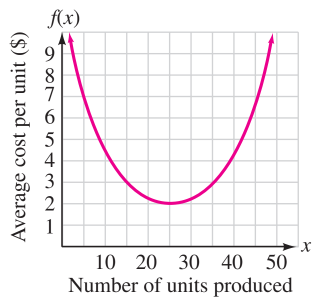
    a. $\frac{x+8}{x}$ does not simplify **b.** $\frac{3a^2+23}{a^2}$ does not simplify

11. MANUFACTURING Each graph shows the average cost to manufacture a certain item for a given number of units produced. Which graph is best described as the graph of a:

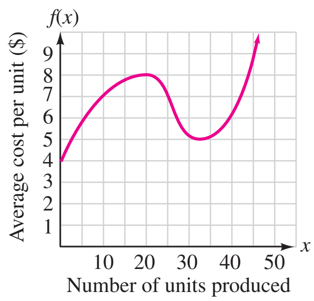
    a. linear function? iii. **b.** quadratic function? i.

    c. rational function? iv. **d.** polynomial function? ii.

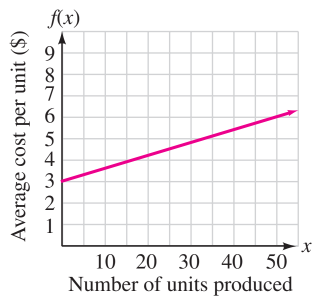
i.



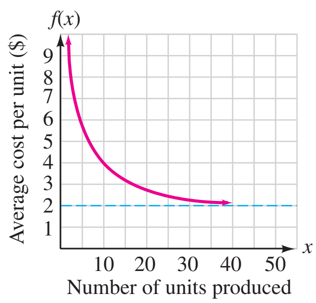
ii.



iii.



iv.



12. Refer to the graphs in Problem 11. Complete the description of the graph by filling in each blank with the word *decreases* or *increases*.

a. Graph **i.** decreases, then steadily increases.

b. Graph **ii.** increases, then decreases, and then steadily increases.

c. Graph **iii.** steadily increases.

d. Graph **iv.** steadily decreases approaching a cost of \$2.00 per unit.

NOTATION

▶ **13.** A student checks his answers with those in the back of his textbook. Determine whether they are equivalent.

Answer	Book's answer	Equivalent?
$\frac{-3}{x+3}$	$-\frac{3}{x+3}$	yes
$\frac{-x+4}{6x+1}$	$\frac{-(x-4)}{6x+1}$	yes
$\frac{x+7}{(x-4)(x+2)}$	$\frac{x+7}{(x+2)(x-4)}$	yes
$-\frac{x-4}{x+4}$	$\frac{4-x}{x+4}$	yes
$\frac{a-3b}{2b-a}$	$\frac{3b-a}{a-2b}$	yes

14. a. In $\frac{(x+5)\overset{1}{\cancel{(x-5)}}}{x\underset{1}{\cancel{(x-5)}}}$, what do the slashes show?

removing a common factor of the numerator and denominator

b. In $\frac{(x-3)\overset{-1}{\cancel{(x-7)}}}{(x+3)\underset{1}{\cancel{(7-x)}}}$, what do the slashes show?

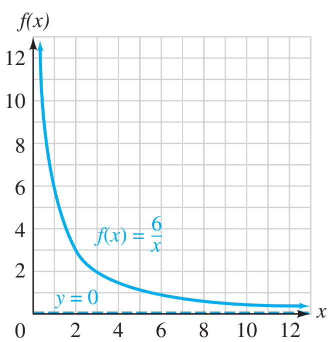
removing factors that are opposites in the numerator and denominator

GUIDED PRACTICE

Complete the table of values for each rational function (round to the nearest hundredth when appropriate). Then graph the function. Each function is defined for $x > 0$. Label the horizontal asymptote. See Example 1 and Objective 3.

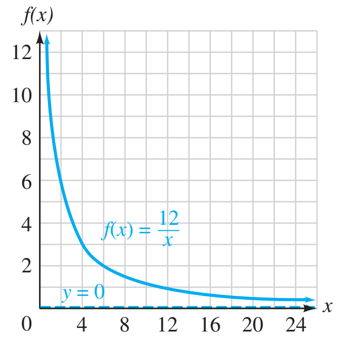
▶ **15.** $f(x) = \frac{6}{x}$

x	$f(x)$
1	6
2	3
4	1.5
6	1
8	0.75
10	0.6
12	0.5



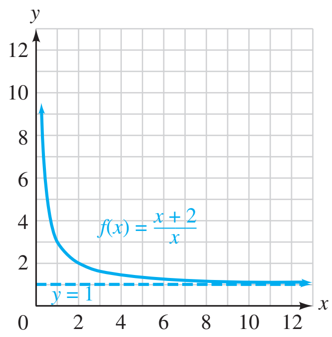
16. $f(x) = \frac{12}{x}$

x	$f(x)$
1	12
4	3
8	1.5
12	1
16	0.75
20	0.6
24	0.5



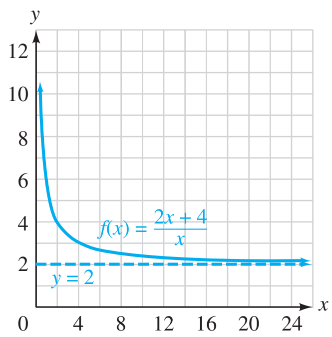
17. $f(x) = \frac{x+2}{x}$

x	$f(x)$
1	3
2	2
4	1.5
6	1.33
8	1.25
10	1.2
12	1.17



18. $f(x) = \frac{2x+4}{x}$

x	$f(x)$
1	6
4	3
8	2.5
12	2.33
16	2.25
20	2.2
24	2.17



Find the domain of each rational function. Express your answer in words and using interval notation. See Example 2.

19. $f(x) = \frac{2}{x}$

all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$

▶ **20.** $f(x) = \frac{8}{x-1}$

all real numbers except 1, $(-\infty, 1) \cup (1, \infty)$

21. $f(x) = \frac{2x}{x+2}$

all real numbers except -2, $(-\infty, -2) \cup (-2, \infty)$

▶ **22.** $f(x) = \frac{2x+1}{x^2-2x}$

all real numbers except 0 and 2, $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

23. $f(x) = \dfrac{3x - 1}{x - x^2}$
all real numbers except 0 and 1, $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

24. $f(x) = \dfrac{x^2 + 36}{x^2 - 36}$
all real numbers except -6 and 6, $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

25. $f(x) = \dfrac{x^2 + 3x + 2}{x^2 - x - 56}$
all real numbers except -7 and 8, $(-\infty, -7) \cup (-7, 8) \cup (8, \infty)$

▶ **26.** $f(x) = \dfrac{2x^2 - 3x - 2}{x^2 + 2x - 24}$
all real numbers except -6 and 4, $(-\infty, -6) \cup (-6, 4) \cup (4, \infty)$

Simplify each rational expression. See Example 3.

27. $\dfrac{12a^3}{18a}$ $\dfrac{2a^2}{3}$

28. $\dfrac{25b^4}{55b}$ $\dfrac{5b^3}{11}$

29. $\dfrac{15a^2}{25a^8}$ $\dfrac{3}{5a^6}$

30. $\dfrac{12x}{16x^7}$ $\dfrac{3}{4x^6}$

31. $\dfrac{27st}{36st^2}$ $\dfrac{3}{4t}$

32. $\dfrac{49xy^2}{21xy}$ $\dfrac{7y}{3}$

33. $\dfrac{24x^3y^{10}}{18x^4y^3}$ $\dfrac{4y^7}{3x}$

▶ **34.** $\dfrac{15a^5b^4}{21a^8b^3}$ $\dfrac{5b}{7a^3}$

Simplify each rational expression. See Example 4.

35. $\dfrac{4x^2}{2x^3 - 12x^2}$ $\dfrac{2}{x - 6}$

36. $\dfrac{15y^2}{5y^3 + 15y^2}$ $\dfrac{3}{y + 3}$

▶ **37.** $\dfrac{24n^4}{16n^4 + 24n^3}$ $\dfrac{3n}{2n + 3}$

38. $\dfrac{18m^4}{36m^4 - 9m^3}$ $\dfrac{2m}{4m - 1}$

39. $\dfrac{2x + 18}{x^2 - 81}$ $\dfrac{2}{x - 9}$

40. $\dfrac{6x - 12}{x^2 - 4}$ $\dfrac{6}{x + 2}$

41. $\dfrac{4a^2 - 25}{20a - 50}$ $\dfrac{2a + 5}{10}$

42. $\dfrac{9b^2 - 16}{21b + 28}$ $\dfrac{3b - 4}{7}$

Simplify each rational expression. See Example 5.

43. $\dfrac{5x^2 - 10x}{x^2 - 4x + 4}$ $\dfrac{5x}{x - 2}$

44. $\dfrac{x^2 + 6x + 9}{2x^2 + 6x}$ $\dfrac{x + 3}{2x}$

45. $\dfrac{x^2 + 2x + 1}{x^2 + 4x + 3}$ $\dfrac{x + 1}{x + 3}$

46. $\dfrac{y^2 - 4y + 4}{y^2 - 8y + 12}$ $\dfrac{y - 2}{y - 6}$

▶ **47.** $\dfrac{3d^2 + 13d + 4}{3d^2 + 7d + 2}$ $\dfrac{d + 4}{d + 2}$

48. $\dfrac{10r^2 + 17r + 3}{2r^2 + 17r + 21}$ $\dfrac{5r + 1}{r + 7}$

49. $\dfrac{2h^2 + 9h - 5}{4h^2 - 4h + 1}$ $\dfrac{h + 5}{2h - 1}$

50. $\dfrac{6x^2 + x - 2}{8x^2 + 2x - 3}$ $\dfrac{3x + 2}{4x + 3}$

51. $\dfrac{t^3 + 27}{t^3 + 3t^2 + 4t + 12}$ $\dfrac{t^2 - 3t + 9}{t^2 + 4}$

52. $\dfrac{m^3 + 64}{m^3 + 4m^2 + 3m + 12}$ $\dfrac{m^2 - 4m + 16}{m^2 + 3}$

53. $\dfrac{s^3 + s^2 - 6s - 6}{s^3 + 1}$ $\dfrac{s^2 - 6}{s^2 - s + 1}$

▶ **54.** $\dfrac{d^3 + 5d^2 - 5d - 25}{d^3 + 125}$ $\dfrac{d^2 - 5}{d^2 - 5d + 25}$

Simplify each rational expression. See Example 6.

55. $\dfrac{3m^2 - 2mn - n^2}{mn - m^2}$ $-\dfrac{3m + n}{m}$ or $\dfrac{-(3m + n)}{m}$

56. $\dfrac{5s^2 - 4st - t^2}{st - s^2}$ $-\dfrac{5s + t}{s}$ or $\dfrac{-(5s + t)}{s}$

57. $\dfrac{b^2 - a^2}{a - b}$ $-b - a$

58. $\dfrac{d^2 - 16c^2}{4c - d}$ $-d - 4c$

59. $\dfrac{4 - x^2}{x^2 - x - 2}$ $-\dfrac{x + 2}{x + 1}$ or $\dfrac{-(2 + x)}{x + 1}$

▶ **60.** $\dfrac{x^2 - 2x - 15}{25 - x^2}$ $-\dfrac{x + 3}{x + 5}$ or $\dfrac{-(x + 3)}{5 + x}$

61. $\dfrac{20x^3 - 20x^4}{x^2 - 2x + 1}$ $-\dfrac{20x^3}{x - 1}$

▶ **62.** $\dfrac{16m^5 - 2m^6}{m^2 - 16m + 64}$ $-\dfrac{2m^5}{m - 8}$

Use a graphing calculator to graph each rational function. From the graph, determine its domain and range. Answer using interval notation. See Using Your Calculator: Finding the Domain and Range of a Rational Function.

63. $f(x) = \dfrac{x}{x - 2}$
D: $(-\infty, 2) \cup (2, \infty)$,
R: $(-\infty, 1) \cup (1, \infty)$

64. $f(x) = \dfrac{x + 2}{x}$
D: $(-\infty, 0) \cup (0, \infty)$,
R: $(-\infty, 1) \cup (1, \infty)$

65. $f(x) = \dfrac{x + 1}{x^2 - 4}$
D: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$,
R: $(-\infty, \infty)$

66. $f(x) = \dfrac{x - 2}{x^2 - 3x - 4}$
D: $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$,
R: $(-\infty, \infty)$

TRY IT YOURSELF

Simplify each expression. If an expression cannot be simplified, write "Does not simplify."

▶ **67.** $\dfrac{x^2 + x - 30}{3x^2 - 3x - 60}$ $\dfrac{x + 6}{3(x + 4)}$

68. $\dfrac{4x^2 + 24x + 32}{16x^2 + 8x - 48}$ $\dfrac{x + 4}{2(2x - 3)}$

▶ **69.** $\dfrac{a^2 - 4}{a^3 - 8}$ $\dfrac{a + 2}{a^2 + 2a + 4}$

70. $\dfrac{x^3 - 27}{3x^2 - 8x - 3}$ $\dfrac{x^2 + 3x + 9}{3x + 1}$

71. $\dfrac{m^3 - mn^2}{mn^2 + m^2n - 2m^3}$ $-\dfrac{m + n}{n + 2m}$ or $\dfrac{-m - n}{n + 2m}$

72. $\dfrac{a^3 - ab^2}{ab^2 - 4a^2b + 3a^3}$ $-\dfrac{a + b}{b - 3a}$ or $\dfrac{a + b}{3a - b}$

73. $\dfrac{sx + 4s - 3x - 12}{sx + 4s + 6x + 24}$ $\dfrac{s - 3}{s + 6}$

74. $\dfrac{ax + by + ay + bx}{a + b}$ $x + y$

75. $\dfrac{2x^2 - 3x - 9}{2x^2 + 3x - 9}$ Does not simplify.

▶ **76.** $\dfrac{6x^2 - 7x - 5}{2x^2 + 5x + 2}$ $\dfrac{3x - 5}{x + 2}$

77. $\dfrac{3x + 6y}{x + 2y}$ 3

78. $\dfrac{y - xy}{xy - x}$ Does not simplify.

79. $\dfrac{x^4 + 3x^3 + 9x^2}{x^3 - 27}$ $\dfrac{x^2}{x - 3}$

80. $\dfrac{x^3 + 8}{x^4 - 2x^3 + 4x^2}$ $\dfrac{x + 2}{x^2}$

81. $\dfrac{2x^2 + 2x - 12}{x^3 + 3x^2 - 4x - 12}$ $\dfrac{2}{x + 2}$

82. $\dfrac{3x^2 - 3y^2}{x^2 + 2y + 2x + yx}$ $\dfrac{3(x - y)}{x + 2}$

83. $\dfrac{4x^2 + 8x + 3}{6 + x - 2x^2}$ $-\dfrac{2x + 1}{x - 2}$ or $\dfrac{2x + 1}{2 - x}$

84. $\dfrac{6x^2 + 13x + 6}{6 - 5x - 6x^2}$ $-\dfrac{3x + 2}{3x - 2}$ or $\dfrac{3x + 2}{2 - 3x}$

85. $\dfrac{x^2 - 6x + 9}{81 - x^4}$ $-\dfrac{x-3}{(9+x^2)(3+x)}$ or $-\dfrac{x-3}{(x^2+9)(x+3)}$

86. $\dfrac{y^2 - 2y + 1}{1 - y^4}$ $-\dfrac{y-1}{(1+y^2)(1+y)}$ or $-\dfrac{y-1}{(y^2+1)(y+1)}$

87. $\dfrac{16p^3q^2}{24pq^8}$ $\dfrac{2p^2}{3q^6}$

88. $\dfrac{30a^3b^{15}}{18a^9b^{10}}$ $\dfrac{5b^5}{3a^6}$

89. $\dfrac{t^3 - 5t^2 + 6t}{9t - t^3}$ $-\dfrac{t-2}{3+t}$ or $-\dfrac{t-2}{t+3}$

90. $\dfrac{a^4 - 27a}{36a - 4a^3}$ $-\dfrac{a^2+3a+9}{4(3+a)}$ or $-\dfrac{a^2+3a+9}{4(a+3)}$

APPLICATIONS

91. ENVIRONMENTAL CLEANUP Suppose the cost (in dollars) of removing p% of the pollution in a river is given by the rational function

$$f(p) = \frac{50{,}000p}{100 - p} \quad \text{where } 0 \le p < 100$$

Find the cost of removing each percent of pollution.

a. 50% \$50,000 **b.** 80% \$200,000

92. DIRECTORY COSTS The average (mean) cost for a service club to publish a directory of its members is given by the rational function

$$f(x) = \frac{1.25x + 700}{x}$$

where x is the number of directories printed. Find the average cost per directory if:

a. 500 directories are printed. \$2.65

b. 2,000 directories are printed. \$1.60

93. UTILITY COSTS An electric company charges \$7.50 per month plus 9¢ for each kilowatt hour (kwh) of electricity used.

a. Find a linear function that gives the total cost of n kwh of electricity. $c(n) = 0.09n + 7.50$

b. Find a rational function that gives the average cost per kwh when using n kwh. $c(n) = \dfrac{0.09n + 7.50}{n}$

c. Find the average cost per kwh when 775 kwh are used. about 10¢

94. SCHEDULING WORK CREWS The rational function

$$f(t) = \frac{t^2 + 2t}{2t + 2}$$

gives the number of days it would take two construction crews, working together, to frame a house that crew 1 (working alone) could complete in t days and crew 2 (working alone) could complete in $t + 2$ days.

a. If crew 1 could frame a certain house in 15 days, how long would it take both crews working together? almost 8 days

b. If crew 2 could frame a certain house in 20 days, how long would it take both crews working together? about 9.5 days

95. FILLING A POOL The rational function

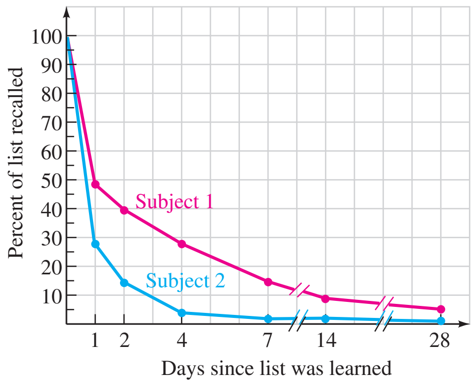
$$f(t) = \frac{t^2 + 3t}{2t + 3}$$

gives the number of hours it would take two pipes, working together, to fill a pool that the larger pipe (working alone) could fill in t hours and the smaller pipe (working alone) could fill in $t + 3$ hours.

a. If the smaller pipe could fill a pool in 7 hours, how long would it take both pipes to fill the pool? about 2.5 hr

b. If the larger pipe could fill a pool in 8 hours, how long would it take both pipes to fill the pool? about 4.6 hr

96. RETENTION STUDY After learning a list of words, two subjects were tested over a 28-day period to see what percent of the list they remembered. In both cases, their percent recall could be modeled by rational functions, as shown in the illustration.



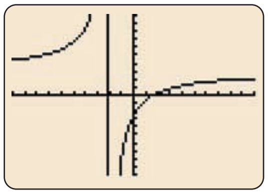
a. Use the graphs to complete the table.

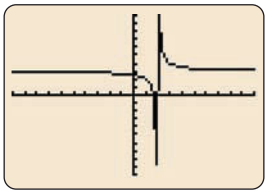
Days since learning	0	1	2	4	7	14	28
% recall—subject 1	100	48	40	28	15	9	5
% recall—subject 2	100	27	15	4	2	2	1

b. After 28 days, which subject had the better recall? subject 1

WRITING

97. A student simplified $\frac{6x^2 - 7x - 5}{2x^2 + 5x + 2}$ and obtained $\frac{3x - 5}{x - 2}$. As a check, she graphed $Y_1 = \frac{6x^2 - 7x - 5}{2x^2 + 5x + 2}$ and $Y_2 = \frac{3x - 5}{x - 2}$. What conclusion can be drawn from the graphs? Explain your answer.





98. Simplify: $\frac{6x^2 + x - 2}{8x^2 + 2x - 3}$. Then explain how the table of values for $Y_1 = \frac{6x^2 + x - 2}{8x^2 + 2x - 3}$ and $Y_2 = \frac{3x + 2}{4x + 3}$ shown in the illustration in the next column can be used to check your result.

X	Y1	Y2
-2	.8	.8
-1	1	1
0	.66667	.66667
1	.71429	.71429
2	.72727	.72727
3	.73333	.73333
4	.73684	.73684

X= -2

REVIEW

Perform each operation.

99. $(a^2 - 4a - 3)(a - 2)$ $a^3 - 6a^2 + 5a + 6$

100. $(3c^2 + 5c) + (7 - c^2 - 5c)$ $2c^2 + 7$

101. $-3mn^2(m^3 - 7mn - 2m^2)$ $-3m^4n^2 + 21m^2n^3 + 6m^3n^2$

102. $(4u^2 + z^2 - 3u^2z^2) - (u^3 + 3z^2 - 3u^2z^2)$ $-u^3 + 4u^2 - 2z^2$

SECTION 6.2
Multiplying and Dividing Rational Expressions

Objectives

1. Multiply rational expressions.
2. Find powers of rational expressions.
3. Divide rational expressions.
4. Perform mixed operations.

In this section, we review the rules for multiplying and dividing arithmetic fractions—fractions whose numerators and denominators are integers. Then we use these rules, in combination with the simplification skills learned in Section 6.1, to multiply and divide rational expressions.

1 Multiply rational expressions.

Recall that to multiply fractions, we multiply the numerators and multiply the denominators. For example,

$$\frac{3}{5} \cdot \frac{2}{7} = \frac{3 \cdot 2}{5 \cdot 7} = \frac{6}{35}$$

$$\frac{4}{7} \cdot \frac{5}{8} = \frac{4 \cdot 5}{7 \cdot 8} = \frac{\overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{2}} \cdot 5}{7 \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{2}} \cdot 2} = \frac{5}{14}$$

Factor 4 as $2 \cdot 2$. Factor 8 as $2 \cdot 2 \cdot 2$. Then simplify.

We use the same procedure to multiply rational expressions.

Multiplying Rational Expressions

To multiply rational expressions, multiply their numerators and their denominators. Then, if possible, factor and simplify.

For any two rational expressions, $\frac{A}{B}$ and $\frac{C}{D}$,

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

Self Check 1

Multiply: $\dfrac{x^7}{16y} \cdot \dfrac{24y}{17x^3}$ $\dfrac{3x^4}{34}$

Now Try **Problem 17**

Teaching Example 1 Multiply: $\dfrac{12x^2}{9y} \cdot \dfrac{y^3}{x^5}$
Answer: $\dfrac{4y^2}{3x^3}$

EXAMPLE 1

Multiply: $\dfrac{25a^3}{11b} \cdot \dfrac{b}{5a}$

Strategy To find the product, we will use the rule for multiplying rational expressions. In the process, we must be prepared to factor the numerators and denominators so that any common factors can be removed.

WHY We want to give the result in simplified form.

Solution

$$\frac{25a^3}{11b} \cdot \frac{b}{5a} = \frac{25a^3 \cdot b}{11b \cdot 5a}$$ Multiply the numerators. Multiply the denominators.

It is obvious that the numerator and denominator of $\frac{25a^3 \cdot b}{11b \cdot 5a}$ have several common factors, such as 5, a, and b. These common factors become more apparent when we factor the numerator and denominator completely.

$$\frac{25a^3 \cdot b}{11b \cdot 5a} = \frac{5 \cdot 5 \cdot a \cdot a \cdot a \cdot b}{11 \cdot b \cdot 5 \cdot a}$$ Factor $25a^3$.

$$= \frac{\overset{1}{\cancel{5}} \cdot 5 \cdot \overset{1}{\cancel{a}} \cdot a \cdot a \cdot \overset{1}{\cancel{b}}}{11 \cdot \underset{1}{\cancel{b}} \cdot \underset{1}{\cancel{5}} \cdot \underset{1}{\cancel{a}}}$$ Simplify by replacing $\frac{5}{5}$, $\frac{a}{a}$, and $\frac{b}{b}$ with the equivalent fraction $\frac{1}{1}$. This removes the factor $\frac{5 \cdot a \cdot b}{5 \cdot a \cdot b} = 1$.

$$= \frac{5a^2}{11}$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

Success Tip We could also use rules for exponents to simplify the product:

$$\frac{25a^3 \cdot b}{11b \cdot 5a} = \frac{\overset{1}{\cancel{5}} \cdot 5 \cdot a^{3-1} \cdot b^{1-1}}{11 \cdot \underset{1}{\cancel{5}}}$$
$$= \frac{5a^2b^0}{11}$$
$$= \frac{5a^2}{11}$$

Self Check 2

Multiply:

a. $\dfrac{a^2 + 6a + 9}{18a} \cdot \dfrac{3a^3}{7a + 21}$

b. $\dfrac{a^2 + a - 56}{a^2 - 49} \cdot \dfrac{a^2 - a - 56}{a^2 - 64}$

Now Try **Problems 19 and 23**

Self Check 2 Answers
a. $\dfrac{a^2(a + 3)}{42}$ **b.** 1

EXAMPLE 2

Multiply:

a. $\dfrac{x^2 - 6x + 9}{20x} \cdot \dfrac{5x^2}{6x - 18}$ **b.** $\dfrac{x^2 - x - 6}{x^2 - 4} \cdot \dfrac{x^2 + x - 6}{x^2 - 9}$

Strategy To find the product, we will use the rule for multiplying rational expressions. In the process, we must be prepared to factor the numerators and denominators so that any common factors can be removed.

WHY We want to give the result in simplified form.

Solution

a. $$\frac{x^2 - 6x + 9}{20x} \cdot \frac{5x^2}{6x - 18} = \frac{(x^2 - 6x + 9)5x^2}{20x(6x - 18)}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{(x - 3)(x - 3)5xx}{4 \cdot 5 \cdot x \cdot 6(x - 3)}$$ Factor the numerator. Factor the denominator.

$$= \frac{\cancel{(x-3)}(x-3)\cancel{5}\cancel{x}x}{4\cdot\cancel{5}\cdot\cancel{x}\cdot 6\cancel{(x-3)}}$$

Simplify by removing common factors of the numerator and denominator.

$$= \frac{x(x-3)}{24}$$

Multiply the remaining monomial factors in the numerator. Multiply the remaining factors in the denominator.

b. $$\frac{x^2-x-6}{x^2-4}\cdot\frac{x^2+x-6}{x^2-9} = \frac{(x^2-x-6)(x^2+x-6)}{(x^2-4)(x^2-9)}$$

Multiply the numerators. Multiply the denominators.

$$= \frac{(x-3)(x+2)(x+3)(x-2)}{(x+2)(x-2)(x+3)(x-3)}$$

Factor the polynomials.

$$= \frac{\cancel{(x-3)}\cancel{(x+2)}\cancel{(x+3)}\cancel{(x-2)}}{\cancel{(x+2)}\cancel{(x-2)}\cancel{(x+3)}\cancel{(x-3)}}$$

Simplify by removing common factors of the numerator and denominator.

$$= 1$$

Teaching Example 2 Multiply:

a. $\frac{a^2+6a+9}{a}\cdot\frac{a^3}{a+3}$

b. $\frac{x^2-2x-15}{x^2-25}\cdot\frac{x^2+9x+20}{x^2+7x+12}$

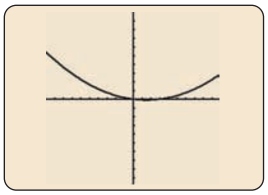
Answers:

a. $a^2(a+3)$ **b.** 1

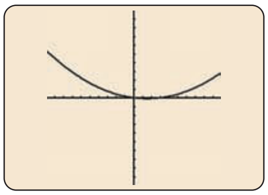
Caution! Note that when all of the factors of the numerator and denominator are removed as shown, the result is 1 and not 0.

Using Your CALCULATOR Checking an Algebraic Simplification

We can check the simplification in Example 2a by graphing the functions $f(x) = \left(\frac{x^2-6x+9}{20x}\right)\left(\frac{5x^2}{6x-18}\right)$, shown in figure (a), and $g(x) = \frac{x(x-3)}{24}$, shown in figure (b), and observing that the graphs are the same, except that 0 and 3 are not included in the domain of the function f.



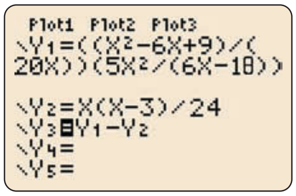
(a)



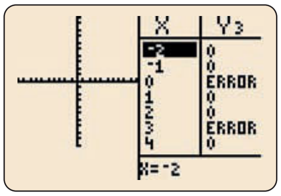
(b)

We can use the split-screen G-T (graph, table) mode to check the result of a multiplication. To set the split-screen feature on a graphing calculator, press $\boxed{\text{MODE}}$, press ▼ seven times, press ▶ twice, then press $\boxed{\text{ENTER}}$. If we enter $Y_3 = Y_1 - Y_2$, use the cursor to highlight the $=$ sign as shown on the next page in figure (c), and then press $\boxed{\text{GRAPH}}$, we get the display shown in figure (d). The zeros under the Y_3 column indicate that the value of $\left(\frac{x^2-6x+9}{20x}\right)\left(\frac{5x^2}{6x-18}\right)$ and the value of $\frac{x(x-3)}{24}$ are the same for different values of x. (The error message is given because when $x = 0$ and $x = 3$, $\left(\frac{x^2-6x+9}{20x}\right)\left(\frac{5x^2}{6x-18}\right)$ is undefined.)

The graph of $Y_3 = Y_1 - Y_2$ is difficult to see because it lies on the x-axis. The graph indicates that for all x-values (except those that make the rational expressions undefined), $Y_3 = 0$, or more specifically, $\left(\frac{x^2-6x+9}{20x}\right)\left(\frac{5x^2}{6x-18}\right) = \frac{x(x-3)}{24}$.



(c)



(d)

Self Check 3

Multiply:

$$\frac{2a^2 + 5ab - 12b^2}{2a^2 + 11ab + 12b^2} \cdot \frac{2a^2 - 3ab - 9b^2}{2a^2 - ab - 3b^2}$$

Now Try **Problems 29 and 31**

Self Check 3 Answer

$$\frac{a - 3b}{a + b}$$

Teaching Example 3 Multiply:

$$\frac{2x^2 + 5xy - 3y^2}{2x^2 + 7xy + 3y^2} \cdot \frac{2x^2 - 9xy - 5y^2}{2x^2 + xy - y^2}$$

Answer:

$$\frac{x - 5y}{x + y}$$

EXAMPLE 3

Multiply: $\dfrac{6x^2 + 5xy - 4y^2}{2x^2 + 5xy + 3y^2} \cdot \dfrac{8x^2 + 6xy - 9y^2}{12x^2 + 7xy - 12y^2}$

Strategy To find the product, we will use the rule for multiplying rational expressions. In the process, we must be prepared to factor the numerators and denominators so that any common factors can be removed.

WHY We want to give the result in simplified form.

Solution

$$\frac{6x^2 + 5xy - 4y^2}{2x^2 + 5xy + 3y^2} \cdot \frac{8x^2 + 6xy - 9y^2}{12x^2 + 7xy - 12y^2}$$

$$= \frac{(6x^2 + 5xy - 4y^2)(8x^2 + 6xy - 9y^2)}{(2x^2 + 5xy + 3y^2)(12x^2 + 7xy - 12y^2)}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{(3x + 4y)(2x - y)(4x - 3y)(2x + 3y)}{(2x + 3y)(x + y)(3x + 4y)(4x - 3y)}$$ Factor the trinomials.

$$= \frac{\overset{1}{\cancel{(3x + 4y)}}(2x - y)\overset{1}{\cancel{(4x - 3y)}}\overset{1}{\cancel{(2x + 3y)}}}{\underset{1}{\cancel{(2x + 3y)}}(x + y)\underset{1}{\cancel{(3x + 4y)}}\underset{1}{\cancel{(4x - 3y)}}}$$ Simplify by removing common factors of the numerator and denominator.

$$= \frac{2x - y}{x + y}$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

Self Check 4

Multiply:

$$\frac{x}{8x^3 - 32x^2 + 8x - 32} \cdot (4x - x^2)$$

Now Try **Problem 39**

Self Check 4 Answer

$$-\frac{x^2}{8(x^2 + 1)}$$

Teaching Example 4 Multiply:

$$(5x - x^2) \cdot \frac{x}{x^2 - 25}$$

Answer:

$$-\frac{x^2}{x + 5}$$

EXAMPLE 4

Multiply: $(2x - x^2) \cdot \dfrac{x}{5x^3 - 10x^2 + 20x - 40}$

Strategy We will write $2x - x^2$ as a rational expression with denominator 1. (Remember, any number divided by 1 remains unchanged.) Then we will use the rule for multiplying rational expressions.

WHY Writing $2x - x^2$ as $\frac{2x - x^2}{1}$ is helpful during the multiplication process when we multiply numerators and multiply denominators.

Solution

$$(2x - x^2) \cdot \frac{x}{5x^3 - 10x^2 + 20x - 40}$$

$$= \frac{2x - x^2}{1} \cdot \frac{x}{5x^3 - 10x^2 + 20x - 40}$$ Write $2x - x^2$ as $\frac{2x - x^2}{1}$.

$$= \frac{(2x - x^2)x}{1(5x^3 - 10x + 20x - 40)}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{x(2 - x)x}{1 \cdot 5(x^3 - 2x^2 + 4x - 8)}$$ Factor out x in the numerator. Factor out 5 in the denominator.

$$= \frac{x(2-x)x}{1 \cdot 5[x^2(x-2) + 4(x-2)]}$$ In the denominator, begin factoring by grouping.

$$= \frac{x(2-x)x}{1 \cdot 5(x-2)(x^2+4)}$$ In the denominator, complete the factoring by grouping. The brackets [ ] are no longer needed.

$$= \frac{x(\overset{-1}{\cancel{2-x}})x}{1 \cdot 5(\underset{1}{\cancel{x-2}})(x^2+4)}$$ Simplify. Recall that the quotient of any nonzero quantity and its opposite is −1: $\frac{2-x}{x-2} = -1$.

$$= \frac{-x^2}{5(x^2+4)}$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

Since the − sign can be written in front of the fraction, this result can be expressed as

$$-\frac{x^2}{5(x^2+4)}$$

Success Tip In Examples 2–4, we would obtain the same answer if we had factored the numerators and denominators first and simplified before we multiplied.

2 Find powers of rational expressions.

EXAMPLE 5 Find: $\left(\frac{x^2+x-1}{2x+3}\right)^2$

Strategy We will find the product $\left(\frac{x^2+x-1}{2x+3}\right)\left(\frac{x^2+x-1}{2x+3}\right)$ using the rule for multiplying rational expressions.

WHY The exponent 2 means the base, $\frac{x^2+x-1}{2x+3}$, should be written as a factor two times.

Solution

$$\left(\frac{x^2+x-1}{2x+3}\right)^2 = \left(\frac{x^2+x-1}{2x+3}\right)\left(\frac{x^2+x-1}{2x+3}\right)$$

$$= \frac{(x^2+x-1)(x^2+x-1)}{(2x+3)(2x+3)}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{x^4+2x^3-x^2-2x+1}{4x^2+12x+9}$$

Self Check 5

Find: $\left(\frac{x+5}{x^2-6x}\right)^2$

Now Try Problem 43

Self Check 5 Answer

$\frac{x^2+10x+25}{x^4-12x^3+36x^2}$

Teaching Example 5 Find: $\left(\frac{3x+y}{x-4}\right)^2$

Answer: $\frac{9x^2+6xy+y^2}{x^2-8x+16}$

3 Divide rational expressions.

Recall that one number is called the **reciprocal** of another if their product is 1. To find the reciprocal of a fraction, we invert its numerator and denominator. We have seen that to divide fractions, we multiply the first fraction by the reciprocal of the second fraction.

$$\frac{3}{5} \div \frac{8}{9} = \frac{3}{5} \cdot \frac{9}{8} \qquad \frac{4}{7} \div \frac{2}{21} = \frac{4}{7} \cdot \frac{21}{2}$$

$$= \frac{3 \cdot 9}{5 \cdot 8} \qquad = \frac{4 \cdot 21}{7 \cdot 2}$$

$$= \frac{27}{40} \qquad = \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot 3 \cdot \overset{1}{\cancel{7}}}{\underset{1}{\cancel{7}} \cdot \underset{1}{\cancel{2}}}$$

Factor 4 as 2 · 2. Factor 21 as 3 · 7. Then simplify.

$$= 6$$

We use the same procedure to divide rational expressions.

Dividing Rational Expressions

To divide two rational expressions, multiply the first by the reciprocal of the second. Then, if possible, factor and simplify.

For any two rational expressions, $\frac{A}{B}$ and $\frac{C}{D}$, where $\frac{C}{D} \neq 0$,

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}$$

Self Check 6

Divide: $\frac{8a^4}{t^5} \div \frac{28a^3}{t^2}$ $\frac{2a}{7t^3}$

Now Try **Problem 51**

Teaching Example 6 Divide: $\frac{15x}{y^2z^3} \div \frac{25x}{y^3z}$
Answer:
$\frac{3y}{5z^2}$

EXAMPLE 6

Divide: $\frac{6}{y^3z^2} \div \frac{20}{yz^3}$

Strategy We will use the rule for dividing rational expressions. After multiplying by the reciprocal, we will use rules for exponents to simplify the result.

WHY We want to give the result in simplified form.

Solution

$$\frac{6}{y^3z^2} \div \frac{20}{yz^3} = \frac{6}{y^3z^2} \cdot \frac{yz^3}{20}$$ Multiply the first rational expression by the reciprocal of the second.

$$= \frac{6yz^3}{20y^3z^2}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{2 \cdot 3 \cdot y^{1-3}z^{3-2}}{2 \cdot 2 \cdot 5}$$ Factor 6 and 20. To divide exponential expressions with the same base, keep the base and subtract the exponents.

$$= \frac{\overset{1}{\cancel{2}} \cdot 3 \cdot y^{-2}z^{1}}{\underset{1}{\cancel{2}} \cdot 2 \cdot 5}$$ Remove the common factor of 2. Simplify the exponents.

$$= \frac{3z}{10y^2}$$ Write the result without the negative exponent.

Self Check 7

Divide: $\frac{x^3 - 8}{9x - 9} \div \frac{x^2 + 2x + 4}{3x^2 - 3x}$

Now Try **Problem 55**

Self Check 7 Answer
$\frac{x(x - 2)}{3}$

Teaching Example 7 Divide: $\frac{a^3 - 125}{2a - 10} \div \frac{a^2 + 5a + 25}{12}$
Answer:
6

EXAMPLE 7

Divide: $\frac{x^3 + 8}{4x + 4} \div \frac{x^2 - 2x + 4}{2x^2 - 2}$

Strategy To find the quotient, we will use the rule for dividing rational expressions. After multiplying by the reciprocal, we will factor each polynomial that is not prime and remove any common factors of the numerator and denominator.

WHY We want to give the result in simplified form.

Solution

$$\frac{x^3 + 8}{4x + 4} \div \frac{x^2 - 2x + 4}{2x^2 - 2}$$

$$= \frac{x^3 + 8}{4x + 4} \cdot \frac{2x^2 - 2}{x^2 - 2x + 4}$$ Multiply the first rational expression by the reciprocal of the second.

$$= \frac{(x^3 + 8)(2x^2 - 2)}{(4x + 4)(x^2 - 2x + 4)}$$

Multiply the numerators. Multiply the denominators.

$$= \frac{(x + 2)\cancel{(x^2 - 2x + 4)}\cancel{2}\cancel{(x + 1)}(x - 1)}{2 \cdot \cancel{2}\cancel{(x + 1)}\cancel{(x^2 - 2x + 4)}}$$

Factor completely. $x^2 - 2x + 4$ is prime. Then simplify.

$$= \frac{(x + 2)(x - 1)}{2}$$

Caution! When dividing rational expressions, always write the result in simplest form by removing any factors common to the numerator and denominator.

EXAMPLE 8

Divide: $\frac{b^3 - 4b}{x - 1} \div (b - 2)$

Strategy We will begin by writing $b - 2$ as a rational expression by inserting a denominator 1. Then we will use the rule for dividing rational expressions.

WHY Writing $b - 2$ over 1 is helpful when we invert its numerator and denominator to find its reciprocal.

Solution

$$\frac{b^3 - 4b}{x - 1} \div (b - 2) = \frac{b^3 - 4b}{x - 1} \div \frac{b - 2}{1}$$

Write $b - 2$ as a fraction with a denominator of 1.

$$= \frac{b^3 - 4b}{x - 1} \cdot \frac{1}{b - 2}$$

Multiply the first rational expression by the reciprocal of the second.

$$= \frac{b^3 - 4b}{(x - 1)(b - 2)}$$

Multiply the numerators. Multiply the denominators.

$$= \frac{b(b + 2)\cancel{(b - 2)}}{(x - 1)\cancel{(b - 2)}}$$

Factor $b^3 - 4b$ and then simplify.

$$= \frac{b(b + 2)}{x - 1}$$

Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

Self Check 8

Divide: $\frac{m^4 - 9m^2}{a^2 - 3a} \div (m^2 + 3m)$

Now Try Problem 63

Self Check 8 Answer
$\frac{m(m - 3)}{a(a - 3)}$

Teaching Example 8 Divide: $\frac{a^3 - 16a}{a + 4} \div (a - 4)$
Answer:
a

4 Perform mixed operations.

EXAMPLE 9

Simplify: $\frac{x^2 + 2x - 3}{6x^2 + 5x + 1} \div \frac{2x^2 - 2}{2x^2 - 5x - 3} \cdot \frac{6x^2 + 4x - 2}{x^2 - 2x - 3}$

Strategy We will consider the division first by multiplying the first rational expression by the reciprocal of the second. Then we will find the product of the three rational expressions.

WHY By the rules for the order of operations, we must perform division and multiplication in order from left to right.

Solution
Since multiplications and divisions are done in order from left to right, we begin by focusing on the division. We introduce grouping symbols to emphasize this. To divide the expressions within the parentheses, we invert $\frac{2x^2 - 2}{2x^2 - 5x - 3}$ and multiply.

Self Check 9

Simplify:
$\frac{x^2 - 25}{4x^2 + 12x + 9} \div \frac{x^2 - 5x}{3x - 1} \cdot \frac{2x + 3}{3x^2 + 14x - 5}$

Now Try Problem 67

Self Check 9 Answer
$\frac{1}{x(2x + 3)}$

Teaching Example 9 Simplify: $\dfrac{x^2-4}{x^2+4x+4} \div \dfrac{x^2-x-6}{x^2+8x+12} \cdot \dfrac{x-3}{x^2+4x-12}$
Answer: $\dfrac{1}{x+2}$

$$\left(\frac{x^2+2x-3}{6x^2+5x+1} \div \frac{2x^2-2}{2x^2-5x-3}\right)\frac{6x^2+4x-2}{x^2-2x-3}$$
$$= \left(\frac{x^2+2x-3}{6x^2+5x+1} \cdot \frac{2x^2-5x-3}{2x^2-2}\right)\frac{6x^2+4x-2}{x^2-2x-3}$$

Next, we multiply the three rational expressions and simplify the result.

$$= \frac{(x^2+2x-3)(2x^2-5x-3)(6x^2+4x-2)}{(6x^2+5x+1)(2x^2-2)(x^2-2x-3)}$$

$$= \frac{(x+3)\overset{1}{\cancel{(x-1)}}\overset{1}{\cancel{(2x+1)}}\overset{1}{\cancel{(x-3)}}\overset{1}{\cancel{2}}(3x-1)\overset{1}{\cancel{(x+1)}}}{(3x+1)\underset{1}{\cancel{(2x+1)}}\underset{1}{\cancel{2}}(x+1)\underset{1}{\cancel{(x-1)}}\underset{1}{\cancel{(x-3)}}\underset{1}{\cancel{(x+1)}}}$$ Factor each polynomial completely and simplify.

$$= \frac{(x+3)(3x-1)}{(3x+1)(x+1)}$$

ANSWERS TO SELF CHECKS

1. $\frac{3x^4}{34}$ **2. a.** $\frac{a^2(a+3)}{42}$ **b.** 1 **3.** $\frac{a-3b}{a+b}$ **4.** $-\frac{x^2}{8(x^2+1)}$ **5.** $\frac{x^2+10x+25}{x^4-12x^3+36x^2}$ **6.** $\frac{2a}{7t^3}$ **7.** $\frac{x(x-2)}{3}$ **8.** $\frac{m(m-3)}{a(a-3)}$ **9.** $\frac{1}{x(2x+3)}$

SECTION 6.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\frac{a^2-9}{a^2-49} \cdot \frac{a-7}{a+3}$ is the product of two rational expressions.

2. The reciprocal of $\frac{a+3}{a+7}$ is $\frac{a+7}{a+3}$.

3. To find the reciprocal of a rational expression, we invert its numerator and denominator.

▶ **4.** To simplify a rational expression, remove any factors common to the numerator and denominator.

CONCEPTS

Fill in the blanks.

5. To multiply rational expressions, multiply their numerators and multiply their denominators. In symbols,

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

6. To divide two rational expressions, multiply the first by the reciprocal of the second. In symbols,

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}$$

NOTATION

Complete each solution.

7. $$\frac{x^2+3x}{5x-25} \cdot \frac{x-5}{x+3} = \frac{(x^2+3x)(x-5)}{(5x-25)(x+3)} = \frac{x(x+3)(x-5)}{5(x-5)(x+3)} = \frac{x}{5}$$

▶ **8.** $$\frac{x^2-x-6}{4x^2+16x} \div \frac{x-3}{x+4} = \frac{x^2-x-6}{4x^2+16x} \cdot \frac{x+4}{x-3} = \frac{(x^2-x-6)(x+4)}{(4x^2+16x)(x-3)} = \frac{(x-3)(x+2)(x+4)}{4x(x+4)(x-3)} = \frac{x+2}{4x}$$

▶ Selected exercises available online at **www.webassign.net/brookscole**

9. A student checks her answers with those in the back of her textbook. Determine whether they are equivalent.

Student's answer	Book's answer	Equivalent?
$\frac{-x^{10}}{y^2}$	$-\frac{x^{10}}{y^2}$	yes
$\frac{x-3}{x+3}$	$\frac{3-x}{x+3}$	no
$\frac{a+b}{(2-x)(c+d)}$	$-\frac{a+b}{(x-2)(c+d)}$	yes

▶ **10. a.** Write $5x^2 + 35x$ as a fraction. $\frac{5x^2+35x}{1}$

b. What is the reciprocal of $5x^2 + 35x$? $\frac{1}{5x^2+35x}$

GUIDED PRACTICE

Multiply, and then simplify, if possible. See Objective 1.

11. $\frac{3}{4} \cdot \frac{11}{3}$ $\frac{11}{4}$

▶ **12.** $\frac{13}{6} \cdot \frac{6}{21}$ $\frac{13}{21}$

13. $\frac{15}{24} \cdot \frac{16}{25}$ $\frac{2}{5}$

14. $\frac{49}{36} \cdot \frac{18}{35}$ $\frac{7}{10}$

Multiply, and then simplify, if possible. See Example 1.

15. $\frac{3a}{10} \cdot \frac{2}{15a^4}$ $\frac{1}{25a^3}$

16. $\frac{4p}{21} \cdot \frac{7}{12p^6}$ $\frac{1}{9p^5}$

17. $\frac{12x^6}{7y^4} \cdot \frac{y}{8x^2}$ $\frac{3x^4}{14y^3}$

▶ **18.** $\frac{b^6}{27a^2} \cdot \frac{18a^4}{5b^9}$ $\frac{2a^2}{15b^3}$

Multiply, and then simplify, if possible. See Example 2.

19. $\frac{y^2+6y+9}{15y} \cdot \frac{3y^2}{2y+6}$ $\frac{y(y+3)}{10}$

20. $\frac{3p^2}{6p+24} \cdot \frac{p^2-16}{6p}$ $\frac{p(p-4)}{12}$

21. $\frac{x^2+x-6}{5x} \cdot \frac{5x-10}{x+3}$ $\frac{(x-2)^2}{x}$

22. $\frac{z^2+4z-5}{25z-25} \cdot \frac{5z}{z+5}$ $\frac{z}{5}$

▶ **23.** $\frac{x^2+2x+1}{9x^3} \cdot \frac{2x^2-2x}{2x^2-2}$ $\frac{x+1}{9x^2}$

24. $\frac{a^4+6a^3}{a^2-16} \cdot \frac{3a-12}{3a+18}$ $\frac{a^3}{a+4}$

25. $\frac{t^2+t-6}{t^2-6t+9} \cdot \frac{t^2-9}{t^2-4}$ $\frac{(t+3)^2}{(t-3)(t+2)}$

26. $\frac{s^2-5s+6}{s^2-10s+16} \cdot \frac{s^2-6s-16}{s^2+2s}$ $\frac{s-3}{s}$

Multiply, and then simplify, if possible. See Example 3.

27. $\frac{2x^2-x-3}{x^2-1} \cdot \frac{x^2+x-2}{2x^2+x-6}$ 1

28. $\frac{2p^2-5p-3}{p^2-9} \cdot \frac{2p^2+5p-3}{2p^2+5p+2}$ $\frac{2p-1}{p+2}$

29. $\frac{3t^2-t-2}{6t^2-5t-6} \cdot \frac{4t^2-9}{2t^2+5t+3}$ $\frac{t-1}{t+1}$

30. $\frac{9x^2+3x-20}{3x^2-7x+4} \cdot \frac{3x^2-5x+2}{9x^2+18x+5}$ $\frac{3x-2}{3x+1}$

31. $\frac{x^2+4xy+4y^2}{2x^2+4xy} \cdot \frac{3x-6y}{x^2-4y^2}$ $\frac{3}{2x}$

32. $\frac{x^2-y^2}{xy} \cdot \frac{x^2}{x^2+2xy+y^2}$ $\frac{x(x-y)}{y(x+y)}$

33. $\frac{3a^2+7ab+2b^2}{a^2+2ab} \cdot \frac{a^2-ab}{3a^2+ab}$ $\frac{a-b}{a}$

▶ **34.** $\frac{a^2+3ab+2b^2}{a^2-3ab-4b^2} \cdot \frac{a^2-4ab}{ab^2+2b^3}$ $\frac{a}{b^2}$

Multiply, and then simplify, if possible. See Example 4.

35. $15x\left(\frac{x+1}{15x}\right)$ $x + 1$

36. $30t\left(\frac{t-7}{30t}\right)$ $t - 7$

37. $12y\left(\frac{y+8}{6y}\right)$ $2y + 16$ or $2(y + 8)$

▶ **38.** $16x\left(\frac{3x+8}{4x}\right)$ $12x + 32$ or $4(3x + 8)$

39. $(6a - a^2) \cdot \frac{a^3}{2a^3-12a^2+6a-36}$ $-\frac{a^4}{2(a^2+3)}$

40. $(10n - n^2) \cdot \frac{n^6}{n^4-10n^3-2n^2+20n}$ $-\frac{n^6}{n^2-2}$

41. $(x^2 + x - 2cx - 2c) \cdot \frac{x^2+3x+2}{4c^2-x^2}$ $-\frac{(x+1)^2(x+2)}{x+2c}$

▶ **42.** $(2ax - 10x + a - 5) \cdot \frac{x}{2x^2+x}$ $a - 5$

Find each power. See Example 5.

43. $\left(\frac{x-3}{x^2+4}\right)^2$ $\frac{x^2-6x+9}{x^4+8x^2+16}$

▶ **44.** $\left(\frac{2t^2+t}{t-1}\right)^2$ $\frac{4t^4+4t^3+t^2}{t^2-2t+1}$

45. $\left(\frac{2m^2-m-3}{x^2-1}\right)^2$ $\frac{4m^4-4m^3-11m^2+6m+9}{x^4-2x^2+1}$

46. $\left(\frac{k^4+3k}{x^2-x+1}\right)^2$ $\frac{k^8+6k^5+9k^2}{x^4-2x^3+3x^2-2x+1}$

Divide, and then simplify, if possible. See Objective 3.

47. $\frac{6}{11} \div \frac{36}{55}$ $\frac{5}{6}$

▶ **48.** $\frac{17}{12} \div \frac{34}{3}$ $\frac{1}{8}$

49. $\frac{12}{5} \div \frac{24}{45}$ $\frac{9}{2}$

50. $\frac{18}{7} \div \frac{54}{35}$ $\frac{5}{3}$

Divide, and then simplify, if possible. See Example 6.

51. $\dfrac{22x^3}{y^2} \div \dfrac{33x^9}{y^7}$ $\quad \dfrac{2y^5}{3x^6}$

52. $\dfrac{24a^6}{b} \div \dfrac{64a^9}{b^2}$ $\quad \dfrac{3b}{8a^3}$

53. $\dfrac{pq^2}{50} \div \dfrac{p^{10}q^2}{15}$ $\quad \dfrac{3}{10p^9}$

54. $\dfrac{s^3t^3}{12} \div \dfrac{s^3t^{11}}{144}$ $\quad \dfrac{12}{t^8}$

Divide, and then simplify, if possible. See Example 7.

55. $\dfrac{x^{12}}{x^3 - 8} \div \dfrac{x^2}{x^2 - 2x}$ $\quad \dfrac{x^{11}}{x^2 + 2x + 4}$

56. $\dfrac{x^9}{x^3 + 125} \div \dfrac{x^4}{x^2 + 5x}$ $\quad \dfrac{x^6}{x^2 - 5x + 25}$

57. $\dfrac{x^2 - 16}{x^2 - 25} \div \dfrac{5x + 20}{10x^2 - 50x}$ $\quad \dfrac{2x(x - 4)}{x + 5}$

58. $\dfrac{a^2 - 9}{a^2 - 49} \div \dfrac{9a^2 + 27a}{3a + 21}$ $\quad \dfrac{a - 3}{3a(a - 7)}$

59. $\dfrac{3n^2 + 5n - 2}{12n^2 - 13n + 3} \div \dfrac{n^2 + 3n + 2}{4n^2 + 5n - 6}$ $\quad \dfrac{n + 2}{n + 1}$

60. $\dfrac{8y^2 - 14y - 15}{6y^2 - 11y - 10} \div \dfrac{4y^2 - 9y - 9}{3y^2 - 7y - 6}$ $\quad 1$

61. $\dfrac{5cd + d^2}{6d^2} \div \dfrac{125c^3 + d^3}{6c + 6d}$ $\quad \dfrac{c + d}{d(25c^2 - 5cd + d^2)}$

62. $\dfrac{6m - 8n}{9m^3} \div \dfrac{27m^3 - 64n^3}{9m + 9n}$ $\quad \dfrac{2(m + n)}{m^3(9m^2 + 12mn + 16n^2)}$

Divide, and then simplify, if possible. See Example 8.

63. $\dfrac{y^3 - 9y}{y + 2} \div (y - 3)$ $\quad \dfrac{y(y + 3)}{y + 2}$

64. $\dfrac{x - 2}{x} \div (x^2 - 4)$ $\quad \dfrac{1}{x(x + 2)}$

65. $(x + 1) \div \dfrac{x^2 + 2x + 1}{2}$ $\quad \dfrac{2}{x + 1}$

66. $(y + 4) \div \dfrac{y^2 + 8y + 16}{ab}$ $\quad \dfrac{ab}{y + 4}$

Perform each operation and simplify, if possible. See Example 9.

67. $\dfrac{6a^2 - 7a - 3}{a^2 - 1} \div \dfrac{4a^2 - 12a + 9}{a^2 - 1} \cdot \dfrac{2a^2 - a - 3}{3a^2 - 2a - 1}$ $\quad \dfrac{a + 1}{a - 1}$

68. $\dfrac{x^2 - x - 12}{x^2 + x - 2} \div \dfrac{x^2 - 6x + 8}{x^2 - 3x - 10} \cdot \dfrac{x^2 - 3x + 2}{x^2 - 2x - 15}$ $\quad 1$

69. $\dfrac{2x^2 - 2x - 4}{x^2 + 2x - 8} \cdot \dfrac{3x^2 + 15x}{x + 1} \div \dfrac{4x^2 - 100}{x^2 - x - 20}$ $\quad \dfrac{3x}{2}$

70. $\dfrac{4a^2 - 10a + 6}{a^4 - 3a^3} \div \dfrac{3 - 2a}{2a^3} \cdot \dfrac{a - 3}{2a - 2}$ $\quad -2$

TRY IT YOURSELF

Perform the operations and simplify.

71. $\dfrac{x^2 - 6x + 9}{4 - x^2} \div \dfrac{x^2 - 9}{x^2 - 8x + 12}$ $\quad -\dfrac{(x - 3)(x - 6)}{(x + 2)(x + 3)}$

72. $\dfrac{x^3 + 1}{4} \div \dfrac{x + 1}{2}$ $\quad \dfrac{x^2 - x + 1}{2}$

73. $\dfrac{2x^2 - 2x - 12}{x^2 - 4} \cdot \dfrac{x^2 - x - 2}{x^3 - 9x}$ $\quad \dfrac{2(x + 1)}{x(x + 3)}$

74. $\dfrac{x^2 + 2x - 35}{12x^3} \cdot \dfrac{x^2 + 4x - 21}{x^2 - 3x}$ $\quad \dfrac{(x + 7)^2(x - 5)}{12x^4}$

75. $\dfrac{p^3 - q^3}{p^2 - q^2} \cdot \dfrac{q^2 + pq}{p^3 + p^2q + pq^2}$ $\quad \dfrac{q}{p}$

76. $\dfrac{x^2 - 4}{2b - bx} \div \dfrac{x^2 + 4x + 4}{2b + bx}$ $\quad -1$

77. $\dfrac{10r^2s}{6rs^2} \cdot \dfrac{3r^3}{2rs}$ $\quad \dfrac{5r^3}{2s^2}$

78. $\dfrac{3a^3b}{25cd^3} \cdot \dfrac{5cd^2}{6ab}$ $\quad \dfrac{a^2}{10d}$

79. $10(h - 9)\dfrac{h - 3}{9 - h}$ $\quad -10h + 30$ or $-10(h - 3)$

80. $r(r - 25)\dfrac{r + 4}{r - 25}$ $\quad r^2 + 4r$ or $r(r + 4)$

81. $\dfrac{2x^2 + 5xy + 3y^2}{3x^2 - 5xy + 2y^2} \div \dfrac{2x^2 + xy - 3y^2}{3x^2 - 5xy + 2y^2}$ $\quad \dfrac{x + y}{x - y}$

82. $\dfrac{2p^2 - 5pq - 3q^2}{p^2 - 9q^2} \div \dfrac{2p^2 + 5pq + 2q^2}{2p^2 + 5pq - 3q^2}$ $\quad \dfrac{2p - q}{p + 2q}$

83. $(4x^2 - 9) \div \dfrac{2x^2 + 5x + 3}{x + 2} \cdot \dfrac{1}{2x - 3}$ $\quad \dfrac{x + 2}{x + 1}$

84. $(4x + 12) \div \dfrac{2x - 6}{x^2} \cdot \dfrac{x - 3}{2}$ $\quad x^2(x + 3)$

85. $\dfrac{x^3 - 3x^2 - 25x + 75}{x^3 - 27} \cdot \dfrac{2x^3 + 6x^2 + 18x}{x^2 + 10x + 25}$ $\quad \dfrac{2x(x - 5)}{x + 5}$

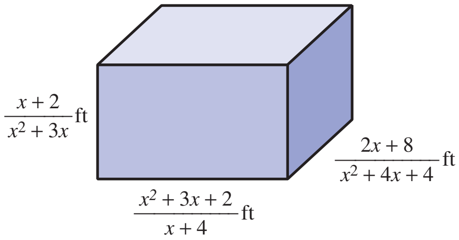
86. $\dfrac{x^2 + 3x + xy + 3y}{x^2 - 9} \cdot \dfrac{3 - x}{x^3 + 3x^2}$ $\quad -\dfrac{x + y}{x^2(x + 3)}$

APPLICATIONS

87. PHYSICS EXPERIMENTS The following table contains data from a physics experiment. Complete the table.

Trial	Rate (m/sec)	Time (sec)	Distance (m)
1	$\dfrac{k_1^2 + 3k_1 + 2}{k_1 - 3}$	$\dfrac{k_1^2 - 3k_1}{k_1 + 1}$	$k_1(k_1 + 2)$
2	$\dfrac{k_2^2 + 6k_2 + 5}{k_2 + 1}$	$k_2 + 6$	$k_2^2 + 11k_2 + 30$

88. GEOMETRY Find a simplified rational expression that represents the volume of the rectangular solid shown here. $\frac{2(x+1)}{x(x+3)}\text{ ft}^3$



WRITING

89. Explain how to multiply two rational expressions.

90. Write some comments to the student who wrote the following solution, explaining the error.

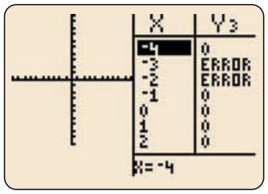
$$\frac{x^2+x-2}{x^2-4}\cdot\frac{x-2}{x-1}=\frac{(x+2)(x-1)(x-2)}{(x+2)(x-2)(x-1)}$$
$$=0$$

91. The graph of $Y_3 = Y_1 - Y_2$, where

$$Y_1=\frac{2x^2-5x-3}{x^2-9}\cdot\frac{2x^2+5x-3}{2x^2+5x+2}$$
$$Y_2=\frac{2x-1}{x+2}$$

is shown. Explain how the graph and table can be used to verify that

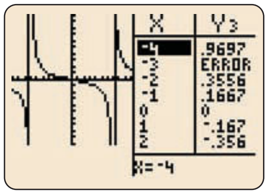
$$\frac{2x^2-5x-3}{x^2-9}\cdot\frac{2x^2+5x-3}{2x^2+5x+2}=\frac{2x-1}{x+2}$$



▶ **92.** A student obtained an answer of $\frac{x+3}{x+7}$ after performing $\frac{x^2-9}{x^2-49}\div\frac{x+3}{x+7}$. As a check, he graphed $Y_3 = Y_1 - Y_2$, where

$$Y_1=\left(\frac{x^2-9}{x^2-49}\right)\div\left(\frac{x+3}{x+7}\right)$$
$$Y_2=\frac{x+3}{x+7}$$

The graph is shown. Explain what conclusion can be drawn from the graph and the table.



REVIEW

Complete the rules for exponents. Assume that there are no divisions by 0.

93. $x^m x^n = x^{m+n}$

94. $(x^m)^n = x^{mn}$

95. $(xy)^n = x^n y^n$

▶ **96.** $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

97. $x^0 = 1$

98. $x^{-n} = \frac{1}{x^n}$

99. $\frac{x^m}{x^n} = x^{m-n}$

100. $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

101. $\frac{x^{-m}}{y^{-n}} = \frac{y^n}{x^m}$

102. $x^1 = x$

SECTION 6.3
Adding and Subtracting Rational Expressions

Objectives

1. Add and subtract rational expressions with like denominators.
2. Add and subtract rational expressions with unlike denominators.
3. Find the least common denominator.
4. Perform mixed operations.

The methods used to add and subtract rational expressions are based on the rules for adding and subtracting arithmetic fractions. In this section, we will add and subtract rational expressions with *like* and *unlike* denominators.

1 Add and subtract rational expressions with like denominators.

To add or subtract fractions with a common denominator, we add or subtract their numerators and write the sum or difference over the common denominator. For example,

$$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} \qquad \frac{3}{7} - \frac{2}{7} = \frac{3-2}{7}$$
$$= \frac{5}{7} \qquad\qquad = \frac{1}{7}$$

We use the same procedure to add and subtract rational expressions with like denominators.

Adding and Subtracting Rational Expressions That Have the Same Denominator

To add (or subtract) rational expressions that have same denominator, add (or subtract) their numerators and write the sum (or difference) over the common denominator. Then, if possible, factor and simplify.

If $\frac{A}{D}$ and $\frac{B}{D}$ are rational expressions,

$$\frac{A}{D} + \frac{B}{D} = \frac{A+B}{D} \quad \text{and} \quad \frac{A}{D} - \frac{B}{D} = \frac{A-B}{D}$$

Self Check 1

Perform the operations:

a. $\frac{17}{22t} + \frac{13}{22t}$ $\frac{15}{11t}$

b. $\frac{a^2}{a^2-2a} - \frac{4}{a^2-2a}$ $\frac{a+2}{a}$

Now Try **Problems 17 and 23**

Teaching Example 1 Perform the operations:

a. $\frac{5}{7x} + \frac{9}{7x}$ **b.** $\frac{a^2}{a^2+a} - \frac{1}{a^2+a}$

Answers:

a. $\frac{2}{x}$ **b.** $\frac{a-1}{a}$

EXAMPLE 1

Perform the operations: **a.** $\frac{4}{3x} + \frac{7}{3x}$ **b.** $\frac{a^2}{a^2-1} - \frac{a}{a^2-1}$

Strategy In part a, we will add the numerators and write the sum over the common denominator. In part b, we will subtract the numerators and write the difference over the common denominator. Then, if possible, we will factor and simplify.

WHY These are the rules for adding and subtracting rational expressions that have the *same* denominator.

Solution

a. $\frac{4}{3x} + \frac{7}{3x} = \frac{4+7}{3x}$ Add the numerators. Write the sum over the common denominator, 3x.

$= \frac{11}{3x}$ The result does not simplify.

b. $\frac{a^2}{a^2-1} - \frac{a}{a^2-1} = \frac{a^2-a}{a^2-1}$ Subtract the numerators. Write the difference over the common denominator, $a^2 - 1$.

We note that the polynomials in the numerator and the denominator of the result factor.

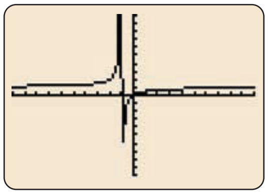
$$= \frac{a(a-1)}{(a+1)(a-1)}$$

$$= \frac{a\cancel{(a-1)}^{1}}{(a+1)\cancel{(a-1)}_{1}} \quad \text{Simplify.}$$

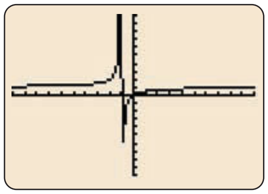
$$= \frac{a}{a+1}$$

Using Your CALCULATOR **Checking Algebra**

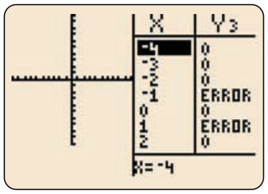
We can check the subtraction in Example 1b by replacing each a with x and graphing the rational functions $f(x) = \frac{x^2}{x^2 - 1} - \frac{x}{x^2 - 1}$, shown in figure (a), and $g(x) = \frac{x}{x + 1}$, shown in figure (b), and observing that the graphs are the same. Note that -1 and 1 are not in the domain of the first function and that -1 is not in the domain of the second function.



(a)



(b)



(c)

Figure (c) shows the display when the G-T mode is used to check the simplification. Here, $Y_3 = Y_1 - Y_2$, where $Y_1 = \frac{x^2}{x^2 - 1} - \frac{x}{x^2 - 1}$ and $Y_2 = \frac{x}{x + 1}$.

2 Add and subtract rational expressions with unlike denominators.

Recall that writing a fraction as an equivalent fraction with a larger denominator is called *building the fraction.* For example, to write $\frac{3}{5}$ as an equivalent fraction with a denominator of 35, we multiply it by 1 in the form of $\frac{7}{7}$. When a number is multiplied by 1, its value does not change.

$$\frac{3}{5} = \frac{3}{5} \cdot \frac{7}{7} = \frac{21}{35}$$

To add and subtract rational expressions with different denominators, we write them as equivalent expressions having a common denominator. To do so, we build rational expressions.

Building Rational Expressions

To build a rational expression, multiply it by 1 in the form of $\frac{c}{c}$, where c is any nonzero number or expression.

The following steps summarize how to add or subtract rational expressions with different denominators.

Adding and Subtracting Rational Expressions with Unlike Denominators

1. Find the LCD.
2. Write each rational expression as an equivalent expression with the LCD as the denominator. To do so, build each rational expression using a form of 1 that involves any factor(s) needed to obtain the LCD.
3. Add or subtract the numerators and write the sum or difference over the LCD.
4. Simplify the resulting rational expression, if possible.

Self Check 2

Add: $\frac{5}{a} + \frac{7}{b}$ $\frac{5b + 7a}{ab}$

Now Try **Problem 25**

Teaching Example 2 Add: $\frac{9}{a} + \frac{3}{b}$

Answer:
$\frac{9b + 3a}{ab}$

EXAMPLE 2

Add: $\frac{3}{x} + \frac{4}{y}$

Strategy The LCD for the rational expressions is xy. We will multiply each one by the appropriate form of 1 to build it into an equivalent rational expression with a denominator of xy.

WHY Since the denominators are different, we cannot add these rational expressions in their present form.

Solution

$$\frac{3}{x} + \frac{4}{y} = \frac{3}{x} \cdot \frac{y}{y} + \frac{4}{y} \cdot \frac{x}{x}$$ Build the rational expressions so that each has a denominator of xy.

$$= \frac{3y}{xy} + \frac{4x}{xy}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{3y + 4x}{xy}$$ Add the numerators. Write the sum over the common denominator, xy.

Self Check 3

Subtract: $\frac{3a}{a + 3} - \frac{5a}{a - 3}$

Now Try **Problems 29 and 31**

Self Check 3 Answer
$-\frac{2a(a + 12)}{(a + 3)(a - 3)}$

Teaching Example 3 Subtract:
$\frac{8x}{x + 3} - \frac{2x}{x + 1}$
Answer:
$\frac{6x^2 + 2x}{(x + 3)(x + 1)}$ or $\frac{2x(3x + 1)}{(x + 3)(x + 1)}$

EXAMPLE 3

Subtract: $\frac{4x}{x + 2} - \frac{7x}{x - 2}$

Strategy The LCD for the rational expressions is $(x + 2)(x - 2)$. We will multiply each one by the appropriate form of 1 to build it into an equivalent rational expression with a denominator of $(x + 2)(x - 2)$.

WHY Since the denominators are different, we cannot subtract these rational expressions in their present form.

Solution

$$\frac{4x}{x + 2} - \frac{7x}{x - 2}$$

$$= \frac{4x}{x + 2} \cdot \frac{x - 2}{x - 2} - \frac{7x}{x - 2} \cdot \frac{x + 2}{x + 2}$$ Build each rational expression.

$$= \frac{4x^2 - 8x}{(x + 2)(x - 2)} - \frac{7x^2 + 14x}{(x + 2)(x - 2)}$$ Multiply the numerators. Multiply the denominators. Leave the denominator in factored form. Subtract the numerators. Write the difference over the common denominator.

This numerator is written within parentheses to make sure that we subtract both of its terms.

$$= \frac{(4x^2 - 8x) - (7x^2 + 14x)}{(x + 2)(x - 2)}$$

$$= \frac{4x^2 - 8x - 7x^2 - 14x}{(x + 2)(x - 2)}$$ To subtract the polynomials in the numerator, add the first and the opposite of the second.

$$= \frac{-3x^2 - 22x}{(x + 2)(x - 2)}$$ Combine like terms in the numerator.

If the common factor of $-x$ is factored out of the terms in the numerator, this result can be written in two other equivalent forms.

$$\frac{-3x^2 - 22x}{(x + 2)(x - 2)} = \frac{-x(3x + 22)}{(x + 2)(x - 2)} = -\frac{x(3x + 22)}{(x + 2)(x - 2)}$$

The result does not simplify.

We can use the following fact to add or subtract rational expressions whose denominators are opposites.

Multiplying by −1

When a polynomial is multiplied by -1, the result is its opposite.

EXAMPLE 4 Add: $\dfrac{x}{x-y} + \dfrac{y}{y-x}$

Strategy Since the denominators are opposites, either one can serve as the LCD. If we choose $x - y$, we can multiply $\frac{y}{y-x}$ by $\frac{-1}{-1}$ to build it into an equivalent rational expression with the denominator $x - y$.

WHY When $y - x$ is multiplied by -1, the subtraction is reversed, and the result is $x - y$.

Solution

$$\frac{x}{x-y} + \frac{y}{y-x} = \frac{x}{x-y} + \frac{y}{y-x}\cdot\frac{-1}{-1}$$ Build $\frac{y}{y-x}$ so that it has a denominator of $x - y$.

$$= \frac{x}{x-y} + \frac{-y}{-y+x}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{x}{x-y} + \frac{-y}{x-y}$$ Write the second denominator, $-y + x$, as $x - y$. The rational expressions now have a common denominator.

$$= \frac{x-y}{x-y}$$ Add the numerators. Write the result over the common denominator, $x - y$.

$$= 1$$ Simplify.

Self Check 4

Add: $\dfrac{2a}{a-b} + \dfrac{b}{b-a}$ $\dfrac{2a-b}{a-b}$

Now Try **Problem 33**

Teaching Example 4 Add: $\dfrac{a}{a-7} + \dfrac{7}{7-a}$

Answer: 1

EXAMPLE 5 Subtract: $3 - \dfrac{7}{x-15}$

Strategy We will begin by writing 3 as $\frac{3}{1}$.

WHY Then we can multiply $\frac{3}{1}$ by the appropriate form of 1 to build it into an equivalent rational expression with a denominator of $x - 15$.

Solution

$$3 - \frac{7}{x-15} = \frac{3}{1} - \frac{7}{x-15}$$ $3 = \frac{3}{1}$

$$= \frac{3}{1}\cdot\frac{x-15}{x-15} - \frac{7}{x-15}$$ Build $\frac{3}{1}$ to a rational expression with a denominator of $x - 15$.

$$= \frac{3x-45}{x-15} - \frac{7}{x-15}$$ Distribute the multiplication by 3.

$$= \frac{3x-45-7}{x-15}$$ Subtract the numerators. Write the difference over the common denominator, $x - 15$.

$$= \frac{3x-52}{x-2}$$ Combine like terms in the numerator. The result does not simplify.

Self Check 5

Subtract: $6 - \dfrac{5y}{6-y}$ $\dfrac{-11y+36}{6-y}$

Now Try **Problem 37**

Teaching Example 5 Subtract: $9 - \dfrac{4}{a+3}$

Answer: $\dfrac{9a+23}{a+3}$

3 Find the least common denominator.

When adding or subtracting rational expressions with unlike denominators, it is easiest if we write the rational expressions in terms of the smallest common denominator possible, called the *least* (or lowest) *common denominator (LCD)*. To find the least common denominator of several rational expressions, we follow these steps.

Finding the LCD

1. Factor each denominator completely.
2. The LCD is a product that uses each different factor obtained in step 1 the greatest number of times it appears in any one factorization.

Self Check 6

Find the LCD of:

a. $\frac{5x}{28z^3}$ and $\frac{1}{21z}$

b. $\frac{a-1}{a^2-25}$ and $\frac{3-a^2}{a^2+7a+10}$

Now Try **Problems 41 and 45**

Self Check 6 Answers

a. $84z^3$ **b.** $(a-5)(a+5)(a+2)$

Teaching Example 6 Find the LCD of:

a. $\frac{13x}{15y^2}$ and $\frac{11x}{25y}$

b. $\frac{x}{x^2-49}$ and $\frac{4}{x^2-14x+49}$

Answers:

a. $75y^2$ **b.** $(x+7)(x-7)^2$

EXAMPLE 6

Find the LCD of:

a. $\frac{5a}{24b}$ and $\frac{11a}{18b^2}$ **b.** $\frac{1}{x^2-12x+36}$ and $\frac{3-x}{x^2-6x}$

Strategy We begin by factoring completely the denominator of each rational expression.

WHY Since the LCD must contain the factors of each denominator, we need to write each denominator in factored form.

Solution

a. We write each denominator as the product of prime numbers and variables.

$$24b = 2 \cdot 2 \cdot 2 \cdot 3 \cdot b = 2^3 \cdot 3 \cdot b$$
$$18b^2 = 2 \cdot 3 \cdot 3 \cdot b \cdot b = 2 \cdot 3^2 \cdot b^2$$

To find the LCD, we form a product using each of these factors the greatest number of times it appears in any one factorization.

The greatest number of times the factor 2 appears is three times.
The greatest number of times the factor 3 appears is twice.
The greatest number of times the factor b appears is twice.

$$\text{LCD} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot b \cdot b = 72b^2$$

b. We factor each denominator completely:

$$x^2 - 12x + 36 = (x-6)(x-6) = (x-6)^2$$
$$x^2 - 6x = x(x-6)$$

To find the LCD, we form a product using the highest power of each of the factors:

The greatest number of times the factor x appears is once.
The greatest number of times the factor x − 6 appears is twice.

$$\text{LCD} = x(x-6)^2$$

Success Tip Note that the highest power of each factor is used to form the LCD:

$$24b = 2^3 \cdot 3 \cdot b$$
$$18b^2 = 2 \cdot 3^2 \cdot b^2$$
$$\text{LCD} = 2^3 \cdot 3^2 \cdot b^2 = 72b^2$$

Self Check 7

Add: $\frac{5x}{28z^3} + \frac{1}{21z}$ $\frac{15x+4z^2}{84z^3}$

Now Try **Problems 49 and 51**

EXAMPLE 7

Add: $\frac{5a}{24b} + \frac{11a}{18b^2}$

Strategy In Example 6, we saw that the LCD of these rational expressions is $72b^2$. We will multiply each one by the appropriate form of 1 to build it into an equivalent rational expression with a denominator of $72b^2$.

WHY Since the denominators are different, we cannot add these rational expressions in their present form.

Solution

$$\frac{5a}{24b} + \frac{11a}{18b^2} = \frac{5a}{24b} \cdot \frac{3b}{3b} + \frac{11a}{18b^2} \cdot \frac{4}{4}$$ Build the rational expressions so that each has a denominator of $72b^2$.

$$= \frac{15ab}{72b^2} + \frac{44a}{72b^2}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{15ab + 44a}{72b^2}$$ Add the numerators. Write the sum over the common denominator. The result does not simplify.

Teaching Example 7 Add: $\frac{7a}{75b^2} + \frac{a}{30b^3}$

Answer: $\frac{14ab + 5a}{150b^3}$

EXAMPLE 8

Subtract: $\frac{x}{x^2 - 2x + 1} - \frac{4}{x^2 - 1}$

Strategy We will factor each denominator, find the LCD, and build the rational expressions so each one has the LCD as its denominator.

WHY Since the denominators are different, we cannot subtract these rational expressions in their present form.

Solution

We factor each denominator to find the LCD:

$$\left.\begin{aligned} x^2 - 2x + 1 &= (x - 1)(x - 1) = (x - 1)^2 \\ x^2 - 1 &= (x + 1)(x - 1) \end{aligned}\right\}$$ The greatest number of times $x - 1$ appears is twice. The greatest number of times $x + 1$ appears is once.

The LCD is $(x - 1)^2(x + 1)$ or $(x - 1)(x - 1)(x + 1)$.

We now write each rational expression with its denominator in factored form. Then we multiply each numerator and denominator by the missing factor, so that each rational expression has a denominator of $(x - 1)(x - 1)(x + 1)$.

$$\frac{x}{x^2 - 2x + 1} - \frac{4}{x^2 - 1}$$

$$= \frac{x}{(x - 1)(x - 1)} - \frac{4}{(x + 1)(x - 1)}$$ Write each denominator in factored form.

$$= \frac{x}{(x - 1)(x - 1)} \cdot \frac{x + 1}{x + 1} - \frac{4}{(x + 1)(x - 1)} \cdot \frac{x - 1}{x - 1}$$ Build each rational expression.

$$= \frac{x^2 + x}{(x - 1)(x - 1)(x + 1)} - \frac{4x - 4}{(x - 1)(x - 1)(x + 1)}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{(x^2 + x) - (4x - 4)}{(x - 1)(x - 1)(x + 1)}$$ Subtract the numerators. Write the difference over the common denominator.

$$= \frac{x^2 + x - 4x + 4}{(x - 1)(x - 1)(x + 1)}$$ In the numerator, subtract the trinomials.

$$= \frac{x^2 - 3x + 4}{(x - 1)(x - 1)(x + 1)}$$ Combine like terms. The result does not simplify.

Self Check 8

Subtract: $\frac{a + 2}{a^2 - 4a + 4} - \frac{a - 3}{a^2 - 4}$

Now Try **Problem 57**

Self Check 8 Answer

$\frac{9a - 2}{(a - 2)(a - 2)(a + 2)}$

Teaching Example 8 Subtract:

a. $\frac{4}{x^2 - x - 6} - \frac{x}{x^2 - 9}$

b. $\frac{x + 1}{x^2 + 5x + 6} - \frac{x - 3}{x^2 - 4}$

Answers:

a. $\frac{-x^2 + 2x + 12}{(x + 3)(x - 3)(x + 2)}$

b. $\frac{-x + 7}{(x + 2)(x + 3)(x - 2)}$

Success Tip To build each rational expression, we use the FOIL method to multiply the numerators. Note that we do not multiply out the denominators. For example, to build the second rational expression, we have:

$$\frac{x-4}{(x+1)(x-1)} \cdot \frac{x-1}{x-1}$$

The result is: $\frac{x^2-5x+4}{(x+1)(x-1)(x-1)}$

4 Perform mixed operations.

Self Check 9

Perform the operations:

$$\frac{5a}{a^2-25} - \frac{7}{a-5} + \frac{2}{a+5}$$

Now Try **Problem 63**

Self Check 9 Answer

$$-\frac{45}{(a+5)(a-5)}$$

Teaching Example 9 Perform the operations:

$$\frac{3a}{a^2-6a+5} - \frac{1}{a^2-1} + \frac{a+2}{a^2-4a-5}$$

Answer:

$$\frac{4a^2+3a+3}{(a-5)(a-1)(a+1)}$$

EXAMPLE 9 Perform the operations: $\frac{2x}{x^2-4} - \frac{1}{x^2-3x+2} + \frac{x+1}{x^2+x-2}$

Strategy We will factor each denominator, find the LCD, and build the rational expressions so each one has the LCD as its denominator.

WHY Since the denominators are different, we cannot add or subtract these rational expressions in their present form.

Solution

We factor each denominator to find the LCD and note that the greatest number of times each factor appears is once.

$$\left.\begin{aligned} x^2-4 &= (x-2)(x+2) \\ x^2-3x+2 &= (x-2)(x-1) \\ x^2+x-2 &= (x-1)(x+2) \end{aligned}\right\} \quad \text{LCD} = (x-2)(x+2)(x-1)$$

We then write each rational expression as an equivalent rational expression with the LCD as its denominator and do the subtraction and addition.

$$\frac{2x}{x^2-4} - \frac{1}{x^2-3x+2} + \frac{x+1}{x^2+x-2}$$

$$= \frac{2x}{(x-2)(x+2)} - \frac{1}{(x-2)(x-1)} + \frac{x+1}{(x-1)(x+2)} \quad \text{Factor the denominators.}$$

$$= \frac{2x}{(x-2)(x+2)} \cdot \frac{x-1}{x-1} - \frac{1}{(x-2)(x-1)} \cdot \frac{x+2}{x+2} + \frac{x+1}{(x-1)(x+2)} \cdot \frac{x-2}{x-2}$$

$$= \frac{2x(x-1) - 1(x+2) + (x+1)(x-2)}{(x+2)(x-2)(x-1)} \quad \text{Write the sum and difference over the common denominator.}$$

$$= \frac{2x^2-2x-x-2+x^2-x-2}{(x+2)(x-2)(x-1)}$$

$$= \frac{3x^2-4x-4}{(x+2)(x-2)(x-1)} \quad \text{Combine like terms.}$$

$$= \frac{(3x+2)\overset{1}{\cancel{(x-2)}}}{(x+2)\underset{1}{\cancel{(x-2)}}(x-1)} \quad \text{Factor the trinomial and simplify.}$$

$$= \frac{3x+2}{(x+2)(x-1)}$$

Caution! Always write the result in simplest form by removing any factors common to the numerator and denominator.

ANSWERS TO SELF CHECKS

1. a. $\frac{15}{11t}$ **b.** $\frac{a+2}{a}$ **2.** $\frac{5b+7a}{ab}$ **3.** $-\frac{2a(a+12)}{(a+3)(a-3)}$ **4.** $\frac{2a-b}{a-b}$ **5.** $\frac{-11y+36}{6-y}$ **6. a.** $84z^3$ **b.** $(a-5)(a+5)(a+2)$ **7.** $\frac{15x+4z^2}{84z^3}$ **8.** $\frac{9a-2}{(a-2)(a-2)(a+2)}$ **9.** $-\frac{45}{(a+5)(a-5)}$

SECTION 6.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. The rational expressions $\frac{7}{6n}$ and $\frac{n+1}{6n}$ have a common denominator of $6n$.

2. The least common denominator of $\frac{x-8}{x+6}$ and $\frac{6-5x}{x}$ is $x(x+6)$.

3. To build a rational expression, we multiply it by a form of 1. For example, $\frac{2}{n^2}\cdot\frac{8}{8}=\frac{16}{8n^2}$.

4. The polynomials $x-y$ and $y-x$ are opposites because their terms are the same but opposite in sign.

CONCEPTS

Fill in the blanks.

5. To add or subtract rational expressions that have the same denominator, add or subtract the numerators, and write the sum or difference over the common denominator.

In symbols, if $\frac{A}{D}$ and $\frac{B}{D}$ are rational expressions,

$$\frac{A}{D}+\frac{B}{D}=\frac{A+B}{D} \quad\text{and}\quad \frac{A}{D}-\frac{B}{D}=\frac{A-B}{D}$$

6. When a number is multiplied by 1, its value does not change.

7. To find the least common denominator of several rational expressions, factor each denominator completely. The LCD is a product that uses each different factor the greatest number of times it appears in any one factorization.

8. $$\frac{x^2+3x}{x-1}-\frac{2x-1}{x-1}=\frac{x^2+3x-(2x-1)}{x-1}$$

9. Consider the following two procedures.

i. $$\frac{x^2-2x}{x^2+4x-12}=\frac{x\overset{1}{\cancel{(x-2)}}}{(x+6)\underset{1}{\cancel{(x-2)}}}=\frac{x}{x+6}$$

ii. $$\frac{x}{x+6}=\frac{x}{x+6}\cdot\frac{x-2}{x-2}=\frac{x^2-2x}{(x+6)(x-2)}$$

a. In which of these procedures are we *building* a rational expression? ii

b. For what type of problem is procedure ii often necessary? adding or subtracting rational expressions

c. What name is used to describe procedure i? simplifying a rational expression

10. The LCD for $\frac{2x+1}{x^2+5x+6}$ and $\frac{3x}{x^2-4}$ is

$$\text{LCD}=(x+2)(x+3)(x-2)$$

If we want to subtract these rational expressions, what form of 1 should be used:

a. to build $\frac{2x+1}{x^2+5x+6}$? $\frac{x-2}{x-2}$

b. to build $\frac{3x}{x^2-4}$? $\frac{x+3}{x+3}$

11. Consider the following factorizations.

$$2\cdot 3\cdot 3\cdot (x-2)$$
$$3(x-2)(x+1)$$

a. What is the greatest number of times the factor 3 appears in any one factorization? twice

b. What is the greatest number of times the factor $x-2$ appears in any one factorization? once

12. The factorizations of the denominators of two rational expressions follow. Find the LCD.

$$2\cdot 3\cdot a\cdot a\cdot a \quad 18a^3$$
$$2\cdot 3\cdot 3\cdot a\cdot a$$

13. Factor each denominator completely.

a. $\frac{17}{40x^2}$ $2\cdot 2\cdot 2\cdot 5\cdot x\cdot x$

b. $\frac{x+25}{2x^2-6x}$ $2x(x-3)$

c. $\frac{n^2+3n-4}{n^2-64}$ $(n+8)(n-8)$

14. By what must $y-4$ be multiplied to obtain $4-y$? -1

NOTATION

Complete each solution.

15. $$\frac{6x-1}{3x-1}+\frac{3x-2}{3x-1}=\frac{6x-1+3x-2}{3x-1}$$
$$=\frac{9x-3}{3x-1}$$
$$=\frac{3(3x-1)}{3x-1}$$
$$=3$$

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16. $\frac{8}{3v} - \frac{1}{4v^2} = \frac{8}{3v} \cdot \frac{4v}{4v} - \frac{1}{4v^2} \cdot \frac{3}{3}$
$= \frac{32v}{12v^2} - \frac{3}{12v^2}$
$= \frac{32v - 3}{12v^2}$

GUIDED PRACTICE

Add or subtract, and then simplify, if possible. See Example 1.

17. $\frac{8}{3x} + \frac{5}{3x}$ $\frac{13}{3x}$

18. $\frac{3}{4y} + \frac{8}{4y}$ $\frac{11}{4y}$

19. $\frac{t}{4r} + \frac{t}{4r}$ $\frac{t}{2r}$

20. $\frac{16x}{3z^2} - \frac{x}{3z^2}$ $\frac{5x}{z^2}$

21. $\frac{4y}{y-4} - \frac{16}{y-4}$ 4

22. $\frac{3x}{2x+2} + \frac{x+4}{2x+2}$ 2

23. $\frac{3x}{x^2-9} - \frac{9}{x^2-9}$ $\frac{3}{x+3}$

24. $\frac{9x}{x^2-1} - \frac{9}{x^2-1}$ $\frac{9}{x+1}$

Add or subtract, and then simplify, if possible. See Example 2.

25. $\frac{15}{p} + \frac{2}{q}$ $\frac{15q + 2p}{pq}$

26. $\frac{2}{a} + \frac{19}{b}$ $\frac{2b + 19a}{ab}$

27. $\frac{7}{2b} - \frac{11}{3a}$ $\frac{21a - 22b}{6ab}$

28. $\frac{5}{3n} - \frac{7}{4m}$ $\frac{20m - 21n}{12mn}$

Add or subtract, and then simplify, if possible. See Example 3.

29. $\frac{3}{x+2} + \frac{5}{x-4}$ $\frac{8x-2}{(x+2)(x-4)}$

30. $\frac{6}{a+4} - \frac{2}{a+3}$ $\frac{4a+10}{(a+4)(a+3)}$

31. $\frac{6x}{x+3} - \frac{4x}{x-3}$ $\frac{2x^2-30x}{(x+3)(x-3)}$

32. $\frac{t}{t+2} + \frac{8}{t-2}$ $\frac{t^2+6t+16}{(t+2)(t-2)}$

Add or subtract, and then simplify, if possible. See Example 4.

33. $\frac{5x}{x-3} + \frac{4x}{3-x}$ $\frac{x}{x-3}$

34. $\frac{8x}{x-4} - \frac{10x}{4-x}$ $\frac{18x}{x-4}$

35. $\frac{9m}{m-n} - \frac{2}{n-m}$ $\frac{9m+2}{m-n}$

36. $\frac{3s}{s-x} + \frac{1}{x-s}$ $\frac{3s-1}{s-x}$

Add or subtract, and then simplify, if possible. See Example 5.

37. $4 + \frac{1}{x-2}$ $\frac{4x-7}{x-2}$

38. $2 - \frac{1}{x+1}$ $\frac{2x+1}{x+1}$

39. $x + \frac{4x}{7x-3}$ $\frac{7x^2+x}{7x-3}$

40. $x - \frac{3x}{3x-2}$ $\frac{3x^2-5x}{3x-2}$

The denominators of several fractions are given. Find the LCD. See Example 6.

41. $12xy, 18x^2y$ $36x^2y$

42. $15ab^2, 27a^2b$ $135a^2b^2$

43. $x^2 + 3x, x^2 - 9$ $x(x+3)(x-3)$

44. $3y^2 - 6y, 3y(y-4)$ $3y(y-2)(y-4)$

45. $x^3 + 27, x^2 + 6x + 9$ $(x+3)^2(x^2-3x+9)$

46. $x^3 - 8, x^2 - 4x + 4$ $(x-2)^2(x^2+2x+4)$

47. $2x^2 + 5x + 3, 4x^2 + 12x + 9, x^2 + 2x + 1$ $(2x+3)^2(x+1)^2$

48. $2x^2 + 5x + 3, 4x^2 + 12x + 9, 4x + 6$ $2(2x+3)^2(x+1)$

Perform the operations and simplify the result when possible. See Example 7.

49. $\frac{11}{5m} - \frac{5}{6m}$ $\frac{41}{30m}$

50. $\frac{5}{9s} - \frac{1}{4s}$ $\frac{11}{36s}$

51. $\frac{3}{4ab^2} - \frac{5}{2a^2b}$ $\frac{3a-10b}{4a^2b^2}$

52. $\frac{1}{5xy^3} - \frac{2}{15x^2y}$ $\frac{3x-2y^2}{15x^2y^3}$

Perform the operations and simplify the result when possible. See Example 8.

53. $\frac{1}{x+3} + \frac{2}{x^2+4x+3}$ $\frac{1}{x+1}$

54. $\frac{4}{y^2+8y+12} + \frac{1}{y+6}$ $\frac{1}{y+2}$

55. $\frac{m}{m^2+9m+20} - \frac{4}{m^2+7m+12}$ $\frac{m-5}{(m+3)(m+5)}$

56. $\frac{t}{t^2+5t+6} - \frac{2}{t^2+3t+2}$ $\frac{t-3}{(t+3)(t+1)}$

57. $\frac{x}{x^2+5x+6} + \frac{x}{x^2-4}$ $\frac{2x^2+x}{(x+3)(x+2)(x-2)}$

58. $\frac{2a}{a^2-2a-8} + \frac{3}{a^2-5a+4}$ $\frac{2a^2+a+6}{(a-4)(a+2)(a-1)}$

59. $\frac{x+2}{6x-42} - \frac{x-3}{5x-35}$ $\frac{-x+28}{30(x-7)}$

60. $\frac{x-1}{4x-24} - \frac{3x-2}{5x-30}$ $\frac{-7x+3}{20(x-6)}$

Perform the operations and simplify the result when possible. See Example 9.

61. $\frac{5x}{x+1} + \frac{3}{x+1} - \frac{2x}{x+1}$ 3

62. $\frac{4}{a+4} - \frac{2a}{a+4} + \frac{3a}{a+4}$ 1

63. $\frac{8}{x^2-9} + \frac{2}{x-3} - \frac{6}{x}$ $\frac{-4x^2+14x+54}{x(x+3)(x-3)}$ or $\frac{-2(2x^2-7x-27)}{x(x+3)(x-3)}$

64. $\frac{x}{x^2-4} - \frac{x}{x+2} + \frac{2}{x}$ $\frac{-x^3+5x^2-8}{x(x+2)(x-2)}$

65. $\dfrac{3x}{2x-1}+\dfrac{x+1}{3x+2}-\dfrac{2x}{6x^3+x^2-2x}$ $\dfrac{11x^2+7x-3}{(2x-1)(3x+2)}$

▶ **66.** $\dfrac{2}{x-2}+\dfrac{3}{x+2}-\dfrac{x-1}{x^2-4}$ $\dfrac{4x-1}{(x+2)(x-2)}$

▶ **67.** $\dfrac{1}{x+y}-\dfrac{1}{x-y}-\dfrac{2y}{y^2-x^2}$ 0

68. $\dfrac{a}{a-b}+\dfrac{b}{a+b}+\dfrac{a^2+b^2}{b^2-a^2}$ $\dfrac{2b}{a+b}$

TRY IT YOURSELF

Perform the operations and simplify the result when possible.

69. $\dfrac{s+7}{s+3}-\dfrac{s-3}{s+7}$ $\dfrac{14s+58}{(s+3)(s+7)}$

▶ **70.** $\dfrac{t+5}{t-5}-\dfrac{t-5}{t+5}$ $\dfrac{20t}{(t-5)(t+5)}$

▶ **71.** $\dfrac{x-y}{2}+\dfrac{x+y}{3}$ $\dfrac{5x-y}{6}$

72. $\dfrac{a+b}{3}+\dfrac{a-b}{7}$ $\dfrac{10a+4b}{21}$ or $\dfrac{2(5a+2b)}{21}$

73. $\dfrac{3x^2+3x}{x^2-5x+6}-\dfrac{3x^2-3x+12}{x^2-5x+6}$ $\dfrac{6}{x-3}$

▶ **74.** $\dfrac{2m^2-7}{m^4-9}+\dfrac{4-m^2}{m^4-9}$ $\dfrac{1}{m^2+3}$

75. $\dfrac{a^2+ab}{a^3-b^3}-\dfrac{b^2}{b^3-a^3}$ $\dfrac{1}{a-b}$

▶ **76.** $\dfrac{y^2-3xy}{x^3-y^3}-\dfrac{x^2+4xy}{y^3-x^3}$ $\dfrac{1}{x-y}$

77. $2x+3+\dfrac{1}{x+1}$ $\dfrac{2x^2+5x+4}{x+1}$

78. $x+1+\dfrac{1}{x-1}$ $\dfrac{x^2}{x-1}$

79. $\dfrac{4}{x^2-2x-3}-\dfrac{x}{3x^2-7x-6}$ $\dfrac{-x^2+11x+8}{(3x+2)(x+1)(x-3)}$

80. $\dfrac{x+3}{2x^2-5x+2}-\dfrac{3x-1}{x^2-x-2}$ $-\dfrac{5x+1}{(2x-1)(x+1)}$

81. $\dfrac{3}{x+1}-\dfrac{2}{x-1}+\dfrac{x+3}{x^2-1}$ $\dfrac{2}{x+1}$

82. $\dfrac{7n^2}{m-n}+\dfrac{3m}{n-m}-\dfrac{3m^2-n}{m^2-2mn+n^2}$ $\dfrac{7mn^2-7n^3-6m^2+3mn+n}{(m-n)^2}$

83. $\dfrac{8}{9y^2}+\dfrac{1}{6y^4}$ $\dfrac{16y^2+3}{18y^4}$

84. $\dfrac{5}{6a^3}+\dfrac{7}{8a^2}$ $\dfrac{21a+20}{24a^3}$

Perform the operations and simplify the result when possible. Be careful to apply the correct method, because these problems involve addition, subtraction, multiplication, and division of rational expressions.

85. $\dfrac{6}{b^2-9}\cdot\dfrac{b+3}{2b+4}$ $\dfrac{3}{(b+2)(b-3)}$

86. $\dfrac{3a^2-22a+7}{a-a^2}\cdot\dfrac{8a^2-8a}{a^2+a-56}$ $-\dfrac{24a-8}{a+8}$ or $-\dfrac{8(3a-1)}{a+8}$

87. $\dfrac{4a}{a-5}+a$ $\dfrac{a^2-a}{a-5}$

88. $\dfrac{10z}{z+4}+z$ $\dfrac{z^2+14z}{z+4}$

89. $\dfrac{2a+1}{3a-2}-\dfrac{a-4}{2-3a}$ $\dfrac{3a-3}{3a-2}$ or $\dfrac{3(a-1)}{3a-2}$

90. $\dfrac{2x+1}{x^4-81}+\dfrac{2-x}{x^4-81}$ $\dfrac{1}{(x^2+9)(x-3)}$

91. $\dfrac{x^2+x}{3x-15}\div\dfrac{(x+1)^2}{6x-30}$ $\dfrac{2x}{x+1}$

92. $\dfrac{z^2-9}{z^2+4z+3}\div\dfrac{z^2-3z}{(z+1)^2}$ $\dfrac{z+1}{z}$

93. $\dfrac{m}{m^2+5m+6}-\dfrac{2}{m^2+3m+2}$ $\dfrac{m-3}{(m+3)(m+1)}$

94. $\dfrac{1}{m+1}+\dfrac{1}{m-1}+\dfrac{2}{m^2-1}$ $\dfrac{2}{m-1}$

95. $\dfrac{27p^4}{35q}\div\dfrac{9p}{21q}$ $\dfrac{9p^3}{5}$

96. $\dfrac{12t}{25s^5}\div\dfrac{10t}{15s^2}$ $\dfrac{18}{25s^3}$

97. $\dfrac{6}{5d^2-5d}-\dfrac{3}{5d-5}$ $\dfrac{6-3d}{5d(d-1)}$

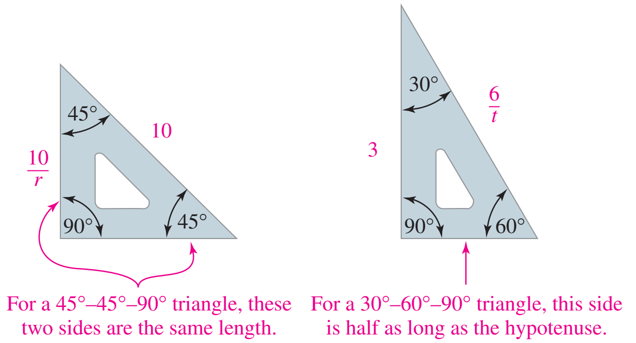
▶ **98.** $\dfrac{9}{2r^2-2r}-\dfrac{5}{2r-2}$ $\dfrac{9-5r}{2r(r-1)}$

99. $\dfrac{s^3t}{4s^2-9t^2}\cdot\dfrac{4s^2-12st+9t^2}{s^3t^2}$ $\dfrac{2s-3t}{t(2s+3t)}$

100. $\dfrac{25x^2-40xy+16y^2}{x^2y^4}\cdot\dfrac{xy^4}{25x^2-16y^2}$ $\dfrac{5x-4y}{x(5x+4y)}$

APPLICATIONS

101. DRAFTING Among the tools used in drafting are the 45°–45°–90° and the 30°–60°–90° triangles shown. Find the perimeter of each triangle. Express each result as a single rational expression. $\frac{10r+20}{r}$; $\frac{3t+9}{t}$



102. THE AMAZON The Amazon River flows in an easterly direction to the Atlantic Ocean. In Brazil, when the river is at low stage, the rate of flow is about 5 mph. Suppose that a river guide can canoe in still water at a rate of r mph.

a. Complete the table to find rational expressions that represent the time it would take the guide to canoe 3 miles downriver and to canoe 3 miles upriver on the Amazon.

	Rate (mph)	Time (hr)	Distance (ml)
Downriver	$r+5$	$\frac{3}{r+5}$	3
Upriver	$r-5$	$\frac{3}{r-5}$	3

b. Find the difference in the times for the trips upriver and downriver. Express the result as a single rational expression. $\frac{30}{(r+5)(r-5)}$ hr

WRITING

103. Explain how to find the least common denominator of a set of rational expressions.

104. Add the rational expressions by expressing them in terms of a common denominator $24b^3$. (*Note:* This is not the LCD.)

$$\frac{r}{4b^2}+\frac{s}{6b}$$

An extra step had to be performed because the lowest common denominator was not used. What was the step?

105. Write some comments to the student who wrote the following solution, explaining his misunderstanding.

$$\text{Multiply: } \frac{1}{x}\cdot\frac{3}{2} = \cancel{\frac{1\cdot 2}{x\cdot 2}\cdot\frac{3\cdot x}{2\cdot x}}$$
$$= \cancel{\frac{2}{2x}\cdot\frac{3x}{2x}}$$
$$= \cancel{\frac{6x}{2x}}$$

106. Write some comments to the student who wrote the following solution, pointing out where she made an error.

$$\text{Subtract: } \frac{1}{x}-\frac{x+1}{x} = \cancel{\frac{1-x+1}{x}}$$
$$= \cancel{\frac{2-x}{x}}$$

REVIEW

Solve each equation.

107. $a(a-6)=-9$ a repeated solution of 3

108. $x^2-\frac{1}{2}(x+1)=0$ $1, -\frac{1}{2}$

109. $y^3+y^2=0$ a repeated solution of $0, -1$

110. $5x^2=6-13x$ $\frac{2}{5}, -3$

Objectives

1. Simplify complex fractions using division.
2. Simplify complex fractions using the LCD.

SECTION 6.4
Simplifying Complex Fractions

A rational expression whose numerator and/or denominator contain rational expressions is called a **complex rational expression** or, more simply, a **complex fraction.** The expression above the main fraction bar of a complex fraction is the numerator, and the expression below the main fraction bar is the denominator. Two examples are:

$$\frac{\frac{3a}{b}}{\frac{6ac}{b^2}} \quad \leftarrow \text{Numerator} \rightarrow \quad \leftarrow \text{Main fraction bar} \rightarrow \quad \leftarrow \text{Denominator} \rightarrow \quad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$$

In this section, we will discuss two methods for simplifying complex fractions. To **simplify a complex fraction** means to write it in the form $\frac{A}{B}$, where A and B are polynomials that have no common factors.

1 Simplify complex fractions using division.

One method for simplifying complex fractions uses the fact that the main fraction bar indicates division.

Simplifying Complex Fractions

Method 1: Using Division

1. Add or subtract in the numerator and/or denominator so that the numerator is a single fraction and the denominator is a single fraction.
2. Perform the indicated division by multiplying the numerator of the complex fraction by the reciprocal of the denominator.
3. Simplify the result, if possible.

EXAMPLE 1

Use Method 1 to simplify: $\dfrac{\frac{3a}{b}}{\frac{6ac}{b^2}}$

Strategy We will perform the division indicated by the main fraction bar using the procedure for dividing rational expressions from Section 6.2.

WHY We can skip the first step of Method 1 and immediately divide because the numerator and the denominator of the complex fraction are already single fractions.

Solution

$$\dfrac{\frac{3a}{b}}{\frac{6ac}{b^2}} = \frac{3a}{b} \div \frac{6ac}{b^2}$$ The main fraction bar of the complex fraction indicates division.

$$= \frac{3a}{b} \cdot \frac{b^2}{6ac}$$ To divide rational expressions, multiply the first by the reciprocal of the second.

$$= \frac{3a \cdot b^2}{b \cdot 6ac}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{\overset{1}{\cancel{3}} \cdot \overset{1}{\cancel{a}} \cdot \overset{1}{\cancel{b}} \cdot b}{\underset{1}{\cancel{b}} \cdot 2 \cdot \underset{1}{\cancel{3}} \cdot \underset{1}{\cancel{a}} \cdot c}$$ Factor the numerator and denominator. Then simplify by removing common factors of the numerator and denominator.

$$= \frac{b}{2c}$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

Success Tip Method 1 works well when a complex fraction is written, or can be easily written, as a quotient of two single rational expressions.

Self Check 1

Use Method 1 to simplify: $\dfrac{\frac{7x^4}{8y^5}}{\frac{21x^3}{20y}}$ $\frac{5x}{6y^4}$

Now Try Problem 11

Teaching Example 1 Use Method 1 to simplify: $\dfrac{\frac{11x}{y}}{\frac{55xz}{y^7}}$

Answer: $\frac{y^6}{5z}$

2 Simplify complex fractions using the LCD.

A second method for simplifying complex fractions uses the concepts of LCD and multiplication by a form of 1. The multiplication by 1 produces a simpler, equivalent expression, which will not contain fractions in its numerator or denominator.

Simplifying Complex Fractions

Method 2: Multiplying by the LCD

1. Find the LCD of all fractions within the complex fraction.
2. Multiply the complex fraction by 1 in the form $\frac{\text{LCD}}{\text{LCD}}$.
3. Perform the operations in the numerator and denominator. No fractional expressions should remain within the complex fraction.
4. Simplify the result, if possible.

Self Check 2

Use Method 2 to simplify: $\dfrac{\frac{7x^4}{8y^5}}{\frac{21x^3}{20y}}$ $\frac{5x}{6y^4}$

Now Try **Problem 11**

Teaching Example 2 Use Method 2 to simplify: $\dfrac{\frac{11x}{y}}{\frac{55xz}{y^7}}$

Answer: $\frac{y^6}{5z}$

EXAMPLE 2

Use Method 2 to simplify: $\dfrac{\frac{3a}{b}}{\frac{6ac}{b^2}}$

Strategy We will use Method 2 to rework Example 1. We can eliminate the fractions in the numerator and denominator of the complex fraction by multiplying it by 1, written in the form $\frac{b^2}{b^2}$.

WHY We use $\frac{b^2}{b^2}$ because b^2 is the LCD of $\frac{3a}{b}$ and $\frac{6ac}{b^2}$.

Solution

$$\dfrac{\frac{3a}{b}}{\frac{6ac}{b^2}} = \dfrac{\frac{3a}{b}}{\frac{6ac}{b^2}} \cdot \frac{b^2}{b^2}$$ Multiply the complex fraction by a form of 1: $\frac{b^2}{b^2} = 1$.

$$= \dfrac{\frac{3ab^2}{b}}{\frac{6acb^2}{b^2}}$$ Multiply the numerators: $\frac{3a}{b} \cdot b^2$. Multiply the denominators: $\frac{6ac}{b^2} \cdot b^2$.

$$= \frac{3ab}{6ac}$$ Simplify the numerator of the complex fraction (highlighted in blue). Simplify the denominator of the complex fraction highlighted in red.

$$= \frac{b}{2c}$$ Simplify the resulting rational expression. This is the same result that was obtained using Method 1.

Success Tip When simplifying a complex fraction, the same result will be obtained regardless of the method used.

Self Check 3

Simplify: $\dfrac{\frac{3}{m} + 2}{m + 2}$ $\frac{3 + 2m}{m^2 + 2m}$

Now Try **Problem 17**

EXAMPLE 3

Simplify: $\dfrac{\frac{2}{x} + 5}{x + 3}$

Strategy We will simplify the complex fraction using both Method 1 and Method 2.

WHY We want to show that the result is the same using either method.

Solution

Method 1

We add in the numerator of the complex fraction to make it a single rational expression.

$$\frac{\frac{2}{x}+5}{x+3} = \frac{\frac{2}{x}+\frac{5}{1}\cdot\frac{x}{x}}{\frac{x+3}{1}}$$ Write 5 as $\frac{5}{1}$ and $x + 3$ as $\frac{x+3}{1}$. Build $\frac{5}{1}$ to have the LCD x (highlighted in blue).

$$= \frac{\frac{2}{x}+\frac{5x}{x}}{\frac{x+3}{1}}$$ Multiply $\frac{5}{1}$ and $\frac{x}{x}$ highlighted in blue.

$$= \frac{\frac{2+5x}{x}}{\frac{x+3}{1}}$$ Add $\frac{2}{x}$ and $\frac{5x}{x}$ to get $\frac{2+5x}{x}$. The numerator is now a single rational expression.

$$= \frac{2+5x}{x} \div \frac{x+3}{1}$$ Write the division indicated by the main fraction bar using a ÷ symbol.

$$= \frac{2+5x}{x} \cdot \frac{1}{x+3}$$ Multiply by the reciprocal of $\frac{x+3}{1}$.

$$= \frac{2+5x}{x^2+3x}$$ Multiply the numerators and multiply the denominators.

Method 2

The LCD of all rational expressions in the complex fraction is x.

$$\frac{\frac{2}{x}+5}{x+3} = \frac{\frac{2}{x}+5}{x+3}\cdot\frac{x}{x}$$ Multiply the complex fraction by 1 in the form $\frac{x}{x}$.

$$= \frac{\left(\frac{2}{x}+5\right)x}{(x+3)x}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{\frac{2}{x}\cdot x + 5\cdot x}{x\cdot x + 3\cdot x}$$ ← In the numerator, distribute the multiplication by x. ← In the denominator, distribute the multiplication by x.

$$= \frac{2+5x}{x^2+3x}$$ Perform each of the four multiplications by x. Notice that no fractional expression remains within the complex fraction.

Teaching Example 3 Simplify: $\frac{\frac{4}{x}+3}{5+x}$
Answer: $\frac{4+3x}{x(5+x)}$

Using Your CALCULATOR Checking Algebra

A check of the simplification done in Example 3 can be performed using a scientific calculator. If

$$\frac{\frac{2}{x}+5}{x+3} = \frac{2+5x}{x^2+3x}$$

the expressions on each side will have identical values when evaluated for a given value of x (say, $x = 4$). To evaluate the expression on the left side, we enter these numbers and press these keys.

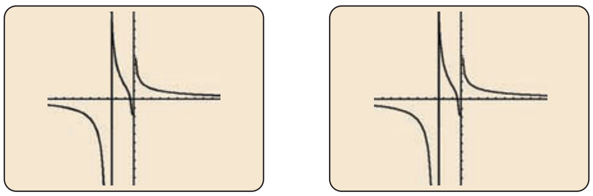
$\boxed{(}\ 2\ \boxed{\div}\ \mathbf{4}\ \boxed{+}\ 5\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ \mathbf{4}\ \boxed{+}\ 3\ \boxed{)}\ \boxed{=}$ $\boxed{0.785714286}$

To evaluate the expression on the right side, we enter these numbers and press these keys.

$\boxed{(}\ 2\ \boxed{+}\ 5\ \boxed{\times}\ \mathbf{4}\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ \mathbf{4}\ \boxed{x^2}\ \boxed{+}\ 3\ \boxed{\times}\ \mathbf{4}\ \boxed{)}\ \boxed{=}$ $\boxed{0.785714286}$

The results are the same, so it appears that the simplification is correct. We say "appears" because checking for only a single value of x is not definitive. The expressions should yield identical values when evaluated for any value of x for which the fractions are defined.

We can also check the simplification in Example 3 by graphing the functions $f(x) = \dfrac{\frac{2}{x} + 5}{3 + x}$ shown in figure (a) and $g(x) = \dfrac{2 + 5x}{x^2 + 3x}$ shown in figure (b) and observing that the graphs are the same. Each graph has window settings of $[-10, 10]$ for x and $[-10, 10]$ for y.



(a) (b)

Self Check 4

Simplify: $\dfrac{\frac{5}{s} - \frac{2}{t}}{\frac{7}{s} + \frac{3}{t}}$ $\dfrac{5t - 2s}{7t + 3s}$

Now Try **Problem 21**

Teaching Example 4 Simplify:
$\dfrac{\frac{2}{x} - \frac{5}{y}}{\frac{3}{x} + \frac{8}{y}}$
Answer:
$\dfrac{2y - 5x}{3y + 8x}$

EXAMPLE 4

Simplify: $\dfrac{\frac{2}{x} + \frac{3}{y}}{\frac{3}{x} - \frac{4}{y}}$

Strategy We will simplify the complex fraction using both Method 1 and Method 2.

WHY We want to show that using either method, the result is the same.

Solution

Method 1

To write the numerator and denominator of the complex fraction as single fractions, we add the rational expressions in the numerator and subtract the rational expressions in the denominator.

$$\frac{\frac{2}{x} + \frac{3}{y}}{\frac{3}{x} - \frac{4}{y}} = \frac{\frac{2}{x} \cdot \frac{y}{y} + \frac{3}{y} \cdot \frac{x}{x}}{\frac{3}{x} \cdot \frac{y}{y} - \frac{4}{y} \cdot \frac{x}{x}}$$

The LCD for the numerator is xy. Build each rational expression so that each has a denominator of xy.
The LCD for the denominator is xy. Build each rational expression so that each has a denominator of xy.

$$= \frac{\frac{2y}{xy} + \frac{3x}{xy}}{\frac{3y}{xy} - \frac{4x}{xy}}$$

Multiply the numerators.
Multiply the denominators.

$$= \frac{\frac{2y + 3x}{xy}}{\frac{3y - 4x}{xy}}$$

Add the rational expressions in the numerator and subtract the rational expressions in the denominator.

$$= \frac{2y + 3x}{xy} \div \frac{3y - 4x}{xy}$$ Write the division indicated by the main fraction bar using a ÷ symbol.

$$= \frac{2y + 3x}{xy} \cdot \frac{xy}{3y - 4x}$$ Multiply by the reciprocal of $\frac{3y - 4x}{xy}$.

$$= \frac{(2y + 3x) \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{y}}}{\underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{y}} \cdot (3y - 4x)}$$ Multiply the rational expressions and simplify the result.

$$= \frac{2y + 3x}{3y - 4x}$$

Method 2

The LCD of the fractions appearing in the complex fraction is xy. We multiply the complex fraction by 1 in the form of $\frac{\text{LCD}}{\text{LCD}}$.

$$\frac{\frac{2}{x} + \frac{3}{y}}{\frac{3}{x} - \frac{4}{y}} = \frac{\frac{2}{x} + \frac{3}{y}}{\frac{3}{x} - \frac{4}{y}} \cdot \frac{xy}{xy}$$ Multiply the complex fraction by a form of 1: $\frac{xy}{xy}$.

$$= \frac{\left(\frac{2}{x} + \frac{3}{y}\right)xy}{\left(\frac{3}{x} - \frac{4}{y}\right)xy}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{\frac{2}{x} \cdot xy + \frac{3}{y} \cdot xy}{\frac{3}{x} \cdot xy - \frac{4}{y} \cdot xy}$$ ← In the numerator, distribute the multiplication by xy. ← In the denominator, distribute the multiplication by xy.

$$= \frac{2y + 3x}{3y - 4x}$$ Perform each of the four multiplications by xy.

If negative exponents occur, we write an equivalent expression involving positive exponents and simplify using Method 1 or Method 2.

EXAMPLE 5

Simplify: $\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}$

Strategy We will use the rule for exponents $a^{-m} = \frac{1}{a^m}$ to write each term of the numerator and denominator without negative exponents.

WHY We can then simplify the resulting complex fraction.

Solution

We write the complex fraction without using negative exponents and use Method 2 to simplify.

$$\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}} = \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}}$$ Use the rule for negative exponents to write x^{-1}, y^{-1}, x^{-2}, and y^{-2} using positive exponents. See the Caution! box.

Self Check 5

Simplify: $\frac{x^{-2} + y^{-2}}{x^{-1} - y^{-1}}$

Now Try Problem 29

Self Check 5 Answer
$\frac{y^2 + x^2}{xy^2 - x^2y}$ or $\frac{x^2 + y^2}{xy(y - x)}$

Teaching Example 5 Simplify: $\frac{2x^{-1} + 3y^{-1}}{x^{-2} + 5y^{-2}}$
Answer:
$\frac{2xy^2 + 3x^2y}{y^2 + 5x^2}$

$$= \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x^2} - \frac{1}{y^2}} \cdot \frac{x^2y^2}{x^2y^2}$$ The LCD of all rational expressions in the complex fraction is x^2y^2. Multiply the complex fraction by $\frac{LCD}{LCD}$.

$$= \frac{\left(\frac{1}{x} + \frac{1}{y}\right)x^2y^2}{\left(\frac{1}{x^2} - \frac{1}{y^2}\right)x^2y^2}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{\frac{1}{x} \cdot x^2y^2 + \frac{1}{y} \cdot x^2y^2}{\frac{1}{x^2} \cdot x^2y^2 - \frac{1}{y^2} \cdot x^2y^2}$$ In the numerator, distribute the multiplication by x^2y^2. In the denominator, distribute the multiplication by x^2y^2.

$$= \frac{xy^2 + yx^2}{y^2 - x^2}$$ Perform each of the four multiplications by x^2y^2.

$$= \frac{xy\overset{1}{\cancel{(y + x)}}}{\underset{1}{\cancel{(y + x)}}(y - x)}$$ Factor the numerator and denominator.

$$= \frac{xy}{y - x}$$ Simplify the rational expression by removing the factor $y + x$ that is common to the numerator and denominator.

Caution! Recall that a *factor* can be moved from the numerator to the denominator if the sign of its exponent is changed. However, in this case, x^{-1}, y^{-1}, x^{-2}, and y^{-2} are *terms*. Thus,

$$\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}} \neq \frac{x^2 + y^2}{x - y}$$

Self Check 6

Simplify: $\dfrac{\frac{b}{b + 4} + \frac{2}{b + 3}}{\frac{b}{b^2 + 7b + 12}}$

Now Try **Problem 33**

Self Check 6 Answer

$\dfrac{b^2 + 5b + 8}{b}$

Teaching Example 6 Simplify:

$\dfrac{\frac{4}{x^2 + 9x + 14}}{\frac{5}{x + 7} - \frac{2}{x + 2}}$

Answer:

$\dfrac{4}{3x - 4}$

EXAMPLE 6

Simplify: $\dfrac{\frac{1}{a^2 - 3a + 2}}{\frac{3}{a - 2} - \frac{2}{a - 1}}$

Strategy We will factor $a^2 - 3a + 2$ and determine the LCD for all the fractions appearing in the complex fraction. Then we will use Method 2 to simplify.

WHY Method 2 works well when the complex fraction has sums and/or differences in the numerator or denominator.

Solution

To find the LCD for all the fractions appearing in the complex fraction, we must factor $a^2 - 3a + 2$.

$$\frac{\frac{1}{a^2 - 3a + 2}}{\frac{3}{a - 2} - \frac{2}{a - 1}} = \frac{\frac{1}{(a - 2)(a - 1)}}{\frac{3}{a - 2} - \frac{2}{a - 1}}$$ Factor the trinomial $a^2 - 3a + 2$.

The LCD of the fractions in the numerator and denominator of the complex fraction is $(a - 2)(a - 1)$. We multiply the numerator and the denominator by the LCD.

$$= \frac{\dfrac{1}{(a-2)(a-1)}}{\dfrac{3}{a-2}-\dfrac{2}{a-1}} \cdot \frac{(a-2)(a-1)}{(a-2)(a-1)}$$

$$= \frac{\left[\dfrac{1}{(a-2)(a-1)}\right](a-2)(a-1)}{\left(\dfrac{3}{a-2}-\dfrac{2}{a-1}\right)(a-2)(a-1)}$$

Multiply the numerators.
Multiply the denominators.

$$= \frac{\dfrac{(a-2)(a-1)}{(a-2)(a-1)}}{\dfrac{3(a-2)(a-1)}{a-2}-\dfrac{2(a-2)(a-1)}{a-1}}$$

Perform the multiplication in the numerator.
In the denominator, distribute the LCD, $(a-2)(a-1)$.

$$= \frac{1}{3(a-1)-2(a-2)}$$

Simplify each of the three rational expressions highlighted in blue.

$$= \frac{1}{3a-3-2a+4}$$

In the denominator, use the distributive property.

$$= \frac{1}{a+1}$$

In the denominator, combine like terms.

ANSWERS TO SELF CHECKS

1. $\frac{5x}{6y^4}$ **2.** $\frac{5x}{6y^4}$ **3.** $\frac{3+2m}{m^2+2m}$ **4.** $\frac{5t-2s}{7t+3s}$ **5.** $\frac{y^2+x^2}{xy^2-x^2y}$ or $\frac{x^2+y^2}{xy(y-x)}$ **6.** $\frac{b^2+5b+8}{b}$

SECTION 6.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\dfrac{\frac{x}{y}+\frac{1}{x}}{\frac{1}{y}+\frac{2}{x}}$ and $\dfrac{\frac{5a^2}{b}}{\frac{b}{2a^3}}$ are examples of complex _rational_ expressions, or more simply, _complex_ fractions.

▶ **2.** To _simplify_ a complex fraction means to express it in the form $\frac{A}{B}$, where A and B are polynomials with no common factors.

CONCEPTS

3. To simplify the following complex fraction, it is multiplied by what form of 1? $\frac{t^2}{t^2}$

$$\frac{\dfrac{4}{t^2}+\dfrac{b}{t}}{\dfrac{3b}{t}} = \frac{\dfrac{4}{t^2}+\dfrac{b}{t}}{\dfrac{3b}{t}} \cdot \frac{t^2}{t^2}$$

▶ **4.** Determine the LCD of the rational expressions appearing in each complex fraction.

a. $\dfrac{1+\frac{4}{c}}{\frac{2}{c}+c}$ c

b. $\dfrac{\frac{6}{m^2}+\frac{1}{2m}}{\frac{m^2-1}{4}}$ $4m^2$

c. $\dfrac{\frac{p}{p+2}+\frac{12}{p+3}}{\frac{p-1}{p^2+5p+6}}$ $(p+2)(p+3)$

d. $\dfrac{2+\frac{3}{x+1}}{\frac{1}{x}+x+x^2}$ $x(x+1)$

▶ Selected exercises available online at
www.webassign.net/brookscole

NOTATION

Complete each solution to simplify the rational expression.

5. $\dfrac{\frac{5m^2}{6}}{\frac{25m}{3}} = \dfrac{5m^2}{6} \div \dfrac{25m}{3}$ Use Method 1

$= \dfrac{5m^2}{6} \cdot \dfrac{3}{25m}$

$= \dfrac{5 \cdot m \cdot m \cdot 3}{2 \cdot 3 \cdot 5 \cdot 5 \cdot m}$

$= \dfrac{m}{10}$

6. $\dfrac{\frac{2}{a} - \frac{1}{b}}{\frac{5}{a} + \frac{3}{b}} = \dfrac{\frac{2}{a} - \frac{1}{b}}{\frac{5}{a} + \frac{3}{b}} \cdot \dfrac{ab}{ab}$ Use Method 2

$= \dfrac{\left(\frac{2}{a} - \frac{1}{b}\right) ab}{\left(\frac{5}{a} + \frac{3}{b}\right) ab}$

$= \dfrac{\frac{2}{a} \cdot ab - \frac{1}{b} \cdot ab}{\frac{5}{a} \cdot ab + \frac{3}{b} \cdot ab}$

$= \dfrac{2b - a}{5b + 3a}$

7. a. Fill in the blank: The expression $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ is equivalent to $\frac{a}{b} \div \frac{c}{d}$.

b. What is the numerator and what is the denominator of the following complex fraction?

$$\dfrac{6 - k - \frac{5}{k}}{k^2 - 9}$$ $6 - k - \frac{5}{k}$; $k^2 - 9$

8. A student checks her answers with those in the back of her textbook. Determine whether they are equivalent.

Student's answer	Book's answer	Equivalent?
$\frac{3 + 2t}{t^2 + 2t}$	$\frac{2t + 3}{t(t + 2)}$	yes
$\frac{5 - 3x^2}{x + x^2}$	$-\frac{3x^2 - 5}{x^2 + x}$	yes
$\frac{3xy(y + x)}{(2y - x)(2y + 3x)}$	$\frac{3xy^2 + 3x^2y}{(2y + x)(2y - 3x)}$	no

GUIDED PRACTICE

Simplify each complex fraction. See Examples 1 and 2.

9. $\dfrac{\frac{a^6}{2}}{\frac{3a}{4}}$ $\frac{2a^5}{3}$

10. $\dfrac{\frac{3b^7}{4}}{\frac{b^9}{2}}$ $\frac{3}{2b^2}$

11. $\dfrac{\frac{20x}{y}}{\frac{36x}{y^2}}$ $\frac{5y}{9}$

12. $\dfrac{\frac{5t^4}{9x^2}}{\frac{2t}{18x}}$ $\frac{5t^3}{x}$

13. $\dfrac{\frac{18x^5}{35y^2}}{\frac{2x^8}{21y^5}}$ $\frac{27y^3}{5x^3}$

14. $\dfrac{\frac{32m}{45n^{10}}}{\frac{8m^7}{3n^{11}}}$ $\frac{4n}{15m^6}$

15. $\dfrac{\frac{16c}{77d^4}}{\frac{28c^7}{55d^0}}$ $\frac{20}{49c^6d^4}$

16. $\dfrac{\frac{8m^{10}}{27n^0}}{\frac{32m}{63n^9}}$ $\frac{7m^9n^9}{12}$

Simplify each complex fraction. See Example 3.

17. $\dfrac{\frac{3}{a} - 2}{a - 3}$ $\frac{3 - 2a}{a^2 - 3a}$

18. $\dfrac{\frac{5}{t} - 5}{t - 5}$ $\frac{5 - 5t}{t^2 - 5t}$

19. $\dfrac{4p - \frac{4}{p}}{12 - \frac{4}{p}}$ $\frac{p^2 - 1}{3p - 1}$

20. $\dfrac{3s - \frac{3}{s}}{6 + \frac{3}{s}}$ $\frac{s^2 - 1}{2s + 1}$

Simplify each complex fraction. See Example 4.

21. $\dfrac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{x} + \frac{1}{y}}$ $y - x$

22. $\dfrac{\frac{y}{x} - \frac{x}{y}}{\frac{1}{y} - \frac{1}{x}}$ $-y - x$

23. $\dfrac{\frac{1}{a} - \frac{1}{b}}{\frac{a}{b} - \frac{b}{a}}$ $-\frac{1}{a + b}$

24. $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{a}{b} - \frac{b}{a}}$ $\frac{1}{a - b}$

25. $\dfrac{\frac{2}{a^2} + \frac{1}{a}}{\frac{2}{a} + \frac{1}{a^2}}$ $\frac{2 + a}{2a + 1}$

26. $\dfrac{\frac{3}{y^2} - \frac{4}{y}}{\frac{1}{y} + \frac{15}{y^2}}$ $\frac{3 - 4y}{y + 15}$

27. $\dfrac{\frac{3}{b^2} - \frac{4}{b} + 1}{1 - \frac{1}{b} - \frac{6}{b^2}}$ $\frac{b - 1}{b + 2}$

28. $\dfrac{\frac{18}{c^2} + \frac{11}{c} + 1}{1 - \frac{3}{c} - \frac{10}{c^2}}$ $\frac{c + 9}{c - 5}$

Simplify each complex fraction. See Example 5.

29. $\dfrac{x^{-2}-y^{-2}}{x^{-1}-y^{-1}}$ $\dfrac{y+x}{xy}$

30. $\dfrac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}$ $\dfrac{y+x}{y-x}$

31. $\dfrac{a-b^{-2}}{b-a^{-2}}$ $\dfrac{a^3b^2-a^2}{a^2b^3-b^2}$ or $\dfrac{a^2(ab^2-1)}{b^2(a^2b-1)}$

32. $\dfrac{m^{-1}+2n}{m^{-1}-n}$ $\dfrac{1+2mn}{1-mn}$

Simplify each complex fraction. See Example 6.

33. $\dfrac{\dfrac{3}{z-3}+\dfrac{2}{z-2}}{\dfrac{5z}{z^2-5z+6}}$ $\dfrac{5z-12}{5z}$

34. $\dfrac{\dfrac{h}{h^2+3h+2}}{\dfrac{4}{h+2}-\dfrac{4}{h+1}}$ $-\dfrac{h}{4}$

35. $\dfrac{\dfrac{2}{x+3}-\dfrac{1}{x-3}}{\dfrac{3}{x^2-9}}$ $\dfrac{x-9}{3}$

36. $\dfrac{2+\dfrac{1}{x^2-1}}{1+\dfrac{1}{x-1}}$ $\dfrac{2x^2-1}{x^2+x}$

TRY IT YOURSELF

Simplify each complex fraction.

37. $\dfrac{1+\dfrac{x}{y}}{1-\dfrac{x}{y}}$ $\dfrac{y+x}{y-x}$

38. $\dfrac{\dfrac{x}{y}+1}{1-\dfrac{x}{y}}$ $\dfrac{x+y}{y-x}$

39. $\dfrac{\dfrac{x^2+5x+6}{3xy}}{\dfrac{9-x^2}{6xy}}$ $\dfrac{2x+4}{3-x}$

40. $\dfrac{\dfrac{x-y}{xy}}{\dfrac{y-x}{x}}$ $-\dfrac{1}{y}$

41. $\dfrac{1+\dfrac{6}{x}+\dfrac{8}{x^2}}{1+\dfrac{1}{x}-\dfrac{12}{x^2}}$ $\dfrac{x+2}{x-3}$

42. $\dfrac{1-x-\dfrac{2}{x}}{\dfrac{6}{x^2}+\dfrac{1}{x}-1}$ $\dfrac{x^3-x^2+2x}{(x-3)(x+2)}$

43. $\dfrac{\dfrac{ac-ad-c+d}{a^3-1}}{\dfrac{c^2-2cd+d^2}{a^2+a+1}}$ $\dfrac{1}{c-d}$

44. $\dfrac{\dfrac{2x-tx+2y-ty}{x^2+2xy+y^2}}{\dfrac{t^3-8}{15x+15y}}$ $-\dfrac{15}{t^2+2t+4}$

45. $\dfrac{\dfrac{1}{a+1}+1}{\dfrac{3}{a-1}+1}$ $\dfrac{a-1}{a+1}$

46. $\dfrac{2+\dfrac{4}{y-7}}{\dfrac{4}{y-7}}$ $\dfrac{y-5}{2}$

47. $\dfrac{5ab^2}{\dfrac{ab}{25}}$ $125b$

48. $\dfrac{\dfrac{6a^2b}{4t}}{3a^2b^2}$ $\dfrac{1}{2bt}$

49. $\dfrac{a-4+\dfrac{1}{a}}{-\dfrac{1}{a}-a+4}$ -1

50. $\dfrac{a+1+\dfrac{1}{a^2}}{\dfrac{1}{a^2}+a-1}$ $\dfrac{a^3+a^2+1}{a^3-a^2+1}$

51. $\dfrac{\dfrac{2}{a+1}+\dfrac{1}{a+3}}{\dfrac{2a}{a^2+4a+3}}$ $\dfrac{3a+7}{2a}$

52. $\dfrac{\dfrac{2}{t-2}-\dfrac{1}{t^2+t-6}}{\dfrac{4}{t+3}}$ $\dfrac{2t+5}{4t-8}$

53. $\dfrac{y}{x^{-1}-y^{-1}}$ $\dfrac{xy^2}{y-x}$

54. $\dfrac{x^{-1}+y^{-1}}{(x+y)^{-1}}$ $\dfrac{(x+y)^2}{xy}$

55. $\dfrac{\dfrac{1}{x^2}-\dfrac{3}{xy}+\dfrac{2}{y^2}}{\dfrac{2}{x^2}-\dfrac{1}{xy}-\dfrac{1}{y^2}}$ $\dfrac{y-2x}{2y+x}$

56. $\dfrac{\dfrac{3}{s^2}+\dfrac{7}{st}+\dfrac{2}{t^2}}{\dfrac{2}{t^2}-\dfrac{5}{st}-\dfrac{3}{s^2}}$ $\dfrac{3t+s}{s-3t}$

57. $\dfrac{5xy}{1+\dfrac{1}{xy}}$ $\dfrac{5x^2y^2}{xy+1}$

58. $\dfrac{3a}{a+\dfrac{1}{a}}$ $\dfrac{3a^2}{a^2+1}$

59. $\dfrac{\dfrac{2}{y-1}-\dfrac{2}{y}}{\dfrac{3}{y-1}-\dfrac{1}{1-y}}$ $\dfrac{1}{2y}$

60. $\dfrac{\dfrac{1}{x}-\dfrac{4}{x-1}}{\dfrac{3}{x-1}+\dfrac{2}{x}}$ $\dfrac{-3x-1}{5x-2}$

61. $\dfrac{\dfrac{t}{x^2-y^2}}{\dfrac{t}{x+y}}$ $\dfrac{1}{x-y}$

62. $\dfrac{\dfrac{7}{a-b}}{\dfrac{b}{a^3-b^3}}$ $\dfrac{7(a^2+ab+b^2)}{b}$

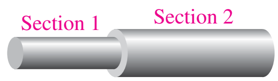
63. $\dfrac{\dfrac{4}{cd}}{c^{-1}+d^{-1}}$ $\dfrac{4}{c+d}$

64. $\dfrac{\dfrac{y}{x}+x^{-1}}{x^{-1}+\dfrac{2x}{y}}$ $\dfrac{y^2+y}{2x^2+y}$

APPLICATIONS

65. ENGINEERING The stiffness k of the shaft shown is given by the formula

$$k=\frac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_2}}$$



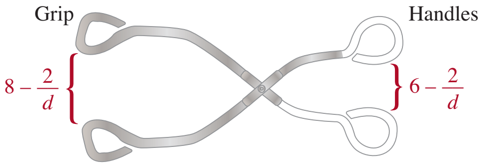
where k_1 and k_2 are the individual stiffnesses of each section. Simplify the complex fraction. $\dfrac{k_1k_2}{k_2+k_1}$

66. TRANSPORTATION If a bus travels a distance d_1 at a speed s_1, and then travels a distance d_2 at a speed s_2, the average (mean) speed $\bar{s}$ is given by the formula

$$\bar{s}=\frac{d_1+d_2}{\dfrac{d_1}{s_1}+\dfrac{d_2}{s_2}}$$

Simplify the complex fraction. $\dfrac{s_1s_2(d_1+d_2)}{d_1s_2+d_2s_1}$

▶ **67.** KITCHEN UTENSILS What is the ratio of the width of the grip of the ice tongs to the width of the opening of the handles? Express the result in simplest form. $\frac{4d-1}{3d-1}$



▶ **68.** DATA ANALYSIS Use the data in the table to find the average measurement for the three-trial experiment. Express the answer as a simplified rational expression. $\frac{7k}{30}$

	Trial 1	Trial 2	Trial 3
Measurement	$\frac{k}{3}$	$\frac{k}{5}$	$\frac{k}{6}$

WRITING

69. What is a complex fraction?

▶ **70.** Which of the two methods for simplifying a complex fraction do you think is simpler? Explain.

71. Evaluate each expression for $x = 2$.

$$\frac{\frac{4}{3} + x}{\frac{x}{2} + \frac{2}{x}} \text{ and } \frac{x^2 + 4}{x + 2}$$

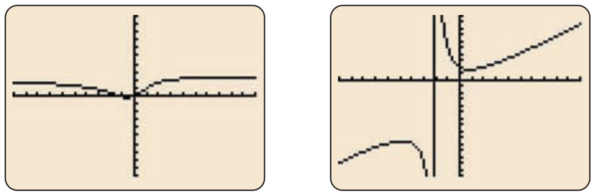
Determine whether the result verifies that

$$\frac{\frac{4}{3} + x}{\frac{x}{2} + \frac{2}{x}} = \frac{x^2 + 4}{x + 2}$$

72. To check a simplification to verify that

$$\frac{\frac{4}{3} + x}{\frac{x}{2} + \frac{2}{x}} = \frac{x^2 + 4}{x + 2},$$

a student graphed each expression's associated function. What conclusion can be made about the simplification from the graphs?



(a) (b)

REVIEW

Solve each equation.

73. $\frac{8(a-5)}{3} = 2(a-4)$ 8

74. $\frac{3t^2}{5} + \frac{7t}{10} = \frac{3t+6}{5}$ $\frac{4}{3}, -\frac{3}{2}$

75. $a^4 - 13a^2 + 36 = 0$ 2, −2, 3, −3

▶ **76.** $|2x - 1| = 9$ 5, −4

Objectives

1 Divide a monomial by a monomial.

2 Divide a polynomial by a monomial.

3 Divide a polynomial by a polynomial.

4 Divide polynomials with missing terms.

SECTION 6.5 Dividing Polynomials

We have discussed addition, subtraction, and multiplication of polynomials. We will now discuss how to divide polynomials. This topic appears in a chapter about rational expressions because rational expressions indicate division of polynomials. For example,

$$\frac{x^2 - 3x + 7}{x + 1} = (x^2 - 3x + 7) \div (x + 1)$$

We begin with the simplest case, dividing a monomial by a monomial.

1 Divide a monomial by a monomial.

To divide monomials, we can use the method for simplifying fractions or the quotient rule for exponents.

EXAMPLE 1 Simplify: **a.** $\dfrac{21x^5}{7x^2}$ **b.** $\dfrac{10r^6s}{6rs^3}$

Write answers using positive exponents.

Strategy First, we will simplify the rational expression by factoring and removing any factors that are common to the numerator and the denominator. Then we will simplify the rational expression using the rules for exponents.

WHY We want to show that the result is the same using either method.

Solution

By simplifying fractions

a. $\dfrac{21x^5}{7x^2} = \dfrac{3 \cdot \overset{1}{\cancel{7}} \cdot \overset{1}{\cancel{x}} \cdot \overset{1}{\cancel{x}} \cdot x \cdot x \cdot x}{\underset{1}{\cancel{7}} \cdot \underset{1}{\cancel{x}} \cdot \underset{1}{\cancel{x}}}$

$= 3x^3$

b. $\dfrac{10r^6s}{6rs^3} = \dfrac{\overset{1}{\cancel{2}} \cdot 5 \cdot \overset{1}{\cancel{r}} \cdot r \cdot r \cdot r \cdot r \cdot r \cdot \overset{1}{\cancel{s}}}{\underset{1}{\cancel{2}} \cdot 3 \cdot \underset{1}{\cancel{r}} \cdot \underset{1}{\cancel{s}} \cdot s \cdot s}$

$= \dfrac{5r^5}{3s^2}$

Using the rules for exponents

$\dfrac{21x^5}{7x^2} = 3x^{5-2}$ Divide the coefficients. Keep the common base x and subtract exponents.

$= 3x^3$

$\dfrac{10r^6s}{6rs^3} = \dfrac{5}{3}r^{6-1}s^{1-3}$ Simplify $\frac{10}{6}$. Keep each base and subtract exponents.

$= \dfrac{5}{3}r^5s^{-2}$

$= \dfrac{5r^5}{3s^2}$ Move s^{-2} to the denominator and change the sign of the exponent.

Self Check 1

Simplify:

a. $\dfrac{30y^4}{5y^2}$ $6y^2$

b. $\dfrac{8c^2d^6}{32c^5d^2}$ $\dfrac{d^4}{4c^3}$

Now Try **Problems 15 and 17**

Teaching Example 1 Simplify: **a.** $\dfrac{3a^2b^3}{2a^3b}$ **b.** $\dfrac{52x^4y}{13xy}$

Answers:

a. $\dfrac{3b^2}{2a}$ **b.** $4x^3$

Success Tip In this section, you will see that regardless of the number of terms involved, every polynomial division is a series of monomial divisions.

2 Divide a polynomial by a monomial.

Recall that to add two fractions with the same denominator, we add their numerators and keep the common denominator.

$$\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$$

We can use this rule in reverse to divide polynomials by monomials.

Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Let A, B, and D represent monomials, where D is not 0,

$$\frac{A+B}{D} = \frac{A}{D} + \frac{B}{D}$$

EXAMPLE 2 Divide: **a.** $\dfrac{9x^2 + 6x}{3x}$ **b.** $\dfrac{12a^4b^3 - 18a^3b^2 + 2a^2}{6a^2b^2}$

Strategy We will divide each term of the polynomial in the numerator by the monomial in the denominator.

Self Check 2

Divide:

a. $\dfrac{50h^3 + 15h^2}{5h^2}$

b. $\dfrac{22s^5t^2 - s^4t^3 + 44s^2t}{11s^2t^2}$

Now Try **Problems 21 and 25**

Self Check 2 Answers

a. $10h + 3$ b. $2s^3 - \dfrac{s^2t}{11} + \dfrac{4}{t}$

Teaching Example 2 Divide:

a. $\dfrac{-3x^2 + 6x}{12x}$

b. $\dfrac{16x^3y^2 - 8x^2y^5 + 20x}{4x^2y^3}$

Answers:

a. $-\dfrac{1}{4}x + \dfrac{1}{2}$ b. $\dfrac{4x}{y} - 2y^2 + \dfrac{5}{xy^3}$

WHY A fraction bar indicates division of the numerator by the denominator.

Solution

a. Here, we have a binomial divided by a monomial.

$$\frac{9x^2 + 6x}{3x} = \frac{9x^2}{3x} + \frac{6x}{3x}$$ Divide each term of the numerator, $9x^2 + 6x$, by the denominator, $3x$.

$$= 3x^{2-1} + 2x^{1-1}$$ Perform each monomial division. Divide the coefficients. Keep each base and subtract the exponents.

$$= 3x + 2$$ Recall that $x^0 = 1$.

Check: We multiply the divisor, $3x$, and the quotient, $3x + 2$. The result should be the dividend, $9x^2 + 6x$.

$$3x(3x + 2) = 9x^2 + 6x$$ The answer checks.

The Language of Algebra The names of the parts of a division statement are

$$\underset{\text{Divisor}}{\overset{\text{Dividend}}{\frac{9x^2 + 6x}{3x}}} = \underset{\text{Quotient}}{3x + 2}$$

b. Here we have a trinomial divided by a monomial.

$$\frac{12a^4b^3 - 18a^3b^2 + 2a^2}{6a^2b^2} = \frac{12a^4b^3}{6a^2b^2} - \frac{18a^3b^2}{6a^2b^2} + \frac{2a^2}{6a^2b^2}$$ Divide each term of the numerator by the denominator, $6a^2b^2$.

$$= 2a^{4-2}b^{3-2} - 3a^{3-2}b^{2-2} + \frac{a^{2-2}}{3b^2}$$ Perform each monomial division. Simplify: $\frac{2}{6} = \frac{1}{3}$.

$$= 2a^2b - 3a + \frac{1}{3b^2}$$

Since the variables in a polynomial must have whole-number exponents, $2a^2b - 3a + \frac{1}{3b^2}$ is not a polynomial because the last term can be written as $\frac{1}{3}b^{-2}$.

Check: $6a^2b^2\left(2a^2b - 3a + \dfrac{1}{3b^2}\right) = 12a^4b^3 - 18a^3b^2 + 2a^2$ The answer checks.

Success Tip The sum, difference, and product of two polynomials are always polynomials. However, as seen in Example 2b, the quotient of two polynomials is not always a polynomial.

3 Divide a polynomial by a polynomial.

To divide a polynomial by a polynomial (other than a monomial), we use a method similar to long division in arithmetic. To use long division to divide $x^2 + 7x + 12$ (the *dividend*) by $x + 4$ (the *divisor*), we proceed as follows:

Step 1
$$\begin{array}{r} x \phantom{{}+7x+12} \\ x + 4\overline{)x^2 + 7x + 12} \end{array}$$
Divide the first term of the dividend by the first term of the divisor: $\frac{x^2}{x} = x$. Write the result, x, above the long division symbol.

Step 2

$$\begin{array}{r} x \phantom{{}+7x+12} \\ x+4\overline{)x^2+7x+12} \\ \underline{x^2+4x} \phantom{{}+12} \end{array}$$

Multiply each term of the divisor by x. Write the result, $x^2 + 4x$, under $x^2 + 7x$, and draw a line. Be sure to align the like terms.

Step 3

$$\begin{array}{r} x \phantom{{}+7x+12} \\ x+4\overline{)x^2+7x+12} \\ -\underline{(x^2+4x)} \phantom{{}+12} \\ 3x+12 \end{array}$$

Subtract $x^2 + 4x$ from $x^2 + 7x$. Work column by column: $x^2 - x^2 = 0$ and $7x - 4x = 3x$.

Bring down the next term, 12.

Step 4

$$\begin{array}{r} x \phantom{{}+7x} + 3 \\ x+4\overline{)x^2+7x+12} \\ -\underline{(x^2+4x)} \phantom{{}+12} \\ 3x+12 \end{array}$$

Divide the first term of $3x + 12$ by the first term of the divisor: $\frac{3x}{x} = 3$. Write $+\ 3$ above the long division symbol to form the second term of the quotient.

Step 5

$$\begin{array}{r} x \phantom{{}+7x} + 3 \\ x+4\overline{)x^2+7x+12} \\ -\underline{(x^2+4x)} \phantom{{}+12} \\ 3x+12 \\ \underline{3x+12} \\ 0 \end{array}$$

Multiply each term of the divisor by 3. Write the result, $3x + 12$, under $3x + 12$ and draw a line. Be sure to align the like terms.

Step 6

$$\begin{array}{r} x \phantom{{}+7x} + 3 \\ x+4\overline{)x^2+7x+12} \\ -\underline{(x^2+4x)} \phantom{{}+12} \\ 3x+12 \\ -\underline{(3x+12)} \\ 0 \end{array}$$

Subtract $3x + 12$ from $3x + 12$. Work vertically: $3x - 3x = 0$ and $12 - 12 = 0$.

This is the remainder.

Step 7 The division process stops when the result of the subtraction is a constant or a polynomial with degree less than the degree of the divisor. Here, the quotient is $x + 3$ and the remainder is 0.

We can check the answer using the fact that for any division:

Divisor · quotient + remainder = dividend

Check: $\overbrace{(x+4)(x+3)}^{\text{Divisor} \cdot \text{quotient}} + \overbrace{0}^{\text{remainder}} = \overbrace{x^2+7x+12}^{\text{dividend}}$ The answer checks.

Success Tip The long division method aligns like terms vertically.

$$\begin{array}{r} x + 3 \\ x+4\overline{)x^2+7x+12} \\ -\underline{(x^2+4x)} \phantom{{}+12} \\ 3x+12 \\ -\underline{(3x+12)} \\ 0 \end{array}$$

x^2-terms

x-terms

constants

Self Check 3

Divide: $\dfrac{4x^3 + 11x^2 + 2x - 3}{4x + 3}$

Now Try **Problems 27 and 33**

Self Check 3 Answer

$x^2 + 2x - 1$

Teaching Example 3 Divide: $\dfrac{2x^3 + 9x^2 + 5x - 6}{2x + 3}$

Answer:

$x^2 + 3x - 2$

EXAMPLE 3 Divide: $\dfrac{2a^3 + 13a^2 + 11a - 6}{2a + 3}$

Strategy We will use the long division method. The dividend is $2a^3 + 13a^2 + 11a - 6$ and the divisor is $2a + 3$.

WHY Since the divisor has more than one term, we must use the long division method to divide the polynomials.

Solution

$$\begin{array}{r} a^2 \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \end{array}$$

Divide the first term of the dividend by the first term of the divisor: $\frac{2a^3}{2a} = a^2$. Write the result, a^2 above the long division symbol.

$$\begin{array}{r} a^2 \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \\ -(2a^3 + 3a^2) \\ \hline 10a^2 + 11a \end{array}$$

Multiply each term in the divisor by a^2 to get $2a^3 + 3a^2$. Subtract $2a^3 + 3a^2$ from $2a^3 + 13a^2$ and bring down the $11a$.

$$\begin{array}{r} a^2 + 5a \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \\ -(2a^3 + 3a^2) \\ \hline 10a^2 + 11a \end{array}$$

Divide the first term of $10a^2 + 11a$ by the first term of the divisor: $\frac{10a^2}{2a} = 5a$. Write $+ 5a$ above the long division symbol to form the second term of the quotient.

$$\begin{array}{r} a^2 + 5a \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \\ -(2a^3 + 3a^2) \\ \hline 10a^2 + 11a \\ -(10a^2 + 15a) \\ \hline -4a - 6 \end{array}$$

Multiply each term in the divisor by $5a$ to get $10a^2 + 15a$. Subtract $10a^2 + 15a$ from $10a^2 + 11a$ and bring down the -6.

$$\begin{array}{r} a^2 + 5a - 2 \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \\ -(2a^3 + 3a^2) \\ \hline 10a^2 + 11a \\ -(10a^2 + 15a) \\ \hline -4a - 6 \end{array}$$

Divide the first term of $-4a - 6$ by the first term of the divisor: $\frac{-4a}{2a} = -2$. Write -2 above the long division symbol to form the third term of the quotient.

$$\begin{array}{r} a^2 + 5a - 2 \\ 2a + 3\overline{)2a^3 + 13a^2 + 11a - 6} \\ 2a^3 + 3a^2 \\ \hline 10a^2 + 11a \\ -(10a^2 + 15a) \\ \hline -4a - 6 \\ -(-4a - 6) \\ \hline 0 \end{array}$$

Multiply each term in the divisor by -2 to get $-4a - 6$. Subtract $-4a - 6$ from $-4a - 6$ to get 0.

This is the remainder.

Since the remainder is 0, the quotient is $a^2 + 5a - 2$. We can check the quotient by verifying that

$$\textbf{\textit{Check:}}\quad \underbrace{(2a+3)(a^2+5a-2)}_{\text{Divisor} \cdot \text{quotient}} + \underbrace{0}_{\text{remainder}} = \underbrace{2a^3 + 13a^2 + 11a - 6}_{\text{dividend}}$$

The quotient checks.

Success Tip The long division process is a series of four steps that are repeated: divide, multiply, subtract, and bring down.

The long division method used in algebra can have a remainder just as long division in arithmetic does.

EXAMPLE 4

Divide: $\dfrac{3x^3 + 2x^2 - 3x - 28}{x - 2}$

Strategy We will use the long division method. The dividend is $3x^3 + 2x^2 - 3x - 28$ and the divisor is $x - 2$.

WHY Since the divisor has more than one term, we must use the long division method to divide the polynomials.

Solution

$$\begin{array}{r|l} & 3x^2 + 8x + 13 \\ \hline x - 2 & 3x^3 + 2x^2 - 3x - 28 \\ & -(3x^3 - 6x^2) \\ \hline & 8x^2 - 3x \\ & -(8x^2 - 16x) \\ \hline & 13x - 28 \\ & -(13x - 26) \\ \hline & -2 \end{array}$$

The first division: $\frac{3x^3}{x} = 3x^2$.

The second division: $\frac{8x^2}{x} = 8x$.

The third division: $\frac{13x}{x} = 13$.

Subtract: $-28 - (-26) = -2$. The remainder is -2.

This division gives a quotient of $3x^2 + 8x + 13$ and a remainder of -2. It is common to form a fraction with the remainder as the numerator and the divisor as the denominator and to write the result as

$$3x^2 + 8x + 13 + \frac{-2}{x-2} \quad \text{or} \quad 3x^2 + 8x + 13 - \frac{2}{x-2}$$

To check, we verify that

$$(x-2)\left(3x^2 + 8x + 13 + \frac{-2}{x-2}\right) = 3x^3 + 2x^2 - 3x - 28.$$

Self Check 4

Divide: $\dfrac{2a^3 + 3a^2 - a - 85}{a - 3}$

Now Try Problem 37

Self Check 4 Answer

$2a^2 + 9a + 26 + \dfrac{-7}{a-3}$ or

$2a^2 + 9a + 26 - \dfrac{7}{a-3}$

Teaching Example 4 Divide:
$\dfrac{3x^3 - 11x^2 - 22x + 35}{x - 5}$

Answer:

$3x^2 + 4x - 2 + \dfrac{25}{x-5}$

Success Tip The long division method for polynomials continues until the degree of the remainder is less than the degree of the divisor. Here, the remainder, -2, has degree 0. The divisor, $x - 2$, has degree 1. Therefore, the division process ends.

The division method works best when the terms of the divisor and the dividend are written in descending powers of the variable. If the powers in the dividend or divisor are not in descending order, we use the commutative property of addition to write them that way.

Self Check 5

Divide:

$2 + 3a\overline{)-4a + 15a^2 + 18a^3 - 4}$

Now Try **Problems 39 and 43**

Self Check 5 Answer

$6a^2 + a - 2$

Teaching Example 5 Divide:

$(-9x + 8x^3 + 10x^2 - 9) \div (3 + 2x)$

Answer:

$4x^2 - x - 3$

EXAMPLE 5

Divide: $(-10x + 12x^3 + x^2 - 3) \div (1 + 3x)$

Strategy We will write the dividend and divisor in descending powers of x and use the long division method.

WHY It is easier to align like terms in columns when the powers of the variable in the dividend and divisor are written in descending order.

Solution

After writing the terms of the dividend and divisor in descending powers of x, we proceed as follows.

$$
\begin{array}{r}
4x^2 - x - 3 \\
3x + 1\overline{)12x^3 + x^2 - 10x - 3} \\
-(12x^3 + 4x^2) \\
\hline
-3x^2 - 10x \\
-(-3x^2 - x) \\
\hline
-9x - 3 \\
-(-9x - 3) \\
\hline
0
\end{array}
$$

The first division: $\frac{12x^3}{3x} = 4x^2$.

The second division: $\frac{-3x^2}{3x} = -x$.

The third division: $\frac{-9x}{3x} = -3$.

Thus,

$$\frac{-10x + 12x^3 + x^2 - 3}{1 + 3x} = 4x^2 - x - 3$$

4 Divide polynomials with missing terms.

If a power of the variable is missing in the dividend, we will insert placeholder terms or leave space. This keeps like terms in the same column, which is necessary when performing the subtraction in vertical form.

Self Check 6

Divide $125a^3 - 1$ by $5a - 1$

Now Try **Problem 47**

Self Check 6 Answer

$25a^2 + 5a + 1$

Teaching Example 6 Divide $8x^3 + 1$ by $2x + 1$.

Answer:

$4x^2 - 2x + 1$

EXAMPLE 6

Divide $64x^3 + 1$ by $4x + 1$.

Strategy The dividend, $64x^3 + 1$, does not have an x^2-term or an x-term. We will insert a $0x^2$ term and a $0x$ term as placeholders and use the long division method.

WHY We insert placeholder terms so that like terms will be aligned in the same column when we subtract.

Solution

After inserting placeholder terms of $0x^2$ and $0x$ in the dividend, we proceed as follows.

$$
\begin{array}{r}
16x^2 - 4x + 1 \\
4x + 1\overline{)64x^3 + \mathbf{0x^2} + \mathbf{0x} + 1} \\
-(64x^3 + 16x^2) \\
\hline
-16x^2 + 0x \\
-(-16x^2 - 4x) \\
\hline
4x + 1 \\
-(4x + 1) \\
\hline
0
\end{array}
$$

The first division: $\frac{64x^3}{4x} = 16x^2$.

The second division: $\frac{-16x^2}{4x} = -4x$.

The third division: $\frac{4x}{4x} = 1$.

Thus,

$$\frac{64x^3 + 1}{4x + 1} = 16x^2 - 4x + 1$$

EXAMPLE 7 Divide: $\dfrac{-17x^2 + 5x + x^4 + 2}{x^2 - 1 + 4x}$

Strategy We will write the terms of the divisor and the dividend in descending powers of x. Then we will insert any needed placeholder terms and use the long division method.

WHY We write the terms of the divisor and the dividend in descending powers of x (and insert necessary placeholder terms) so that like terms will be aligned in the same column when we subtract.

Solution

After writing the divisor and the dividend in descending powers of x and inserting $0x^3$ for the missing term in the dividend, we proceed as follows:

$$
\begin{array}{r@{}l}
 & x^2 - 4x \\
x^2 + 4x - 1 & \overline{)\,x^4 + 0x^3 - 17x^2 + 5x + 2} \\
 & -(x^4 + 4x^3 - x^2) \\
 & \phantom{x^4 + {}} -4x^3 - 16x^2 + 5x \\
 & \phantom{x^4 + {}} -(-4x^3 - 16x^2 + 4x) \\
 & \phantom{x^4 + 0x^3 - 17x^2 + {}} x + 2
\end{array}
$$

The first division: $\frac{x^4}{x^2} = x^2$.

The second division: $\frac{-4x^3}{x^2} = -4x$.

This is the remainder.

The degree of the remainder, $x + 2$, is 1 and the degree of the divisor, $x^2 + 4x - 1$, is 2. Since the degree of the remainder is less than the divisor, the division process stops. Thus,

$$\frac{-17x^2 + 5x + x^4 + 2}{x^2 - 1 + 4x} = x^2 - 4x + \frac{x + 2}{x^2 + 4x - 1}$$

Self Check 7

Divide:
$$\frac{2a^2 + 3a^3 + a^4 - 6 + a}{a^2 + 1 - 2a}$$

Now Try **Problem 51**

Self Check 7 Answer

$a^2 + 5a + 11 + \dfrac{18a - 17}{a^2 - 2a + 1}$

Teaching Example 7 Divide:
$\dfrac{x - 7x^2 + x^4 - 5}{-3x + x^2 + 4}$

Answer:

$x^2 + 3x - 2 + \dfrac{-17x + 3}{x^2 - 3x + 4}$

ANSWERS TO SELF CHECKS

1. a. $6y^2$ **b.** $\frac{d^4}{4c^3}$ **2. a.** $10h + 3$ **b.** $2s^3 - \frac{s^2t}{11} + \frac{4}{t}$ **3.** $x^2 + 2x - 1$
4. $2a^2 + 9a + 26 + \frac{-7}{a - 3}$ or $2a^2 + 9a + 26 - \frac{7}{a - 3}$ **5.** $6a^2 + a - 2$ **6.** $25a^2 + 5a + 1$
7. $a^2 + 5a + 11 + \frac{18a - 17}{a^2 - 2a + 1}$

SECTION 6.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. The expression $\frac{18x^7}{9x^4}$ is a monomial divided by a __monomial__. The expression $\frac{6x^3 - 4x^2 + 8x - 2}{2x^4}$ is a __polynomial__ divided by a monomial. The expression $\frac{x^2 - 8x + 12}{x - 6}$ is a trinomial divided by a __binomial__.

▶ **2.** The powers of x in $2x^4 + 3x^3 + 4x^2 - 7x - 8$ are written in __descending__ order.

3. Fill in the names of the parts of the following division.

Divisor — Quotient

$$
\begin{array}{r@{}l}
 & x - 2 \\
x - 6 & \overline{)\,x^2 - 8x - 4} \\
 & -(x^2 - 6x) \\
 & -2x - 4 \\
 & -(-2x + 12) \\
 & -16
\end{array}
$$

Dividend

Remainder

▶ Selected exercises available online at **www.webassign.net/brookscole**

4. Since $5x^2 + 6$ is missing an x-term, we insert a _placeholder_ $0x$ term in a division and write the polynomial as $5x^2 + 0x + 6$.

CONCEPTS

Fill in the blanks.

5. a. To divide a polynomial by a monomial, divide each _term_ of the polynomial by the monomial.

b. $\dfrac{18x + 9}{9} = \dfrac{18x}{\boxed{9}} + \dfrac{9}{\boxed{9}}$

c. $\dfrac{30x^2 + 12x - 24}{6} = \dfrac{30x^2}{\boxed{6}} + \dfrac{12x}{\boxed{6}} - \dfrac{24}{\boxed{6}}$

6. Divisor · _quotient_ + remainder = dividend

7. Suppose that after dividing $2x^3 + 5x^2 - 11x + 4$ by $2x - 1$, you obtain $x^2 + 3x - 4$. Show how multiplication can be used to check the result. $(2x - 1)(x^2 + 3x - 4) = 2x^3 + 5x^2 - 11x + 4$

8. Consider the first step of the division process for

$$2x^2 - 1\overline{)4x^4 + 0x^3 + 0x^2 + 0x - 1}$$

How many times does $2x^2$ divide $4x^4$? $2x^2$

NOTATION

Complete each solution.

9.
$$\begin{array}{r} x + 7 \\ x + 4\overline{)x^2 + 11x + 28} \\ -(\boxed{x^2} + 4x) \\ \boxed{7x} + 28 \\ -(7x + \boxed{28}) \\ 0 \end{array}$$

10.
$$\begin{array}{r} 2x - 1 \\ 3x + 4\overline{)6x^2 + 5x - 4} \\ -(6x^2 + \boxed{8x}) \\ \boxed{-3x} - 4 \\ -(-3x - \boxed{4}) \\ \boxed{0} \end{array}$$

11. If a polynomial is divided by $3a - 2$ and the quotient is $3a^2 + 5$ with a remainder of 6, how do we write the result?
$3a^2 + 5 + \frac{6}{3a - 2}$

12. A polynomial is divided by $3a - 2$. The quotient is $3a^2 + 5$ with a remainder of -6. Write the answer to the division in two ways.
$3a^2 + 5 - \frac{6}{3a - 2}$, $3a^2 + 5 + \frac{-6}{3a - 2}$

13. List three ways we can use symbols to write $x^2 - x - 12$ divided by $x - 4$.
$\frac{x^2 - x - 12}{x - 4}$, $x - 4\overline{)x^2 - x - 12}$, $(x^2 - x - 12) \div (x - 4)$

14. Is the following statement true or false? Justify your answer.

$$2x^3 - 9 = 2x^3 + 0x^2 + 0x - 9$$ True; $0x^2 = 0$, $0x = 0$

GUIDED PRACTICE

Simplify. Write answers using positive exponents. See Example 1.

15. $\dfrac{4x^2y^3}{8x^5y^2}$ $\frac{y}{2x^3}$

16. $\dfrac{25x^4y^7}{5xy^9}$ $\frac{5x^3}{y^2}$

17. $\dfrac{33a^2b^2}{44a^4b^2}$ $\frac{3}{4a^2}$

18. $\dfrac{63a^4}{81a^6b^3}$ $\frac{7}{9a^2b^3}$

Perform each division. See Example 2.

19. $\dfrac{4x^4 + 6x}{2}$ $2x^4 + 3x$

20. $\dfrac{11a^3 - 99a^2}{11}$ $a^3 - 9a^2$

21. $\dfrac{4x^2 - x^3}{6x}$ $\frac{2x}{3} - \frac{x^2}{6}$

22. $\dfrac{5y^4 + 45y^3}{15y^2}$ $\frac{y^2}{3} + 3y$

23. $\dfrac{54a^3y^2 - 18a^4y^3}{27a^2y^2}$ $2a - \frac{2a^2y}{3}$

24. $\dfrac{12x^2y^3 + x^3y^2}{6xy}$ $2xy^2 + \frac{x^2y}{6}$

25. $\dfrac{24x^6y^7 - 12x^5y^{12} + 36xy}{-48x^2y^3}$ $-\frac{x^4y^4}{2} + \frac{x^3y^9}{4} - \frac{3}{4xy^2}$

26. $\dfrac{9x^4y^3 + 18x^2y - 27xy^4}{-9x^3y^3}$ $-x - \frac{2}{xy^2} + \frac{3y}{x^2}$

Perform each division. See Objective 3 and Example 3.

27. $\dfrac{x^2 + 5x + 6}{x + 3}$ $x + 2$

28. $\dfrac{x^2 + 10x + 21}{x + 7}$ $x + 3$

29. $\dfrac{x^2 - 10x + 21}{x - 7}$ $x - 3$

30. $\dfrac{x^2 - 5x + 6}{x - 3}$ $x - 2$

31. $\dfrac{16x^2 - 16x - 5}{4x + 1}$ $4x - 5$

32. $\dfrac{6x^2 - x - 12}{2x - 3}$ $3x + 4$

33. $\dfrac{6x^3 - x^2 - 6x - 9}{2x - 3}$ $3x^2 + 4x + 3$

34. $\dfrac{16x^3 + 16x^2 - 9x - 5}{4x + 5}$ $4x^2 - x - 1$

Perform each division. See Example 4.

35. $\dfrac{t^3 + 8t^2 + 13t + 9}{t + 6}$ $t^2 + 2t + 1 + \frac{3}{t + 6}$

36. $\dfrac{s^3 + 10s^2 + 17s + 12}{s + 8}$ $s^2 + 2s + 1 + \frac{4}{s + 8}$

37. $\dfrac{6x^3 + 11x^2 - 19x - 2}{3x - 2}$ $2x^2 + 5x - 3 + \frac{-8}{3x - 2}$

38. $\dfrac{6x^3 + 11x^2 - 9x - 20}{2x + 3}$ $3x^2 + x - 6 + \frac{-2}{2x + 3}$

Perform each division. See Example 5.

39. $(2a + 1 + a^2) \div (a + 1)$ $a + 1$

40. $(a - 15 + 6a^2) \div (2a - 3)$ $3a + 5$

41. $(6y - 4 + 10y^2) \div (5y - 2)$ $2y + 2$

42. $(-10x + x^2 + 16) \div (x - 2)$ $x - 8$

43. $\dfrac{3x^2 + 9x^3 + 4x + 4}{2 + 3x}$ $3x^2 - x + 2$

44. $\dfrac{3 + 5x + 6x^3 + 11x^2}{3 + 2x}$ $3x^2 + x + 1$

45. $\dfrac{13x + 16x^4 + 3x^2 + 3}{3 + 4x}$ $4x^3 - 3x^2 + 3x + 1$

46. $\dfrac{4x^3 - 12x^2 + 17x - 12}{2x - 3}$ $2x^2 - 3x + 4$

Perform each division. See Example 6.

47. Divide $8a^3 + 1$ by $2a + 1$. $4a^2 - 2a + 1$

48. Divide $27a^3 - 8$ by $3a - 2$. $9a^2 + 6a + 4$

49. Divide $15a^3 - 29a^2 + 16$ by $3a - 4$. $5a^2 - 3a - 4$

50. Divide $15c^3 - 19c^2 + 4$ by $5c + 2$. $3c^2 - 5c + 2$

Perform each division. See Example 7.

51. Divide $7x^2 - x + x^4 + 5x^3 - 12$ by $x^2 - 3 + 2x$.
$x^2 + 3x + 4$

52. Divide $x^4 + 2 + 4x^2 + 3x + 2x^3$ by $x^2 + 2 + x$.
$x^2 + x + 1$

53. $3x^2 - 7x + 4\overline{)7x - 1 + 6x^3 - 5x^2}$
$2x + 3 + \frac{20x - 13}{3x^2 - 7x + 4}$

54. $3m^2 - m + 4\overline{)5m - 11 + 9m^3 - 6m^2}$
$3m - 1 + \frac{-8m - 7}{3m^2 - m + 4}$

TRY IT YOURSELF

Perform each division.

55. $y - 2\overline{)-24y + 24 + 6y^2}$ $6y - 12$

56. $a - 3\overline{)54 - 21a + a^2}$ $a - 18$

57. $\dfrac{4a^3 + a^2 - 3a + 7}{a + 1}$ $4a^2 - 3a + \frac{7}{a + 1}$

58. $\dfrac{3x^3 - 2x^2 + x - 6}{x - 1}$ $3x^2 + x + 2 + \frac{-4}{x - 1}$

59. $(x^6 - x^4 + 2x^2 - 8) \div (x^2 - 2)$ $x^4 + x^2 + 4$

60. $(x^3 + 3x + 5x^2 + 6 + x^4) \div (x^2 + 3)$ $x^2 + x + 2$

61. $\dfrac{5a^5 - 10a}{25a^3}$ $\frac{a^2}{5} - \frac{2}{5a^2}$

62. $\dfrac{24b^7 - 32b^2}{16b^5}$ $\frac{3b^2}{2} - \frac{2}{b^3}$

63. Divide $2s^2 + 13s + 5$ by $2s + 3$. $s + 5 + \frac{-10}{2s + 3}$

64. Divide $4s^2 + 6s + 1$ by $2s - 1$. $2s + 4 + \frac{5}{2s - 1}$

65. $\dfrac{40m^{17}n^{20}}{35m^{15}n^{30}}$ $\frac{8m^2}{7n^{10}}$

66. $\dfrac{34s^{30}t^{15}}{14s^{40}t^{12}}$ $\frac{17t^3}{7s^{10}}$

67. Divide $m^3 - 4m^2 + 2m - 1$ by $m^2 + 1$. $m - 4 + \frac{m + 3}{m^2 + 1}$

68. Divide $6m^3 + 2m^2 + m + 4$ by $2m^2 - 3$.
$3m + 1 + \frac{10m + 7}{2m^2 - 3}$

69. $(y^3 - 64) \div (y - 4)$
$y^2 + 4y + 16$

70. $(8w^3 + 1) \div (2w + 1)$
$4w^2 - 2w + 1$

71. $\dfrac{a^8 + a^6 - 4a^4 + 5a^2 - 3}{a^4 + 2a^2 - 3}$ $a^4 - a^2 + 1$

72. $\dfrac{2x^4 + 3x^3 + 3x^2 - 5x - 3}{2x^2 - x - 1}$ $x^2 + 2x + 3$

73. $\dfrac{40x^3z^2 - 8x^2z - 4z}{4xz}$
$10x^2z - 2x - \frac{1}{x}$

74. $\dfrac{22a^2b^2 - 18a^2b - 52a}{2ab}$
$11ab - 9a - \frac{26}{b}$

75. $\dfrac{x^5 + 3x + 2}{x^3 + 1 + 2x}$
$x^2 - 2 + \frac{-x^2 + 7x + 4}{x^3 + 2x + 1}$

76. $\dfrac{9a^4 + 6a^3 + 55a^2 + 18a + 81}{3a^2 + a + 9}$
$3a^2 + a + 9$

77. Divide $11x^2 - 4x + 8x^4 - 6x^3 + 3$ by $3 + 4x^2 - x$.
$2x^2 - x + 1$

78. Divide $4x - 2 + x^4 - x^2 - x^3$ by $x - 1 + x^2$.
$x^2 - 2x + 2$

79. $\dfrac{15x^2 + 9x - 3}{27}$
$\frac{5x^2}{9} + \frac{x}{3} - \frac{1}{9}$

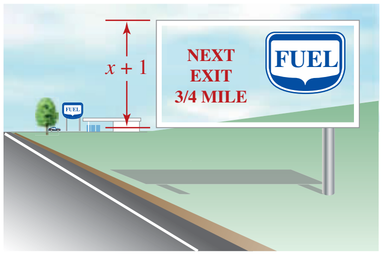
80. $\dfrac{8x^2 + 12x + 9}{6}$
$\frac{4x^2}{3} + 2x + \frac{3}{2}$

81. $x^2 + 3\overline{)x^6 + 2x^4 - 6x^2 - 9}$
$x^4 - x^2 - 3$

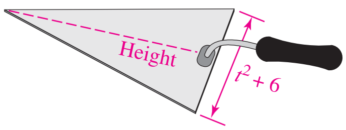
82. $m^2 - 2\overline{)m^4 - 3m^2 + 10}$
$m^2 - 1 + \frac{8}{m^2 - 2}$

APPLICATIONS

83. ADVERTISING Find the length of one of the longer sides of the rectangular-shaped billboard if its area is represented by $x^3 - 4x^2 + x + 6$. $x^2 - 5x + 6$



84. MASONRY The trowel shown is in the shape of an isosceles triangle. Find the height if its area is represented by $6 + 18t + t^2 + 3t^3$. $6t + 2$



▶ **85.** WINTER TRAVEL Complete the following table, which lists the rate (mph), time traveled (hr), and distance traveled (mi) by an Alaskan trail guide using two different means of travel.

	r ·	t =	d
Dog sled	$3x - 2$	$4x + 7$	$12x^2 + 13x - 14$
Snowshoes	$3x + 4$	$x + 5$	$3x^2 + 19x + 20$

▶ **86.** PRICING Complete the table for two items sold at a produce store.

	Price per lb ·	Number of lb =	Total Value
Cashews	$x^2 + 2x + 4$	$x^2 - 2x + 4$	$x^4 + 4x^2 + 16$
Sunflower seeds	$x^2 - 7$	$x^2 + 6$	$x^4 - x^2 - 42$

WRITING

87. Explain how to check to determine if

$$(3x^2 - 15) \div (x + 3) = 3x - 9 + \frac{12}{x+3}.$$

88. Explain the error in the following long division. Use the word *degree* in your answer.

$$\begin{array}{r@{}l} & \cancel{x + \dfrac{9x+3}{3x+1}} \\ 3x + 1 \big) & \overline{3x^2 + 10x + 3} \\ & \underline{-(3x^2 + x)} \\ & \phantom{3x^2 +{}} 9x + 3 \end{array}$$

REVIEW

Simplify each expression.

89. $2(x^2 + 4x - 1) + 3(2x^2 - 2x + 2)$
$8x^2 + 2x + 4$

▶ **90.** $3(2a^2 - 3a + 2) - 4(2a^2 + 4a - 7)$
$-2a^2 - 25a + 34$

91. $-2(3y^3 - 2y + 7) - (y^2 + 2y - 4) + 4(y^3 + 2y - 1)$
$-2y^3 - y^2 + 10y - 14$

92. $3(4y^3 + 3y - 2) + 2(3y^2 - y + 3) - 5(2y^3 - y^2 - 2)$
$2y^3 + 11y^2 + 7y + 10$

Objectives

1 Perform synthetic division.

2 Use the remainder theorem to evaluate polynomials.

3 Use the factor theorem to factor polynomials.

SECTION 6.6
Synthetic Division

We have discussed how to divide polynomials by polynomials using a long division process. We will now discuss a shortcut method, called **synthetic division,** that we can use to divide a polynomial by a binomial of the form $x - k$.

1 Perform synthetic division.

To see how synthetic division works, we consider the division of $4x^3 - 5x^2 - 11x + 20$ by $x - 2$.

$$\begin{array}{r@{}l} & \phantom{4x^3 - {}}4x^2 + 3x - 5 \\ x - 2 \big) & \overline{4x^3 - 5x^2 - 11x + 20} \\ & \underline{-(4x^3 - 8x^2)} \\ & \phantom{4x^3 - {}} 3x^2 - 11x \\ & \phantom{4x^3 {}} \underline{-(3x^2 - 6x)} \\ & \phantom{4x^3 - 5x^2 {}} - 5x + 20 \\ & \underline{-(-5x + 10)} \\ & \phantom{4x^3 - 5x^2 - 11x + {}} 10 \quad \text{(remainder)} \end{array} \qquad \begin{array}{r@{}l} & \phantom{4 - {}} 4 \quad 3 - 5 \\ 1 - 2 \big) & \overline{4 - 5 - 11 \quad 20} \\ & \underline{-(\mathbf{4} - 8)} \\ & \phantom{4 - {}} 3 \; \mathbf{- 11} \\ & \underline{-(\mathbf{3} - 6)} \\ & \phantom{4 - 5 {}} - 5 \quad \mathbf{20} \\ & \underline{-(\mathbf{- 5} \quad 10)} \\ & \phantom{4 - 5 - 11 \quad {}} 10 \quad \text{(remainder)} \end{array}$$

On the left is the long division, and on the right is the same division with the variables and their exponents removed and the coefficients of the quotient moved to the left. We can remember the various powers of x without actually writing them, because the exponents of the terms in the divisor, dividend, and quotient were written in descending order.

We can further shorten the version on the right. The numbers printed in color need not be written, because they are duplicates of the numbers above them. If we remember to perform subtraction at the proper times, the minus symbols and the parentheses can also be dropped. Thus, we can write the division in the following form:

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 4 & 3 & -5 & \\ \cline{2-5}
1-2 & 4 & -5 & -11 & 20 \\
 & & -8 & & \\ \cline{2-3}
 & & 3 & & \\
 & & & -6 & \\ \cline{3-4}
 & & & -5 & \\
 & & & & 10 \\ \cline{3-5}
 & & & & 10
\end{array}
$$

We can shorten the process further by compressing the work vertically and eliminating the 1 (the coefficient of x in the divisor):

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 4 & 3 & -5 & \\ \cline{2-5}
-2 & 4 & -5 & -11 & 20 \\
 & & -8 & -6 & 10 \\ \cline{2-5}
 & & 3 & -5 & 10
\end{array}
$$

If we write the 4 in the quotient on the bottom line, the bottom line gives the coefficients of the quotient and the remainder. If we eliminate the top line, the division appears as follows:

$$
\begin{array}{r|rrrr}
-2 & 4 & -5 & -11 & 20 \\
 & & -8 & -6 & 10 \\ \cline{2-5}
 & 4 & 3 & -5 & 10
\end{array}
$$

The bottom line is obtained by subtracting the middle line from the top line. If we replace the -2 in the divisor by 2, the division process will reverse the signs of every entry in the middle line, and then the bottom line can be obtained by addition. This gives the final form of the synthetic division.

$$
\begin{array}{r|rrrr}
2 & 4 & -5 & -11 & 20 \\
 & & 8 & 6 & -10 \\ \cline{2-5}
 & 4 & 3 & -5 & 10 \\
 & \downarrow & \downarrow & \downarrow & \downarrow
\end{array}
$$

These are the coefficients of the dividend.

These are the coefficients of the quotient and the remainder.

$$4x^2 + 3x - 5 + \frac{10}{x-2}$$

Read the result from the bottom row.

Thus,

$$\frac{4x^3 - 5x^2 - 11x + 20}{x-2} = 4x^2 + 3x - 5 + \frac{10}{x-2}$$

The Language of Algebra Synthetic division is used to divide a polynomial by a binomial of the form $x - k$. We call k the **synthetic divisor.** In the above example, we are dividing by $x - 2$, so k is 2.

Self Check 1

Use synthetic division to divide: $5x^2 - 4x - 33$ by $x - 3$ $5x + 11$

Now Try **Problem 15**

Teaching Example 1 Divide: $(8x^2 - 11x - 39) \div (x - 3)$
Answer:
$8x + 13$

EXAMPLE 1 Divide: $(6x^2 - 29x - 5) \div (x - 5)$

Strategy We will use synthetic division to divide.

WHY When a polynomial is divided by a binomial of the form $x - k$, synthetic division produces the quotient (and possible remainder) with less effort than long division.

Solution

We write the coefficients in the dividend and the 5 in the divisor in the following form:

Since we are dividing the polynomial by $x - 5$, the synthetic divisor is 5. → $\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ \hline & & & \end{array}$ This represents the dividend $6x^2 - 29x - 5$.

Then we follow these steps:

$$\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ & \downarrow & & \\ \hline & 6 & & \end{array}$$ Begin by bringing down the 6.

$$\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ & & 30 & \\ \hline & \boxed{6} & & \end{array}$$ Multiply 5 by 6 to get 30.

$$\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ & & 30 & \\ \hline & 6 & \boxed{1} & \end{array}$$ Add −29 and 30 to get 1.

$$\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ & & 30 & 5 \\ \hline & 6 & \boxed{1} & \end{array}$$ Multiply 1 by 5 to get 5.

$$\begin{array}{r|rrr} 5 & 6 & -29 & -5 \\ & & 30 & 5 \\ \hline & 6 & 1 & 0 \end{array}$$ Add −5 and 5 to get 0.

The remainder is 0.

The numbers 6 and 1 represent the quotient $6x + 1$, and 0 is the remainder. Thus,

$$\frac{6x^2 - 29x - 5}{x - 5} = 6x + 1$$

Self Check 2

Use synthetic division to divide: $\dfrac{x^3 + 3x - 62}{x - 4}$

Now Try **Problems 21 and 25**

Self Check 2 Answer

$x^2 + 4x + 19 + \dfrac{14}{x - 4}$

Teaching Example 2 Divide: $\dfrac{x^4 - x^2 + 1}{x - 2}$
Answer:
$x^3 + 2x^2 + 3x + 6 + \dfrac{13}{x - 2}$

EXAMPLE 2 Divide: $\dfrac{x^3 + x^2 - 1}{x - 3}$

Strategy We will use synthetic division to divide.

WHY When a polynomial is divided by a binomial of the form $x - k$, synthetic division produces the quotient (and possible remainder) with less effort than long division.

Solution

We begin by writing

$$\begin{array}{r|rrrr} 3 & 1 & 1 & 0 & -1 \\ \hline & & & & \end{array}$$ Write 0 for the coefficient of x, the missing term.

and complete the division as follows.

$$\begin{array}{r|rrrr} 3 & 1 & 1 & 0 & -1 \\ & & 3 & & \\ \hline & 1 & 4 & & \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & 1 & 0 & -1 \\ & & 3 & 12 & \\ \hline & 1 & 4 & 12 & \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & 1 & 0 & -1 \\ & & 3 & 12 & 36 \\ \hline & 1 & 4 & 12 & 35 \end{array}$$

Multiply, then add. Multiply, then add. Multiply, then add.

The numbers 1, 4, and 12 represent the quotient $x^2 + 4x + 12$ and 35 is the remainder. Thus,

$$\frac{x^3 + x^2 - 1}{x - 3} = x^2 + 4x + 12 + \frac{35}{x - 3}$$

EXAMPLE 3 Divide $5a^2 + 6a^3 + 2 - 4a$ by $a + 2$.

Strategy Here, the dividend and the divisor are polynomials in the variable a. Since $a + 2$ can be written as $a - (-2)$, the divisor can be written in the form $a - k$ and we can use synthetic division.

WHY We will use synthetic division because it produces the quotient (and possible remainder) with less effort than long division.

Solution

First, we write the dividend with the powers of x in descending order.

$$6a^3 + 5a^2 - 4a + 2$$

Then we write the divisor in $a - k$ form: $a - (\mathbf{-2})$. Thus, $k = -2$. Using synthetic division, we begin by writing

This represents division by $a + 2$. → $$\begin{array}{r|rrrr} -2 & 6 & 5 & -4 & 2 \\ & & & & \\ \hline & & & & \end{array}$$

and complete the division.

$$\begin{array}{r|rrrr} -2 & 6 & 5 & -4 & 2 \\ & & -12 & 14 & -20 \\ \hline & 6 & -7 & 10 & -18 \end{array}$$

The remainder is negative.

Thus,

$$\frac{5a^2 + 6a^3 + 2 - 4a}{a + 2} = 6a^2 - 7a + 10 + \frac{-18}{a + 2}$$

Success Tip Because the remainder is negative, we can also write the result as

$$6a^2 - 7a + 10 - \frac{18}{a + 2}$$

Self Check 3

Use synthetic division to divide $2a - 4a^2 + 3a^3 - 3$ by $a + 1$.

Now Try **Problems 29 and 33**

Self Check 3 Answer

$3a^2 - 7a + 9 + \frac{-12}{a + 1}$

Teaching Example 3 Divide $7x^2 + 2x - 4 + 2x^3$ by $x + 3$.
Answer:
$2x^2 + x - 1 - \frac{1}{x + 3}$

2 Use the remainder theorem to evaluate polynomials.

Synthetic division is important because of the **remainder theorem.**

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - k$, the remainder is $P(k)$.

It follows from the remainder theorem that we can evaluate polynomials using synthetic division. We illustrate this in the following example.

Self Check 4

Let $P(x) = 5x^3 - 3x^2 + x + 6$. Find each value:

a. $P(1)$ 9

b. use synthetic division to find the remainder when $P(x)$ is divided by $x - 1$ 9

Now Try **Problem 41**

Teaching Example 4
Let $P(x) = -x^3 + 2x^2 - 3x + 5$. Find each value:

a. $P(4)$

b. the remainder when $P(x)$ is divided by $x - 4$

Answers:
a. $P(4) = -39$ **b.** -39

EXAMPLE 4 Let $P(x) = 2x^3 - 3x^2 - 2x + 1$. Find each value:

a. $P(3)$ **b.** the remainder when $P(x)$ is divided by $x - 3$

Strategy To find $P(3)$, we will substitute 3 for x in $P(x)$ and simplify. To find the remainder when $P(x)$ is divided by $x - 3$, we will use synthetic division.

WHY After finding the remainder in two ways, we will see that the method using synthetic division is easier.

Solution

To find $P(3)$ we evaluate the function for $x = 3$.

a.
$$\begin{aligned} P(x) &= 2x^3 - 3x^2 - 2x + 1 \\ P(3) &= 2(3)^3 - 3(3)^2 - 2(3) + 1 \quad \text{Substitute 3 for } x. \\ &= 2(27) - 3(9) - 6 + 1 \\ &= 54 - 27 - 6 + 1 \\ &= 22 \end{aligned}$$

Thus, $P(3) = 22$.

b. We use synthetic division to find the remainder when $2x^3 - 3x^2 - 2x + 1$ is divided by $x - 3$.

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -2 & 1 \\ & & 6 & 9 & 21 \\ \hline & 2 & 3 & 7 & 22 \end{array}$$

Thus, the remainder is 22.

The same results in parts a and b show that instead of substituting 3 for x in $P(x) = 2x^3 - 3x^2 - 2x + 1$, we can divide the polynomial $2x^3 - 3x^2 - 2x + 1$ by $x - 3$ and note the remainder to find $P(3)$.

Success Tip It is often easier to find $P(k)$ by using synthetic division than by substituting k for x in $P(x)$. This is especially true if k is a decimal.

3 Use the factor theorem to factor polynomials.

If two quantities are multiplied, each is called a *factor* of the product. Thus, $x - 2$ is a factor of $6x - 12$, because $6(x - 2) = 6x - 12$. A theorem, called the *factor theorem,* tells us how to find one factor of a polynomial if the remainder of a certain division is 0.

Factor Theorem

If $P(x)$ is a polynomial in x, then

$P(k) = 0$ if and only if $x - k$ is a factor of $P(x)$

If $P(x)$ is a polynomial in x and if $P(k) = 0$, k is called a **zero of the polynomial function.**

EXAMPLE 5 Let $P(x) = 3x^3 - 5x^2 + 3x - 10$. Show that:

a. $P(2) = 0$ **b.** $x - 2$ is a factor of $P(x)$

Strategy We will substitute 2 for x in $P(x)$ to verify that $P(2) = 0$. We will then use synthetic division to divide $P(x)$ by $x - 2$.

WHY Since $P(x) = 0$, the division has a remainder of 0. This means that the divisor and the quotient are factors of the dividend.

Solution

a. Use the remainder theorem to evaluate $P(2)$ by dividing $P(x) = 3x^3 - 5x^2 + 3x - 10$ by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 3 & -5 & 3 & -10 \\ & & 6 & 2 & 10 \\ \hline & 3 & 1 & 5 & \mathbf{0} \end{array}$$

The remainder in this division is 0. By the remainder theorem, the remainder is $P(2)$. Thus, $P(2) = 0$, and 2 is a zero of the polynomial.

b. Because the remainder is 0, the numbers 3, 1, and 5 in the synthetic division in part a represent the quotient $3x^2 + x + 5$. Thus,

$$\underbrace{(x - 2)}_{\text{Divisor}} \cdot \underbrace{(3x^2 + x + 5)}_{\text{quotient}} + \underbrace{0}_{\text{remainder}} = \underbrace{3x^3 - 5x^2 + 3x - 10}_{\text{the dividend, } P(x)}$$

or

$$(x - 2)(3x^2 + x + 5) = 3x^3 - 5x^2 + 3x - 10$$

Thus, $x - 2$ is a factor of $3x^3 - 5x^2 + 3x - 10$.

Self Check 5

Let $P(x) = x^3 - 4x^2 + x + 6$. Show that $x + 1$ is a factor of $P(x)$ using synthetic division.

Now Try **Problem 65**

Self Check 5 Answer

Since $P(-1) = 0$, $x + 1$ is a factor of $P(x)$.

Teaching Example 5 Let $P(x) = 5x^3 + 7x^2 - 28x - 12$. Show that:

a. $P(-3) = 0$

b. $x + 3$ is a factor of $P(x)$

Answers:

a. $5(-3)^3 + 7(-3)^2 - 28(-3) - 12 = 0$

b. $(x + 3)(5x^2 - 8x - 4) + 0$

The result in Example 5 is true because the remainder, $P(2)$, is 0. If the remainder had not been 0, then $x - 2$ would not have been a factor of $P(x)$.

Using Your CALCULATOR **Approximating Zeros of Polynomials**

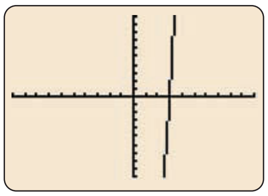
We can use a graphing calculator to approximate the real zeros of a polynomial function. For example, to find the real zeros of $f(x) = 2x^3 - 6x^2 + 7x - 21$, we graph the function as in the figure.

It is clear from the display that the function f has a zero at $x = 3$.

$$\begin{aligned} f(\mathbf{3}) &= 2(\mathbf{3})^3 - 6(\mathbf{3})^2 + 7(\mathbf{3}) - 21 && \text{Substitute 3 for } x. \\ &= 2(27) - 6(9) + 21 - 21 \\ &= 0 \end{aligned}$$

From the factor theorem, we know that $x - 3$ is a factor of the polynomial. To find the other factor, we can synthetically divide by 3.

$$\begin{array}{r|rrrr} 3 & 2 & -6 & 7 & -21 \\ & & 6 & 0 & 21 \\ \hline & 2 & 0 & 7 & 0 \end{array}$$



Thus, $f(x) = (x - 3)(2x^2 + 7)$. Since $2x^2 + 7$ cannot be factored over the real numbers, we can conclude that 3 is the only real zero of the polynomial function.

ANSWERS TO SELF CHECKS

1. $5x + 11$ **2.** $x^2 + 4x + 19 + \frac{14}{x - 4}$ **3.** $3a^2 - 7a + 9 + \frac{-12}{a + 1}$ **4. a.** 9 **b.** 9
5. Since $P(-1) = 0$, $x + 1$ is a factor of $P(x)$.

SECTION 6.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. The method of dividing $x^2 + 2x - 9$ by $x - 4$ shown below is called synthetic division.

$$\begin{array}{r|rrr} 4 & 1 & 2 & -9 \\ & & 4 & 24 \\ \hline & 1 & 6 & 15 \end{array}$$

2. Synthetic division is used to divide a polynomial by a binomial of the form $x - k$.

3. In Problem 1, the synthetic divisor is 4.

4. By the remainder theorem, if a polynomial $P(x)$ is divided by $x - k$, the remainder is $P(k)$.

5. The factor theorem tells us how to find one factor of a polynomial if the remainder of a certain division is 0.

6. If $P(x)$ is a polynomial and if $P(k) = 0$, then k is called a zero of the polynomial.

CONCEPTS

7. a. What division is represented below? $(5x^3 + x - 3) \div (x + 2)$

b. What is the answer? $5x^2 - 10x + 21 + \frac{-45}{x + 2}$

$$\begin{array}{r|rrrr} -2 & 5 & 0 & 1 & -3 \\ & & -10 & 20 & -42 \\ \hline & 5 & -10 & 21 & -45 \end{array}$$

Fill in the blanks.

8. In the synthetic division process, numbers below the line are multiplied by the synthetic divisor, and that product is carried above the line to the next column. Numbers above the horizontal line are added.

9. Rather than substituting 8 for x in $P(x) = 6x^3 - x^2 - 17x + 9$, we can divide the polynomial $6x^3 - x^2 - 17x + 9$ by $x - 8$ to find $P(8)$.

10. For $P(x) = x^3 - 4x^2 + x + 6$, suppose we know that $P(3) = 0$. Then $x - 3$ is a factor of $x^3 - 4x^2 + x + 6$.

NOTATION

Complete each synthetic division.

11. Divide $6x^3 + x^2 - 23x + 2$ by $x - 2$.

$$\begin{array}{r|rrrr} 2 & 6 & 1 & -23 & 2 \\ & & 12 & 26 & 6 \\ \hline & 6 & 13 & 3 & 8 \end{array}$$

12. Divide $2x^3 - 4x^2 - 25x + 15$ by $x + 3$.

$$\begin{array}{r|rrrr} -3 & 2 & -4 & -25 & 15 \\ & & -6 & 30 & -15 \\ \hline & 2 & -10 & 5 & 0 \end{array}$$

GUIDED PRACTICE

Use synthetic division to perform each division. See Example 1.

13. $(2x^2 + x - 3) \div (x - 1)$ $2x + 3$

14. $(4x^2 - 5x - 6) \div (x - 2)$ $4x + 3$

15. $(5x^2 - 27x + 10) \div (x - 5)$ $5x - 2$

16. $(6x^2 - 29x + 20) \div (x - 4)$ $6x - 5$

17. $(3x^2 - 13x + 12) \div (x - 3)$ $3x - 4$

18. $(2x^2 - 23x + 63) \div (x - 7)$ $2x - 9$

19. $(5x^2 - 24x - 36) \div (x - 6)$ $5x + 6$

20. $(3x^2 - 14x - 24) \div (x - 6)$ $3x + 4$

Use synthetic division to perform each division. See Example 2.

21. $\dfrac{a^3 - 3a^2 + 4}{a - 2}$ $a^2 - a - 2$

22. $\dfrac{a^3 - 2a^2 - 9}{a - 3}$ $a^2 + a + 3$

23. $\dfrac{3a^3 - 47a - 4}{a - 4}$ $3a^2 + 12a + 1$

24. $\dfrac{2a^3 - 7a + 5}{a - 1}$ $2a^2 + 2a - 5$

25. $\dfrac{3b^3 - 31b + 13}{b - 3}$ $3b^2 + 9b - 4 + \frac{1}{b - 3}$

26. $\dfrac{4c^3 - 107c + 37}{c - 5}$ $4c^2 + 20c - 7 + \frac{2}{c - 3}$

27. $\dfrac{4t^3 - t - 18}{t - 2}$ $4t^2 + 8t + 15 + \frac{12}{t - 2}$

28. $\dfrac{m^3 + 2m + 5}{m - 2}$ $m^2 + 2m + 6 + \frac{17}{m - 2}$

Use synthetic division to perform each division. See Example 3.

29. Divide $x - 4x^2 + x^3 + 6$ by $x + 1$. $x^2 - 5x + 6$

30. Divide $4x^2 - 10x + 12 + x^3$ by $x + 6$. $x^2 - 2x + 2$

31. Divide $20x^2 - 36x - 42 + 3x^3$ by $x + 8$. $3x^2 - 4x - 4 + \frac{-10}{x+8}$

32. Divide $3x - 6x^2 + 5x^3 + 10$ by $x + 1$. $5x^2 - 11x + 14 + \frac{-4}{x+1}$

33. Divide $8 - 3x + 7x^2 + 2x^3$ by $x + 5$. $2x^2 - 3x + 12 + \frac{-52}{x+5}$

▶ **34.** Divide $1 - 4x + 7x^2 + 3x^3$ by $x + 3$. $3x^2 - 2x + 2 + \frac{-5}{x+3}$

35. Divide $27 + x^3 - 17x + 8x^2$ by $x + 10$. $x^2 - 2x + 3 + \frac{-3}{x+10}$

36. Divide $1 + x^3 - 23x + 5x^2$ by $x + 8$. $x^2 - 3x + 1 + \frac{-7}{x+8}$

Use a calculator and synthetic division to perform each division. See Examples 1–3.

37. $\dfrac{7.2x^2 - 2.1x + 0.5}{x - 0.2}$ $7.2x - 0.66 + \frac{0.368}{x-0.2}$

▶ **38.** $\dfrac{2.7x^2 + x - 5.2}{x + 1.7}$ $2.7x - 3.59 + \frac{0.903}{x+1.7}$

39. $\dfrac{9x^3 - 25}{x + 57}$ $9x^2 - 513x + 29{,}241 - \frac{1{,}666{,}762}{x+57}$

40. $\dfrac{0.5x^3 + x}{x - 2.3}$ $0.5x^2 + 1.15x + 3.645 + \frac{8.3835}{x-2.3}$

Let $P(x) = 2x^3 - 4x^2 + 2x - 1$. Evaluate $P(x)$ by substituting the given value of x into the polynomial and simplifying. Then evaluate the polynomial by using the remainder theorem and synthetic division. See Example 4.

41. $P(1)$ -1

42. $P(2)$ 3

43. $P(-2)$ -37

▶ **44.** $P(-1)$ -9

▶ **45.** $P(3)$ 23

46. $P(-4)$ -201

47. $P(0)$ -1

48. $P(4)$ 71

Let $Q(x) = x^4 - 3x^3 + 2x^2 + x - 3$. Evaluate $Q(x)$ by substituting the given value of x into the polynomial and simplifying. Then evaluate the polynomial by using the remainder theorem and synthetic division. See Example 4.

49. $Q(-1)$ 2

50. $Q(1)$ -2

51. $Q(2)$ -1

▶ **52.** $Q(-2)$ 43

53. $Q(3)$ 18

▶ **54.** $Q(0)$ -3

55. $Q(-3)$ 174

56. $Q(-4)$ 473

Use the remainder theorem and synthetic division to find each function value. See Example 4.

57. $P(x) = x^3 - 4x^2 + x - 2$; find $P(2)$ -8

58. $P(x) = x^3 - 3x^2 + x + 1$; find $P(1)$ 0

59. $P(x) = 2x^3 + x + 2$; find $P(3)$ 59

▶ **60.** $P(x) = x^3 + x^2 + 1$; find $P(-2)$ -3

61. $P(x) = x^4 - 2x^3 + x^2 - 3x + 2$; find $P(-2)$ 44

▶ **62.** $P(x) = x^5 + 3x^4 - x^2 + 1$; find $P(-1)$ 2

63. $P(x) = 3x^5 + 1$; find $P\left(-\frac{1}{2}\right)$ $\frac{29}{32}$

▶ **64.** $P(x) = 5x^7 - 7x^4 + x^2 + 1$; find $P(2)$ 533

Use the factor theorem and determine whether the first expression is a factor of $P(x)$. See Example 5.

65. $x - 3$; $P(x) = x^3 - 3x^2 + 5x - 15$ yes

▶ **66.** $x + 1$; $P(x) = x^3 + 2x^2 - 2x - 3$ (*Hint:* Write $x + 1$ as $x - (-1)$.) yes

67. $x + 2$; $P(x) = 3x^2 - 7x + 4$ (*Hint:* Write as $x - (-2)$.) no

▶ **68.** x; $P(x) = 7x^3 - 5x^2 - 8x$ (*Hint:* $x = x - 0$.) yes

TRY IT YOURSELF

Use synthetic division to perform each division.

69. $\dfrac{5x^2 + 4 + 6x^3}{x + 1}$ $6x^2 - x + 1 + \frac{3}{x+1}$

70. $\dfrac{-4 + 3x^2 - x}{x - 4}$ $3x + 11 + \frac{40}{x-4}$

71. $(x^2 - 5x + 14) \div (x + 2)$ $x - 7 + \frac{28}{x+2}$

72. $(x^2 + 13x + 42) \div (x + 6)$ $x + 7$

▶ **73.** Divide $a^5 - 1$ by $a - 1$. $a^4 + a^3 + a^2 + a + 1$

74. Divide $b^4 - 81$ by $b - 3$. $b^3 + 3b^2 + 9b + 27$

75. $\dfrac{-6c^5 + 14c^4 + 38c^3 + 4c^2 + 25c - 36}{c - 4}$ $-6c^4 - 10c^3 - 2c^2 - 4c + 9$

76. $\dfrac{-5x^5 + 4x^4 + 30x^3 + 2x^2 + 20x + 3}{x - 3}$ $-5x^4 - 11x^3 - 3x^2 - 7x - 1$

77. $\dfrac{9a^3 + 3a^2 - 21a - 7}{a + \frac{1}{3}}$ $9a^2 - 21$

78. $\dfrac{8t^3 - 4t^2 + 2t - 1}{t - \frac{1}{2}}$ $8t^2 + 2$

79. $\dfrac{4x^4 + 12x^3 - x^2 - x + 12}{x + 3}$ $4x^3 - x + 2 + \frac{6}{x+3}$

▶ **80.** $\dfrac{x^4 - 9x^3 + x^2 - 7x - 20}{x - 9}$ $x^3 + x + 2 + \frac{-2}{x-9}$

81. $\dfrac{3x^3 - 25x^2 + 10x - 16}{x - 8}$ $3x^2 - x + 2$

82. $\dfrac{2x^3 + 3x^2 - 8x + 3}{x + 3}$ $2x^2 - 3x + 1$

83. $(2x^3 - 50 - 16x^2 - 35x) \div (x - 10)$ $2x^2 + 4x + 5$

84. $(m^3 - m^2 - m - 1) \div (m - 1)$ $m^2 - 1 + \frac{-2}{m-1}$

▶ **85.** $(4x^3 - 1 + 5x^2) \div (x + 2)$ $4x^2 - 3x + 6 + \frac{-13}{x+2}$

86. $(t^3 + t^2 + t + 2) \div (t + 1)$ $t^2 + 1 + \frac{1}{t+1}$

87. Divide $8a^3 - 10a^2 - 32a - 15$ by $a + \dfrac{3}{4}$. $8a^2 - 16a - 20$

88. Divide $4a^3 - 2a^2 - 18a - 9$ by $a + \dfrac{3}{2}$. $4a^2 - 8a - 6$

WRITING

89. When dividing a polynomial by a binomial of the form $x - k$, synthetic division is considered to be faster than long division. Explain why.

90. Let $P(x) = x^3 - 6x^2 - 9x + 4$. You now know two ways to find $P(6)$. What are they? Which method do you prefer?

91. Explain the factor theorem.

▶ **92.** This section includes a feature entitled *Using Your Calculator: Approximating Zeros of Polynomials.* What is a *zero* of a polynomial?

REVIEW

Evaluate each expression for $x = -3$, $y = -5$, and $z = 0$.

93. $x^2z(y^3 - z)$ 0

94. $|y^3 - z|$ 125

95. $\dfrac{x - y^2}{2y - 1 + x}$ 2

▶ **96.** $\dfrac{2y + 1}{x} - x$ 6

SECTION 6.7
Solving Rational Equations

Objectives

1 Solve rational equations.

2 Solve rational equations with extraneous solutions.

3 Solve formulas for a specified variable.

In Chapter 1, we solved equations such as $\frac{1}{6}x + \frac{5}{2} = \frac{1}{3}$ by multiplying both sides by the LCD. With this approach, the equation that results is equivalent to the original equation, but easier to solve because it is cleared of fractions.

In this section, we will extend the fraction-clearing strategy to solve another type of equation, called a *rational equation.*

1 Solve rational equations.

If an equation contains one or more rational expressions, it is called a **rational equation.** Rational equations often have a variable in a denominator. Some examples are:

$$\frac{3}{5} + \frac{7}{x + 2} = 2, \qquad \frac{x + 3}{x - 3} = \frac{2}{x^2 - 4}, \qquad \text{and} \qquad \frac{-x^2 + 10}{x^2 - 1} + \frac{3x}{x - 1} = \frac{2x}{x + 1}$$

To solve a rational equation, we find all the values of the variable that make the equation true. Any value of the variable that makes a denominator in a rational equation equal to 0 cannot be a solution of the equation. Such a number must be rejected, because division by 0 is undefined.

Self Check 1

Solve: $\dfrac{2}{5} + \dfrac{8}{x - 4} = 2$ 9

Now Try **Problem 13**

EXAMPLE 1

Solve: $\dfrac{3}{5} + \dfrac{7}{x + 2} = 2$

Strategy This equation contains a rational expression that has a variable in the denominator. We begin by asking, "What value(s) of x make that denominator 0?"

WHY If a number makes the denominator of a rational expression 0, that number cannot be a solution of the equation because division by 0 is undefined.

Teaching Example 1 Solve: $\frac{1}{5} + \frac{2}{x-1} = 3$

Answer: $\frac{12}{7}$

Solution

We note that x cannot be -2, because this would produce a 0 in the denominator of $\frac{7}{x+2}$.

Since the denominators of the rational expressions in the equation are 5 and $x + 2$, we multiply both sides by the LCD, $5(x + 2)$, to clear the equation of fractions.

$$\frac{3}{5} + \frac{7}{x+2} = 2 \quad \text{This is the equation to solve.}$$

$$5(x+2)\left(\frac{3}{5} + \frac{7}{x+2}\right) = 5(x+2)(2) \quad \text{Write each side of the equation within parentheses and then multiply both sides by the LCD.}$$

$$5(x+2)\left(\frac{3}{5}\right) + 5(x+2)\left(\frac{7}{x+2}\right) = 5(x+2)(2) \quad \text{On the left side, distribute the multiplication by } 5(x+2).$$

$$\cancel{5}^{1}(x+2)\left(\frac{3}{\cancel{5}_{1}}\right) + 5(x\cancel{+2})^{1}\left(\frac{7}{(x\cancel{+2})_{1}}\right) = 5(x+2)(2) \quad \text{On the left side, simplify: } \frac{5}{5} = 1 \text{ and } \frac{x+2}{x+2} = 1.$$

$$3(x+2) + 5(7) = 10(x+2) \quad \text{Simplify each side.}$$

The resulting equation does not contain any fractions. We now solve this linear equation for x.

$$3x + 6 + 35 = 10x + 20 \quad \text{Use the distributive property and simplify.}$$
$$3x + 41 = 10x + 20 \quad \text{Combine like terms.}$$
$$-7x = -21 \quad \text{Subtract 10x and 41 from both sides.}$$
$$x = 3 \quad \text{Divide both sides by } -7.$$

The solution is 3 and the solution set is $\{3\}$. To check, we substitute 3 for x in the original equation and simplify:

Check:
$$\frac{3}{5} + \frac{7}{x+2} = 2$$
$$\frac{3}{5} + \frac{7}{3+2} \stackrel{?}{=} 2$$
$$\frac{3}{5} + \frac{7}{5} \stackrel{?}{=} 2$$
$$2 = 2 \quad \text{True}$$

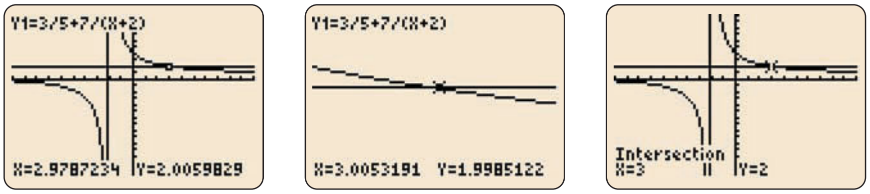
Success Tip To *simplify the expression* $\frac{3}{5} + \frac{7}{x+2}$, we build each fraction to have the LCD $5(x + 2)$, add the numerators, and write the sum over the LCD.

To *solve the equation* $\frac{3}{5} + \frac{7}{x+2} = 2$, we multiply both sides by the LCD $5(x + 2)$ to eliminate the denominators.

Using Your CALCULATOR Solving Rational Equations Graphically

To use a graphing calculator to solve $\frac{3}{5} + \frac{7}{x+2} = 2$, we graph the functions $f(x) = \frac{3}{5} + \frac{7}{x+2}$ and $g(x) = 2$. If we trace and move the cursor closer to the intersection point of the two graphs, we will get the approximate value of x shown in figure (a) on the next page. If we zoom twice and trace again, we get the results shown in figure (b). As we saw in Example 1, the exact solution is 3.

An alternate way of finding the point of intersection of the two graphs is to use the INTERSECT feature. In figure (c), the display shows that the graphs intersect at the point (3, 2). This implies that the solution of the rational equation is 3.



(a) (b) (c)

Self Check 2

Solve: $\dfrac{2}{x-3} = \dfrac{-x}{x^2-9} + \dfrac{4}{x+3}$

Now Try **Problem 17**

Self Check 2 Answer
18

Teaching Example 2 Solve:
$\dfrac{20}{x^2-x-6} + \dfrac{x}{x-3} = \dfrac{x}{x+2}$
Answer:
-4

EXAMPLE 2

Solve: $\dfrac{-x^2+10}{x^2-1} + \dfrac{3x}{x-1} = \dfrac{2x}{x+1}$

Strategy We will begin by factoring the first denominator.

WHY To determine any restrictions on the variable and to find the LCD, we need to write $x^2 - 1$ in factored form.

Solution

Since $x^2 - 1$ factors as $(x+1)(x-1)$, we can write the given equation as:

$$\frac{-x^2+10}{(x+1)(x-1)} + \frac{3x}{x-1} = \frac{2x}{x+1} \qquad \text{Factor the denominator } x^2-1.$$

We see that -1 and 1 cannot be solutions of the equation because they make rational expressions in the equation undefined.

We can clear the equation of fractions by multiplying both sides by $(x+1)(x-1)$, which is the LCD of the three rational expressions.

$$(x+1)(x-1)\left[\frac{-x^2+10}{(x+1)(x-1)} + \frac{3x}{x-1}\right] = (x+1)(x-1)\left(\frac{2x}{x+1}\right)$$

$$(x+1)(x-1)\left[\frac{-x^2+10}{(x+1)(x-1)}\right] + (x+1)(x-1)\left(\frac{3x}{x-1}\right) = (x+1)(x-1)\left(\frac{2x}{x+1}\right)$$

$$\overset{1}{\cancel{(x+1)}}\overset{1}{\cancel{(x-1)}}\left[\frac{-x^2+10}{\underset{1}{\cancel{(x+1)}}\underset{1}{\cancel{(x-1)}}}\right] + (x+1)\overset{1}{\cancel{(x-1)}}\left(\frac{3x}{\underset{1}{\cancel{x-1}}}\right) = \overset{1}{\cancel{(x+1)}}(x-1)\left(\frac{2x}{\underset{1}{\cancel{x+1}}}\right)$$

$$\begin{aligned}
-x^2 + 10 + 3x(x+1) &= 2x(x-1) && \text{Simplify. The resulting equation does not contain any fractions.}\\
-x^2 + 10 + 3x^2 + 3x &= 2x^2 - 2x && \text{Use the distributive property.}\\
2x^2 + 10 + 3x &= 2x^2 - 2x && \text{Combine like terms on each side.}\\
10 + 3x &= -2x && \text{Subtract } 2x^2 \text{ from both sides.}\\
10 + 5x &= 0 && \text{Add } 2x \text{ to both sides.}\\
5x &= -10 && \text{Subtract 10 from both sides.}\\
x &= -2 && \text{Divide both sides by 5.}
\end{aligned}$$

The solution is -2. Verify that it satisfies the original equation.

Caution! After multiplying both sides by the LCD and simplifying, the equation should not contain any fractions. If it does, check for an algebraic error, or perhaps your LCD is incorrect.

We can summarize the procedure used to solve rational equations.

Solving Rational Equations

1. Factor all denominators.
2. Determine which numbers cannot be solutions of the equation.
3. Multiply both sides of the equation by the LCD of all rational expressions in the equation.
4. Use the distributive property to remove parentheses, remove any factors equal to 1, and write the result in simplified form.
5. Solve the resulting equation.
6. Check all possible solutions in the original equation.

Using Your CALCULATOR Checking Apparent Solutions

We can use a scientific calculator to check the solution -2 found in Example 2 by evaluating

$$\frac{-x^2+10}{x^2-1}+\frac{3x}{x-1} \quad \text{and} \quad \frac{2x}{x+1}$$

In each case, the result is 4. Since the results are the same, -2 is a solution of the equation.

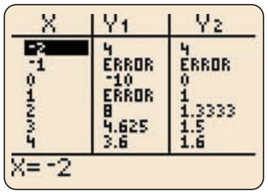
We can also check by using a graphing calculator. One way of doing this is to enter

$$Y_1=\frac{-x^2+10}{x^2-1}+\frac{3x}{x-1} \quad \text{and} \quad Y_2=\frac{2x}{x+1}$$

and compare the values of the expressions when $x = -2$ in the table mode. See the figure. We know that -2 is a solution of

$$\frac{-x^2+10}{x^2-1}+\frac{3x}{x-1}=\frac{2x}{x+1}$$

because the value of Y1 and Y2 are the same (namely, 4) for $x = -2$.



EXAMPLE 3

Solve: $\frac{a}{2}=\frac{a-6}{3a-9}-\frac{1}{3}$

Strategy We will begin by factoring the second denominator.

WHY To determine any restrictions on the variable and to find the LCD, we need to write $3a - 9$ in factored form.

Self Check 3

Solve: $\frac{b}{5}=\frac{b-14}{2b-16}-\frac{1}{2}$ 3, 5

Now Try **Problem 25**

Teaching Example 3 Solve: $\frac{x+1}{5} - 2 = -\frac{4}{x}$

Answer: 4, 5

Solution

Since the binomial $3a - 9$ factors as $3(a - 3)$, we can write the given equation as:

$$\frac{a}{2} = \frac{a-6}{3(a-3)} - \frac{1}{3} \quad \text{Factor the denominator } 3a - 9.$$

We see that 3 cannot be a solution of the equation, because it makes one of the rational expressions in the equation undefined.

We can clear the equation of fractions by multiplying both sides by $2 \cdot 3 \cdot (a - 3)$, which is the LCD of the three rational expressions.

$$\mathbf{2 \cdot 3 \cdot (a-3)}\left(\frac{a}{2}\right) = \mathbf{2 \cdot 3 \cdot (a-3)}\left[\frac{a-6}{3(a-3)} - \frac{1}{3}\right] \quad \text{Multiply both sides by the LCD, } 2 \cdot 3 \cdot (a-3).$$

$$\mathbf{2 \cdot 3 \cdot (a-3)}\left(\frac{a}{2}\right) = \mathbf{2 \cdot 3 \cdot (a-3)}\left[\frac{a-6}{3(a-3)}\right] - \mathbf{2 \cdot 3 \cdot (a-3)}\left(\frac{1}{3}\right) \quad \text{On the right side, distribute } 2 \cdot 3 \cdot (a-3).$$

$$\overset{1}{\cancel{2}} \cdot 3(a-3)\left(\frac{a}{\underset{1}{\cancel{2}}}\right) = 2 \cdot \overset{1}{\cancel{3}}\overset{1}{\cancel{(a-3)}}\left[\frac{a-6}{\underset{1}{\cancel{3}}\underset{1}{\cancel{(a-3)}}}\right] - 2 \cdot \overset{1}{\cancel{3}}(a-3)\left(\frac{1}{\underset{1}{\cancel{3}}}\right) \quad \text{Remove common factors of the numerator and denominator.}$$

$$\begin{aligned} 3a(a-3) &= 2(a-6) - 2(a-3) \\ 3a^2 - 9a &= 2a - 12 - 2a + 6 && \text{Use the distributive property.} \\ 3a^2 - 9a &= -6 && \text{Combine like terms.} \end{aligned}$$

To use factoring to solve the resulting quadratic equation, we must write it in standard form $ax^2 + bx + c = 0$.

$$\begin{aligned} 3a^2 - 9a + 6 &= 0 && \text{To get 0 on the right side, add 6 to both sides.} \\ a^2 - 3a + 2 &= 0 && \text{Divide both sides by 3.} \\ (a-1)(a-2) &= 0 && \text{Factor the trinomial.} \end{aligned}$$

$$\begin{aligned} a - 1 &= 0 \quad \text{or} \quad a - 2 = 0 && \text{Set each factor equal to 0.} \\ a &= 1 \qquad\qquad\; a = 2 \end{aligned}$$

Verify that 1 and 2 both satisfy the original equation.

Recall that the quotient of a polynomial and its opposite is -1. For example, $\frac{y-1}{1-y} = -1$. We can use this fact when solving rational equations whose denominators contain factors that are opposites.

Self Check 4

Solve: $\frac{1}{2h-8} + \frac{11}{4-h} = \frac{3}{2}$ -3

Now Try **Problem 33**

Teaching Example 4 Solve: $\frac{3}{2x-2} - \frac{1}{x-1} = \frac{1}{2}$

Answer: 2

EXAMPLE 4

Solve: $\frac{1}{6y-6} + \frac{1}{1-y} = \frac{1}{6}$

Strategy We will begin by factoring the first denominator.

WHY To determine any restrictions on the variable and to find the LCD, we need to write $6y - 6$ in factored form.

Solution

Since the binomial $6y - 6$ factors as $6(y - 1)$,we can write the given equation as:

$$\frac{1}{6(y-1)} + \frac{1}{1-y} = \frac{1}{6}$$

We see that 1 cannot be a solution of the equation because it makes two of the rational expressions in the equation undefined.

We note that $y - 1$ and $1 - y$ are opposites. We can clear the equation of fractions by multiplying both sides by $6(y - 1)$.

$$6(y-1)\left[\frac{1}{6(y-1)} + \frac{1}{1-y}\right] = 6(y-1)\left(\frac{1}{6}\right)$$ Multiply both sides by the LCD, $6(y - 1)$.

$$6(y-1)\left[\frac{1}{6(y-1)}\right] + 6(y-1)\left(\frac{1}{1-y}\right) = 6(y-1)\left(\frac{1}{6}\right)$$ On the left side, distribute $6(y - 1)$.

$$\overset{1}{\cancel{6}}\overset{1}{\cancel{(y-1)}}\left[\frac{1}{\underset{1}{\cancel{6}}\underset{1}{\cancel{(y-1)}}}\right] + 6\overset{-1}{\cancel{(y-1)}}\left(\frac{1}{\underset{1}{\cancel{1-y}}}\right) = \overset{1}{\cancel{6}}(y-1)\left(\frac{1}{\underset{1}{\cancel{6}}}\right)$$ Simplify: $\frac{y-1}{1-y} = -1$.

$$1 - 6 = y - 1$$
$$-5 = y - 1$$ Combine like terms.
$$-4 = y$$ Add 1 to both sides.

The solution is -4. Verify that it satisfies the original equation.

2 Solve rational equations with extraneous solutions.

When we multiply both sides of an equation by a quantity that contains a variable, we can get false solutions, called *extraneous solutions*. This happens when we multiply both sides of an equation by 0 and get a solution that gives a 0 in the denominator of a rational expression. Extraneous solutions must be discarded.

EXAMPLE 5

Solve: $3 - \frac{1-2t}{t+2} = \frac{t-3}{t+2}$

Strategy We will clear the equation of fractions by multiplying both sides by the LCD, $t + 2$.

WHY Equations that contain only integers are usually easier to solve than equations that contain fractions.

Solution

We note that t cannot be -2, because this would give a 0 in a denominator.

$$3 - \frac{1-2t}{t+2} = \frac{t-3}{t+2}$$

$$(t+2)\left(3 - \frac{1-2t}{t+2}\right) = (t+2)\left(\frac{t-3}{t+2}\right)$$ Multiply both sides by the LCD.

$$(t+2)(3) - (t+2)\left(\frac{1-2t}{t+2}\right) = (t+2)\left(\frac{t-3}{t+2}\right)$$ On the left side, distribute the multiplication by $t + 2$.

$$(t+2)(3) - \overset{1}{\cancel{(t+2)}}\left(\frac{1-2t}{\underset{1}{\cancel{t+2}}}\right) = \overset{1}{\cancel{(t+2)}}\left(\frac{t-3}{\underset{1}{\cancel{t+2}}}\right)$$ Remove common factors of the numerator and denominator.

$$3t + 6 - (1 - 2t) = t - 3$$ Simplify.
$$3t + 6 - 1 + 2t = t - 3$$
$$5t + 5 = t - 3$$ Combine like terms.
$$4t = -8$$
$$t = -2$$

Since t cannot be -2, it is an extraneous solution and must be discarded. This equation has no solution.

Self Check 5

Solve: $2 - \frac{2a}{a-1} = \frac{3a-5}{a-1}$

Now Try **Problem 37**

Self Check 5 Answer
No solution; 1 is extraneous.

Teaching Example 5 Solve: $\frac{2(x+1)}{x-3} = \frac{x+5}{x-3}$
Answer:
No solution; 3 is extraneous.

The Language of Algebra *Extraneous* means not a vital part. Mathematicians speak of *extraneous* solutions. Rock groups don't want any *extraneous* sounds (like humming or feedback) coming from their amplifiers. Artists erase any *extraneous* marks on their sketches.

3 Solve formulas for a specified variable.

Many formulas involve rational expressions. We can use the fraction-clearing method of this section to solve such formulas for a specified variable.

Self Check 6

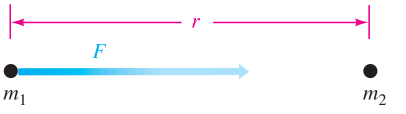
Solve the law of gravitation formula for r^2. $r^2 = \frac{Gm_1m_2}{F}$

Now Try **Problem 41**

Teaching Example 6 Solve $F = \dfrac{Gm_1m_2}{r^2}$ for m_1.
Answer:
$m_1 = \dfrac{Fr^2}{Gm_2}$

EXAMPLE 6 ***Physics*** The *law of gravitation,* formulated by Sir Isaac Newton in 1684, states that if two masses, m_1 and m_2, are separated by a distance of r, the force F exerted by one mass on the other is

$$F = \frac{Gm_1m_2}{r^2}$$



where G is the gravitational constant. Solve for m_2.

Strategy To solve for m_2, we will treat it as if it were the only variable in the equation. To isolate this variable, we will use the same strategy that we used in previous examples to solve rational equations in one variable.

WHY We can solve a formula as if it were an equation in one variable because all the other variables are treated as if they were numbers (constants).

Solution

$$F = \frac{Gm_1m_2}{r^2}$$

$$r^2(F) = \overset{1}{\cancel{r^2}}\left(\frac{Gm_1m_2}{\underset{1}{\cancel{r^2}}}\right) \quad \text{Multiply both sides by the LCD, } r^2\text{. Simplify: } \frac{r^2}{r^2} = 1.$$

$$\frac{r^2F}{Gm_1} = \frac{Gm_1m_2}{Gm_1} \quad \text{To isolate } m_2\text{, divide both sides by } Gm_1.$$

$$\frac{r^2F}{Gm_1} = m_2 \quad \text{Simplify the right side by removing the factors } G \text{ and } m_1\text{, which are common to the numerator and denominator.}$$

$$m_2 = \frac{r^2F}{Gm_1} \quad \text{Reverse the sides of the equation so that } m_2 \text{ is on the left.}$$

Self Check 7

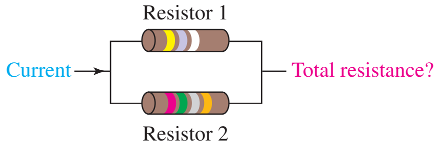
Solve $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{z}$ for z. $z = \dfrac{xy}{y - x}$

Now Try **Problem 49**

Teaching Example 7 Solve $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ for R_1.
Answer:
$R_1 = \dfrac{RR_2}{R_2 - R}$

EXAMPLE 7 ***Electronics*** In electronic circuits, resistors oppose the flow of an electric current. The total resistance R of a parallel combination of two resistors as shown is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



where R_1 is the resistance of the first resistor and R_2 is the resistance of the second resistor. Solve for R.

Strategy To solve for R, we will treat it as if it were the only variable in the equation. To isolate this variable, we will use the same strategy that we used in previous examples to solve rational equations in one variable; we will clear the equation of fractions.

WHY We can solve a formula as if it were an equation in one variable because all the other variables are treated as if they were numbers (constants).

Solution

We begin by clearing the equation of fractions by multiplying both sides by the LCD, which is RR_1R_2.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$RR_1R_2\left(\frac{1}{R}\right) = RR_1R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad \text{Multiply both sides by the LCD.}$$

$$RR_1R_2\left(\frac{1}{R}\right) = RR_1R_2\left(\frac{1}{R_1}\right) + RR_1R_2\left(\frac{1}{R_2}\right) \quad \text{On the right side, distribute } RR_1R_2.$$

$$\overset{1}{\cancel{R}}R_1R_2\left(\frac{1}{\underset{1}{\cancel{R}}}\right) = R\overset{1}{\cancel{R_1}}R_2\left(\frac{1}{\underset{1}{\cancel{R_1}}}\right) + RR_1\overset{1}{\cancel{R_2}}\left(\frac{1}{\underset{1}{\cancel{R_2}}}\right) \quad \text{Remove common factors of the numerator and denominator.}$$

$$R_1R_2 = RR_2 + RR_1 \quad \text{Simplify.}$$

$$R_1R_2 = R(R_2 + R_1) \quad \text{Factor out } R \text{ on the right side.}$$

$$\frac{R_1R_2}{R_2 + R_1} = R \quad \text{To isolate } R\text{, divide both sides by } R_2 + R_1.$$

$$R = \frac{R_1R_2}{R_2 + R_1} \quad \text{Reverse the sides of the equation to write } R \text{ on the left side.}$$

ANSWERS TO SELF CHECKS

1. 9 **2.** 18 **3.** 3, 5 **4.** -3 **5.** No solution; 1 is extraneous **6.** $r^2 = \frac{Gm_1m_2}{F}$ **7.** $z = \frac{xy}{y - x}$

SECTION 6.7 STUDY SET

VOCABULARY

Fill in the blanks.

1. Equations that contain one or more rational expressions, such as $\frac{x}{x+2} = 4 + \frac{10}{x+1}$, are called rational equations.

▶ **2.** When solving a rational equation, if we obtain a number that does not satisfy the original equation, the number is called an extraneous solution.

CONCEPTS

3. Is 2 a solution of the following equations?

a. $\frac{x+2}{x+3} + \frac{1}{x^2 + 2x - 3} = 1$ yes

b. $\frac{x+2}{x-2} + \frac{1}{x^2 - 4} = 1$ no

4. Consider the rational equation $\frac{x}{x-3} = \frac{1}{x} + \frac{2}{x-3}$.

a. What values of x make a denominator 0? 3, 0

b. What values of x make a rational expression undefined? 3, 0

c. What numbers can't be solutions of the equation? 3, 0

▶ **5.** To clear the following equation of fractions, by what should both sides be multiplied?

$$\frac{4}{10} + y = \frac{4y - 50}{5y - 25}$$ the LCD, $10(y - 5)$

▶ Selected exercises available online at **www.webassign.net/brookscole**

6. Perform each multiplication.

a. $4x\left(\frac{3}{4x}\right)$ 3

b. $(x + 6)(x - 2)\left(\frac{3}{x - 2}\right)$ $3(x + 6) = 3x + 18$

c. $8(x + 4)\left(\frac{7x}{2(x + 4)}\right)$ $28x$

d. $6(m - 5)\left(\frac{7}{5 - m}\right)$ -42

NOTATION

Complete each solution.

7.
$$\frac{10}{3y} - \frac{7}{30} = \frac{9}{2y}$$
$$30y\left(\frac{10}{3y} - \frac{7}{30}\right) = 30y\left(\frac{9}{2y}\right)$$
$$30y\left(\frac{10}{3y}\right) - 30y\left(\frac{7}{30}\right) = 30y\left(\frac{9}{2y}\right)$$
$$100 - 7y = 135$$
$$-7y = 35$$
$$y = -5$$

8.
$$\frac{2}{u - 1} + \frac{1}{u} = \frac{1}{u^2 - u}$$
$$\frac{2}{u - 1} + \frac{1}{u} = \frac{1}{u(u - 1)}$$
$$u(u-1)\left(\frac{2}{u - 1} + \frac{1}{u}\right) = u(u-1)\left[\frac{1}{u(u - 1)}\right]$$
$$u(u-1)\left(\frac{2}{u - 1}\right) + u(u-1)\left(\frac{1}{u}\right) = u(u-1)\left[\frac{1}{u(u - 1)}\right]$$
$$2u + u - 1 = 1$$
$$3u = 2$$
$$u = \frac{2}{3}$$

GUIDED PRACTICE

Solve each equation. See Example 1.

9. $\frac{1}{4} + \frac{9}{x} = 1$ 12

10. $\frac{1}{3} - \frac{10}{x} = -3$ 3

11. $\frac{1}{a} = \frac{1}{3} - \frac{2}{3a}$ 5

12. $\frac{1}{b} = \frac{1}{8} - \frac{3}{8b}$ 11

13. $\frac{18}{y + 1} + \frac{2}{5} = 4$ 4

14. $\frac{2}{3} + \frac{10}{a + 2} = 4$ 1

15. $\frac{1}{2} + \frac{x}{x - 1} = 3$ $\frac{5}{3}$

16. $\frac{2}{3} + \frac{a}{a - 2} = 5$ $\frac{13}{5}$

Solve each equation. See Example 2.

17. $\frac{4}{t + 3} + \frac{8}{t^2 - 9} = \frac{2}{t - 3}$ 5

18. $\frac{5}{x - 1} = \frac{1}{x^2 - 1} + \frac{1}{x - 1}$ $-\frac{3}{4}$

19. $\frac{4}{x^2 - 4} - \frac{5}{x - 2} = \frac{1}{x + 2}$ $-\frac{2}{3}$

20. $\frac{1}{m + 3} - \frac{m}{m^2 - 9} = \frac{-2}{m - 3}$ $-\frac{3}{2}$

21. $\frac{2}{x - 2} + \frac{10}{x + 5} = \frac{2x}{x^2 + 3x - 10}$ 1

22. $\frac{2}{a + 4} + \frac{2a - 1}{a^2 + 2a - 8} = \frac{1}{a - 2}$ 3

23. $\frac{1}{n + 2} - \frac{2}{n - 3} = \frac{-2n}{n^2 - n - 6}$ 7

24. $\frac{2x}{x^2 + 9x + 20} - \frac{3}{x + 4} = \frac{2}{x + 5}$ $-\frac{23}{3}$

Solve each equation. See Example 3.

25. $\frac{2}{5x - 5} + \frac{x - 2}{15} = \frac{4}{5x - 5}$ 4, −1

26. $\frac{3}{2x + 4} = \frac{x - 2}{2} + \frac{x - 5}{2x + 4}$ −4, 3

27. $\frac{p - 1}{2} + 1 = \frac{3}{p}$ 2, −3

28. $\frac{b + 1}{2} - \frac{3}{2} = \frac{4}{b}$ 4, −2

29. $\frac{16}{t + 3} + \frac{7}{t - 2} = 3$ $-\frac{1}{3}, 7$

30. $\frac{17}{s - 4} - \frac{10}{s + 2} = 2$ $-\frac{9}{2}, 10$

31. $\frac{5}{x - 2} = 2 - \frac{6}{x + 2}$ $-\frac{1}{2}, 6$

32. $\frac{-10}{t + 3} = 1 - \frac{11}{t - 3}$ −8, 9

Solve each equation. See Example 4.

33. $\frac{1}{3x - 18} + \frac{5}{6 - x} = \frac{1}{3}$ −8

34. $\frac{1}{2x - 16} + \frac{14}{8 - x} = \frac{3}{2}$ −1

35. $\frac{7}{3x - 9} + \frac{1}{3 - x} = \frac{4}{9}$ 6

36. $\frac{1}{2d - 4} - \frac{1}{2 - d} = \frac{1}{4}$ 8

Solve each equation. If a solution is extraneous, so indicate. See Example 5.

37. $4 - \frac{3x}{x - 9} = \frac{5x - 72}{x - 9}$ No solution; 9 is extraneous.

38. $2 - \frac{2x}{x - 10} = \frac{4x - 60}{x - 10}$ No solution; 10 is extraneous.

39. $\frac{6}{x + 3} + \frac{48}{x^2 - 2x - 15} - \frac{7}{x - 5} = 0$ No solution; −3 is extraneous.

40. $\frac{3}{x - 4} + \frac{2}{x + 5} + \frac{18}{x^2 + x - 20} = 0$ No solution; −5 is extraneous.

Solve each formula for the specified variable.
See Examples 6 and 7.

41. $Q = \dfrac{A - I}{L}$ for A (from banking)
$A = LQ + I$

42. $z = \dfrac{x - \overline{x}}{s}$ for x (from statistics)
$x = sz + \overline{x}$

43. $I = \dfrac{E}{R_L + r}$ for r (from physics)
$r = \frac{E - IR_L}{I}$

▶ **44.** $P = \dfrac{R - C}{n}$ for C (from business)
$C = R - nP$

45. $\mu_R = \dfrac{n_1(n_1 + n_2 + 1)}{2}$ for n_2 (from statistics)
$n_2 = \frac{2\mu_R - n_1^2 - n_1}{n_1}$

46. $\dfrac{P_1V_1}{T_1} = \dfrac{P_2V_2}{T_2}$ for T_2 (from chemistry)
$T_2 = \frac{P_2T_1V_2}{P_1V_1}$

47. $P = \dfrac{Q_1}{Q_2 - Q_1}$ for Q_1 (from refrigeration/heating)
$Q_1 = \frac{PQ_2}{1 + P}$

48. $S = \dfrac{a - \ell r}{1 - r}$ for r (from mathematics)
$r = \frac{S - a}{S - \ell}$

▶ **49.** $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$ for R (from electronics)
$R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

50. $\dfrac{x}{a} + \dfrac{y}{b} = 1$ for a (from mathematics)
$a = \frac{bx}{b - y}$

▶ **51.** $\dfrac{E}{e} = \dfrac{R + r}{r}$ for r (from engineering)
$r = \frac{eR}{E - e}$

▶ **52.** $P + \dfrac{a}{V^2} = \dfrac{RT}{V - b}$ for b (from physics)
$b = \frac{PV^3 + aV - RTV^2}{PV^2 + a}$

TRY IT YOURSELF

Solve each equation. If a solution is extraneous, so indicate.

53. $\dfrac{x + 2}{x + 3} - 1 = \dfrac{-1}{x^2 + 2x - 3}$ 2

54. $\dfrac{m + 6}{3m - 12} + \dfrac{5}{4 - m} = \dfrac{2}{3}$ -1

55. $\dfrac{3}{y} + \dfrac{7}{2y} = 13$ $\frac{1}{2}$

56. $\dfrac{2}{x} + \dfrac{1}{2} = \dfrac{7}{2x}$ 3

57. $\dfrac{3}{r} + \dfrac{12}{r^2 - 4r} = -\dfrac{7}{r - 4}$ No solution; 0 is extraneous.

58. $\dfrac{4t^2 + 36}{t^2 - 9} - \dfrac{4t}{t + 3} = \dfrac{-12}{t - 3}$
No solution; -3 is extraneous.

59. $\dfrac{x + 4}{2x + 14} - \dfrac{x}{2x + 6} = \dfrac{3}{16}$
1, -11

60. $\dfrac{30}{y - 2} + \dfrac{24}{y - 5} = 13$
8, $\frac{41}{13}$

61. $\dfrac{3}{m} = 2 - \dfrac{m}{m - 2}$ 1, 6

62. $\dfrac{n}{2} = 1 + \dfrac{12}{n}$ -4, 6

63. $\dfrac{x + 2}{2x - 6} + \dfrac{3}{3 - x} = \dfrac{x}{2}$
a repeated solution of 2

64. $\dfrac{3}{4x - 8} = \dfrac{1}{36} - \dfrac{2}{6 - 3x}$
5

65. $\dfrac{2}{x} + \dfrac{1}{2} = \dfrac{9}{4x} - \dfrac{1}{2x}$
$-\frac{1}{2}$

66. $\dfrac{7}{5x} - \dfrac{1}{2} = \dfrac{5}{6x} + \dfrac{1}{3}$
$\frac{17}{25}$

67. $\dfrac{3 - 5y}{2 + y} = \dfrac{-5y - 3}{y - 2}$
0

68. $\dfrac{a - 3}{a + 1} = \dfrac{a - 6}{a + 5}$
$\frac{9}{7}$

▶ **69.** $\dfrac{21}{x^2 - 4} - \dfrac{14}{x + 2} = \dfrac{3}{2 - x}$
5

70. $\dfrac{-5}{c + 2} = \dfrac{3}{2 - c} + \dfrac{2c}{c^2 - 4}$
4

71. $\dfrac{x - 4}{x - 3} - \dfrac{x - 2}{3 - x} = x - 3$
5; 3 is extraneous.

72. $\dfrac{5}{x + 4} + \dfrac{1}{x + 4} = x - 1$
2, -5

73. $\dfrac{a + 2}{a + 1} = \dfrac{a - 4}{a - 3}$ 1

74. $\dfrac{z + 2}{z + 8} = \dfrac{z - 3}{z - 2}$ 4

▶ **75.** $\dfrac{5}{y - 1} + \dfrac{3}{y - 3} = \dfrac{8}{y - 2}$
6

76. $\dfrac{3 + 2a}{a^2 + 6 + 5a} + \dfrac{2 - 5a}{a^2 - 4} = \dfrac{2 - 3a}{a^2 - 6 + a}$
$-\frac{2}{5}$

77. $\dfrac{3}{s - 2} + \dfrac{s - 14}{2s^2 - 3s - 2} - \dfrac{4}{2s + 1} = 0$
1

78. $\dfrac{1}{y^2 - 2y - 3} + \dfrac{1}{y^2 - 4y + 3} - \dfrac{1}{y^2 - 1} = 0$
-3

79. $\dfrac{x}{x + 2} = 1 - \dfrac{3x + 2}{x^2 + 4x + 4}$
2

80. $\dfrac{a - 1}{a + 3} - \dfrac{1 - 2a}{3 - a} = \dfrac{2 - a}{a - 3}$
0

81. $\dfrac{5}{2z^2 + z - 3} - \dfrac{2}{2z + 3} = \dfrac{z + 1}{z - 1} - 1$
$\frac{1}{6}$

82. $\dfrac{x}{x-5} + \dfrac{5}{x} = \dfrac{11}{6}$ 2, 15

83. $\dfrac{5}{3x+12} - \dfrac{1}{9} = \dfrac{x-1}{3x}$ $-\frac{3}{2}$, 2

84. $\dfrac{1}{y+5} = \dfrac{1}{3y+6} - \dfrac{y+2}{y^2+7y+10}$ $-\frac{7}{5}$

For each expression in part a, perform the indicated operations and then simplify, if possible. Solve each equation in part b and check the result.

85. a. $\dfrac{11}{12} - \dfrac{3}{2x} + \dfrac{4}{x}$ $\frac{11x+30}{12x}$ **b.** $\dfrac{11}{12} - \dfrac{3}{2x} = \dfrac{4}{x}$ 6

86. a. $\dfrac{1}{6x} - \dfrac{2}{x-6}$ $\frac{-11x-6}{6x(x-6)}$ **b.** $\dfrac{1}{6x} = \dfrac{2}{x-6}$ $-\frac{6}{11}$

87. a. $\dfrac{m}{m-2} - \dfrac{1}{m-3}$ $\frac{m^2-4m+2}{(m-2)(m-3)}$ **b.** $\dfrac{m}{m-2} - \dfrac{1}{m-3} = 1$ 4

88. a. $\dfrac{a^2+1}{a^2-a} - \dfrac{a}{a-1}$ $\frac{1}{a(a-1)}$ **b.** $\dfrac{a^2+1}{a^2-a} - \dfrac{a}{a-1} = \dfrac{1}{a}$ 2

APPLICATIONS

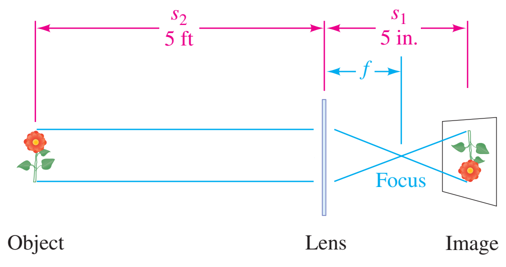
89. PHOTOGRAPHY The illustration shows the relationship between distances when taking a photograph. The design of a camera lens uses the equation

$$\frac{1}{f} = \frac{1}{s_1} + \frac{1}{s_2}$$

which relates the focal length f of a lens to the image distance s_1 and the object distance s_2.

a. Solve the formula for f. $f = \frac{s_1 s_2}{s_1 + s_2}$

b. Find the focal length of the lens in the illustration. (*Hint:* Convert feet to inches.) $4\frac{8}{13}$ in.



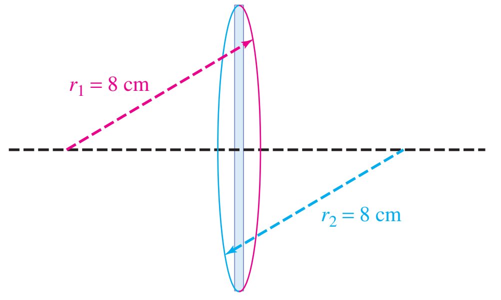
90. OPTICS See the illustration in the next column. The focal length, f, of a lens is given by the lensmaker's formula,

$$\frac{1}{f} = 0.6\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

where f is the focal length of the lens and r_1 and r_2 are the radii of the two circular surfaces.

a. Solve the formula for f. $f = \frac{r_1 r_2}{0.6r_1 + 0.6r_2}$

b. Find the focal length of the lens in the illustration. $\frac{20}{3}$ cm



91. ACCOUNTING As a piece of equipment gets older, its value usually lessens. One way to calculate *depreciation* is to use the formula

$$V = C - \left(\frac{C-S}{L}\right)N$$

where V denotes the value of the equipment at the end of year N, L is its useful lifetime (in years), C is its cost new, and S is its salvage value at the end of its useful life.

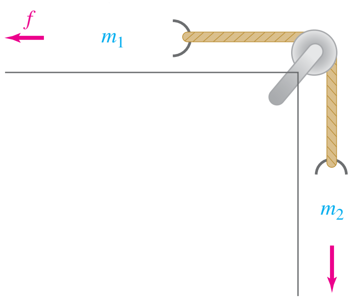
a. Solve the formula for L. $L = \frac{SN - CN}{V - C}$

b. Determine what an accountant considered the useful lifetime of a forklift that cost \$25,000 new, was worth \$13,000 after 4 years, and has a salvage value of \$1,000. 8 yr

92. ENGINEERING The equation

$$a = \frac{9.8m_2 - f}{m_2 + m_1}$$

models the system shown, where a is the acceleration of the suspended block, m_1 and m_2 are the masses of the blocks, and f is the friction force. Solve for m_2. $m_2 = \frac{am_1 + f}{9.8 - a}$



WRITING

93. Why is it necessary to check the solutions of a rational equation?

94. Explain what it means to *clear* a rational equation of fractions. Give an example.

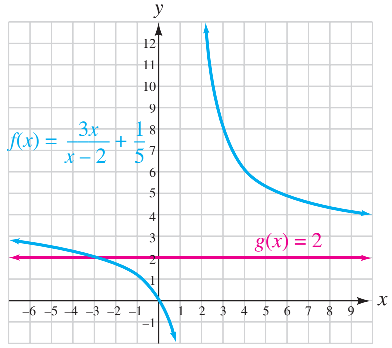
95. Would you use the same approach to answer the following problems? Explain why or why not.

$$\text{Simplify: } \frac{x^2-10}{x^2-1} - \frac{3x}{x-1} - \frac{2x}{x+1}$$

$$\text{Solve: } \frac{x^2-10}{x^2-1} - \frac{3x}{x-1} = -\frac{2x}{x+1}$$

96. Explain how to solve the rational equation graphically:

$$\frac{3x}{x-2} + \frac{1}{5} = 2$$



REVIEW

Write each italicized number in scientific notation.

97. OIL The total cost of the Alaskan pipeline, running 800 miles from Prudhoe Bay to Valdez, was *\$9,000,000,000.* 9.0×10^9

98. NATURAL GAS The TransCanada Pipeline transported a record *2,352,000,000,000* cubic feet of gas in 1995. 2.352×10^{12}

99. RADIOACTIVITY The least stable radioactive isotope is lithium 5, which decays in *0.00000000000000000000044* second. 4.4×10^{-22}

100. BALANCES The finest balances in the world are made in Germany. They can weigh objects to an accuracy of 35×10^{-11} ounce. 3.5×10^{-10}

SECTION 6.8
Problem Solving Using Rational Equations

Objectives

1. Solve shared-work problems.
2. Solve uniform motion problems.

In this section, we will use rational equations to model situations involving work and uniform motion.

1 Solve shared-work problems.

Problems in which two or more people (or machines) work together to complete a job are called **shared-work problems.** To solve such problems, we must determine the **rate of work** for each person or machine involved. For example, suppose it takes you 4 hours to clean your house. Your rate of work can be expressed as $\frac{1}{4}$ of the job is completed per hour. If someone else takes 5 hours to clean the same house, they complete $\frac{1}{5}$ of the job per hour. In general, a rate of work can be determined in the following way.

Rate of Work

If a job can be completed in x hours, the rate of work can be expressed as:

$$\frac{1}{x} \text{ of the job is completed per hour}$$

If a job is completed in some other unit of time, such as x minutes or x days, the rate of work is expressed in that unit.

To solve shared-work problems, we must also determine the *amount of work* completed. To do this, we use a formula similar to the distance formula $d = rt$ used for motion problems.

Work completed = rate of work · time worked or $W = rt$

Self Check 1

PAINTING A HOUSE One crew can paint a house in 3 days and another crew can paint the same house in 4 days. If both crews work together, how long will it take them to paint the house?

Now Try **Problem 11**

Self Check 1 Answer

If both crews work together, it will take $1\frac{5}{7}$ days to paint the house.

Teaching Example 1 LAWN CARE

One crew can mow a lawn in 6 hours and another crew can mow the lawn in 4 hours. If both crews work together, how long will it take them to mow the lawn?

Answer:

If both crews work together, it will take $2\frac{2}{5}$ hours to mow the lawn.

EXAMPLE 1 ***Home Construction*** One crew can drywall a house in 4 days and another crew can drywall the same house in 5 days. If both crews work together, how long will it take to drywall the house?

© Jim Craigmyle/Corbis

Analyze It is helpful to organize the facts of a shared-work problem in a table.

Form Let x = the number of days it will take to drywall the house if both crews work together. Since the crews will be working for the same amount of time, enter x as the time worked for each crew.

If the first crew can drywall the house in 4 days, its rate working alone is $\frac{1}{4}$ of the job per day. If the second crew can drywall the house in 5 days, its rate working alone is $\frac{1}{5}$ of the job per day. To determine the work completed by each crew, multiply the rate by the time.

	Rate	· Time	= Work completed
1st crew	$\frac{1}{4}$	x	$\frac{x}{4}$
2nd crew	$\frac{1}{5}$	x	$\frac{x}{5}$

Enter this information first.

Multiply to get each of these entries; $W = rt$.

In shared-work problems, the number 1 represents one whole job completed. So we have

The part of the job done by 1st crew	plus	the part of the job done by 2nd crew	equals	1 job completed.
$\frac{x}{4}$	$+$	$\frac{x}{5}$	$=$	1

Solve

$$\frac{x}{4} + \frac{x}{5} = 1$$

$$20\left(\frac{x}{4} + \frac{x}{5}\right) = 20(1) \quad \text{Clear the equation of fractions by multiplying both sides by the LCD, 20.}$$

$$20\left(\frac{x}{4}\right) + 20\left(\frac{x}{5}\right) = 20 \quad \text{Distribute the multiplication by 20.}$$

$$\overset{1}{\cancel{4}} \cdot 5 \cdot \left(\frac{x}{\underset{1}{\cancel{4}}}\right) + 4 \cdot \overset{1}{\cancel{5}} \cdot \left(\frac{x}{\underset{1}{\cancel{5}}}\right) = 20 \quad \text{Factor 20 as } 4 \cdot 5\text{, and remove common factors.}$$

$$5x + 4x = 20 \quad \text{Simplify.}$$

$$9x = 20 \quad \text{Combine like terms.}$$

$$x = \frac{20}{9} \quad \text{Divide both sides by 9.}$$

State If both crews work together, it will take $\frac{20}{9}$ or $2\frac{2}{9}$ days to drywall the house.

Check In $\frac{20}{9}$ days, the first crew drywalls $\frac{1}{4} \cdot \frac{20}{9} = \frac{5}{9}$ of the house and the second crew drywalls $\frac{1}{5} \cdot \frac{20}{9} = \frac{4}{9}$ of the house. The sum of these efforts, $\frac{4}{9} + \frac{5}{9}$, is $\frac{9}{9}$ or 1 house drywalled. The result checks.

Example 1 can be solved in a different way by considering the amount of work done by each crew in 1 day. As before, if we let $x =$ the number of days it will take to drywall the house if both crews work together, then together, in 1 day, they will complete $\frac{1}{x}$ of the job. If we add what the first crew can do in 1 day to what the second crew can do in 1 day, the sum is what they can do together in 1 day.

What 1st crew can do in 1 day	plus	what 2nd crew can do in 1 day	equals	what they can do together in 1 day.
$\frac{1}{4}$	$+$	$\frac{1}{5}$	$=$	$\frac{1}{x}$

To solve the equation, begin by clearing it of fractions.

$$20x\left(\frac{1}{4} + \frac{1}{5}\right) = 20x\left(\frac{1}{x}\right)$$ Multiply both sides by the LCD, $20x$.

$$5x + 4x = 20$$ Distribute the multiplication by $20x$ and simplify.

$$9x = 20$$ Combine like terms.

$$x = \frac{20}{9}$$ Divide both sides by 9.

This is the same as the solution obtained in Example 1.

EXAMPLE 2 *Setting up Seating*

It takes the head custodian at a school 30 minutes less time than his assistant to set up the chairs for a program in the auditorium. Working together, they can set up the chairs in 20 minutes. How long would it take each person working alone to set up the chairs?

Analyze We will organize the facts of this shared-work problem in a table.

Form Let $x =$ the number of minutes that it takes the assistant to set up the chairs. Then $x - 30 =$ the number of minutes that it takes the head custodian to set up the chairs.

If the assistant can set up the chairs in x minutes, his rate working alone is $\frac{1}{x}$ of the job per minute. If the head custodian can set up the chairs in $(x - 30)$ minutes, his rate working alone is $\frac{1}{x-30}$ of the job per minute. Each rate is entered in the table below.

If they work together, it takes them 20 minutes to complete the job. We enter 20 minutes for the time worked for each person. To determine the work completed by each person, we multiply the rate by the time, as shown in the table.

	Rate	· Time	= Work completed
Assistant	$\frac{1}{x}$	20	$\frac{20}{x}$
Head custodian	$\frac{1}{x-30}$	20	$\frac{20}{x-30}$

Enter this information first.

Multiply to get each of these entries; $W = rt$.

Self Check 2

FILLING A POOL It takes a small garden hose 1 day longer to fill a swimming pool than a larger hose. Working together, they can fill the pool in $1\frac{1}{5}$ days. How long would it take each hose used individually to fill the pool?

Now Try **Problem 21**

Self Check 2 Answer
The larger hose would take 2 days and the smaller hose would take 3 days to fill the pool.

Teaching Example 2 ROOFING A HOUSE A homeowner estimates that it will take him 3 more days to roof his house than a professional roofer would take. If working together, the homeowner and professional roofer can complete the job in $2\frac{6}{11}$ days, how long would it take the homeowner working alone?
Answer:
It will take the homeowner 7 days to roof his house.

In shared-work problems, the number 1 represents one whole job completed. So we have

The part of the job done by the assistant	plus	the part of the job done by the head custodian	equals	1 job completed.
$\frac{20}{x}$	$+$	$\frac{20}{x-30}$	$=$	1

Solve

$$x(x-30)\left(\frac{20}{x}+\frac{20}{x-30}\right)=x(x-30)(1)$$ Clear the equation of fractions by multiplying both sides by the LCD, $x(x-30)$.

$$\cancel{x}(x-30)\left(\frac{20}{\cancel{x}}\right)+x\cancel{(x-30)}\left(\frac{20}{\cancel{x-30}}\right)=x(x-30)(1)$$ Distribute $x(x-30)$ and then remove common factors.

$$(x-30)(20)+x(20)=x(x-30)(1)$$ Simplify.

$$20x-600+20x=x^2-30x$$ Perform the multiplication on each side. This is a quadratic equation.

$$40x-600=x^2-30x$$ Combine like terms.

$$0=x^2-70x+600$$ To get 0 on the left side, subtract $40x$ and add 600 to both sides.

$$0=(x-10)(x-60)$$ Factor the trinomial.

$$\cancel{x-10=0} \quad \text{or} \quad x-60=0$$ Set each factor equal to 0.

$$\cancel{x=10} \qquad\qquad x=60$$

State The solutions of the equation are 10 and 60. If it takes them 20 minutes, working together, to set up the chairs, it does not make sense that it would take the assistant 10 minutes working alone. We reject that solution. Thus, we have found that, working alone, it takes the assistant 60 minutes and it takes the head custodian $60-30=30$ minutes to set up the chairs.

Check In 20 minutes, the assistant completes $\frac{20}{60}=\frac{1}{3}$ of the job while the head custodian completes $\frac{20}{30}=\frac{2}{3}$ of the job. The sum of these efforts, $\frac{1}{3}+\frac{2}{3}$, is $\frac{3}{3}$ or 1 job completed. The results check.

2 Solve uniform motion problems.

In the next two examples, rational equations are used to model situations involving **uniform motion.**

EXAMPLE 3 ***Road Trips*** A doctor drove 200 miles to attend a national convention. Because of poor weather, her average speed on the return trip was 10 mph less than her average speed going to the convention. If the return trip took 1 hour longer, how fast did she drive in each direction?

Analyze We need to find her rates of speed going to and returning from the convention. They can be represented using a variable. The distance traveled was 200 miles each way. To describe the travel times, we note that

$rt = d$ *r is the rate of speed, t is the time traveled, and d is the distance.*

or

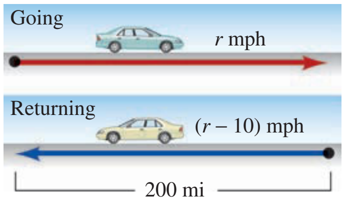
$t = \frac{d}{r}$ *Divide both sides by r.*

Form Let $r =$ the average rate of speed (in mph) going to the meeting. Then $r - 10 =$ the average rate of speed on the return trip. We can organize the facts of the problem in the table.

	Rate	· Time	= Distance
Going	r	$\frac{200}{r}$	200
Returning	$r - 10$	$\frac{200}{r-10}$	200

Enter this information first.

We obtained these entries by dividing the distance by the rate: $t = \frac{d}{r}$.



Because the return trip took 1 hour longer, we can form the following equation:

The time it took to travel to the convention	plus	1	equals	the time it took to return.
$\frac{200}{r}$	$+$	1	$=$	$\frac{200}{r-10}$

Solve We can solve the equation as follows:

$$r(r-10)\left(\frac{200}{r} + 1\right) = r(r-10)\left(\frac{200}{r-10}\right)$$ *Multiply both sides by the LCD, $r(r - 10)$.*

$$\overset{1}{\cancel{r}}(r-10)\frac{200}{\underset{1}{\cancel{r}}} + r(r-10)1 = r\overset{1}{\cancel{(r-10)}}\left(\frac{200}{\underset{1}{\cancel{r-10}}}\right)$$ *Distribute $r(r - 10)$ and then remove common factors.*

$$200(r-10) + r(r-10) = 200r$$

$$200r - 2{,}000 + r^2 - 10r = 200r$$ *Distribute 200 and r. This is a quadratic equation.*

$$190r - 2{,}000 + r^2 = 200r$$ *Combine like terms.*

$$r^2 - 10r - 2{,}000 = 0$$ *To get 0 on the right side, subtract $200r$ from both sides.*

$$(r-50)(r+40) = 0$$ *Factor $r^2 - 10r - 2{,}000$.*

$$r - 50 = 0 \quad \text{or} \quad \cancel{r + 40 = 0}$$ *Set each factor equal to 0.*

$$r = 50 \quad \Big| \quad \cancel{r = -40}$$

State We must exclude the solution of -40, because a speed cannot be negative. Thus, the doctor averaged 50 mph going to the convention, and she averaged $50 - 10$ or 40 mph returning.

Check At 50 mph, the 200-mile trip took 4 hours. At 40 mph, the return trip took 5 hours, which is 1 hour longer. The results check.

Self Check 3

ROAD TRIPS A caravan of students traveled 150 miles to an academic competition. On the return trip, their average speed was 20 mph less than going, due to road construction. If the return trip took 2 hours longer, how fast did they drive in each direction?

Now Try **Problem 31**

Self Check 3 Answer
The caravan of students averaged 50 mph going to the competition and 30 mph returning.

Teaching Example 3 ROAD TRIPS
A group of college students traveled 300 miles to spend the weekend at home. Because of traffic conditions, their average speed on the return trip was 10 mph less than their average speed going. If the return trip took 1 hour more, how fast did they drive each direction?
Answer:
They averaged 60 mph going home, and 50 mph returning to college.

Self Check 4

WATER TRAVEL A motorboat goes 5 miles upstream in the same time it requires to go 7 miles downstream. If the river flows at 2 miles per hour, find the speed of the boat in still water.

Now Try **Problem 39**

Self Check 4 Answer

The speed of the boat in still water is 12 mph.

Teaching Example 4 DINNER CRUISE A dinner cruise boat can travel 9 miles down the river and return in a total of 3 hours. If the boat can travel 8 mph in still water, find the speed of the river's current.

Answer:

The river's current is 4 mph.

EXAMPLE 4 ***Riverboat Cruises*** The Forest City Queen can make a 9-mile trip down the Rock River and return in a total of 1.6 hours. If the riverboat travels 12 mph in still water, find the speed of the current in the Rock River.

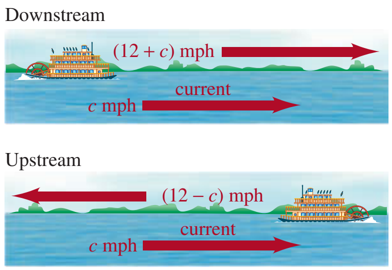
Analyze We can represent the upstream and downstream rates of speed using a variable. In each case, the distance traveled is 9 miles. To write an expression for the time traveled, divide the distance by the rate of speed.

Form We can let $c =$ the speed of the current (in mph). Since the boat travels 12 mph and a current of c mph pushes the boat while it is going downstream, the speed of the boat going downstream is $(12 + c)$ mph. On the return trip, the current pushes against the boat, and its speed is $(12 - c)$ mph. Since $t = \frac{d}{r}\left(\text{time} = \frac{\text{distance}}{\text{rate}}\right)$, the time required for the downstream leg of the trip is $\frac{9}{12 + c}$ hours, and the time required for the upstream leg of the trip is $\frac{9}{12 - c}$ hours. We can organize this information in the table.

	Rate	· Time	= Distance
Going downstream	$12 + c$	$\frac{9}{12 + c}$	9
Going upstream	$12 - c$	$\frac{9}{12 - c}$	9

Enter this information first.

Divide the distance by the rate.



We also know that the total time required for the round trip is 1.6 or $\frac{8}{5}$ hours.

The time it takes to travel downstream	plus	the time it takes to travel upstream	is	the total time for the round trip.
$\frac{9}{12 + c}$	$+$	$\frac{9}{12 - c}$	$=$	$\frac{8}{5}$

Solve We multiply both sides of the equation by $5(12 + c)(12 - c)$ to clear it of fractions.

$$\mathbf{5(12 + c)(12 - c)}\left(\frac{9}{12 + c} + \frac{9}{12 - c}\right) = \mathbf{5(12 + c)(12 - c)}\left(\frac{8}{5}\right)$$

$$5\overset{1}{\cancel{(12 + c)}}(12 - c)\left(\frac{9}{\underset{1}{\cancel{12 + c}}}\right) + 5(12 + c)\overset{1}{\cancel{(12 - c)}}\left(\frac{9}{\underset{1}{\cancel{12 - c}}}\right) = \overset{1}{\cancel{5}}(12 + c)(12 - c)\left(\frac{8}{\underset{1}{\cancel{5}}}\right)$$ Distribute $5(12 + c)(12 - c)$ and remove common factors.

$$45(12 - c) + 45(12 + c) = 8(12 + c)(12 - c)$$ Simplify.

$$540 - 45c + 540 + 45c = 8(144 - c^2)$$ On the left side, distribute. On the right side, use the FOIL method.

$$1{,}080 = 1{,}152 - 8c^2$$ Combine like terms and multiply. This is a quadratic equation.

$$8c^2 - 72 = 0$$ To get 0 on the right side, add $8c^2$ and subtract 1,152 from both sides.

$$c^2 - 9 = 0$$ Divide both sides by 8.

$$(c + 3)(c - 3) = 0$$ Factor the difference of two squares.

$$\cancel{c + 3 = 0} \quad \text{or} \quad c - 3 = 0$$ Set each factor equal to 0.

$$\cancel{c = -3} \qquad\qquad c = 3$$

State Since the current cannot be negative, the solution -3 must be discarded. The current in the Rock River is 3 mph.

Check The downstream trip is at $12 + 3 = 15$ mph for $\frac{9}{12+3} = \frac{3}{5}$ hr. Thus, the distance traveled is $15 \cdot \frac{3}{5} = 9$ miles. The upstream trip is at $12 - 3 = 9$ mph for $\frac{9}{12-3} = 1$ hr. Thus, the distance traveled is $9 \cdot 1 = 9$ miles. Since both distances are 9 miles, the result checks.

ANSWERS TO SELF CHECKS

1. If both crews work together, it will take $1\frac{5}{7}$ days to paint the house.
2. The larger hose would take 2 days and the smaller hose would take 3 days to fill the pool.
3. The caravan of students averaged 50 mph going to the competition and 30 mph returning.
4. The speed of the boat in still water is 12 mph.

SECTION 6.8 STUDY SET

VOCABULARY

Fill in the blanks.

1. In this section, we call problems that involve:
   - people or machines completing jobs, shared-_work_ problems.
   - moving vehicles, uniform _motion_ problems.
2. When a boat travels _downstream_, the speed of the boat is increased by the current. When a boat travels _upstream_, the speed of the boat is decreased by the current.

CONCEPTS

3. Fill in the blank: If a job can be completed in x hours, then the rate of work can be expressed as $\frac{1}{x}$ of the job is completed per hour.
4. a. It takes a night security officer 35 minutes to check each of the doors in an office building to make sure they are locked. What is the officer's rate of work? $\frac{1}{35}$ of the job per minute
   b. It takes a high school mathematics teacher 4 hours to make out the semester report cards. What part of the job does she complete in x hours? $\frac{x}{4}$
5. Complete the table.

	Rate	· Time	= Work completed
1st crew	$\frac{1}{15}$	x	$\frac{x}{15}$
2nd crew	$\frac{1}{8}$	x	$\frac{x}{8}$

6. Solve $d = rt$ for t. $t = \frac{d}{r}$

7. Complete the table.

	r	· t	$= d$
Running	x	$\frac{12}{x}$	12
Bicycling	$x + 15$	$\frac{12}{x+15}$	12

8. A boat can cruise at 30 mph in still water.
   a. What is its cruising speed upstream against a current of 4 mph? 26 mph
   b. What is its cruising speed downstream with a current of 4 mph? 34 mph

NOTATION

9. Write $\frac{41}{9}$ hours using a mixed number. $4\frac{5}{9}$ hr
10. Fill in the blanks: In the formula $W = rt$, the variable W stands for the _work_ completed, r is the _rate_, and t is the _time_.

APPLICATIONS

11. ROOFING A homeowner estimates that it will take him 7 days to roof his house. A professional roofer estimates that he could roof the house in 4 days. How long will it take if the homeowner helps the roofer? $2\frac{6}{11}$ days
12. DECORATING One crew can put up holiday decorations in a department store in 12 hours. A second crew can put up the decorations in 15 hours. How long will it take if both crews work together to decorate the store? $6\frac{2}{3}$ hours

Selected exercises available online at
www.webassign.net/brookscole

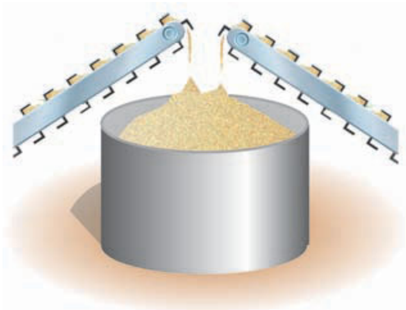
13. HOUSEPAINTING The illustration shows two bids to paint a house.

a. To get the job done quicker, the homeowner hired both the painters who submitted bids. How long will it take them to paint the house working together? $1\frac{7}{8}$ days

b. What will the homeowner have to pay each painter? Santos: $412.50, Mays: $375

▶ **14.** GROUNDSKEEPING It takes a groundskeeper 45 minutes to prepare a Little League baseball field for a game. It takes his assistant 55 minutes to prepare the same field. How long will it take if they work together to prepare the field? $24\frac{3}{4}$ min

▶ **15.** FARMING In 10 minutes, a conveyor belt can move 1,000 bushels of corn into the storage bin shown. A smaller belt can move 1,000 bushels to the storage bin in 14 minutes. If both belts are used, how long will it take to move 1,000 bushels to the storage bin? $5\frac{5}{6}$ min



16. BOTTLING At a packaging plant, the older of two machines can fill 5,000 bottles of shampoo in 6 hours. A newer machine can fill 5,000 bottles in 4 hours. If both machines are used, how long will it take to fill 5,000 bottles of shampoo? $2\frac{2}{5}$ hr

▶ **17.** THRILL RIDES At the end of an amusement park ride, a boat lands in a pool, splashing out a lot of water. Three inlet pipes, each working alone, can fill the pool in 10 seconds, 15 seconds, and 20 seconds, respectively. How long would it take to fill the pool if all three inlet pipes are used? $4\frac{8}{13}$ sec

18. SMOKE DAMAGE Three ventilation fans, each working alone, can clear the smoke out of a room in 12 hours, 16 hours, and 24 hours, respectively. How long would it take to clear out the smoke in the room if all three fans are used? $5\frac{1}{3}$ hr

19. FILLING PONDS One pipe can fill a pond in 3 weeks, and a second pipe can fill it in 5 weeks. However, evaporation and seepage can empty the pond in 10 weeks. If both pipes are used, how long will it take to fill the pond? $2\frac{4}{13}$ weeks

▶ **20.** HOUSECLEANING Sally can clean the house in 6 hours, her father can clean the house in 4 hours, and her younger brother, Dennis, can completely mess up the house in 8 hours. If Sally and her father clean and Dennis plays, how long will it take to clean the house? $3\frac{3}{7}$ hr

21. FINE DINING It takes a waiter 5 minutes less time than a busboy to fold the napkins used for the dinner seating in an upscale restaurant. Working together, they can fold the napkins in 6 minutes. How long would it take each person working alone to fold the napkins? waiter: 10 min, busboy: 15 min

▶ **22.** FIRE DRILL If the east and west exit doors of a banquet hall are open, the occupants can clear out in 2 minutes. It takes 3 minutes longer to clear the hall if just the east door is open as it does if just the west door is open. How long does it take to clear the hall if just the west door is open? 3 min

▶ **23.** FUND-RAISING LETTERS Working together, two secretaries can stuff the envelopes for a political fund-raising letter in 4 hours. Working alone, it takes the slower worker 6 hours longer to do the job than the faster worker. How long does it take each to do the job alone? faster worker: 6 hr, slower worker: 12 hr

24. SURVEYS It takes one team 9 days less than another to survey 1,000 people. If the teams work together, it takes them 20 days to complete such a survey. How long will it take each to do the survey alone? faster team: 36 days, slower team: 45 days

25. PLUMBING An experienced plumber can install the plumbing in a new apartment twice as fast as his apprentice. Working together, they can complete the plumbing job in 4 days. How long would it take each, working alone, to complete the plumbing? experienced plumber: 6 days, apprentice: 12 days

▶ **26.** NEWSLETTERS An elementary school teacher can assemble and staple the weekly newsletter three times faster than her student aide. Working together, they can assemble and staple the letters in 12 minutes. How long would it take each, working alone, to complete the job? teacher: 16 min, student aide: 48 min

27. DETAILING A CAR It takes a man 3 hours to wash and wax the family car. If his teenage son helps him, it only takes 1 hour. How long would it take the son, working alone, to wash and wax the car?
$1\frac{1}{2}$ hours

28. CLEANUP CREWS It takes one crew 4 hours to clean an auditorium after an event. If a second crew helps, it only takes 1.5 hours. How long would it take the second crew, working alone, to clean the auditorium?
$2\frac{2}{5}$ hours

29. OYSTERS According to the *Guinness Book of World Records,* the record for opening oysters is 100 in 140 seconds by Mike Racz in Invercargill, New Zealand, on July 16, 1990. If it would take a novice $8\frac{1}{2}$ minutes to perform the same task, how long would it take them working together to open 100 oysters? (*Hint:* Work in terms of seconds.)
about 110 sec

30. END ZONES One groundskeeper can paint the end zone of a football field in 2 hours. Another can paint it in 1 hour 20 minutes. How many minutes will it take them working together to paint the end zone?
48 min

AP Photo/Brian Garfinkel

31. TRUCK DELIVERIES A trucker drove 120 miles to make a delivery and returned home on the same route. Because of foggy conditions, his average speed on the return trip was 10 mph less than his average speed going. If the return trip took 1 hour longer, how fast did he drive in each direction?
going: 40 mph, returning: 30 mph

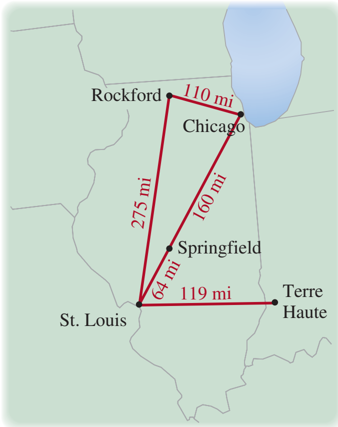
32. MOVING HOUSES A house mover towed a historic Victorian home 45 miles to locate it on a new site. On his return, without the heavy house in tow, his average speed was 30 mph faster and the trip was 2 hours shorter. How fast did he drive in each direction?
going: 15 mph, returning 45 mph

33. TRAIN TRAVEL A train traveled 120 miles from Freeport to Chicago and returned the same distance in a total time of 5 hours. If the train traveled 20 mph slower on the return trip, how fast did the train travel in each direction?
going: 60 mph, returning: 40 mph

34. BOXING For his morning workout, a boxer bicycles for 8 miles and then jogs back to camp along the same route. If he bicycles 6 mph faster than he jogs, and the entire workout lasts 2 hours, how fast does he jog?
6 mph

35. RATES OF SPEED Two trains made the same 315-mile run. Since one train traveled 10 mph faster than the other, it arrived 2 hours earlier. Find the speed of each train. 35 mph and 45 mph

36. DELIVERIES A FedEx delivery van traveled from Rockford to Chicago in 3 hours less time than it took a second van to travel from Rockford to St. Louis. If the vans traveled at the same average speed, use the information in the map to help determine how long the first driver was on the road. 2 hr



37. COMPARING TRAVEL A plane can fly 600 miles in the same time as it takes a car to go 240 miles. If the car travels 90 mph slower than the plane, find the speed of the plane. 150 mph

38. COMPARING TRAVEL A bicyclist can travel 40 miles in the same time that a motorcyclist can travel 60 miles. If the bicyclist travels 12 mph slower than the motorcyclist, find the speed of the motorcyclist. 36 mph

39. BOATING It takes 6 hours for a boater to travel 16 miles upstream and back. If the speed of the boat in still water is 6 mph, what is the speed of the current? 2 mph

40. RIVER TOURS A wave runner trip begins by going 60 miles upstream against a current. There, the driver turns around and returns with the current. If the still-water speed of the wave runner is set at 25 mph and the entire trip takes 5 hours, what is the speed of the current? 5 mph


Image Copyright Crok Photography, 2009. Used under license from Shutterstock.com

▶ **41.** BOATING A man can drive a motorboat 45 miles down the Colorado River in the same amount of time that he can drive 27 miles upstream. Find the speed of the current if the speed of the boat is 12 mph in still water. 3 mph

▶ **42.** CROP DUSTING A helicopter spraying fertilizer over a field can fly 0.5 mile downwind in the same time as it can fly 0.4 mile upwind. Find the speed of the wind if the helicopter travels 45 mph in still air when dusting crops. 5 mph

WRITING

43. In Example 1, one crew could drywall a house in 4 days, and another crew could drywall the same house in 5 days. We were asked to find how long it would take them to drywall the house working together. Explain why each of the following approaches is incorrect.

The time it would take to drywall the house

- ~~is the *sum* of the lengths of time it takes each crew to drywall the house: 4 days + 5 days = 9 day.~~
- ~~is the *difference* in lengths of time it takes each crew to drywall the house: 5 days − 4 days = 1 day.~~
- ~~is the *average* of the lengths of time it takes each crew to drywall the house:~~ $\frac{4 \text{ days} + 5 \text{ days}}{2} = \frac{9}{2} \text{ days} = 4\frac{1}{2} \text{ days}.$

44. Write a shared-work problem that can be modeled by the equation

$$\frac{x}{3} + \frac{x}{4} = 1$$

REVIEW

Simplify each expression. Write answers using positive exponents.

45. $\left(\frac{m^{10}}{n}\right)^8$ $\frac{m^{80}}{n^8}$

46. $\left(\frac{g^{20}}{t^{30}}\right)^{-4}$ $\frac{t^{120}}{g^{80}}$

47. $-w^{-2}$ $-\frac{1}{w^2}$

48. $-3s^0t$ $-3t$

49. $-\frac{4x^{-9} \cdot x^{-3}}{x^{-12}}$ -4

50. $\frac{y^{-3}y^{-4}y^0}{(2y^{-2})^3}$ $\frac{1}{8y}$

51. $(-x^2)^5y^7y^3x^{-2}y^0$ $-x^8y^{10}$

52. $5^2r^{-5}(r^6)^3$ $25r^{13}$

SECTION 6.9
Proportion and Variation

Objectives

1. Identify ratios, rates, and proportions.
2. Solve proportions.
3. Use proportions to solve problems.
4. Solve problems involving similar triangles.
5. Solve problems involving direct variation.
6. Solve problems involving inverse variation.
7. Solve problems involving joint variation.
8. Solve problems involving combined variation.

In this section, we will discuss the *ratio-proportion model* and four *variation models*. They can be used to solve a variety of application problems.

1 Identify ratios, rates, and proportions.

The quotient of two numbers or two quantities with the same units is often called a **ratio.** For example, $\frac{2}{3}$ can be read as "the ratio of 2 to 3." The notation 2:3 (read as "2 is to 3") is another way to denote a ratio. Some more examples of ratios are

$\frac{4x}{7y}$ The ratio of 4x to 7y and $\frac{x-2}{3x}$ The ratio of x − 2 to 3x

When we compare two quantities having different units, we call the comparison a **rate,** and we can write it as a fraction. One example is an average rate of speed.

A distance traveled → 372 miles
in a period of time → 6 hours

$$\frac{372 \text{ miles}}{6 \text{ hours}} = 62 \text{ mph} \leftarrow \text{The average rate of speed}$$

Rates are often used to express **unit costs,** such as the cost per pound of ground beef.

The cost of a package of beef → \$18.95
The weight of the package → 5 lb

$$\frac{\$18.95}{5 \text{ lb}} = \$3.79 \text{ per lb} \leftarrow \text{The cost per pound}$$

An equation indicating that two ratios or rates are equal is called a **proportion.** Two examples are

$$\frac{1}{4} = \frac{2}{8} \quad \text{and} \quad \frac{4}{7} = \frac{12}{21}$$

In the proportion $\frac{a}{b} = \frac{c}{d}$, the terms a and d are called the **extremes** of the proportion, and the terms b and c are called the **means.**

To develop a fundamental property of proportions, we suppose that

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion and multiply both sides by bd to obtain

$$\begin{array}{rll} bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right) & & \text{To clear the fractions, multiply both sides by the LCD, } bd. \\ \overset{1}{\cancel{b}}d \cdot \frac{a}{\underset{1}{\cancel{b}}} = b\overset{1}{\cancel{d}} \cdot \frac{c}{\underset{1}{\cancel{d}}} & & \text{Remove common factors of the numerator and denominator.} \\ ad = bc & & \text{Simplify.} \end{array}$$

Since $ad = bc$, the product of the extremes equals the product of the means.

The same products ad and bc can be found by multiplying diagonally in the proportion $\frac{a}{b} = \frac{c}{d}$. We call ad and bc **cross products.**

$$\overset{ad}{} \qquad \overset{bc}{}$$
$$\frac{a}{b} = \frac{c}{d}$$

The Fundamental Property of Proportions

In a proportion, the product of the extremes is equal to the product of the means.

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ and if $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$.

2 Solve proportions.

To solve a proportion, we can use the fundamental property of proportions.

EXAMPLE 1 Solve for x: $\frac{x+3}{x} = \frac{x}{x+6}$

Strategy To solve for x, we will set the cross products equal.

WHY This equation is a proportion, and in a proportion the product of the means equals the product of the extremes.

Solution

$$\begin{array}{rll} \frac{x+3}{x} = \frac{x}{x+6} & & \text{This is the proportion to solve.} \\ (x+3)(x+6) = x \cdot x & & \text{Find each cross product and set them equal.} \\ x^2 + 9x + 18 = x^2 & & \text{Perform the multiplications.} \\ 9x + 18 = 0 & & \text{Subtract } x^2 \text{ from both sides.} \\ 9x = -18 & & \text{Subtract 18 from both sides.} \\ x = -2 & & \text{To isolate } x\text{, divide both sides by 9.} \end{array}$$

Self Check 1

Solve for x: $\frac{x-1}{x} = \frac{x}{x+3}$ $\frac{3}{2}$

Now Try **Problem 19**

Teaching Example 1 Solve for x: $\frac{x-2}{x} = \frac{x+2}{x+3}$

Answer: -6

The solution is -2 and the solution set is $\{-2\}$. To check, we substitute -2 for x in the proportion and simplify each side.

Check:
$$\frac{x+3}{x} = \frac{x}{x+6}$$
$$\frac{-2+3}{-2} \stackrel{?}{=} \frac{-2}{-2+6}$$
$$\frac{1}{-2} \stackrel{?}{=} \frac{-2}{4}$$
$$-\frac{1}{2} = -\frac{1}{2} \qquad \text{True}$$

Caution! The expression $\frac{x+3}{x}$ is undefined if x is 0, because division by 0 would be indicated. Similarly, $\frac{x}{x+6}$ is undefined if x is -6. Thus, we can rule out 0 and -6 as possible solutions of $\frac{x+3}{x} = \frac{x}{x+6}$.

Self Check 2

Solve: $\dfrac{3x+1}{12} = \dfrac{x}{x+2}$ $\frac{2}{3}, 1$

Now Try **Problem 27**

Teaching Example 2 Solve: $\frac{3}{x} = \frac{x+3}{6}$
Answer:
$-6, 3$

EXAMPLE 2

Solve: $\dfrac{5a+2}{2a} = \dfrac{18}{a+4}$

Strategy To solve for a, we will set the cross products equal.

WHY This equation is a proportion, and in a proportion the product of the means equals the product of the extremes.

Solution

Because $\frac{5a+2}{2a}$ is undefined if $a = 0$, we have the restriction that $a \neq 0$. Because $\frac{18}{a+4}$ is undefined if $a = -4$, we also have the restriction that $a \neq -4$.

$$\frac{5a+2}{2a} = \frac{18}{a+4} \qquad \text{This is the proportion to solve.}$$
$$(5a+2)(a+4) = 2a(18) \qquad \text{Find each cross product and set them equal.}$$
$$5a^2 + 22a + 8 = 36a \qquad \text{Perform the multiplications.}$$
$$5a^2 - 14a + 8 = 0 \qquad \text{To get 0 on the right side, subtract } 36a \text{ from both sides.}$$
$$(5a-4)(a-2) = 0 \qquad \text{Factor to solve the quadratic equation.}$$
$$5a - 4 = 0 \quad \text{or} \quad a - 2 = 0 \qquad \text{Set each factor equal to 0.}$$
$$5a = 4 \quad \Big| \quad a = 2 \qquad \text{Solve each linear equation.}$$
$$a = \frac{4}{5}$$

The solutions are $\frac{4}{5}$ and 2. Check each of them in the original equation.

3 Use proportions to solve problems.

We can use proportions to solve many application problems. If we are given a ratio (or rate) comparing two quantities, the words of the problem can be translated into a proportion and we can solve it to find the unknown.

EXAMPLE 3 ***Gourmet Cooking*** To make a dessert, a chef needs to purchase 14 pears. If they are on sale at 6 for \$2.34, what will 14 cost?

Strategy We will use the facts in the problem to set up a proportion.

WHY Three of the entries of the proportion (14, 6, and 2.34) are given. We can find the fourth entry, the unknown cost of 14 pears, by solving the proportion.

Solution

First, we let c = the cost of 14 pears in dollars. The price per pear when purchasing 6 pears is $\frac{\$2.34}{6}$, and the price per pear when purchasing 14 pears is $\frac{\$c}{14}$. Since these rates are equal, we have the following proportion:

\$2.34 is to 6 pears as \$c is to 14 pears.

$$\underset{\text{6 pears}\rightarrow}{\overset{\text{Cost of 6 pears}\rightarrow}{}}\frac{2.34}{6} = \frac{c}{14}\underset{\leftarrow\text{14 pears}}{\overset{\leftarrow\text{Cost of 14 pears}}{}}$$

$$\begin{aligned} 14(2.34) &= 6c && \text{Find each cross product and set them equal.} \\ 32.76 &= 6c && \text{Multiply.} \\ \frac{32.76}{6} &= c && \text{To isolate } c\text{, divide both sides by 6.} \\ c &= 5.46 && \text{Perform the division.} \end{aligned}$$

Fourteen pears will cost \$5.46.

Self Check 3

MOVIE TICKETS Five adult admission tickets to a movie cost \$42.50. What will 8 tickets cost? \$68

Now Try **Problem 63**

Teaching Example 3 PRICING A sign at the grocery store advertises salad dressing on sale at 3 for \$5.00. How much will 8 bottles of salad dressing cost?
Answer:
\$13.33

Success Tip Since proportions are rational equations, they can also be solved by multiplying both sides by the LCD. For Example 3, an alternate approach is to multiply both sides by the LCD of 6 and 14, which is 42.

$$42\left(\frac{2.34}{6}\right) = 42\left(\frac{c}{14}\right)$$

4 Solve problems involving similar triangles.

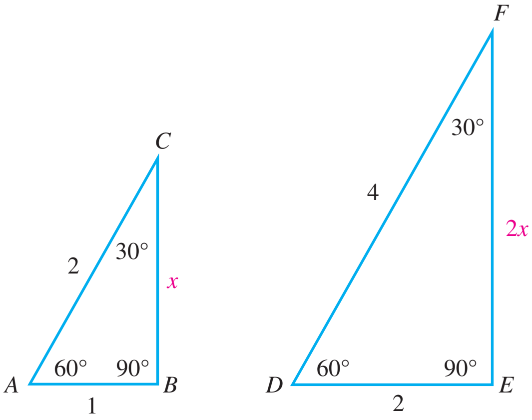
If two angles of one triangle have the same measure as two angles of a second triangle, the triangles will have the same shape. In this case, we say that the triangles are **similar triangles.** Here are some facts about similar triangles.

Similar Triangles

If two triangles are similar, then

1. The three angles of the first triangle have the same measure, respectively, as the three angles of the second triangle.
2. The lengths of all corresponding sides are in proportion.

The following triangles are similar triangles.



Corresponding angles have the same measure.

The corresponding sides are in proportion:

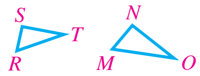
$\frac{2}{4} = \frac{x}{2x}$

$\frac{x}{2x} = \frac{1}{2}$

$\frac{1}{2} = \frac{2}{4}$

Success Tip Similar triangles do not have to be positioned the same. When they are placed differently, be careful to match their corresponding letters correctly. For example, in the illustration below,

ΔRST is similar to ΔMNO



The properties of similar triangles often enable us to find the lengths of the sides of triangles indirectly. For example, we can find the height of a tree and stay safely on the ground.

Self Check 4

HEIGHT OF A TREE Suppose the tree casts a shadow of 32 feet at the same time the yardstick casts a shadow of 4 feet. Find the height of the tree. 24 ft

Now Try **Problem 71**

Teaching Example 4 HEIGHT OF A LIGHT POLE A light pole casts a 12-foot shadow at the same time a 5-foot stick casts a shadow of 2 feet. Find the height of the light pole.
Answer:
30 ft

EXAMPLE 4 ***Height of a Tree*** A tree casts a shadow of 29 feet at the same time as a vertical yardstick casts a shadow of 2.5 feet. Find the height of the tree.

Strategy We will use the facts in the problem to set up a proportion.

WHY Three of the entries of the proportion are given. We can find the fourth entry, the unknown height of the tree, by solving the proportion.

Solution

Refer to the figure, which shows the triangles determined by the tree and its shadow and the yardstick and its shadow. Because the triangles have the same shape, they are similar, and the measures of their corresponding sides are in proportion. If we let h = the height of the tree in feet, we can find h by setting up and solving the following proportion: *h is to 3 as 29 is to 2.5.*

Height of the tree → h ; Height of the yardstick → 3 ; Length of the tree's shadow ← 29 ; Length of the yardstick's shadow ← 2.5

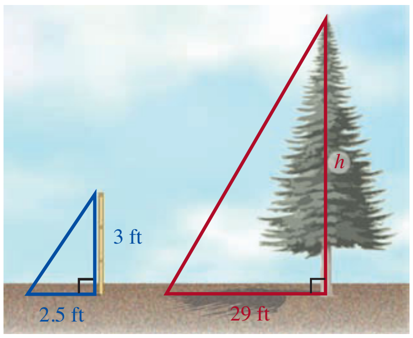
$$\frac{h}{3} = \frac{29}{2.5}$$

$2.5h = 3(29)$ Find each cross product and set them equal.

$2.5h = 87$ Multiply.

$h = 34.8$ To isolate h, divide both sides by 2.5.

The tree is about 35 feet tall.



5 Solve problems involving direct variation.

To introduce direct variation, we consider the formula for the circumference of a circle

$$C = \pi D$$

where C is the circumference, D is the diameter, and $\pi \approx 3.14159$. If we double the diameter of a circle, we determine another circle with a larger circumference C_1 such that

$$C_1 = \pi(2D) = 2\pi D = 2C$$

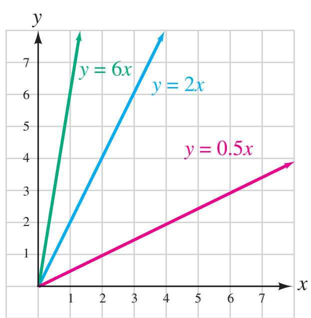
Thus, doubling the diameter results in doubling the circumference. Likewise, if we triple the diameter, we will triple the circumference.

In the formula, $C = \pi D$, we say that the variables C and D *vary directly,* or that they are *directly proportional.* This is because C is always found by multiplying D by a constant. In this example, the constant π is called the *constant of variation* or the *constant of proportionality.*

Direct Variation

The words "y varies directly as x" or "y is directly proportional to x" means that $y = kx$ for some nonzero constant k. The constant k is called the **constant of variation** or the **constant of proportionality.**

Since the formula for direct variation ($y = kx$) defines a linear function, its graph is always a line with a y-intercept at the origin. The graph of $y = kx$ where $x \geq 0$ appears in the figure for three positive values of k.



One example of direct variation is Hooke's law from physics. Hooke's law states that the distance a spring will stretch varies directly as the force that is applied to it.

If d represents a distance and f represents a force, this verbal model of Hooke's law can be expressed as

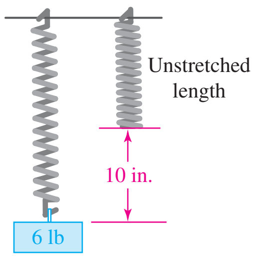
$$d = kf$$ The direct variation model can be read as "d is directly proportional to f."

where k is the constant of variation. Suppose we know that a certain spring stretches 10 inches when a weight of 6 pounds is attached (see the figure). We can find k as follows:

$$d = kf$$

$$10 = k(6)$$ Substitute 10 for d and 6 for f.

$$\frac{5}{3} = k$$



To find the force required to stretch the spring a distance of 35 inches, we can solve the equation $d = kf$ for f, with $d = 35$ and $k = \frac{5}{3}$.

$$\begin{aligned} d &= kf \\ 35 &= \frac{5}{3}f && \text{Substitute 35 for } d \text{ and } \tfrac{5}{3} \text{ for } k. \\ 105 &= 5f && \text{Multiply both sides by 3.} \\ 21 &= f && \text{Divide both sides by 5.} \end{aligned}$$

Thus, the force required to stretch the spring a distance of 35 inches is 21 pounds.

Self Check 5

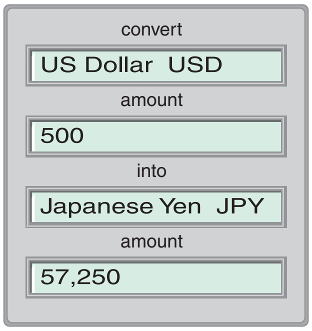
CURRENCY EXCHANGE When exchanging currencies, the number of British pounds received is directly proportional to the number of U.S. dollars to be exchanged. If \$800 converts to 392 pounds, how many pounds will be received if \$1,500 is exchanged? 735 British pounds

Now Try **Problem 79**

Teaching Example 5 CURRENCY EXCHANGE When exchanging currencies, the number of euros received is directly proportional to the number of U.S. dollars to be exchanged. If \$500 converts to 350 euros, how many euros will be received if \$700 is exchanged?
Answer:
490 euros

EXAMPLE 5 *Currency Exchange*

The currency calculator shown here converts from U.S. dollars to Japanese yen. When exchanging these currencies, the number of yen received is directly proportional to the number of dollars to be exchanged. How many yen will an exchange of \$1,200 bring?



Strategy We will use a direct variation model to solve this problem.

WHY The words *the number of yen received is directly proportional to the number of dollars to be exchanged* indicate that this type of model should be used.

Solution

The verbal model can be represented by the equation

$$y = kd \qquad \text{This is a direct variation model.}$$

where y is the number of yen, k is the constant of variation, and d is the number of dollars. From the illustration, we see that an exchange of \$500 brings 57,250 yen. To find k, we substitute 500 for d and 57,250 for y, and then we solve for k.

$$\begin{aligned} y &= kd \\ 57{,}250 &= k(500) \\ 114.5 &= k && \text{To isolate } k \text{, divide both sides by 500.} \end{aligned}$$

To find how many yen an exchange of \$1,200 will bring, we substitute 114.5 for k and 1,200 for d in the direct variation model, and then we evaluate the right side.

$$\begin{aligned} y &= kd \\ y &= 114.5(1{,}200) \\ y &= 137{,}400 \end{aligned}$$

An exchange of \$1,200 will bring 137,400 yen.

We can use the following steps to solve variation problems.

Solving Variation Problems

To solve a variation problem:

1. Translate the verbal model into an equation.
2. Substitute the first set of values into the equation from step 1 to determine the value of k.
3. Substitute the value of k into the equation from step 1.
4. Substitute the remaining set of values into the equation from step 3 and solve for the unknown.

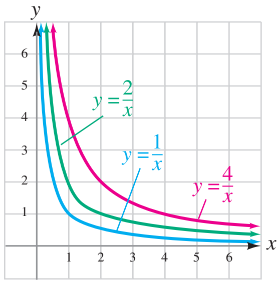
6 Solve problems involving inverse variation.

In the formula $w = \frac{12}{l}$, w gets smaller as l gets larger, and w gets larger as l gets smaller. Since these variables vary in opposite directions in a predictable way, we say that the variables *vary inversely,* or that they are *inversely proportional.* The constant 12 is the constant of variation.

Inverse Variation

The words "y varies inversely as x" or "y is inversely proportional to x" mean that $y = \frac{k}{x}$ for some nonzero constant k. The constant k is called the **constant of variation.**

The formula for inverse variation, $y = \frac{k}{x}$, defines a rational function whose graph will have the x- and y-axes as asymptotes. The graph of $y = \frac{k}{x}$ where $x > 0$ appears in the figure for three positive values of k.



In an elevator, the amount of floor space per person varies inversely as the number of people in the elevator. If f represents the amount of floor space per person and n the number of people in the elevator, the relationship between f and n can be expressed by the equation.

$$f = \frac{k}{n}$$ This inverse variation model can also be read as "f is inversely proportional to n."

The figure shows 6 people in an elevator; each has 8.25 square feet of floor space. To determine how much floor space each person would have if 15 people were in the elevator, we begin by determining k.

$$f = \frac{k}{n}$$

$$8.25 = \frac{k}{6}$$ Substitute 8.25 for f and 6 for n.

$$k = 49.5$$ Multiply both sides by 6 to solve for k.

To find the amount of floor space per person if 15 people are in the elevator, we proceed as follows:

$$f = \frac{k}{n}$$

$$f = \frac{49.5}{15}$$ Substitute 49.5 for k and 15 for n.

$$f = 3.3$$ Do the division.

If 15 people were in the elevator, each would have 3.3 square feet of floor space.

Success Tip If we multiply both sides of $y = \frac{k}{x}$ by x, we get $xy = k$. Thus, for the inverse variation model, k is simply the product of one pair of values of x and y. (Assume $x \neq 0$.)

Self Check 6

PHOTOGRAPHY Find the intensity when the photographer is 8 feet away from the subject. 16 foot-candles

Now Try **Problem 81**

Teaching Example 6
PHOTOGRAPHY In Example 6, find the intensity when the photographer is 2 feet away from the subject.
Answer:
256 foot-candles

EXAMPLE 6 ***Photography*** The intensity I of light received from a light source varies inversely as the square of the distance from the light source. If a photographer, 16 feet away from his subject, has a light meter reading of 4 foot-candles of luminance, what will the meter read if the photographer moves in for a close-up 4 feet away from the subject?

Strategy We will use the inverse variation model of the form $I = \frac{k}{d^2}$, where I represents the intensity and d^2 represents the square of the distance from the light source.

WHY The words *intensity varies inversely as the square of the distance* indicate that this type of model should be used.

Solution

$$I = \frac{k}{d^2}$$ This inverse variation model can also be read as "is inversely proportional to d^2."

To find k, we substitute 4 for I and 16 for d and solve for k.

$$\begin{aligned} I &= \frac{k}{d^2} \\ 4 &= \frac{k}{16^2} \\ 4 &= \frac{k}{256} \\ 1{,}024 &= k \end{aligned}$$ To isolate k, multiply both sides by 256.

To find the intensity when the photographer is 4 feet away from the subject, we substitute 4 for d and 1,024 for k and simplify.

$$\begin{aligned} I &= \frac{k}{d^2} \\ I &= \frac{1{,}024}{4^2} \\ &= 64 \end{aligned}$$

The intensity at 4 feet is 64 foot-candles.

Success Tip The constant of variation is usually positive, because most real-life applications involve only positive quantities. However, the definitions of *direct, inverse, joint,* and *combined variation* allow for a negative constant of variation.

7 Solve problems involving joint variation.

There are times when one variable varies as the product of several variables. For example, the area of a triangle varies directly with the product of its base and height:

$$A = \frac{1}{2}bh$$

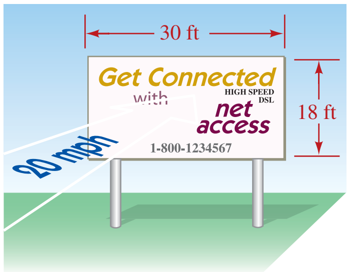
Such variation is called *joint variation.*

Joint Variation

If one variable varies directly as the product of two or more variables, the relationship is called **joint variation.** If y varies jointly with x and z, then $y = kxz$. The nonzero constant k is called the **constant of variation.**

EXAMPLE 7 *Force of the Wind*

The force of the wind on a billboard varies jointly as the area of the billboard and the square of the wind velocity. When the wind is blowing at 20 mph, the force on a billboard 30 feet wide and 18 feet high is 972 pounds. Find the force on a billboard having an area of 300 square feet caused by a 40-mph wind.



Strategy We will use the joint variation model $f = kAv^2$, where f represents the force of the wind, A represents the area of the billboard, and v^2 represents the square of the velocity of the wind.

WHY The words *the force of the wind on a billboard varies jointly as the area of the billboard and the square of the wind velocity* indicate that this type of model should be used.

Solution

$f = kAv^2$ The joint variation model can also be read as "f is directly proportional to the product of A and v^2."

Since the billboard is 30 feet wide and 18 feet high, it has an area of $30 \cdot 18 = 540$ square feet. We can find k by substituting 972 for f, 540 for A, and 20 for v.

$$f = kAv^2$$
$$972 = k(540)(20)^2$$
$$972 = k(216{,}000) \quad \text{Evaluate: } (20)^2 = 400\text{. Then do the multiplication.}$$
$$0.0045 = k \quad \text{Divide both sides by 216,000 to solve for } k\text{.}$$

To find the force exerted on a 300-square-foot billboard by a 40-mph wind, we use the formula $f = 0.0045Av^2$ and substitute 300 for A and 40 for v.

$$f = 0.0045Av^2$$
$$= 0.0045(300)(40)^2$$
$$= 2{,}160$$

The 40-mph wind exerts a force of 2,160 pounds on the billboard.

Self Check 7

FORCE OF THE WIND Refer to Example 7. Find the force of a 25 mph wind on a billboard having an area of 375 square feet. approx. 1,055 lb

Now Try Problem 85

Teaching Example 7 ENERGY Kinetic energy of an object varies jointly with its mass and the square of its velocity. A 25-gram mass moving at the rate of 30 centimeters per second has a kinetic energy of 11,250 dyne-centimeters. Find the kinetic energy of a 10-gram mass that is moving at 40 centimeters per second.
Answer:
8,000 dyne-centimeters

8 Solve problems involving combined variation.

Many applied problems involve a combination of direct and inverse variation. Such variation is called **combined variation.**

EXAMPLE 8 *Highway Construction*

The time it takes to build a highway varies directly as the length of the road, and inversely as the number of workers. If it takes 100 workers 4 weeks to build 2 miles of highway, how long will it take 80 workers to build 10 miles of highway?

Self Check 8

HIGHWAY CONSTRUCTION How long will it take 60 workers to build 6 miles of highway?

Now Try Problem 91

Self Check 8 Answer
20 weeks

Teaching Example 8 HIGHWAY CONSTRUCTION Refer to Example 8. How long will it take 100 workers to build 20 miles of highway?
Answer:
40 weeks

Strategy We will use the combined variation model $t = \frac{kl}{w}$, where t represents the time in days, l represents the length of road built in miles, and w represents the number of workers.

WHY The words *the time it takes to build a highway varies directly as the length of the road, and inversely with the number of workers* indicate that this type of model should be used.

Solution

The relationship between these variables can be expressed by the equation

$$t = \frac{kl}{w} \quad \text{This is a combined variation model.}$$

We substitute 4 for t, 100 for w, and 2 for l to find k:

$$4 = \frac{k(2)}{100}$$

$$400 = 2k \quad \text{Multiply both sides by 100.}$$

$$200 = k \quad \text{Divide both sides by 2 to solve for } k.$$

We now substitute 80 for w, 10 for l, and 200 for k in the equation $t = \frac{kl}{w}$ and simplify:

$$t = \frac{kl}{w}$$

$$t = \frac{200(10)}{80}$$

$$= 25$$

It will take 25 weeks for 80 workers to build 10 miles of highway.

ANSWERS TO SELF CHECKS

1. $\frac{3}{2}$ **2.** $\frac{2}{3}, 1$ **3.** \$68 **4.** 24 ft **5.** 735 British pounds **6.** 16 foot-candles
7. approx. 1,055 lb **8.** 20 weeks

SECTION 6.9 STUDY SET

VOCABULARY

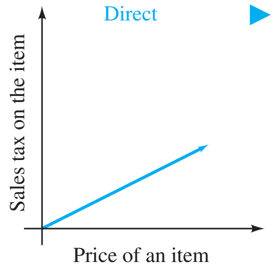
Fill in the blanks.

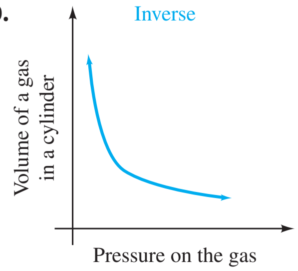
1. A ratio is the quotient of two numbers or two quantities with the same units.
2. An equation that states that two ratios are equal, such as $\frac{1}{2} = \frac{4}{8}$, is called a proportion.
3. In $\frac{50}{3} = \frac{x}{9}$, the terms 50 and 9 are called the extremes and the terms 3 and x are called the means of the proportion. In a proportion, the product of the extremes is equal to the product of the means.
4. The cross products for the proportion $\frac{10}{3} = \frac{5}{x}$ are $10x$ and 15.
5. If two angles of one triangle have the same measure as two angles of a second triangle, the triangles are similar.
6. The equation $y = kx$ defines direct variation: As x increases, y increases.
7. The equation $y = \frac{k}{x}$ defines inverse variation: As x increases, y decreases.
8. The equation $y = kxz$ defines joint variation, and $y = \frac{kz}{x}$ defines combined variation.

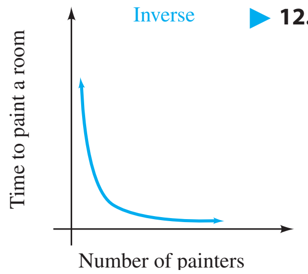
Selected exercises available online at **www.webassign.net/brookscole**

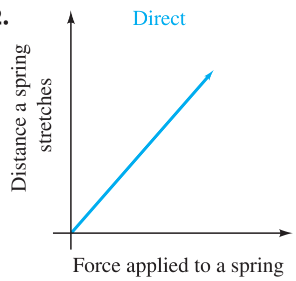
CONCEPTS

Determine whether direct or inverse variation applies and sketch a possible graph for the situation.

9. 

10. 

11. 

12. 

NOTATION

Complete each solution.

13. Solve: $\dfrac{7}{6} = \dfrac{x+3}{12}$

$\boxed{7}(12) = \boxed{6}(x+3)$

$84 = 6x + \boxed{18}$

$\boxed{66} = 6x$

$\boxed{11} = x$

14. Solve: $\dfrac{18}{2x+1} = \dfrac{3}{14}$

$\boxed{18}(14) = (\boxed{2x+1})3$

$252 = \boxed{6x} + 3$

$249 = \boxed{6x}$

$\boxed{41.5} = x$

GUIDED PRACTICE

Solve each proportion. See Example 1.

15. $\dfrac{x}{5} = \dfrac{15}{25}$ 3

16. $\dfrac{4}{y} = \dfrac{6}{27}$ 18

17. $\dfrac{r-2}{3} = \dfrac{r}{5}$ 5

18. $\dfrac{x+1}{x-1} = \dfrac{6}{4}$ 5

19. $\dfrac{5}{5z+3} = \dfrac{3}{2z+6}$ $\frac{21}{5}$

20. $\dfrac{9t+6}{t} = \dfrac{7}{3}$ $-\frac{9}{10}$

21. $\dfrac{x-2}{x} = \dfrac{x+1}{x+2}$ -4

22. $\dfrac{a}{a+1} = \dfrac{a+2}{a}$ $-\frac{2}{3}$

Solve each proportion. See Example 2.

23. $\dfrac{2}{3x} = \dfrac{6x}{36}$ $2, -2$

24. $\dfrac{y}{4} = \dfrac{4}{y}$ $4, -4$

25. $\dfrac{2}{c} = \dfrac{c-3}{2}$ $4, -1$

26. $\dfrac{2}{x+6} = \dfrac{-2x}{5}$ $-5, -1$

27. $\dfrac{1}{x+3} = \dfrac{-2x}{x+5}$ $-\frac{5}{2}, -1$

28. $\dfrac{x-1}{x+1} = \dfrac{2}{3x}$ $-\frac{1}{3}, 2$

29. $\dfrac{2b}{b+5} = \dfrac{-b}{3b+8}$ $0, -3$

30. $\dfrac{-3c}{c-2} = \dfrac{c}{c+2}$ $0, -1$

Express each verbal model in symbols. See Objectives 5 and 6.

31. A varies directly as the square of p. $A = kp^2$

32. t varies directly as s. $t = ks$

33. z varies inversely as the cube of t. $z = \frac{k}{t^3}$

34. v varies inversely as the square of r. $v = \frac{k}{r^2}$

Express each verbal model in symbols. See Objectives 7 and 8.

35. C varies jointly as x, y, and z. $C = kxyz$

36. d varies jointly as r and t. $d = krt$

37. P varies directly as the square of a and inversely as the cube of j. $P = \frac{ka^2}{j^3}$

38. M varies inversely as the cube of n and jointly as x and the square of z. $M = \frac{kxz^2}{n^3}$

Express each variation model in words. In each equation, k is the constant of variation. See Objectives 5 and 6.

39. $r = kt$ r varies directly as t.

40. $A = kr^3$ A varies directly as the cube of r.

41. $b = \dfrac{k}{h}$ b varies inversely as h.

42. $d = \dfrac{k}{W^4}$ d varies inversely as the fourth power of W.

Express each variation model in words. In each equation, k is the constant of variation. See Objectives 7 and 8.

43. $U = krs^2t$ U varies jointly as r, the square of s, and t.

44. $L = kmn$ L varies jointly as m and n.

45. $P = \dfrac{km}{n}$ P varies directly as m and inversely as n.

46. $R = \dfrac{kL}{d^2}$ R varies directly as L and inversely as d^2.

TRY IT YOURSELF

Solve each proportion.

47. $\dfrac{b+4}{5} = \dfrac{3b-6}{3}$ $\frac{7}{2}$

48. $\dfrac{2y+6}{3} = \dfrac{4y-16}{5}$ 39

49. $\dfrac{5}{b+3} = \dfrac{b}{2}$ $-5, 2$

50. $\dfrac{p+2}{p+5} = \dfrac{p-3}{p-2}$ $\frac{11}{2}$

51. $\frac{9z + 6}{z^2 + 3z} = \frac{7}{z + 3}$ no solution

▶ **52.** $\frac{3}{n^2 + 3n} = \frac{2}{n^2 + 4n + 3}$ no solution

53. $\frac{h^2}{5} = \frac{h}{2h - 9}$ $-\frac{1}{2}, 0, 5$

54. $\frac{b^2}{5} = \frac{b}{6b - 13}$ $-\frac{1}{3}, 0, \frac{5}{2}$

55. $\frac{x}{x + 2} = \frac{6}{x + 2}$ 6

56. $\frac{a}{a - 3} = \frac{5}{a - 3}$ 5

57. $\frac{t^2 - 1}{5} = \frac{1 - t^2}{2t}$ $-\frac{5}{2}, -1, 1$

▶ **58.** $\frac{n^2}{6} = \frac{n}{n - 1}$ $-2, 0, 3$

59. $\frac{2.5x + 1}{2} = \frac{4.5}{12}$ -0.1

60. $\frac{2}{5} = \frac{1.5x - 2}{0.25}$ 1.4

61. $\frac{t}{10} = \frac{10}{t}$ $-10, 10$

62. $\frac{-6}{m} = \frac{m}{-6}$ $-6, 6$

APPLICATIONS

Use a proportion to solve each problem.

63. CAFFEINE Many convenience stores sell super-size 44-ounce soft drinks in refillable cups. For each of the products listed in the table, find the amount of caffeine contained in one of the large cups. Round to the nearest milligram. 202 mg, 139 mg, 125 mg

Soft drink, 12 oz	Caffeine (mg)
Mountain Dew	55
Pepsi	38
Coca-Cola Classic	34

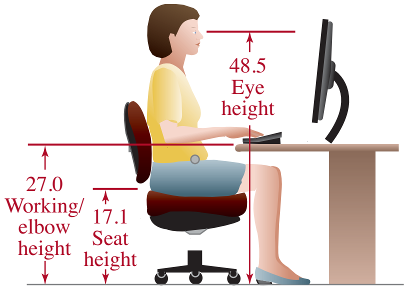
64. TELEPHONES As of 2007, Luxembourg, in Europe, had 1,500 mobile cellular telephones per 1,000 people—the highest rate of any country in the world. If Luxembourg's population is about 480,200, how many mobile cellular telephones does the country have? 720,300

65. WALLPAPERING Read the instructions on the label of wallpaper adhesive. Estimate the amount of adhesive needed to paper 500 square feet of kitchen walls if a heavy wallpaper will be used. about 2 gal

COVERAGE: One-half gallon will hang approximately 4 single rolls (140 sq ft), depending on the weight of the wall covering and the condition of the wall.

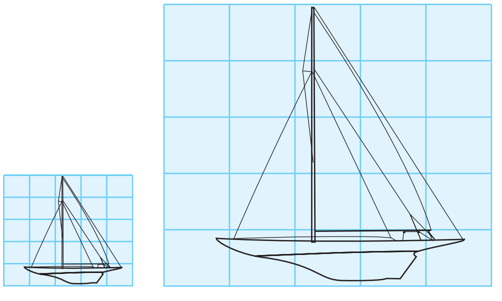
▶ **66.** RECOMMENDED DOSAGES The recommended child's dose of the sedative hydroxine is 0.006 gram per kilogram of body mass. Find the dosage for a 30-kg child in grams and in milligrams. 0.18 g = 180 mg

67. ERGONOMICS The science of ergonomics coordinates the design of working conditions with the requirements of the worker. The illustration gives guidelines for the dimensions (in inches) of a computer workstation to be used by a person whose height is 69 inches. Find a set of workstation dimensions for a person 5 feet 11 inches tall. Round to the nearest tenth. eye: 49.9 in., seat: 17.6 in., elbow: 27.8 in.



▶ **68.** SHOPPING A recipe for guacamole dip calls for 5 avocados. If they are advertised at 3 for \$1.98, what will 5 avocados cost? \$3.30

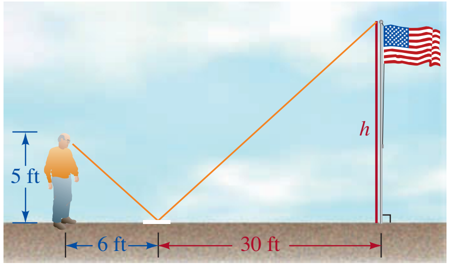
69. DRAWING See the illustration. To make an enlargement of the sailboat, an artist drew a grid over the smaller picture and transferred the contents of each small box to its corresponding larger box on another sheet of paper. If the smaller picture is 3 in. × 5 in. and if the width of the enlargement is 7.5 in., what is the length of the enlargement? 12.5 in.



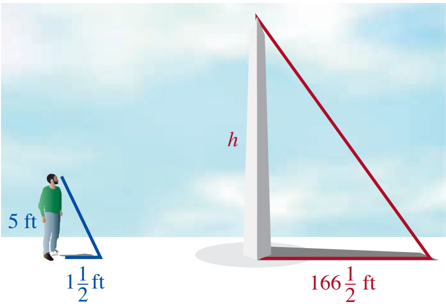
▶ **70.** DRAFTING In a scale drawing, a 280-foot antenna tower is drawn $7\frac{1}{2}$ inches high. The building next to it is drawn $2\frac{1}{4}$ inches high. How tall is the actual building? 84 ft

Use similar triangles to solve each problem.

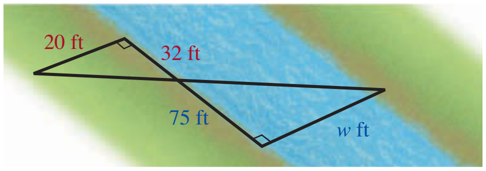
71. FLAGPOLES A man places a mirror on the ground and sees the reflection of the top of a flagpole, as in the illustration. The two triangles in the illustration are similar. Find the height h of the flagpole. 25 ft



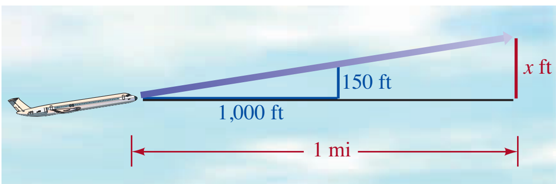
72. WASHINGTON, D.C. The Washington Monument casts a shadow of $166\frac{1}{2}$ feet at the same time as a 5-foot-tall tourist casts a shadow of $1\frac{1}{2}$ feet. Find the height of the monument. 555 ft



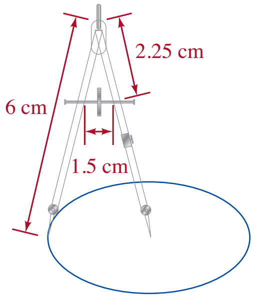
73. WIDTH OF A RIVER Use the dimensions in the illustration to find w, the width of the river. The two triangles in the illustration are similar. $46\frac{7}{8}$ ft



74. FLIGHT PATHS An airplane ascends 150 feet as it flies a horizontal distance of 1,000 feet. How much altitude will it gain as it flies a horizontal distance of 1 mile? (*Hint:* 5,280 feet = 1 mile.) 792 ft



75. GRAPHIC ARTS The compass in the illustration is used to draw circles with different radii (plural for radius). For the setting shown, what radius will the resulting circle have? 4 cm



76. SKI RUNS A ski course with $\frac{1}{2}$ mile of horizontal run falls 100 feet in every 300 feet of run. Find the height of the hill. 880 ft

77. The language of variation is often used to describe various aspects of the Internet and websites. Determine whether each statement, generally speaking, is true or false.

from Campus to Careers

Webmaster

Reggie Casagrande/Getty Images

a. The dollar amount of sales that an Internet website receives is inversely proportional to the amount of Internet traffic that visits the website. false

b. The download time of an Internet website varies directly with the bandwidth being used. false

c. Search engines like Google place a value on a website that is directly proportional to the number of sites that link to it. true

Solve each problem by writing a variation model.

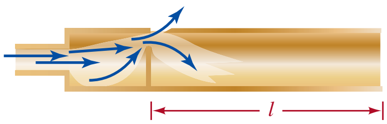
78. GRAVITY The force of gravity acting on an object varies directly as the mass of the object. The force on a mass of 5 kilograms is 49 newtons. What is the force acting on a mass of 12 kilograms? 117.6 newtons

79. FREE FALL An object in free fall travels a distance s that is directly proportional to the square of the time t. If an object falls 1,024 feet in 8 seconds, how far will it fall in 10 seconds? 1,600 ft

80. FINDING DISTANCE The distance that a car can go varies directly as the number of gallons of gasoline it consumes. If a car can go 288 miles on 12 gallons of gasoline, how far can it go on a full tank of 18 gallons? 432 mi

81. FARMING The number of days that a given number of bushels of corn will last when feeding cattle varies inversely as the number of animals. If x bushels will feed 25 cows for 10 days, how long will the feed last for 10 cows? 25 days

82. ORGAN PIPES The frequency of vibration of air in an organ pipe is inversely proportional to the length of the pipe. If a pipe 2 feet long vibrates 256 times per second, how many times per second will a 6-foot pipe vibrate? $85\frac{1}{3}$



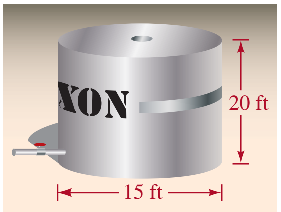
83. GAS PRESSURE Under constant temperature, the volume occupied by a gas varies inversely to the pressure applied. If the gas occupies a volume of 20 cubic inches under a pressure of 6 pounds per square inch, find the volume when the gas is subjected to a pressure of 10 pounds per square inch. 12 in.^{3}

84. REAL ESTATE The following table shows the listing price for three homes in the same general locality. Write the variation model (direct or inverse) that describes the relationship between the listing price and the number of square feet of a house in this area. $P = 105f$

Number of square feet	Listing price
1,720	$180,600
1,205	$126,525
1,080	$113,400

85. TRUCKING COSTS The costs of a trucking company vary jointly as the number of trucks in service and the number of hours they are used. When 4 trucks are used for 6 hours each, the costs are $1,800. Find the costs of using 10 trucks, each for 12 hours. $9,000

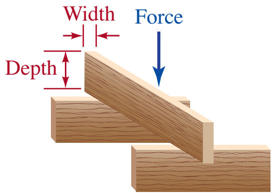
86. OIL STORAGE The number of gallons of oil that can be stored in a cylindrical tank varies jointly as the height of the tank and the square of the radius of its base. The constant of proportionality is 23.5. Find the number of gallons that can be stored in the cylindrical tank shown. 26,437.5 gal



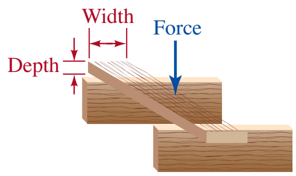
87. ELECTRONICS The voltage (in volts) measured across a resistor is directly proportional to the current (in amperes) flowing through the resistor. The constant of variation is the resistance (in ohms). If 6 volts is measured across a resistor carrying a current of 2 amperes, find the resistance. 3 ohms

88. ELECTRONICS The power (in watts) lost in a resistor (in the form of heat) varies directly as the square of the current (in amperes) passing through it. The constant of proportionality is the resistance (in ohms). What power is lost in a 5-ohm resistor carrying a 3-ampere current? 45 w

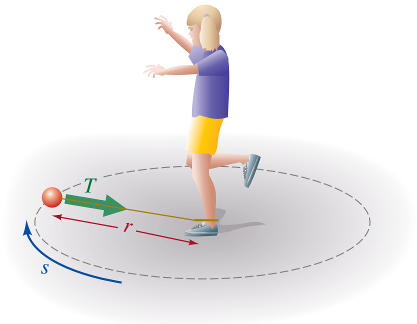
89. STRUCTURAL ENGINEERING The deflection of a beam is inversely proportional to its width and the cube of its depth. If the deflection of a 4-inch-wide by 4-inch-deep beam is 1.1 inches, find the deflection of a 2-inch-wide by 8-inch-deep beam positioned as in the illustration. 0.275 in.



90. STRUCTURAL ENGINEERING Find the deflection of the beam in Exercise 89 when the beam is positioned as in the illustration. 4.4 in.



91. TENSION IN A STRING When playing with a Skip It toy, a child swings a weighted ball on the end of a string in a circular motion around one leg while jumping over the revolving string with the other leg. See the illustration. The tension T in the string is directly proportional to the square of the speed s of the ball and inversely proportional to the radius r of the circle. If the tension in the string is 6 pounds when the speed of the ball is 6 feet per second and the radius is 3 feet, find the tension when the speed is 8 feet per second and the radius is 2.5 feet. 12.8 lb



92. GAS PRESSURE The pressure of a certain amount of gas is directly proportional to the temperature (measured on the Kelvin scale) and inversely proportional to the volume. A sample of gas at a pressure of 1 atmosphere occupies a volume of 1 cubic meter at a temperature of 273 Kelvin. When heated, the gas expands to twice its volume, but the pressure remains constant. To what temperature is it heated? 546 Kelvin

WRITING

93. Distinguish between a *ratio* and a *proportion*.

94. Give examples of two quantities from everyday life that vary directly and two quantities that vary inversely.

REVIEW

Perform the indicated operations.

95. $\left(\frac{5}{2}w^3 + \frac{1}{4}w^2 + \frac{3}{5}\right) - \left(\frac{1}{3}w^3 + \frac{1}{2}w^2 - \frac{1}{5}\right)$
$\frac{13}{6}w^3 - \frac{1}{4}w^2 + \frac{4}{5}$

96. $(6a^2x^3 - 2ax^2 + 3a^3) + (-4a^2x^3 - 2a^3)$
$2a^2x^3 - 2ax^2 + a^3$

97. $(3y + 1)(2y^2 + 3y + 2)$
$6y^3 + 11y^2 + 9y + 2$

98. $(5k - 6m^2)^2$
$25k^2 - 60km^2 + 36m^4$

STUDY SKILLS CHECKLIST

Preparing for the Chapter 6 Test

There are several common mistakes that students make when working with the topics of Chapter 6. To make sure you are prepared for the test over this material, read the list below to help you avoid these mistakes.

☐ When simplifying fractions that have more than one term in the numerator and/or the denominator, you must factor the numerator and denominator using the factoring strategies from Chapter 5. When you have completely factored the numerator and denominator, remove all common factors.

$$\frac{a^2 - 3a - 10}{a^2 - 25} = \frac{\cancel{(a-5)}(a+2)}{\cancel{(a-5)}(a+5)} = \frac{a+2}{a+5}$$

☐ To multiply fractions, you do not get common denominators. Simply factor across the numerators and factor across the denominators and remove common factors.

$$\frac{x^2 - x}{x^2 - 1} \cdot \frac{x^2 - x - 2}{x - 2} = \frac{x\cancel{(x-1)}\cancel{(x-2)}\cancel{(x+1)}}{\cancel{(x+1)}\cancel{(x-1)}\cancel{(x-2)}} = \frac{x}{1} = x$$

☐ To divide fractions, make sure to multiply the first fraction by the reciprocal of the *second* fraction.

$$\frac{4a^2}{3b^2} \div \frac{8a^2}{3b} = \frac{4a^2}{3b^2} \cdot \frac{3b}{8a^2} = \frac{\cancel{4} \cdot \cancel{a^2} \cdot \cancel{3} \cdot \cancel{b}}{\cancel{3} \cdot \cancel{b} \cdot b \cdot 2 \cdot \cancel{4} \cdot \cancel{a^2}} = \frac{1}{2b}$$

☐ When adding or subtracting fractions, you must have a common denominator. Then you add or subtract across the numerator and *keep the common denominator.*

$$\frac{2}{x+5} + \frac{6}{x} = \frac{2x}{x(x+5)} + \frac{6(x+5)}{x(x+5)} = \frac{2x + 6x + 30}{x(x+5)} = \frac{8x + 30}{x(x+5)}$$

☐ When subtracting a fraction with more than one term in the numerator, make sure you distribute the negative through the entire numerator being subtracted.

$$\frac{5x + 7}{x(x-2)} - \frac{3x - 5}{x(x-2)} = \frac{5x + 7 - (3x - 5)}{x(x-2)} = \frac{5x + 7 - 3x + 5}{x(x-2)} = \frac{2x + 12}{x(x-2)}$$

☐ When *solving equations* that involve rational expressions, you can remove the fractions by multiplying both sides by the LCD of the entire equation. When working with *rational expressions,* you must work with the fractions that are in the expression.

☐ Multiplying both sides of an equation by a quantity that contains a variable can lead to extraneous solutions. All possible solutions of a rational equation must be checked.

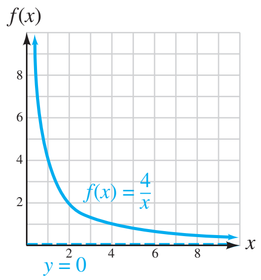
CHAPTER 6 SUMMARY AND REVIEW

SECTION 6.1 Rational Functions and Simplifying Rational Expressions

DEFINITIONS AND CONCEPTS	EXAMPLES
A **rational expression** is an expression of the form $\frac{A}{B}$, where A and B are polynomials and B does not equal 0.	Rational expressions: $\frac{3x^2}{xy}$, $\frac{5b-15}{b^2-25}$, and $\frac{a+2}{a^2-3a-4}$
A **rational function** is a function whose equation is defined by a rational expression in one variable.	Rational functions: $f(x) = \frac{6x}{x-2}$ and $f(n) = \frac{n+3}{n^3+2n-9}$
Since division by 0 is undefined, any values that make the denominator 0 in a rational function must be excluded from the **domain** of the function.	Find the domain of the rational function: $f(x) = \frac{x+3}{x^2-4}$ $x^2 - 4 = 0$ Set the denominator equal to 0. $(x+2)(x-2) = 0$ Factor the difference of two squares. $x + 2 = 0$ or $x - 2 = 0$ Set each factor equal to 0. $x = -2$ | $x = 2$ Solve each equation. The domain of the function is the set of all real numbers except -2 and 2. In interval notation, the domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
To simplify a rational expression: **1.** Factor the numerator and denominator completely. **2.** Remove factors equal to 1 by replacing each pair of factors common to the numerator and denominator with the equivalent fraction $\frac{1}{1}$. **3.** Multiply the remaining factors in the numerator and in the denominator.	Simplify: $\frac{x^2-4}{2x+4} = \frac{\overset{1}{\cancel{(x+2)}}(x-2)}{2\underset{1}{\cancel{(x+2)}}} = \frac{x-2}{2}$ $\frac{2a^3-5a^2-12a}{2a^3-11a^2+12a} = \frac{\overset{1}{\cancel{a}}(2a+3)\overset{1}{\cancel{(a-4)}}}{\underset{1}{\cancel{a}}(2a-3)\underset{1}{\cancel{(a-4)}}} = \frac{2a+3}{2a-3}$
The quotient of any nonzero expression and its **opposite** is -1.	$\frac{7x-6}{6-7x} = -1$ Because $7x - 6$ and $6 - 7x$ are opposites. Simplify: $\frac{10-2b}{b^2-5b} = \frac{2(5-b)}{b(b-5)} = \frac{2\overset{-1}{\cancel{(5-b)}}}{b\underset{1}{\cancel{(b-5)}}} = -\frac{2}{b}$

REVIEW EXERCISES

1. Complete the table of values for the rational function $f(x) = \frac{4}{x}$ where $x > 0$. Round to the nearest hundredth when appropriate. Then graph the function. Label the horizontal asymptote.



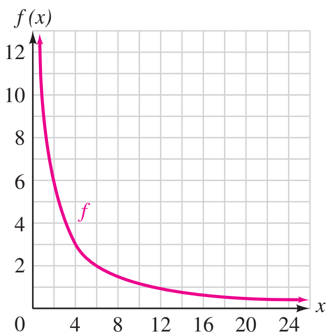
x	$f(x)$
$\frac{1}{2}$	8
1	4
2	2
3	1.33
4	1
5	0.8
6	0.67
7	0.57
8	0.5

2. Use the graph of function f to find each of the following:

a. $f(12)$ 1

b. The value(s) of x for which $f(x) = 6$ 2

c. The domain and range of f
D: $(0, \infty)$, R: $(0, \infty)$



3. Find the domain of the rational function $f(x) = \frac{2x^2 + 8x}{x^2 + 2x - 24}$. Express your answer in words and using interval notation.
The domain is the set of all real numbers except -6 and 4: $(-\infty, -6) \cup (-6, 4) \cup (4, \infty)$.

4. Use a graphing calculator to graph the rational function $f(x) = \frac{3x + 2}{x}$. From the graph, determine the equations of the horizontal and vertical asymptotes and the domain and range.
$y = 3, x = 0$; D: $(-\infty, 0) \cup (0, \infty)$, R: $(-\infty, 3) \cup (3, \infty)$

Simplify each rational expression, if possible.

5. $\dfrac{48x^2y}{76xy^8}$ $\quad \frac{12x}{19y^7}$

6. $\dfrac{x^2 - 49}{x^2 + 14x + 49}$ $\quad \frac{x - 7}{x + 7}$

7. $\dfrac{x^2 - 2x + 4}{2x^5 + 16x^2}$ $\quad \frac{1}{2x^2(x + 2)}$

8. $\dfrac{x^2 + 6x + 36}{x^7 - 216x^4}$ $\quad \frac{1}{x^4(x - 6)}$

9. $\dfrac{5ac - 5ad + 5bc - 5bd}{5d^2 - 5c^2}$ $\quad \frac{-a - b}{c + d}$

10. $\dfrac{m^3 + m^2n - 2mn^2}{2m^3 - mn^2 - m^2n}$ $\quad \frac{m + 2n}{2m + n}$

11. $\dfrac{6x^2 - 5x - 4}{9x^2 - 24x + 16}$ $\quad \frac{2x + 1}{3x - 4}$

12. $\dfrac{2m - 2n}{n - m}$ $\quad -2$

13. $\dfrac{s^2 + t^2}{s - t}$ does not simplify

14. $\dfrac{3m^2 - 10m + 8}{6 - m - m^2}$ $\quad -\frac{3m - 4}{m + 3}$ or $\frac{4 - 3m}{m + 3}$

SECTION 6.2 Multiplying and Dividing Rational Expressions

DEFINITIONS AND CONCEPTS	EXAMPLES
To **multiply rational expressions,** multiply the numerators and multiply the denominators. $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$ Then simplify, if possible.	Multiply, and then simplify, if possible. $\frac{8z^2}{y^3} \cdot \frac{y}{4z} = \frac{8z^2 \cdot y}{y^3 \cdot 4z} = \frac{2 \cdot \overset{1}{\cancel{4}} \cdot \overset{1}{\cancel{z}} \cdot z \cdot \overset{1}{\cancel{y}}}{\underset{1}{\cancel{y}} \cdot y \cdot y \cdot \underset{1}{\cancel{4}} \cdot \underset{1}{\cancel{z}}} = \frac{2z}{y^2}$ $\frac{x^2 - 4}{x + 3} \cdot \frac{3x + 9}{x + 2} = \frac{(x^2 - 4)(3x + 9)}{(x + 3)(x + 2)}$ Multiply the numerators. Multiply the denominators.

$$= \frac{\overset{1}{\cancel{(x+2)}}(x-2)\cdot 3\cdot \overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x+3)}}\underset{1}{\cancel{(x+2)}}}$$ Factor completely and then simplify.

$$= 3(x-2)$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

To **divide rational expressions,** multiply the first by the reciprocal of the second.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C} = \frac{AD}{BC}$$

Then simplify, if possible.

Divide, and then simplify, if possible.

$$\frac{x^2+4x+3}{x^2+3x} \div \frac{3}{x} = \frac{x^2+4x+3}{x^2+3x} \cdot \frac{x}{3}$$ Multiply the first rational expression by the reciprocal of the second.

$$= \frac{(x^2+4x+3)\cdot x}{(x^2+3x)\cdot 3}$$ Multiply the numerators. Multiply the denominators.

$$= \frac{(x+1)\overset{1}{\cancel{(x+3)}}\cdot \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}}\underset{1}{\cancel{(x+3)}}\cdot 3}$$ Factor completely and then simplify.

$$= \frac{x+1}{3}$$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.

REVIEW EXERCISES

Perform the operations and simplify when possible.

15. $\frac{3x^3y^4}{35} \cdot \frac{10}{21x^5y^4}$
$\frac{2}{49x^2}$

16. $\frac{x^3+4x^2+4x}{x^2-x-6} \cdot \frac{9-x^2}{x^2+5x+6}$
$-x$

17. $\frac{2a^2-5a-3}{4a^3-36a} \div \frac{2a^2+5a+2}{2a^2+5a-3}$
$\frac{2a-1}{4a(a+2)}$

18. $\frac{t^4-4t^2}{t} \div (t^3+2t^2)$
$\frac{t-2}{t}$

19. $\left(\frac{h-2}{h^3+4}\right)^2$
$\frac{h^2-4h+4}{h^6+8h^3+16}$

20. $\frac{m^2+3m+9}{m^2+4m+mr+4r} \div \frac{m^3-27}{am+ar+6m+6r}$
$\frac{a+6}{(m+4)(m-3)}$

21. $\frac{8m^2+6mn-9n^2}{2m^2+5mn+3n^2} \cdot \frac{6m^2+5mn-4n^2}{12m^2+7mn-12n^2}$
$\frac{2m-n}{m+n}$

22. $\frac{x^3+3x^2+2x}{2x^2-2x-12} \div \frac{x^3-3x^2-4x}{3x^2-3x} \cdot \frac{2x^2-4x-16}{x^2+3x+2}$
$\frac{3x(x-1)}{(x-3)(x+1)}$

SECTION 6.3 Adding and Subtracting Rational Expressions

DEFINITIONS AND CONCEPTS	EXAMPLES
To **add (or subtract) two rational expressions with like denominators,** add (or subtract) the numerators and keep the common denominator.	Add: $\frac{x^2-26}{x-5} + \frac{1}{x-5} = \frac{x^2-26+1}{x-5}$ Add the numerators. Write the sum over the common denominator, $x-5$.

	$= \dfrac{x^2 - 25}{x - 5}$ Combine like terms. $= \dfrac{(x+5)\cancel{(x-5)}^{1}}{\cancel{x-5}_{1}}$ To simplify the result, factor the numerator and remove the factor common to the numerator and denominator. $= x + 5$
To find the **LCD** of several rational expressions, factor each denominator and use each factor the greatest number of times that it appears in any one denominator. The product of these factors is the LCD.	The denominators of two rational expressions are given. Find the LCD: $\left.\begin{aligned} x^2 - 8x + 16 &= (x-4)(x-4) \\ 9x - 36 &= 3 \cdot 3 \cdot (x-4) \end{aligned}\right\}$ $\begin{aligned} \text{LCD} &= 3 \cdot 3 \cdot (x-4)(x-4) \\ &= 9(x-4)^2 \end{aligned}$
To **add or subtract rational expressions with unlike denominators,** find the LCD and express each rational expression with a denominator that is the LCD. Add (or subtract) the resulting fractions and simplify the result, if possible.	Subtract: $\dfrac{2x}{x+5} - \dfrac{1}{x} = \dfrac{2x}{x+5} \cdot \dfrac{x}{x} - \dfrac{1}{x} \cdot \dfrac{x+5}{x+5}$ Build each rational expression to have the LCD of $x(x + 5)$. $= \dfrac{2x^2}{x(x+5)} - \dfrac{x+5}{x(x+5)}$ Multiply the numerators. Multiply the denominators. $= \dfrac{2x^2 - (x+5)}{x(x+5)}$ Subtract the numerators. Write the difference over the common denominator. $= \dfrac{2x^2 - x - 5}{x(x+5)}$ The result does not simplify.

REVIEW EXERCISES

Perform the operations and simplify when possible.

23. $\dfrac{5y}{x-y} - \dfrac{3}{x-y}$ $\dfrac{5y-3}{x-y}$

24. $\dfrac{d^2}{c^3-d^3} + \dfrac{c^2+cd}{c^3-d^3}$ $\dfrac{1}{c-d}$

25. $\dfrac{4}{t-3} + \dfrac{6}{3-t}$ $-\dfrac{2}{t-3}$

26. $\dfrac{p+3}{p^2+13p+12} - \dfrac{2p+4}{p^2+13p+12}$ $-\dfrac{1}{p+12}$

The denominators of some rational expressions are given. Find the LCD.

27. $15a^2h, 20ah^3$ $60a^2h^3$

28. $ab^2 - ab, ab^2, b^2 - b$ $ab^2(b-1)$

29. $x^2 - 4x - 5, x^2 - 25$ $(x-5)(x+5)(x+1)$

30. $m^2 - 4m + 4, m^3 - 8$ $(m^2+2m+4)(m-2)^2$

Perform the operations and simplify when possible.

31. $9 - \dfrac{1}{a+1}$ $\dfrac{9a+8}{a+1}$

32. $\dfrac{5x}{14z^2} + \dfrac{y^2}{16z}$ $\dfrac{40x+7y^2z}{112z^2}$

33. $\dfrac{4x}{x-4} - \dfrac{3}{x+3}$ $\dfrac{4x^2+9x+12}{(x-4)(x+3)}$

34. $\dfrac{2a+4}{3} - \dfrac{9}{a+2}$ $\dfrac{2a^2+8a-19}{3(a+2)}$

35. $\dfrac{y+7}{y+3} - \dfrac{y-3}{y+7}$ $\dfrac{14y+58}{(y+3)(y+7)}$

36. $\dfrac{4}{3xy-6y} - \dfrac{4}{10-5x}$ $\dfrac{12y+20}{15y(x-2)}$

37. $\dfrac{6}{a^2-9} - \dfrac{5}{a^2-a-6}$ $\dfrac{1}{(a+3)(a+2)}$

38. $\dfrac{a}{a+2} - \dfrac{3}{a^2+2a} + \dfrac{1}{2a+4}$ $\dfrac{2a-3}{2a}$

SECTION 6.4 Simplifying Complex Fractions

DEFINITIONS AND CONCEPTS	EXAMPLES
Complex fractions contain fractions in their numerators and/or their denominators.	Complex fractions: $\dfrac{\frac{2}{t}}{\frac{5}{4t}}$, $\dfrac{\frac{3}{m}+\frac{m}{4}}{\frac{m}{2}}$, and $\dfrac{a^2-b^2}{\frac{1}{a}+\frac{1}{b}}$
Two methods are used to simplify **complex fractions.** **Method 1:** Write the numerator and denominator as single fractions. Then divide the fractions and simplify. This method works well when a complex fraction is written, or can be easily written, as a quotient of two single rational expressions.	Simplify: $\dfrac{\frac{4x^2}{y^3}}{\frac{14x}{y}} = \dfrac{4x^2}{y^3} \div \dfrac{14x}{y}$ The main fraction bar of the complex fraction indicates division. $= \dfrac{4x^2}{y^3} \cdot \dfrac{y}{14x}$ To divide rational expressions, multiply the first by the reciprocal of the second. $= \dfrac{4x^2 \cdot y}{y^3 \cdot 14x}$ Multiply the numerators. Multiply the denominators. $= \dfrac{\overset{1}{\cancel{2}} \cdot 2 \cdot \overset{1}{\cancel{x}} \cdot x \cdot \overset{1}{\cancel{y}}}{\underset{1}{\cancel{y}} \cdot y \cdot y \cdot \underset{1}{\cancel{2}} \cdot 7 \cdot \underset{1}{\cancel{x}}}$ Factor the numerator and denominator. Then simplify by removing common factors of the numerator and denominator. $= \dfrac{2x}{7y^2}$ Multiply the remaining factors in the numerator. Multiply the remaining factors in the denominator.
Method 2: Determine the LCD of all the rational expressions in the complex fraction and multiply the complex fraction by 1, written in the form $\frac{\text{LCD}}{\text{LCD}}$. This method works well when the complex fraction has sums and/or differences in the numerator or denominator.	Simplify: $\dfrac{\frac{1}{x}-y}{\frac{5}{2x}} = \dfrac{\frac{1}{x}-y}{\frac{5}{2x}} \cdot \dfrac{2x}{2x}$ The LCD of all the rational expressions in the complex fraction is 2x. Multiply the complex fraction by 1 in the form $\frac{2x}{2x}$. $= \dfrac{\left(\frac{1}{x}-y\right)2x}{\left(\frac{5}{2x}\right)2x}$ Multiply the numerators. Multiply the denominators. $= \dfrac{\frac{1}{x} \cdot 2x - y \cdot 2x}{5}$ In the numerator, distribute the multiplication by 2x. In the denominator, perform the multiplication by 2x. $= \dfrac{2-2xy}{5}$ In the numerator, perform each multiplication by 2x.

REVIEW EXERCISES

Simplify each complex fraction.

39. $\dfrac{\frac{4a^3b^2}{9c}}{\frac{14a^3b}{9c^4}}$ $\quad \frac{2bc^3}{7}$

40. $\dfrac{\frac{p^2-9}{6pt}}{\frac{p^2+5p+6}{3pt}}$ $\quad \frac{p-3}{2(p+2)}$

41. $\dfrac{\frac{1}{a}+\frac{2}{b}}{\frac{2}{a}-\frac{1}{b}}$ $\quad \frac{b+2a}{2b-a}$

42. $\dfrac{1-\frac{1}{x}-\frac{2}{x^2}}{1+\frac{4}{x}+\frac{3}{x^2}}$ $\quad \frac{x-2}{x+3}$

43. $\dfrac{(x-y)^{-2}}{x^{-2}-y^{-2}}$ $\quad \frac{x^2y^2}{(x-y)^2(y^2-x^2)}$

44. $\dfrac{1+\frac{1}{b+d}}{\frac{1}{b+d}-1}$ $\quad \frac{1+b+d}{1-b-d}$

45. $\dfrac{\frac{2b}{b-1}-\frac{3}{b}}{\frac{1}{b-1}+\frac{2}{b}}$ $\quad \frac{2b^2-3b+3}{3b-2}$

46. $\dfrac{\frac{8}{r+3}}{\frac{4}{r-2}-\frac{2}{r^2+r-6}}$ $\quad \frac{4r-8}{2r+5}$

SECTION 6.5 Dividing Polynomials

DEFINITIONS AND CONCEPTS

To divide monomials, use the method for simplifying fractions or use the rules for exponents.

EXAMPLES

Divide the monomials:

$$\frac{8p^2q}{20pq^3}=\frac{2\cdot\overset{1}{\cancel{4}}\cdot\overset{1}{\cancel{p}}\cdot p\cdot\overset{1}{\cancel{q}}}{\underset{1}{\cancel{4}}\cdot 5\cdot\underset{1}{\cancel{p}}\cdot\underset{1}{\cancel{q}}\cdot q\cdot q}=\frac{2p}{5q^2}$$

or

$$\frac{8p^2q}{20pq^3}=\frac{2}{5}p^{2-1}q^{1-3}$$

Keep each base and subtract the exponents.

$$=\frac{2p^1q^{-2}}{5}$$

$$=\frac{2p}{5q^2}$$

Move q^{-2} to the denominator and change the sign of the exponent.

DEFINITIONS AND CONCEPTS

To divide a polynomial by a monomial, divide each term of the numerator by the denominator.

EXAMPLES

Divide: $\dfrac{9c^9d^4-12c^3d^7}{\mathbf{27cd^5}}=\dfrac{9c^9d^4}{\mathbf{27cd^5}}-\dfrac{12c^3d^7}{\mathbf{27cd^5}}$

$$=\frac{c^8}{3d}-\frac{4c^2d^2}{9}$$

Do each monomial division.

DEFINITIONS AND CONCEPTS

Long division can be used to **divide a polynomial by a polynomial** (other than a monomial). The long division method is a series of four steps that are repeated: Divide, multiply, subtract, and bring down the next term.

When the division has a remainder, write the answer in the form Quotient $+\frac{\text{remainder}}{\text{divisor}}$.

EXAMPLES

Divide $6x^3-x^2+6x+5$ by $2x+1$.

$$\begin{array}{r}
3x^2-2x+4 \\
2x+1\overline{)6x^3-x^2+6x+5} \\
-(6x^3+3x^2) \\
\hline
-4x^2+6x \\
-(-4x^2-2x) \\
\hline
8x+5 \\
-(8x+4) \\
\hline
1
\end{array}$$

The first division: $\frac{6x^3}{2x}=3x^2$.

The second division: $\frac{-4x^2}{2x}=-2x$.

The third division: $\frac{8x}{2x}=4$.

The remainder is 1.

Thus $\frac{6x^3-x^2+6x+5}{2x+1}=3x^2-2x+4+\frac{1}{2x+1}$.

The long division method works best when the terms of the divisor and the dividend are written in **descending powers of the variable.**

To divide $\frac{5x + x^3 + 3 + 3x^2}{x + 1}$, write: $x + 1\overline{)x^3 + 3x^2 + 5x + 3}$

When the dividend has **missing terms,** insert such terms with a coefficient of 0, or leave a blank space.

To divide $\frac{x^2 - 9}{x - 3}$, write: $x - 3\overline{)x^2 + 0x - 9}$

REVIEW EXERCISES

Perform each division. Write answers using positive exponents.

47. $\frac{25h^4k^7}{55hk^9}$ $\frac{5h^3}{11k^2}$

48. $(5x^3y^3z^{10}) \div (10x^3y^6z^{20})$ $\frac{1}{2y^3z^{10}}$

Perform each division.

49. $\frac{36a + 32}{6}$ $6a + \frac{16}{3}$

50. $\frac{30x^3y^2 - 15x^2y - 10xy^2}{-10xy}$ $-3x^2y + \frac{3x}{2} + y$

51. $b + 5\overline{)b^2 + 9b + 20}$ $b + 4$

52. $\frac{-33v - 8v^2 + 3v^3 - 10}{1 + 3v}$ $v^2 - 3v - 10$

53. $x + 2\overline{)x^3 + 8}$ $x^2 - 2x + 4$

54. Divide $(8m^2 - 18m - 9)$ by $4m + 1$. $2m - 5 + \frac{-4}{4m + 1}$

55. $(3a^3 - 2a^2 - 8) \div (a^2 + 5)$ $3a - 2 + \frac{-15a + 2}{a^2 + 5}$

56. $\frac{m^8 + m^6 - 4m^4 + 5m^2 - 1}{m^4 + 2m^2 - 3}$ $m^4 - m^2 + 1 + \frac{2}{m^4 + 2m^2 - 3}$

SECTION 6.6 Synthetic Division

DEFINITIONS AND CONCEPTS / EXAMPLES

Synthetic division is used to divide a polynomial by a binomial of the form $x - k$.

Use synthetic division to divide: $\frac{5x^3 - x^2 + 4x - 3}{x + 1}$

We write the divisor in $x - k$ form as $x - (-1)$ to determine that $k = -1$ and proceed as follows:

This represents division by x + 1. →

$$\begin{array}{r|rrrr} -1 & 5 & -1 & 4 & -3 \\ & & -5 & 6 & -10 \\ \hline & 5 & -6 & 10 & -13 \end{array}$$

← This is the remainder.

Thus, $\frac{5x^3 - x^2 + 4x - 3}{x + 1} = 5x^2 - 6x + 10 + \frac{-13}{x + 1}$.

Remainder theorem: If a polynomial $P(x)$ is divided by $x - k$, the remainder is $P(k)$.

It follows from the remainder theorem that a polynomial can be evaluated using synthetic division.

Since the remainder in the previous division is -13, the remainder theorem guarantees that $P(-1) = -13$. We can check this by evaluating the function at $x = -1$:

$$\begin{aligned} P(x) &= 5x^3 - x^2 + 4x - 3 \\ P(-1) &= 5(-1)^3 - (-1)^2 + 4(-1) - 3 \\ &= -5 - 1 - 4 - 3 \\ &= -13 \end{aligned}$$

Factor theorem: If $P(x)$ is divided by $x - k$, then $P(k) = 0$, if and only if $x - k$ is a factor of $P(x)$.

Determine whether $x - 3$ is a factor of $2x^3 - 8x^2 + 9x - 9$.

We can use synthetic division to find the remainder when the polynomial is divided by $x - 3$. If the remainder is 0, $x - 3$ is a factor. If the remainder is not 0, $x - 3$ is not a factor.

$$\begin{array}{r|rrrr} 3 & 2 & -8 & 9 & -9 \\ & & 6 & -6 & 9 \\ \hline & 2 & -2 & 3 & 0 \end{array}$$

Since the remainder is 0, $x - 3$ is a factor of $2x^3 - 8x^2 + 9x - 9$.

REVIEW EXERCISES

Use synthetic division to perform each division.

57. $(x^2 - 13x + 42) \div (x - 6)$
$x - 7$

58. Divide $m^3 - 6m^2 + 11m - 6$ by $m - 3$.
$m^2 - 3m + 2$

59. $\dfrac{-3n^5 + 10n^4 + 7n^3 + 2n^2 + 9n - 4}{n - 4}$
$-3n^4 - 2n^3 - n^2 - 2n + 1$

60. $\dfrac{4x^3 + 5x^2 - 1}{x + 2}$
$4x^2 - 3x + 6 + \frac{-13}{x + 2}$

61. Divide $3a - a^2 + 3a^3 + 3a^4 + 10$ by $a + 1$.
$3a^3 - a + 4 + \frac{6}{a + 1}$

62. $\dfrac{x^4 + 1}{x - 3}$
$x^3 + 3x^2 + 9x + 27 + \frac{82}{x - 3}$

63. Let $P(x) = x^4 - 2x^3 + x^2 - 3x + 12$. Use the remainder theorem and synthetic division to find $P(-2)$.
54

64. Let $P(x) = x^3 - 13x^2 - 27$. Use the remainder theorem and synthetic division to find $P(5)$.
-227

Use the factor theorem to determine whether the first expression is a factor of $P(x)$.

65. $x - 5$; $P(x) = x^3 - 3x^2 - 8x - 10$
yes

66. $x + 5$; $P(x) = x^3 + 4x^2 - 5x + 5$
Hint: Write $x + 5$ as $x - (-5)$.
no

SECTION 6.7 Solving Rational Equations

DEFINITIONS AND CONCEPTS	EXAMPLES
If an equation contains one or more rational expressions, it is called a **rational equation.**	Rational equations: $\frac{4}{5} = \frac{x}{x - 1}$ and $\frac{2a}{a^2 + 9a + 20} = \frac{2}{a + 5} + \frac{3}{a + 4}$
To **solve a rational equation:** **1.** Factor all denominators. **2.** Determine which numbers cannot be solutions of the equation. **3.** Multiply both sides of the equation by the LCD of all rational expressions in the equation. **4.** Use the distributive property to remove parentheses, remove any factors equal to 1, and write the result in simplified form. **5.** Solve the resulting equation. **6.** Check all possible solutions in the original equation.	Solve: $\frac{3}{2} + \frac{1}{a - 4} = \frac{5}{2a - 8}$ If we factor the last denominator, the equation can be written as: $\frac{3}{2} + \frac{1}{a - 4} = \frac{5}{2(a - 4)}$ We see that 4 cannot be a solution of the equation, because it makes at least one of the rational expressions in the equation undefined. We can clear the equation of fractions by multiplying both sides by $2(a - 4)$, which is the LCD of the three rational expressions. $\mathbf{2(a - 4)}\left(\frac{3}{2} + \frac{1}{a - 4}\right) = \mathbf{2(a - 4)}\left[\frac{5}{2(a - 4)}\right]$ Multiply both sides by the LCD. $\mathbf{2(a - 4)}\left(\frac{3}{2}\right) + \mathbf{2(a - 4)}\left(\frac{1}{a - 4}\right) = \mathbf{2(a - 4)}\left[\frac{5}{2(a - 4)}\right]$ Distribute. $\overset{1}{\cancel{2}}(a - 4)\left(\frac{3}{\underset{1}{\cancel{2}}}\right) + 2\overset{1}{\cancel{(a - 4)}}\left(\frac{1}{\underset{1}{\cancel{a - 4}}}\right) = \overset{1}{\cancel{2}}\overset{1}{\cancel{(a - 4)}}\left[\frac{5}{\underset{1}{\cancel{2}}\underset{1}{\cancel{(a - 4)}}}\right]$ Remove common factors.

	$(a-4)3+2=5$ Simplify. $3a-12+2=5$ Distribute. $3a-10=5$ Combine like terms. $3a=15$ Add 10 to both sides. $a=5$ Divide both sides by 3. The solution is 5. Verify that it satisfies the original equation.
Multiplying both sides of an equation by a quantity that contains a variable can lead to **extraneous solutions.** All possible solutions of a rational equation must be checked.	Solve: $\frac{x+6}{x-1}=2+\frac{7}{x-1}$ $(x-1)\left(\frac{x+6}{x-1}\right)=(x-1)\left(2+\frac{7}{x-1}\right)$ Multiply both sides by the LCD, $x-1$. $(x-1)\left(\frac{x+6}{x-1}\right)=(x-1)(2)+(x-1)\left(\frac{7}{x-1}\right)$ Distribute. $\overset{1}{\cancel{(x-1)}}\left(\frac{x+6}{\underset{1}{\cancel{x-1}}}\right)=(x-1)(2)+\overset{1}{\cancel{(x-1)}}\left(\frac{7}{\underset{1}{\cancel{x-1}}}\right)$ Remove common factors. $x+6=(x-1)(2)+7$ Simplify. $x+6=2x-2+7$ Distribute. $x+6=2x+5$ Combine like terms. $1=x$ Solve for x. Since 1 makes the denominator of a rational expression 0, it does not check. It is an extraneous solution. The equation has no solution.

REVIEW EXERCISES

Solve each equation, if possible.

67. $\frac{4}{x}-\frac{1}{10}=\frac{7}{2x}$ 5

68. $\frac{11}{t}=\frac{6}{t-7}$ $\frac{77}{5}$

69. $\frac{3}{y}-\frac{2}{y+1}=\frac{1}{2}$ $-2, 3$

70. $\frac{2}{3x+15}-\frac{1}{18}=\frac{1}{3x+12}$ $-1, -2$

71. $\frac{3}{x+2}=\frac{1}{2-x}+\frac{2}{x^2-4}$ $\frac{3}{2}$

72. $\frac{x+3}{x-5}+\frac{2x^2+6}{x^2-7x+10}=\frac{3x}{x-2}$ 0

73. $\frac{5a}{a-3}-7=\frac{15}{a-3}$ no solution; 3 is extraneous.

74. **a.** Simplify: $\frac{10}{x^2-4x}-\frac{4}{x}+\frac{5}{x-4}$ $\frac{x+26}{x(x-4)}$

b. Solve: $\frac{10}{x^2-4x}-\frac{4}{x}=\frac{5}{x-4}$ $\frac{26}{9}$

Solve each formula for the indicated variable or expression.

75. $H=\frac{2ab}{a+b}$ for b $b=\frac{Ha}{2a-H}$

76. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for y^2 $y^2=\frac{x^2b^2-a^2b^2}{a^2}$

77. $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$ for R $R=\frac{R_1R_2}{R_2+R_1}$

78. $k=\frac{ma}{F}$ for F $F=\frac{ma}{k}$

SECTION 6.8 Problem Solving Using Rational Equations

DEFINITIONS AND CONCEPTS

Rate of Work: If a job can be completed in x units of time, the rate of work can be expressed as $\frac{1}{x}$ of the job is completed per unit of time.

Shared-work problems:

Work completed = rate of work · time worked

EXAMPLES

PRINTERS Working alone, a 300-A model printer can print a company's payroll checks in 30 minutes. A 500-X model can print the same checks in 20 minutes. How long will it take if the printers work together to print the checks?

Analyze Let x = the number of minutes it will take the printers, working together, to print the checks. Enter the data in a table.

	Rate	· Time	= Work completed
Model 300-A	$\frac{1}{30}$	x	$\frac{x}{30}$
Model 500-X	$\frac{1}{20}$	x	$\frac{x}{20}$

Form The part of the job done by the 300-A model plus the part of the job done by the 500-X model equals 1 job completed.

$$\frac{x}{30} + \frac{x}{20} = 1$$

Solve

$$60\left(\frac{x}{30} + \frac{x}{20}\right) = 60(1) \quad \text{Multiply both sides by the LCD, 60.}$$

$$60\left(\frac{x}{30}\right) + 60\left(\frac{x}{20}\right) = 60(1) \quad \text{Distribute the multiplication by 60.}$$

$$2x + 3x = 60 \quad \text{Perform each multiplication by 60.}$$

$$5x = 60 \quad \text{Combine like terms.}$$

$$x = \frac{60}{5} = 12$$

$$x = 12$$

State Working together, it will take the printers 12 minutes to print the checks.

Check In 12 minutes, the 300-A model will do $\frac{12}{30} = \frac{2}{5}$ of the job and the 500-X model will do $\frac{12}{20} = \frac{3}{5}$ of the job. Together they will do $\frac{2}{5} + \frac{3}{5} = 1$ whole job. The result checks.

Uniform motion problems:

$$\text{Time} = \frac{\text{distance}}{\text{rate}}$$

When a boat travels *downstream,* the speed of the boat is increased by the current. When a boat travels *upstream,* the speed of the boat is decreased by the current.

See pages 554–557 for examples.

REVIEW EXERCISES

79. **a.** If a painter can complete a job in 10 hours, what is the painter's rate of work? 1/10 of the job per hour

b. If the painter works for x hours, how much of the job is completed? $\frac{x}{10}$ of the job is completed

80. DRAINING A TANK If one outlet pipe can drain a tank in 24 hours and another pipe can drain the tank in 36 hours, how long will it take for both pipes to drain the tank? $14\frac{2}{5}$ hr

81. ELECTRICIANS It takes an apprentice 5 days more than an experienced electrician to wire a 1,500-square foot house. Working together, they can wire such a house in 6 days. How long would it take each person working alone to wire the house? experienced electrician: 10 days, apprentice: 15 days

82. SINKS A faucet can fill a garage sink in 2 minutes. It takes 3 minutes for the drain to empty the sink when it is full. How long will it take to fill the sink if the drain is open and the faucet is on? 6 min

83. ADVERTISING A small plane pulling a banner can fly at a rate of 75 mph in calm air. Flying down the coast, with a tailwind, the plane flew 40 miles in the same time that it took to fly 35 miles up the coast, into a headwind. Find the rate of the wind. 5 mph

84. TRIP LENGTH Heavy traffic reduced a driver's usual speed by 10 mph, which lengthened her 200-mile trip by 1 hour. Find the driver's usual speed. 50 mph

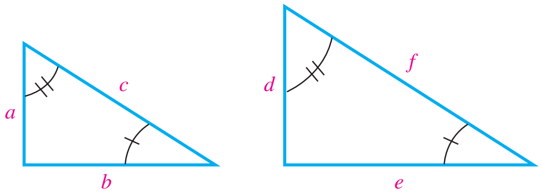
SECTION 6.9 Proportion and Variation

DEFINITIONS AND CONCEPTS	EXAMPLES
A **proportion** is a statement that two ratios are equal. In the proportion $\frac{a}{b} = \frac{c}{d}$, a and d are the **extremes** and b and c are the **means.**	Proportion: $\frac{4}{9} = \frac{28}{63}$ Extremes: 4 and 63 Means: 9 and 28
In any proportion, the product of the extremes is equal to the product of the means. (The **cross products** are equal.)	Proportion: $\frac{4}{9} = \frac{28}{63}$ Cross product: $4 \cdot 63 = 252$ Cross product: $9 \cdot 28 = 252$
To **solve a proportion,** set the product of the extremes equal to the product of the means and solve the resulting equation.	Solve the proportion: $\frac{x+2}{4} = \frac{6}{x}$ $(x+2)x = 4 \cdot 6$ In a proportion, the product of the extremes is equal to the product of the means. $x^2 + 2x = 24$ $x^2 + 2x - 24 = 0$ Subtract 24 from both sides. $(x+6)(x-4) = 0$ Factor the trinomial. $x + 6 = 0$ or $x - 4 = 0$ Set each factor equal to 0. $x = -6$ | $x = 4$ The solutions are -6 and 4. Both solutions check.

If two angles of one triangle have the same measure as two angles of a second triangle, the triangles are similar. The lengths of corresponding sides of **similar triangles** are proportional.

Similar triangles have the same shape but not necessarily the same size.

In these similar triangles: $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$



See page 564 for an example.

The words ***y varies directly as x*** or ***y is directly proportional to x*** mean that $y = kx$ for some nonzero constant k, called the **constant of variation.**

The distance d that a spring stretches ***varies directly*** as the force f attached to the spring: $d = kf$.

The words ***y varies inversely as x*** or ***y is inversely proportional to x*** mean that

$$y = \frac{k}{x} \text{ for some nonzero constant } k.$$

If the voltage in an electric circuit is kept constant, the current I ***varies inversely*** as the resistance R: $I = \frac{k}{R}$.

Strategy for Solving Variation Problems

1. Translate the verbal model into an equation.
2. Substitute the first set of values into the equation from step 1 to determine the value of k.
3. Substitute the value of k into the equation from step 1.
4. Substitute the remaining set of values into the equation from step 3 and solve for the unknown.

Suppose d *varies inversely as* h. If $d = 5$ when $h = 4$, find d when $h = 10$.

1. The words d varies inversely as h translate to $d = \frac{k}{h}$.
2. If we substitute 5 for d and 4 for h, we have

$$5 = \frac{k}{4}$$

$$20 = k \quad \text{Multiply both sides by 4.}$$

3. Since $k = 20$, the inverse variation equation is $d = \frac{20}{h}$.
4. To answer the final question, we substitute 10 for h.

$$d = \frac{20}{10} = 2$$

Joint variation: One variable varies as the product of several variables. For example, $y = kxz$ (k is a constant).

Combined variation: A combination of direct and inverse variation. For example,

$$y = \frac{kx}{z} \text{ (} k \text{ is a constant)}$$

The number of gallons g of oil that can be stored in a cylindrical tank ***varies jointly*** as the height h of the tank and the square of the radius r of its base: $g = khr^2$.

The gravitational force F between two objects with masses m_1 and m_2 ***varies directly*** as the product of their masses and ***inversely*** as the square of the distance d between them: $F = \frac{km_1m_2}{d^2}$.

REVIEW EXERCISES

Solve each proportion.

85. $\frac{x + 1}{8} = \frac{4x - 2}{24}$ 5

86. $\frac{1}{x + 6} = \frac{x + 10}{12}$ $-4, -12$

87. $\frac{-r}{r + 2} = \frac{3r}{r - 2}$ $0, -1$

88. $\frac{18}{t^2} = \frac{3t - 3}{t}$ $-2, 3$

89. SIMILAR TRIANGLES Find the height of a tree if it casts a 44-foot shadow when a 4-foot shrub casts a $2\frac{1}{2}$-foot shadow. 70.4 ft

90. COOKING A recipe for spaghetti sauce requires four 16-ounce bottles of ketchup to make 2 gallons of sauce. How many bottles of ketchup are needed to make 10 gallons of sauce? 20

91. SCALE MODELS A model of a playhouse was made using a 1/12th scale. If the scale model is 5.5 inches tall, how tall is the playhouse? 66 in.

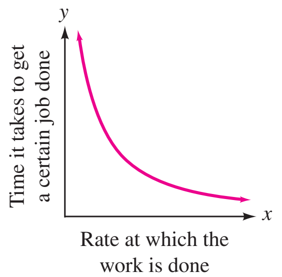
92. PROPERTY TAX The property tax in a certain county varies directly as assessed valuation. If a tax of \$1,575 is charged on a single-family home assessed at \$90,000, determine the property tax on an apartment complex assessed at \$312,000. \$5,460

93. ELECTRICITY For a fixed voltage, the current in an electrical circuit varies inversely as the resistance in the circuit. If a certain circuit has a current of $2\frac{1}{2}$ amps when the resistance is 150 ohms, find the current in the circuit when the resistance is doubled. 1.25 amps

94. Assume that x_1 varies directly with the third power of t and inversely with x_2. Find the constant of variation if $x_1 = 1.6$ when $t = 8$ and $x_2 = 64$. 0.2

95. HURRICANE WINDS The wind force on a vertical surface varies jointly as the area of the surface and the square of the wind's velocity. If a 10-mph wind exerts a force of 1.98 pounds on the sign shown in the illustration, find the force on the sign if the wind is blowing at 80 mph. 126.72 lb

96. Does the graph show direct or inverse variation? inverse variation



CHAPTER 6 TEST

1. Fill in the blanks.
   a. A quotient of two polynomials, such as $\frac{x-8}{x^2-2x-3}$, is called a rational expression.
   b. The reciprocal of $\frac{x+1}{x-7}$ is $\frac{x-7}{x+1}$.
   c. $\dfrac{\frac{x}{4}+\frac{1}{x}}{\frac{1}{8}+\frac{1}{x}}$ is an example of a complex fraction.
   d. When solving a rational equation, if we obtain a number that does not satisfy the original equation, the number is called an extraneous solution.
   e. An equation that states that two ratios are equal, such as $\frac{1}{2}=\frac{3}{6}$, is called a proportion.

2. Explain the error that was made in the solution shown below.

   Simplify:

$$\frac{2(x+2)+3(x-3)}{x+2} = \frac{2\cancelto{1}{(x+2)}+3(x-3)}{\cancelto{1}{x+2}}$$
$$= \frac{2+3x-9}{1}$$
$$= 3x-7$$

Simplify each rational expression.

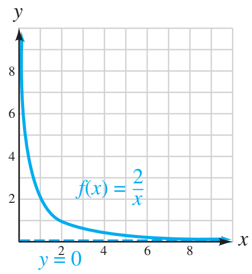
3. $\dfrac{12x^2y^3z^2}{18x^3y^4z^2}$ $\frac{2}{3xy}$

4. $\dfrac{2x+4}{x^2-4}$ $\frac{2}{x-2}$

5. $\dfrac{3y-6z}{2z-y}$ -3

6. $\dfrac{2x^2+7xy+3y^2}{4xy+12y^2}$ $\frac{2x+y}{4y}$

7. Graph the rational function $f(x)=\frac{2}{x}$ for $x>0$. Label the horizontal asymptote.



8. Find the domain of the rational function $f(x)=\frac{x^2+6x+5}{x-x^2}$. Express it using interval notation. $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

Perform the operations and simplify when necessary. Write all answers using positive exponents only.

9. $\dfrac{x^2}{x^3z^2y^2} \cdot \dfrac{x^2z^4}{y^2z}$ $\frac{xz}{y^4}$

10. $\dfrac{a^2+5a+6}{a^2-4} \cdot \dfrac{a^2-5a+6}{a^2-9}$ 1

11. $\dfrac{xu+2u+3x+6}{u^2-9} \cdot \dfrac{13u-39}{x^2+3x+2}$ $\frac{13}{x+1}$

12. $\dfrac{x^3+y^3}{16x^2} \div \dfrac{x^2-xy+y^2}{8x^2+8xy}$ $\frac{(x+y)^2}{2x}$

13. $\dfrac{a^2+7a+12}{a+3} \div \dfrac{16-a^2}{a-4}$ -1

14. $\dfrac{(2x-3)^3}{x^2-2x+1} \div \dfrac{3x^2+7x+2}{3x^2-2x-1} \cdot \dfrac{x^2+x-2}{2x^7-3x^6}$ $\frac{(2x-3)^2}{x^6}$

15. $\dfrac{-3t+4}{t^2+t-20} + \dfrac{6+5t}{t^2+t-20}$ $\frac{2}{t-4}$

16. $\dfrac{3wx}{wx-5} + \dfrac{wx+10}{5-wx}$ 2

17. $8b-5+\dfrac{5b+4}{3b+1}$ $\frac{24b^2-2b-1}{3b+1}$

18. $\dfrac{a+3}{a^2-a-2} - \dfrac{a-4}{a^2-2a-3}$ $\frac{6a-17}{(a+1)(a-2)(a-3)}$

Simplify.

19. $\dfrac{\frac{2u^2w^3}{v^2}}{\frac{4uw^4}{uv}}$ $\frac{u^2}{2vw}$

20. $\dfrac{\frac{4}{3k}+\frac{k}{k+1}}{\frac{k}{k+1}-\frac{3}{k}}$ $\frac{3k^2+4k+4}{3k^2-9k-9}$

21. Divide: $\dfrac{18x^2y^3 - 12x^3y^2 + 9xy}{-3xy^4}$ $-\frac{6x}{y} + \frac{4x^2}{y^2} - \frac{3}{y^3}$

22. Divide: $(y^3 - 48) \div (y + 2)$ $y^2 - 2y + 4 - \frac{56}{y+2}$

23. Let $P(x) = 4x^3 + 3x^2 + 2x - 7$. Use synthetic division to find $P(2)$. 41

24. Use the factor theorem to decide whether $x + 3$ is a factor of $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63$.
$x + 3$ is a factor of $P(x)$.

Solve each equation.

25. $\dfrac{34}{x^2} + \dfrac{13}{20x} = \dfrac{3}{2x}$ 40

26. $\dfrac{u-2}{u-3} + 3 = u + \dfrac{u-4}{3-u}$ 5; 3 is extraneous.

27. $\dfrac{3}{x-2} = \dfrac{x+3}{2x}$ 6, −1

28. $\dfrac{4}{m^2-9} + \dfrac{5}{m^2-m-12} = \dfrac{7}{m^2-7m+12}$ 26

Solve each formula for the indicated variable or expression.

29. $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ for r_2 $r_2 = \frac{rr_1}{r_1 - r}$

30. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ for a^2 $a^2 = \frac{x^2b^2}{b^2 - y^2}$

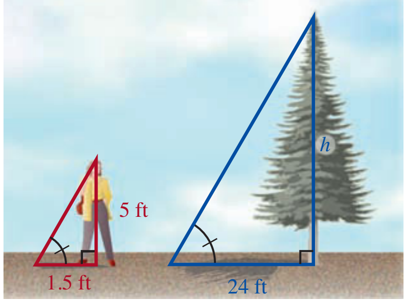
31. ROOFING One crew can finish a 2,800-square-foot roof in 12 hours, and another crew can do the job in 10 hours. If they work together, can they finish before a predicted rain in 5 hours? If not, how long will they have to work in the rain? no, $\frac{5}{11}$ of an hour

32. HOSPITALS It takes a technician 45 minutes longer than his supervisor to clean an outpatient surgery room. Working together, they can clean the room in 30 minutes. How long would it take each person working alone to clean the room? supervisor: 45 min, technician: 90 min

33. TOURING THE COUNTRYSIDE A man bicycles 5 mph faster than he can walk. He bicycles a distance of 24 miles and then hikes back along the same route. If the entire trip takes 11 hours, how fast does he walk? 3 mph

34. RIVER CRUISES A paddleboat can make an 8-mile trip up the Mississippi River and return in a total of 3 hours. If the boat travels 6 mph in still water, find the speed of the current. 2 mph

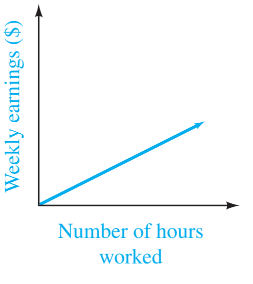
35. SHADOWS Refer to the illustration. Find the height of the tree. 80 ft



36. ANNIVERSARY GIFTS A florist sells a dozen long-stemmed red roses for $57.99. In honor of their 16th wedding anniversary, a man wants to buy 16 roses for his wife. What will the roses cost? $77.32

37. SOUND Sound intensity (loudness) varies inversely as the square of the distance from the source. If a rock band has a sound intensity of 100 decibels 30 feet away from the amplifier, find the sound intensity 60 feet away from the amplifier. 25 decibels

38. Draw a possible graph showing that the weekly earnings of a person *varies directly* with the number of hours worked during the week. Label the axes.



CHAPTERS 1–6 CUMULATIVE REVIEW

1. Evaluate: $12 - 6[(130 - 4^3) - 2]$ [Section 1.3] -372

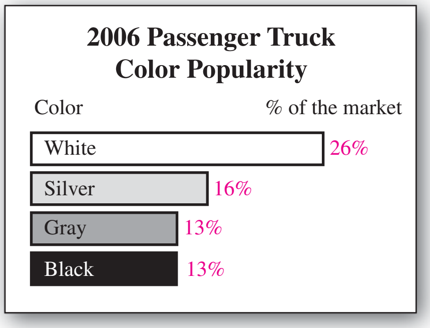
2. Simplify: $9(a^3 + 3a) - 5(3a - a^3) - 8(-a - a^3)$
[Section 1.4] $22a^3 + 20a$

3. Solve: $\frac{3x - 4}{6} - \frac{x - 2}{2} = \frac{-2x - 3}{3}$ [Section 1.5] -2

4. Solve $l = a + (n - 1)d$ for n. [Section 1.6] $n = \frac{l - a + d}{d}$

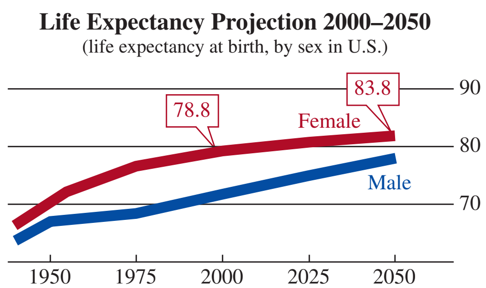
5. Find the area of a circle with diameter 25 inches. Round to one decimal place. [Section 1.6] 490.9 in.^{2}

6. TRUCK SALES See the following graph. A truck dealership is going to order 150 new Toyota Tundra trucks. According to the survey, exactly how many silver trucks should be purchased to meet the expected customer demand? [Section 1.7] 24



Source: Infoplease.com

7. LIFE EXPECTANCY Determine the predicted rate of change in the life expectancy of females during the years 2000–2050, as shown in the graph. [Section 2.3]
Life expectancy will increase 0.1 year each year during this period.



Based on data from the Social Security Administration, Office of Chief Actuary

Find an equation of the line with the given properties. Write the equation in slope–intercept form.

8. Passing through (7, 5) and perpendicular to the line whose equation is $y = \frac{1}{7}x - 4$ [Section 2.4] $y = -7x + 54$

9. Passing through $(-4, 5)$ and $(2, -6)$
[Section 2.4] $y = -\frac{11}{6}x - \frac{7}{3}$

10. The graph of a line is shown on the graphing calculator screen below. The axes are scaled in units of 1.

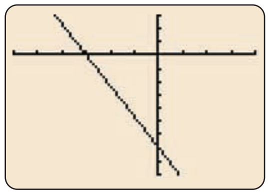
a. Give the x- and y-intercepts of the line.
[Section 2.2] $(-3, 0), (0, -7)$

b. What is the slope of the line? [Section 2.3] $-\frac{7}{3}$

c. The line doesn't lie in one quadrant. Which quadrant is that? [Section 2.1] I

d. What is the equation of the line?
[Section 2.4] $y = -\frac{7}{3}x - 7$

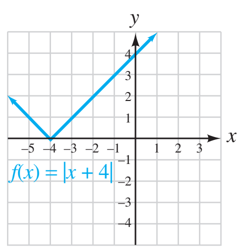
e. Does the line pass through $(15, -42)$?
[Section 2.2] yes



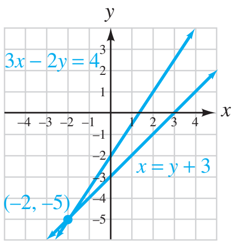
11. If $g(x) = -3x^3 + x - 4$, find $g(-2)$. [Section 2.5] 18

12. Determine whether the equation $x = |y|$ defines y to be a function of x. If it does not, find two ordered pairs where more than one value of y corresponds to a single value of x. [Section 2.5] no; (1, 1), (1, −1)

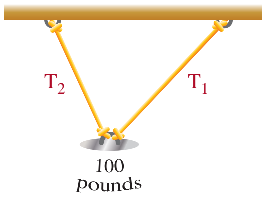
13. Graph $f(x) = |x + 4|$ and give its domain and range in interval notation.
[Section 2.6] D: $(-\infty, \infty)$, R: $[0, \infty)$



14. Solve the system $\begin{cases} x = y + 3 \\ \frac{1}{4}x - \frac{1}{6}y = \frac{1}{3} \end{cases}$ by graphing. [Section 3.1] $(-2, -5)$



15. ENGINEERING The tensions T_1 and T_2 (in pounds) in each of the ropes shown in the illustration can be found by solving the system



$$\begin{cases} 0.6T_1 - 0.8T_2 = 0 \\ 0.8T_1 + 0.6T_2 = 100 \end{cases}$$

$T_1 = 80$, $T_2 = 60$

Find T_1 and T_2. [Section 3.2]

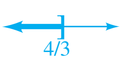
16. JEWELRY A jeweler wants to make angel pins from an alloy that has a 40% silver content. The jeweler has on hand two alloys, one with 50% silver content and one with a 25% silver content. How many ounces of each alloy should be used to make 20 ounces of the 40% silver alloy? Use two variables to solve the problem. [Section 3.3] 12 oz 50% alloy, 8 oz 25% alloy

17. Solve: $\begin{cases} x + 2y + 3z = 11 \\ 5x - y = 13 \\ 2x - 3z = -11 \end{cases}$ [Section 3.4] $(2, -3, 5)$

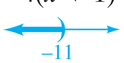
18. Evaluate: $\begin{vmatrix} 3 & -2 \\ -2 & 4 \end{vmatrix}$ [Section 3.7] 8

Solve. Write the solution set in interval notation and then graph it.

19. $\frac{1}{2}x + 6 \geq 4 + 2x$ [Section 4.1]
$\left(-\infty, \frac{4}{3}\right]$



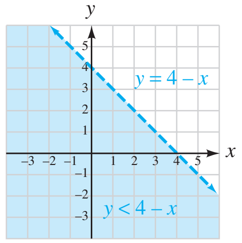
20. $5(x + 2) \leq 4(x + 1)$ and $11 + x < 0$ [Section 4.2]
$(-\infty, -11)$



21. $-4(x + 2) \geq 12$ or $3x + 8 < 11$ [Section 4.2]
$(-\infty, 1)$

22. $\left|\frac{3a}{5} - 2\right| + 1 \geq \frac{6}{5}$ [Section 4.3]
$(-\infty, 3] \cup \left[\frac{11}{3}, \infty\right)$

23. Graph: $y < 4 - x$ [Section 4.4]



24. Without graphing, determine if $(4, -2)$ is a solution of the system

$$\begin{cases} 3x + 2y > 6 \\ x + y \leq 2 \end{cases}$$

[Section 4.5] yes

Simplify each expression. Write answers using positive exponents only.

25. $(3bb^2b^3c^0)^4$ [Section 5.1] $81b^{24}$

26. $9^2d^{-8}(d^9)^2$ [Section 5.1] $81d^{10}$

27. $-\dfrac{x^{-9}}{20k^{-7}}$ [Section 5.1] $-\frac{k^7}{20x^9}$

28. $\left(\dfrac{2x^{-2}y^3}{x^2x^3y^4}\right)^{-3}$ [Section 5.1] $\frac{x^{21}y^3}{8}$

29. MONEY Express the *dollar value* of each type of U.S. coin and currency shown in the illustration, using a power of 10. For example, one hundred dollars can be expressed as $\$10^2$. [Section 5.2]
penny: $\$10^{-2}$, dime: $\$10^{-1}$, one-dollar bill: $\$10^0$, one-hundred-thousand-dollar bill: $\$10^5$

30. Write each number in scientific notation and perform the indicated operation: $\frac{6{,}150{,}000{,}000}{0.003}$
[Section 5.2] 2.05×10^{12}

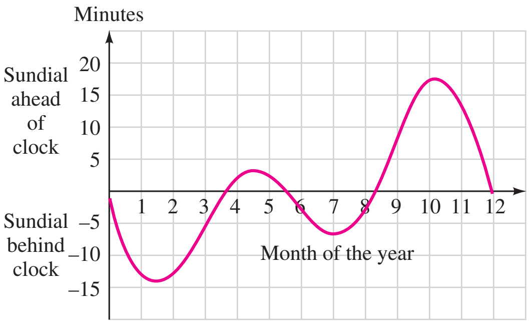
31. SUNDIALS Refer to the illustration. The graph shows the correction that must be made to a sundial reading to obtain accurate clock time. The difference is caused by the Earth's orbit and tilted axis. [Section 5.3]

a. Is this the graph of a function? yes

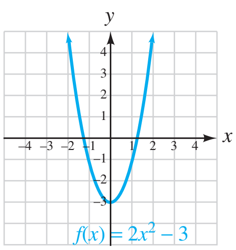
b. During the year, what is the maximum number of minutes the sundial reading gets ahead of a clock? about 17

c. During the year, what is the maximum number of minutes the sundial reading falls behind a clock? about 14

d. How many times during a year is the sundial reading exactly the same as a clock? 4



32. Graph $f(x) = 2x^2 - 3$ and give its domain and range in interval notation. [Section 5.3]
D: $(-\infty, \infty)$, R: $[-3, \infty)$



33. Find the degree of $17x^3y^4 + x^2y + 3$. [Section 5.3] 7

34. Simplify the polynomial: $\frac{9}{4}c^2 - \frac{5}{3}c - \frac{1}{2}c^2 + \frac{5}{6}c$
[Section 5.3] $\frac{7}{4}c^2 - \frac{5}{6}c$

Perform the indicated operations.

35. $(x^3 + 3x^2 - 2x + 7) + (x^3 - 2x^2 + 2x + 5)$ [Section 5.3]
$2x^3 + x^2 + 12$

36. $(2m^2n^2 + 2m - n) - (-2m^2n^2 - 2m + n)$ [Section 5.3]
$4m^2n^2 + 4m - 2n$

37. $\left(\frac{1}{16}r^9s^{10}\right)(32r^2s^{10})$ [Section 5.4]
$2r^{11}s^{20}$

38. $-3a^8(4a^4 + 3a^3 - 4a^2)$ [Section 5.4]
$-12a^{12} - 9a^{11} + 12a^{10}$

39. $(2x^3 - 1)^2$ [Section 5.4]
$4x^6 - 4x^3 + 1$

40. $(a + b + c)(2a - b - 2c)$ [Section 5.4]
$2a^2 + ab - b^2 - 3bc - 2c^2$

Factor each expression.

41. $3r^2s^3 - 6rs^4$ [Section 5.5]
$3rs^3(r - 2s)$

42. $5(x - y) - a(x - y)$ [Section 5.5]
$(x - y)(5 - a)$

43. $xu + yv + xv + yu$ [Section 5.5]
$(x + y)(u + v)$

44. $81x^4 - 16y^4$ [Section 5.6]
$(9x^2 + 4y^2)(3x + 2y)(3x - 2y)$

45. $8x^3 - 27y^6$ [Section 5.6]
$(2x - 3y^2)(4x^2 + 6xy^2 + 9y^4)$

46. $3 - 10x^2 + 8x^4$ [Section 5.7]
$(4x^2 - 3)(2x^2 - 1)$

47. $(x - y)^2 + 3(x - y) - 10$ [Section 5.7]
$(x - y + 5)(x - y - 2)$

48. $x^2 + 10x + 25 - 16z^8$ [Section 5.8]
$(x + 5 + 4z^4)(x + 5 - 4z^4)$

49. Solve $b^2x^2 + a^2y^2 = a^2b^2$ for b^2. [Section 5.5]
$b^2 = \frac{a^2y^2}{a^2 - x^2}$

50. Solve: $6x^2 + 7 = -23x$ [Section 5.9]
$-\frac{1}{3}, -\frac{7}{2}$

51. Solve: $x^3 - 4x = 0$ [Section 5.9]
$0, 2, -2$

52. CAMPING The rectangular-shaped cooking surface of a small camping stove is 108 in.^{2}. If its length is 3 inches longer than its width, what are its dimensions? [Section 5.9]
9 in. by 12 in.

53. Simplify: $\dfrac{2x^2y + xy - 6y}{3x^2y + 5xy - 2y}$ [Section 6.1]
$\frac{2x - 3}{3x - 1}$

54. Find the domain of the function $f(x) = \frac{2x + 1}{x^2 - 2x}$. Express your answer in words and using interval notation. [Section 6.1]
all real numbers except 0 and 2; $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$

Perform the indicated operations and then simplify, if possible.

55. $(10n - n^2) \cdot \dfrac{n^6}{n^4 - 10n^3 - 2n^2 + 20n}$ [Section 6.2]
$-\frac{n^6}{(n^2 - 2)}$

56. $\dfrac{p^3 - q^3}{q^2 - p^2} \div \dfrac{p^3 + p^2q + pq^2}{q^2 + pq}$ [Section 6.2]
$-\frac{q}{p}$

57. $\dfrac{2}{x + y} + \dfrac{3}{x - y} - \dfrac{x - 3y}{x^2 - y^2}$ [Section 6.3]
$\frac{4}{x - y}$

58. $\dfrac{\frac{18}{c^2} + \frac{11}{c} + 1}{1 - \frac{3}{c} - \frac{10}{c^2}}$ [Section 6.4]
$\frac{c + 9}{c - 5}$

Perform the division.

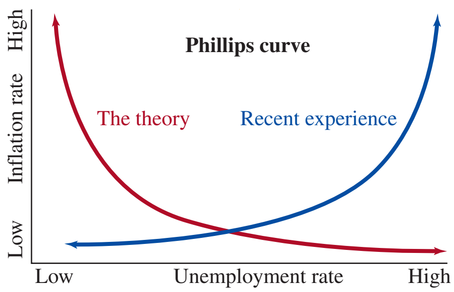
59. $\dfrac{5y^4 + 45y^3}{15y^2}$ [Section 6.5] $\frac{y^2}{3} + 3y$

60. $\dfrac{16x^3 + 16x^2 - 9x - 5}{4x + 5}$ [Section 6.5] $4x^2 - x - 1$

61. Solve: $\dfrac{3}{x - 2} + \dfrac{x^2}{x^2 + x - 6} = \dfrac{x + 4}{x + 3}$ [Section 6.7] -17

62. Solve: $\dfrac{5x - 3}{x + 2} = \dfrac{5x + 3}{x - 2}$ [Section 6.9] 0

63. ECONOMICS The controversial Phillips curve shown below depicts the trade-off between unemployment and inflation as seen by one school of economists. If unemployment drops to very low levels, what does the theoretical model predict will happen to the inflation rate? [Section 6.9] It will rise sharply.



64. ECONOMICS See Exercise 63. The graph shows that economic factors have not followed the Phillips model in recent years. As unemployment has dropped to very low levels, what has happened to the inflation rate? [Section 6.9] It has dropped.

65. COOKING A recipe for brownies calls for 4 eggs and $1\frac{1}{2}$ cups of flour. If the recipe makes 15 brownies, how many cups of flour will be needed to make 130 brownies? [Section 6.9] 13 cups

66. FARMING The number of days a given number of bushels of corn will last when feeding chickens varies inversely with the number of animals. If a certain number of bushels will feed 300 chickens for 4 days, how long will the feed last for 1,200 chickens? [Section 6.9] 1 day

Radical Expressions, Equations, and Functions

7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from Campus to Careers

General Contractor

The growing popularity of remodeling has created a boom for general contractors. If it's an additional bedroom you need or a makeover of a dated kitchen or bathroom, they can provide design and construction expertise, as well as knowledge of local building code requirements. From the planning stages of a project through its completion, general contractors use mathematics every step of the way.

In **Problem 138** of **Study Set 7.5,** you will use concepts from this chapter to examine the movement of construction materials through a tight hallway.

JOB TITLE:
General Contractor
EDUCATION: Courses in mathematics, science, drafting, business math, and English are important. Certificate programs are also available.
JOB OUTLOOK: In general, employment is expected to increase between 9% to 17% through the year 2014.
ANNUAL EARNINGS: Mean annual salary $76,699
FOR MORE INFORMATION:
http://www.bls.gov/oco/ocos005.htm

Objectives

1. Find square roots.
2. Find square roots of expressions containing variables.
3. Graph the square root function.
4. Find cube roots.
5. Graph the cube root function.
6. Find *n*th roots.

SECTION 7.1
Radical Expressions and Radical Functions

In this section, we will reverse the squaring process and learn how to find *square roots* of numbers. We will then generalize the concept of root and consider cube roots, fourth roots, fifth roots, and so on. We will also discuss a new family of functions, called *radical functions*.

1 Find square roots.

When solving problems, we must often find what number must be squared to obtain a second number a. If such a number can be found, it is called a **square root of *a*.** For example,

- 0 is a square root of 0, because $0^2 = 0$.
- 4 is a square root of 16, because $4^2 = 16$.
- -4 is a square root of 16, because $(-4)^2 = 16$.
- $7xy$ is a square root of $49x^2y^2$, because $(7xy)^2 = 49x^2y^2$.
- $-7xy$ is a square root of $49x^2y^2$, because $(-7xy)^2 = 49x^2y^2$.

The preceding examples illustrate the following definition.

Square Root of *a*

The number b is a **square root** of the number a if $b^2 = a$.

All positive numbers have two real-number square roots, one that is positive and one that is negative.

Self Check 1

Find the square roots of 144.

Now Try **Problem 21**

Self Check 1 Answer
12, -12

Teaching Example 1 Find the two square roots of 49.
Answer:
7, -7

EXAMPLE 1

Find the two square roots of 121.

Strategy We will determine the numbers that, when squared, produce 121.

WHY This is the definition of square root.

Solution

The two square roots of 121 are 11 and -11, because

$$11^2 = 121 \quad \text{and} \quad (-11)^2 = 121$$

In the following definition, the symbol $\sqrt{}$ is called a **radical symbol,** and the number a within the radical symbol is called a **radicand.** An expression containing a radical is called a **radical expression.**

$$\text{Radical symbol} \rightarrow \underbrace{\sqrt{a}}_{\text{Radical}} \leftarrow \text{radicand}$$

Square Root Notation

If a is a positive number,

1. $\sqrt{a}$ represents the **positive** or **principal square root** of a. It is the positive number we square to get a.
2. $-\sqrt{a}$ represents the **negative square root** of a. It is the opposite of the principal square root of a: $-\sqrt{a} = -1 \cdot \sqrt{a}$.
3. The principal square root of 0 is 0: $\sqrt{0} = 0$.

To evaluate square root expressions, it is helpful to memorize the natural numbers that are perfect squares.

$\mathbf{1} = 1^2$	$\mathbf{25} = 5^2$	$\mathbf{81} = 9^2$	$\mathbf{169} = 13^2$	$\mathbf{289} = 17^2$
$\mathbf{4} = 2^2$	$\mathbf{36} = 6^2$	$\mathbf{100} = 10^2$	$\mathbf{196} = 14^2$	$\mathbf{324} = 18^2$
$\mathbf{9} = 3^2$	$\mathbf{49} = 7^2$	$\mathbf{121} = 11^2$	$\mathbf{225} = 15^2$	$\mathbf{361} = 19^2$
$\mathbf{16} = 4^2$	$\mathbf{64} = 8^2$	$\mathbf{144} = 12^2$	$\mathbf{256} = 16^2$	$\mathbf{400} = 20^2$

EXAMPLE 2

Evaluate each square root: **a.** $\sqrt{1}$ **b.** $\sqrt{81}$ **c.** $-\sqrt{81}$ **d.** $-\sqrt{225}$ **e.** $\sqrt{\frac{1}{4}}$ **f.** $-\sqrt{\frac{16}{121}}$ **g.** $\sqrt{0.04}$ **h.** $-\sqrt{0.0009}$

Strategy In each case, we will determine what positive number, when squared, produces the radicand. Then we will apply the sign that is in front of the radical.

WHY The symbol $\sqrt{}$ indicates that the positive square root of the number written under it should be found. The symbol $-\sqrt{}$ indicates that the negative square root of the number written under it should be found.

Solution

a. $\sqrt{1} = 1$

b. $\sqrt{81} = 9$

c. $-\sqrt{81} = -9$

d. $-\sqrt{225} = -15$

e. $\sqrt{\frac{1}{4}} = \frac{1}{2}$

f. $-\sqrt{\frac{16}{121}} = -\frac{4}{11}$

g. $\sqrt{0.04} = 0.2$

h. $-\sqrt{0.0009} = -0.03$

Self Check 2

Evaluate each square root:

a. $-\sqrt{49}$ -7

b. $\sqrt{\frac{25}{49}}$ $\frac{5}{7}$

Now Try **Problems 27 and 29**

Teaching Example 2 Evaluate each square root:
a. $\sqrt{121}$ **b.** $-\sqrt{25}$
c. $\sqrt{\frac{169}{225}}$ **d.** $-\sqrt{0.36}$
Answers:
a. 11 **b.** -5 **c.** $\frac{13}{15}$ **d.** -0.6

Numbers such as 1, 81, $\frac{1}{4}$, $\frac{16}{121}$, and 0.04 are called **perfect squares,** because each one is the square of a rational number. In Example 2, we saw that the square root of a perfect square is a rational number.

The square roots of many positive numbers are not rational numbers. For example, $\sqrt{11}$ is an *irrational number.* To find an approximate value of $\sqrt{11}$, we enter 11 into a calculator and press the square root key $\boxed{\sqrt{}}$. (On some calculators, the $\boxed{\sqrt{}}$ key must be pressed first.)

$$\sqrt{11} \approx 3.31662479$$

Caution! Square roots of negative numbers are not real numbers. For example, $\sqrt{-9}$ is not a real number, because no real number squared equals -9. Square roots of negative numbers come from a set called the **imaginary numbers,** which we will discuss in Section 7.7.

2 Find square roots of expressions containing variables.

If $x \neq 0$, the positive number x^2 has x and $-x$ for its two square roots. To denote the positive square root of $\sqrt{x^2}$, we must know whether x is positive or negative.

If x is positive, we can write

$\sqrt{x^2} = x$ $\quad$ $\sqrt{x^2}$ represents the positive square root of x^2, which is x.

If x is negative, then $-x > 0$, and we can write

$\sqrt{x^2} = -x$ $\quad$ $\sqrt{x^2}$ represents the positive square root of x^2, which is $-x$.

If we don't know whether x is positive or negative, we can use absolute value symbols to guarantee that $\sqrt{x^2}$ is positive.

Definition of $\sqrt{x^2}$

For any real number x,

$$\sqrt{x^2} = |x|$$

We use this definition to *simplify* square root radical expressions.

Self Check 3

Simplify:

a. $\sqrt{25a^2}$ $5|a|$

b. $\sqrt{16a^4}$ $4a^2$

Now Try **Problems 34 and 39**

Teaching Example 3 Simplify:
a. $\sqrt{81x^2}$ **b.** $\sqrt{36a^4}$
c. $\sqrt{x^2 + 14x + 49}$
Answers:
a. $9|x|$ **b.** $6a^2$ **c.** $|x + 7|$

EXAMPLE 3 Simplify: **a.** $\sqrt{16x^2}$ **b.** $\sqrt{x^2 + 2x + 1}$ **c.** $\sqrt{m^4}$

Strategy In each case, we will determine what positive expression, when squared, produces the radicand.

WHY The symbol $\sqrt{}$ indicates that the positive square root of the expression written under it should be found.

Solution

If x and m can be any real number, we have

a. $\sqrt{16x^2} = \sqrt{(4x)^2}$ $\quad$ Write the radicand $16x^2$ as $(4x)^2$.

$= |4x|$ $\quad$ Because $(|4x|)^2 = 16x^2$. Since x could be negative, absolute value symbols are needed.

$= 4|x|$ $\quad$ Since 4 is a positive constant in the product $4x$, we can write it outside the absolute value symbols.

b. $\sqrt{x^2 + 2x + 1}$

$= \sqrt{(x + 1)^2}$ $\quad$ Factor the radicand: $x^2 + 2x + 1 = (x + 1)^2$.

$= |x + 1|$ $\quad$ Since $x + 1$ can be negative (for example, when $x = -5$, $x + 1$ is -4), absolute value symbols are needed.

c. $\sqrt{m^4} = m^2$ $\quad$ Because $(m^2)^2 = m^4$. Since $m^2 \geq 0$, no absolute value symbols are needed.

If we are told that x represents a *positive* real number in parts a and b of Example 3, we do not need to use absolute value symbols to guarantee that the answers are positive. For example, if $x > 0$, then

$\sqrt{16x^2} = 4x$ $\quad$ If x is positive, $4x$ is positive.

$\sqrt{x^2 + 2x + 1} = x + 1$ $\quad$ If x is positive, $x + 1$ is positive.

3 Graph the square root function.

Since there is one principal square root for every nonnegative real number x, the equation $f(x) = \sqrt{x}$ determines a function, called a **square root function.** Square root functions belong to a larger family of functions known as **radical functions.**

EXAMPLE 4 Graph $f(x) = \sqrt{x}$ and find its domain and range.

Strategy We will graph the function by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

To graph this square root function, we will evaluate it for several values of x. We begin with $x = 0$, since 0 is the smallest input for which $\sqrt{x}$ is defined.

$$f(x) = \sqrt{x}$$

$$f(0) = \sqrt{0} = 0 \quad \text{Substitute 0 for } x.$$

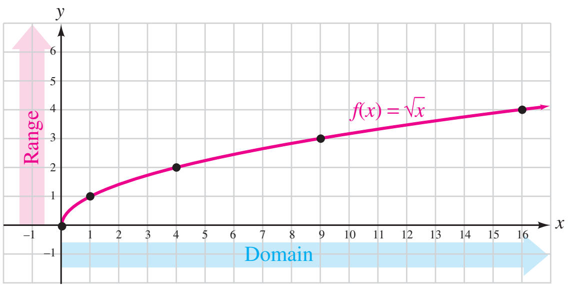
We enter the ordered pair (0, 0) in the table of values below. Then we continue the evaluation process for $x = 1, 4, 9$, and 16 and list the results in the table. After plotting all the ordered pairs, we draw a smooth curve through the points. This is the graph of function f (see figure (a)). Since the equation defines a function, its graph passes the vertical line test.

We can use a graphing calculator with window settings of $[-1, 9]$ for x and $[-1, 9]$ for y to get the graph shown in figure (b). From either graph, we can see that the domain and the range are the set of nonnegative real numbers. Expressed in interval notation, the domain is $[0, \infty)$, and the range is $[0, \infty)$.

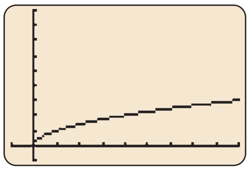
$f(x) = \sqrt{x}$

x	$f(x)$	$(x, f(x))$
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)
16	4	(16, 4)

Select values of x that are perfect squares.



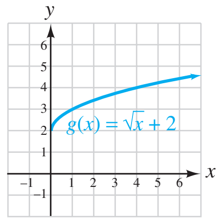
(a)



(b)

Self Check 4

Graph $f(x) = \sqrt{x} + 2$ and find its domain and range.



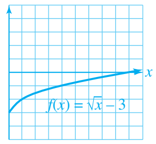
Now Try Problem 42

Self Check 4 Answer

D: $[0, \infty)$, R: $[2, \infty)$

Teaching Example 4 Graph $f(x) = \sqrt{x} - 3$ and find its domain and range.

Answer:



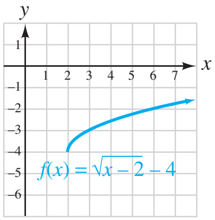
D: $[0, \infty)$, R: $[-3, \infty)$

The graphs of many radical functions are translations or reflections of the square root function, $f(x) = \sqrt{x}$. For example, if $k > 0$,

- The graph of $f(x) = \sqrt{x} + k$ is the graph of $f(x) = \sqrt{x}$ translated k units up.
- The graph of $f(x) = \sqrt{x} - k$ is the graph of $f(x) = \sqrt{x}$ translated k units down.
- The graph of $f(x) = \sqrt{x + k}$ is the graph of $f(x) = \sqrt{x}$ translated k units to the left.
- The graph of $f(x) = \sqrt{x - k}$ is the graph of $f(x) = \sqrt{x}$ translated k units to the right.
- The graph of $f(x) = -\sqrt{x}$ is the graph of $f(x) = \sqrt{x}$ reflected about the x-axis.

Self Check 5

Graph $f(x) = \sqrt{x - 2} - 4$ and find its domain and range.



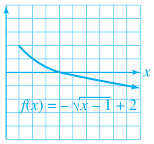
Now Try **Problem 48**

Self Check 5 Answer

D: $[2, \infty)$, R: $[-4, \infty)$

Teaching Example 5 Graph $f(x) = -\sqrt{x - 1} + 2$ and find its domain and range.

Answer:



D: $[1, \infty)$, R: $(-\infty, 2]$

EXAMPLE 5

Graph $f(x) = -\sqrt{x + 4} - 2$ and find its domain and range.

Strategy We will graph this function by translating and reflecting the graph $f(x) = \sqrt{x}$.

WHY $f(x) = -\sqrt{x + 4} - 2$ is the reflection of $f(x) = \sqrt{x}$ about the x-axis, translated 4 units left and 2 units down.

Solution

The graph is shown in figure (a). We can confirm this graph by using a graphing calculator with window settings of $[-5, 6]$ for x and $[-6, 2]$ for y to get the graph shown in figure (b).

We can determine the domain of the function algebraically. Since the expression $\sqrt{x + 4}$ is not a real number when $x + 4$ is negative, we must require that

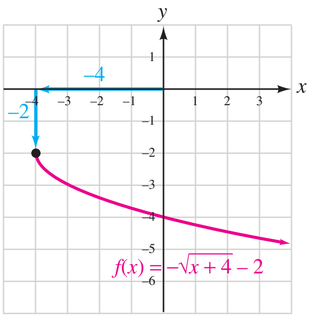
$x + 4 \geq 0$ *Because we cannot find the real square root of a negative number.*

Solving for x, we have

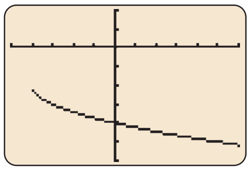
$x \geq -4$ *The x-inputs must be real numbers greater than or equal to -4.*

Thus, the domain of $f(x) = -\sqrt{x + 4} - 2$ is the interval $[-4, \infty)$.

From either graph, we can see that the domain is the interval $[-4, \infty)$ and that the range is the interval $(-\infty, -2]$.



(a)



(b)

Self Check 6

PENDULUMS To the nearest hundredth, find the period of a pendulum that is 3 feet long.

Now Try **Problem 108**

Self Check 6 Answer

1.92 sec

Teaching Example 6 PENDULUMS To the nearest tenth, find the period of a pendulum that is 4 feet long.

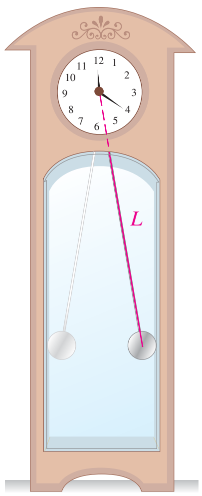
Answer:

2.2 sec

EXAMPLE 6

Pendulums The **period of a pendulum** is the time required for the pendulum to swing back and forth to complete one cycle. The period (in seconds) is a function of the pendulum's length L (in feet) and is given by

$$f(L) = 2\pi\sqrt{\frac{L}{32}}$$



Find the period of the 5-foot-long pendulum of the clock. Round the result to the nearest tenth.

Strategy To find the period of the pendulum we will find $f(5)$.

WHY The notation $f(5)$ represents the period (in seconds) of a pendulum whose length L is 5 feet.

Solution

$$f(L) = 2\pi\sqrt{\frac{L}{32}}$$

$$f(5) = 2\pi\sqrt{\frac{5}{32}} \qquad \text{Substitute 5 for } L.$$

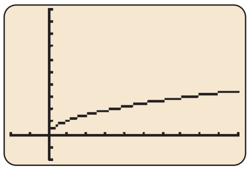
$$\approx 2.483647066 \qquad \text{Use a calculator to find an approximation.}$$

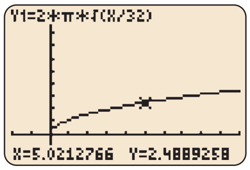
The period is approximately 2.5 seconds.

Using Your CALCULATOR Evaluating a Square Root Function

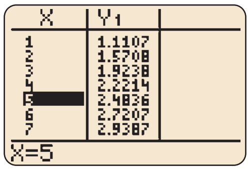
To solve Example 6 with a graphing calculator with window settings of $[-2, 10]$ for x and $[-2, 10]$ for y, we graph the function $f(x) = 2\pi\sqrt{\frac{x}{32}}$, as in figure (a). We then trace and move the cursor toward an x-value of 5 until we see the coordinates shown in figure (b). The period is given by the y-value shown on the screen. By zooming in, we can get better results.

After entering $Y_1 = 2\pi\sqrt{\frac{x}{32}}$, we can also use the TABLE mode to evaluate the function. See figure (c).


(a)



(b)



(c)

4 Find cube roots.

When we raise a number to the third power, we are cubing it, or finding its **cube.** We can reverse the cubing process to find **cube roots** of numbers. To find the cube root of 8, we ask "What number, when cubed, is equal to 8?" It follows that 2 is a cube root of 8, because $2^3 = 8$.

In general, we have this definition.

Cube Root of *a*

The number b is a **cube root** of the real number a if $b^3 = a$.

We note that 64 has two real-number square roots, 8 and -8. However, 64 has only one real-number cube root 4, because 4 is the only real number whose cube is 64. All real numbers have one real cube root. A positive number has a positive cube root, a negative number has a negative cube root, and the cube root of 0 is 0.

Cube Root Notation

The **cube root of *a*** is denoted $\sqrt[3]{a}$. By definition

$$\sqrt[3]{a} = b \quad \text{if} \quad b^3 = a$$

Earlier we determined, that the cube root of 8 is 2. In symbols we can write: $\sqrt[3]{8} = 2$. The number 3 is called the **index**

Index
↓
$\sqrt[3]{8}$

Success Tip Since every real number has exactly one real cube root, it is unnecessary to use absolute value symbols when simplifying cube roots.

Definition of $\sqrt[3]{x^3}$

For any real number x,

$$\sqrt[3]{x^3} = x$$

To simplify cube root radical expressions, it is helpful to memorize the whole numbers that are perfect cubes.

$\mathbf{1} = 1^3$	$\mathbf{27} = 3^3$	$\mathbf{125} = 5^3$	$\mathbf{343} = 7^3$	$\mathbf{729} = 9^3$
$\mathbf{8} = 2^3$	$\mathbf{64} = 4^3$	$\mathbf{216} = 6^3$	$\mathbf{512} = 8^3$	$\mathbf{1{,}000} = 10^3$

Self Check 7

Simplify:

a. $\sqrt[3]{1{,}000}$ 10

b. $\sqrt[3]{\dfrac{1}{27}}$ $\frac{1}{3}$

c. $\sqrt[3]{-125a^3}$ $-5a$

Now Try Problems 50 and 53

Teaching Example 7 Simplify:

a. $\sqrt[3]{64}$ **b.** $\sqrt[3]{\dfrac{8}{27}}$ **c.** $\sqrt[3]{-216a^3}$

Answers:

a. 4 **b.** $\frac{2}{3}$ **c.** $-6a$

EXAMPLE 7 Simplify: **a.** $\sqrt[3]{125}$ **b.** $\sqrt[3]{\dfrac{1}{8}}$ **c.** $\sqrt[3]{-27x^3}$ **d.** $\sqrt[3]{-\dfrac{8a^3}{27b^3}}$ **e.** $\sqrt[3]{0.216x^3y^6}$

Strategy In each case, we will determine what number or expression, when cubed, produces the radicand.

WHY The symbol $\sqrt[3]{}$ indicates that the cube root of the number written under it should be found.

Solution

a. $\sqrt[3]{125} = 5$ Because $5^3 = 5 \cdot 5 \cdot 5 = 125$.

b. $\sqrt[3]{\dfrac{1}{8}} = \dfrac{1}{2}$ Because $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

c. $\sqrt[3]{-27x^3} = -3x$ Because $(-3x)^3 = (-3x)(-3x)(-3x) = -27x^3$.

d. $\sqrt[3]{-\dfrac{8a^3}{27b^3}} = -\dfrac{2a}{3b}$ Because $\left(-\frac{2a}{3b}\right)^3 = \left(-\frac{2a}{3b}\right)\left(-\frac{2a}{3b}\right)\left(-\frac{2a}{3b}\right) = -\frac{8a^3}{27b^3}$.

e. $\sqrt[3]{0.216x^3y^6} = 0.6xy^2$ Because $(0.6xy^2)^3 = (0.6xy^2)(0.6xy^2)(0.6xy^2) = 0.216x^3y^6$.

5 Graph the cube root function.

Since there is one cube root for every real number x, the equation $f(x) = \sqrt[3]{x}$ defines a function, called the **cube root function.** Like square root functions, cube root functions belong to the family of radical functions.

EXAMPLE 8 Consider $f(x) = \sqrt[3]{x}$. **a.** Graph the function.
b. Find its domain and range. **c.** Graph: $g(x) = \sqrt[3]{x} - 2$

Strategy We will graph the function by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve though the plotted points, we will have the graph. The answers to parts b and c can then be determined from the graph.

Solution

a. To graph the function, we select several values for x and find the corresponding values of $f(x)$. We begin with $x = -8$.

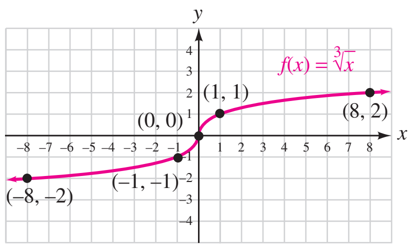
$$f(x) = \sqrt[3]{x}$$
$$f(-8) = \sqrt[3]{-8} \quad \text{Substitute } -8 \text{ for } x.$$
$$f(-8) = -2$$

We enter -8 for x and -2 for $f(x)$ in the table. Then we let $x = -1, 0, 1$, and 8, and list each corresponding function value in the table. After plotting the ordered pairs, we draw a smooth curve through the points to get the graph shown in figure (a).

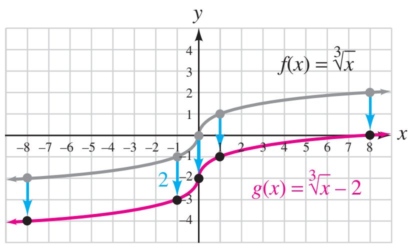
$f(x) = \sqrt[3]{x}$

x	$f(x)$	
-8	-2	→ $(-8, -2)$
-1	-1	→ $(-1, -1)$
0	0	→ $(0, 0)$
1	1	→ $(1, 1)$
8	2	→ $(8, 2)$

Select values of x that are perfect cubes.



(a)



(b)

b. From the graph in figure (a), we see that the domain and the range of function f are the set of real numbers. Thus, the domain is $(-\infty, \infty)$ and the range is $(-\infty, \infty)$.

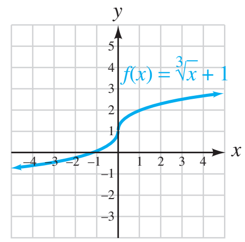
c. Refer to figure (b). The graph of $g(x) = \sqrt[3]{x} - 2$ is the graph of $f(x) = \sqrt[3]{x}$, translated 2 units downward.

Self Check 8

Consider $f(x) = \sqrt[3]{x} + 1$.

a. Graph the function.

b. Find its domain and range.



Now Try **Problem 58**

Self Check 8 Answer

D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

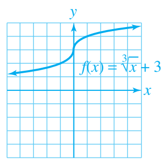
Teaching Example 8 Consider $f(x) = \sqrt[3]{x} + 3$.

a. Graph the function.

b. Find its domain and range.

Answers:

a.



b. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

6 Find *n*th roots.

Just as there are square roots and cube roots, there are fourth roots, fifth roots, sixth roots, and so on. In general, we have the following definition.

*n*th Roots of *a*

The ***n*th root of *a*** is denoted by $\sqrt[n]{a}$, and

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a$$

The number n is called the **index** (or **order**) of the radical. If n is an even natural number, a must be positive or zero and b must be positive.

When n is an odd natural number, the expression $\sqrt[n]{x}$ where $n > 1$ represents an **odd root.** Since every real number has just one real nth root when n is odd, we don't need to worry about absolute value symbols when finding odd roots. For example,

$$\sqrt[5]{243} = \sqrt[5]{3^5} = 3 \qquad \text{Because } 3^5 = 243.$$

$$\sqrt[7]{-128x^7} = \sqrt[7]{(-2x)^7} = -2x \qquad \text{Because } (-2x)^7 = -128x^7.$$

When n is an even natural number, the expression $\sqrt[n]{x}$, where $n > 1$ and $x > 0$, represents an **even root.** In this case, there will be one positive and one negative real nth root. For example, the real sixth roots of 729 are 3 and -3, because $3^6 = 729$ and $(-3)^6 = 729$. When finding even roots, we can use absolute value symbols to guarantee that the nth root is positive.

$$\sqrt[4]{(-3)^4} = |-3| = 3$$

We could also simplify this as follows: $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$.

$$\sqrt[6]{729x^6} = \sqrt[6]{(3x)^6} = |3x| = 3|x|$$

The absolute value symbols guarantee that the sixth root is positive.

In general, we have the following rules.

Rules for $\sqrt[n]{x^n}$

If x represents a real number and $n > 1$, then

If n represents an odd natural number, $\sqrt[n]{x^n} = x$.

If n represents an even natural number, $\sqrt[n]{x^n} = |x|$.

In the radical expression $\sqrt[n]{x}$, n is called the **index** (or **order**) of the radical. When the index is 2, the radical is a square root, and we usually do not write the index.

$$\sqrt{x} = \sqrt[2]{x}$$

Caution! When n is even, where $n > 1$ and $x < 0$, $\sqrt[n]{x}$ is not a real number. For example, $\sqrt[4]{-81}$ is not a real number, because no real number raised to the fourth power is -81.

Self Check 9

Evaluate each radical expression, if possible:

a. $\sqrt[4]{\frac{1}{81}}$ $\frac{1}{3}$

b. $\sqrt[5]{10^5}$ 10

c. $\sqrt[6]{-64}$ not a real number

Now Try **Problems 62, 66, and 70**

EXAMPLE 9 Evaluate each radical expression, if possible:

a. $\sqrt[4]{625}$ **b.** $\sqrt[4]{-1}$ **c.** $\sqrt[5]{-32}$ **d.** $\sqrt[6]{\frac{1}{64}}$ **e.** $\sqrt[7]{10^7}$

Strategy In each case, we will determine what number, when raised to the fourth, fifth, sixth, or seventh power, produces the radicand.

WHY The symbols $\sqrt[4]{\ }$, $\sqrt[5]{\ }$, $\sqrt[6]{\ }$, and $\sqrt[7]{\ }$ indicate that the fourth, fifth, sixth, or seventh root of the number written under it should be found.

Solution

a. $\sqrt[4]{625} = 5$, because $5^4 = 625$ — Read $\sqrt[4]{625}$ as "the fourth root of 625."

b. $\sqrt[4]{-1}$ is not a real number. — This is an even root of a negative number.

c. $\sqrt[5]{-32} = -2$, because $(-2)^5 = -32$ — Read $\sqrt[5]{-32}$ as "the fifth root of -32."

d. $\sqrt[6]{\frac{1}{64}} = \frac{1}{2}$, because $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$ — Read $\sqrt[6]{\frac{1}{64}}$ as "the sixth root of $\frac{1}{64}$."

e. $\sqrt[7]{10^7} = 10$, because $10^7 = 10^7$ — Read $\sqrt[7]{10^7}$ as "the seventh root of 10^7."

Teaching Example 9 Evaluate each radical expression, if possible:

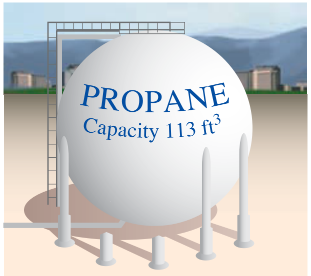
a. $\sqrt[5]{\frac{1}{32}}$ **b.** $\sqrt[4]{-625}$
c. $\sqrt[5]{-243}$ **d.** $-\sqrt[6]{64}$

Answers:
a. $\frac{1}{2}$ **b.** not a real number
c. -3 **d.** -2

Using Your CALCULATOR Finding Roots

The square root key $\sqrt{\ }$ on a reverse entry scientific calculator can be used to evaluate square roots. To evaluate roots with an index greater than 2, we can use the root key $\boxed{\sqrt[x]{y}}$. For example, the function

$$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$$



gives the radius of a sphere with volume V. To find the radius of the spherical propane tank shown in the figure, we substitute 113 for V to get

$$r(V) = \sqrt[3]{\frac{3V}{4\pi}}$$

$$r(113) = \sqrt[3]{\frac{3(113)}{4\pi}}$$

To evaluate a root, we enter the radicand and press the root key $\boxed{\sqrt[x]{y}}$ followed by the index of the radical, which in this case is 3.

3 $\boxed{\times}$ 113 $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{\times}$ $\boxed{\pi}$ $\boxed{)}$ $\boxed{=}$ $\boxed{\text{2nd}}$ $\boxed{\sqrt[x]{y}}$ $\boxed{3}$ $\boxed{=}$

`2.999139118`

To evaluate the cube root of $\frac{3(113)}{4\pi}$ using a direct entry calculator, we enter these numbers and press these keys.

3 $\boxed{\text{2nd}}$ $\boxed{\sqrt[x]{y}}$ $\boxed{(}$ 3 $\boxed{\times}$ 113 $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{\times}$ $\boxed{\pi}$ $\boxed{)}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

To evaluate the cube root of $\frac{3(113)}{4\pi}$ using a graphing calculator, we enter these numbers and press these keys.

$\boxed{\text{MATH}}$ 4 $\boxed{(}$ 3 $\boxed{\times}$ 113 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 4 $\boxed{\times}$ $\boxed{\text{2nd}}$ $\boxed{\pi}$ $\boxed{)}$ $\boxed{)}$ $\boxed{\text{ENTER}}$

```
³√((3*113)/(4*π)
)
              2.999139118
```

The radius of the propane tank is about 3 feet.

Self Check 10

Simplify each expression. Assume all variables are unrestricted.

a. $\sqrt[4]{16a^8}$ $2a^2$

b. $\sqrt[5]{(a+5)^5}$ $a+5$

c. $\sqrt{(x^2+4x+4)}$ $|x+2|$

Now Try **Problems 73, 75, and 82**

Teaching Example 10 Simplify each expression. Assume all variables are unrestricted.

a. $\sqrt[7]{x^7}$ **b.** $\sqrt[4]{x^8}$

c. $\sqrt[6]{64x^6}$ **d.** $\sqrt[8]{(x+1)^8}$

Answers:

a. x **b.** x^2 **c.** $2|x|$ **d.** $|x+1|$

EXAMPLE 10 Simplify each expression. Assume that x can be any real number.

a. $\sqrt[5]{x^5}$ **b.** $\sqrt[4]{16x^4}$ **c.** $\sqrt[6]{(x+4)^6}$ **d.** $\sqrt[3]{(x+1)^3}$

Strategy When the index n is odd, we will determine what expression, when raised to the nth power, produces the radicand. When the index n is even, we will determine what positive expression, when raised to the nth power produces the radicand.

WHY This is the definition of the nth root.

Solution

a. $\sqrt[5]{x^5} = x$ Since n is odd, absolute value symbols aren't needed.

b. $\sqrt[4]{16x^4} = |2x| = 2|x|$ Since n is even and x can be negative, absolute value symbols are needed to guarantee that the result is positive.

c. $\sqrt[6]{(x+4)^6} = |x+4|$ Absolute value symbols are needed to guarantee that the result is positive.

d. $\sqrt[3]{(x+1)^3} = x+1$ Since n is odd, absolute value symbols aren't needed.

If we know that x represents a positive real number in parts b and c of Example 10, we do not need to use absolute value symbols to guarantee that the results are positive.

$\sqrt[4]{16x^4} = 2x$ If x is positive, $2x$ is positive.

$\sqrt[6]{(x+4)^6} = x+4$ If x is positive, $x+4$ is positive.

We summarize the definitions concerning $\sqrt[n]{x}$ as follows.

Summary of the definitions of $\sqrt[n]{x}$

If n represents a natural number greater than 1 and x represents a real number, then

If $x > 0$, then $\sqrt[n]{x}$ is the positive number such that $\left(\sqrt[n]{x}\right)^n = x$.

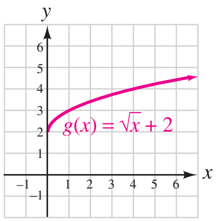
If $x = 0$, then $\sqrt[n]{x} = 0$.

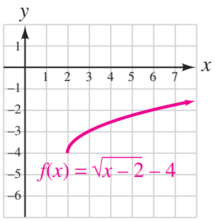
If $x < 0$ $\begin{cases} \text{and } n \text{ is odd, then } \sqrt[n]{x} \text{ is the real number such that } \left(\sqrt[n]{x}\right)^n = x. \\ \text{and } n \text{ is even, then } \sqrt[n]{x} \text{ is not a real number.} \end{cases}$

ANSWERS TO SELF CHECKS

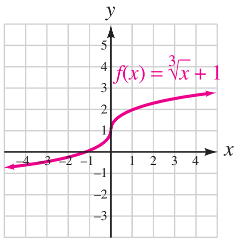
1. $12, -12$ **2. a.** -7 **b.** $\frac{5}{7}$ **3. a.** $5|a|$ **b.** $4a^2$

4. D: $[0, \infty)$, R: $[2, \infty)$ **5.** D: $[2, \infty)$, R: $[-4, \infty)$ **6.** 1.92 sec **7. a.** 10 **b.** $\frac{1}{3}$ **c.** $-5a$





8. a.



b. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

9. a. $\frac{1}{3}$ **b.** 10 **c.** not a real number

10. a. $2a^2$ **b.** $a+5$ **c.** $|x+2|$

SECTION 7.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. $5x^2$ is the square root of $25x^4$, because $(5x^2)^2 = 25x^4$.
2. $f(x) = \sqrt{x}$ and $g(t) = \sqrt[3]{t}$ are radical functions.
3. The symbol $\sqrt{\ }$ is called a radical symbol.
4. In the expression $\sqrt[3]{27x^6}$, 3 is the index and $27x^6$ is the radicand.
5. When n is an odd number, $\sqrt[n]{x}$ represents an odd root.
6. When n is an even number, $\sqrt[n]{x}$ represents an even root.
7. When we write $\sqrt{b^2 + 6b + 9} = |b + 3|$, we say that we have simplified the radical.
8. 6 is the cube root of 216 because $6^3 = 216$.

CONCEPTS

Fill in the blanks.

9. b is a square root of a if $b^2 = a$.
10. $\sqrt{0} = 0$ and $\sqrt[3]{0} = 0$
11. The number 25 has two square roots. The principal square root of 25 is the positive square root of 25.
12. $\sqrt{-4}$ is not a real number, because no real number squared equals -4.
13. $\sqrt[3]{x} = y$ if $y^3 = x$
14. $\sqrt{x^2} = |x|$ and $\sqrt[3]{x^3} = x$
15. The graph of $f(x) = \sqrt{x} + 3$ is the graph of $f(x) = \sqrt{x}$ translated 3 units up.
16. The graph of $f(x) = \sqrt{x + 5}$ is the graph of $f(x) = \sqrt{x}$ translated 5 units to the left.

NOTATION

Translate each sentence into mathematical symbols.

17. The square root of x squared is the absolute value of x. $\sqrt{x^2} = |x|$
18. The cube root of x cubed is x. $\sqrt[3]{x^3} = x$
19. f of x equals the square root of the quantity x minus five. $f(x) = \sqrt{x - 5}$
20. The fifth root of negative thirty-two is negative two. $\sqrt[5]{-32} = -2$

GUIDED PRACTICE

Find the two square roots of each number. See Example 1.

21. 81 9, −9
22. 169 13, −13
23. 256 16, −16
24. 196 14, −14

Evaluate each square root. See Example 2.

25. $\sqrt{121}$ 11
26. $\sqrt{144}$ 12
27. $-\sqrt{64}$ −8
28. $-\sqrt{1}$ −1
29. $\sqrt{\frac{1}{9}}$ $\frac{1}{3}$
30. $-\sqrt{\frac{4}{25}}$ $-\frac{2}{5}$
31. $-\sqrt{0.25}$ −0.5
32. $\sqrt{0.16}$ 0.4

Find each square root. Assume that all variables are unrestricted, and use absolute value symbols when necessary. See Example 3.

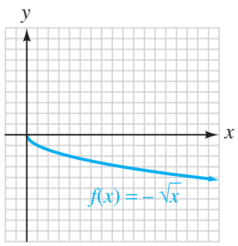
33. $\sqrt{4x^2}$ $2|x|$
34. $\sqrt{16y^4}$ $4y^2$
35. $\sqrt{(t + 5)^2}$ $|t + 5|$
36. $\sqrt{(a + 6)^2}$ $|a + 6|$
37. $\sqrt{(-5b)^2}$ $5|b|$
38. $\sqrt{(-8c)^2}$ $8|c|$
39. $\sqrt{a^2 + 6a + 9}$ $|a + 3|$
40. $\sqrt{x^2 + 10x + 25}$ $|x + 5|$

Complete each table, graph the function, and determine its domain and range. See Example 4.

41. $f(x) = -\sqrt{x}$

x	$f(x)$
0	0
1	−1
4	−2
9	−3
16	−4

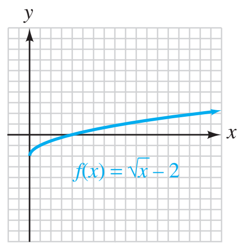
D: $[0, \infty)$, R: $(-\infty, 0]$



42. $f(x) = \sqrt{x} - 2$

x	$f(x)$
0	−2
1	−1
4	0
9	1
16	2

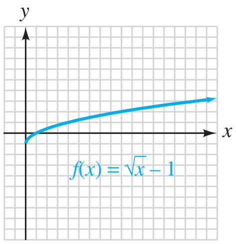
D: $[0, \infty)$, R: $[-2, \infty)$



43. $f(x) = \sqrt{x} - 1$

x	$f(x)$
0	-1
1	0
4	1
9	2
16	3

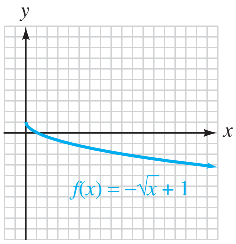
D: $[0, \infty)$, R: $[-1, \infty)$



44. $f(x) = -\sqrt{x} + 1$

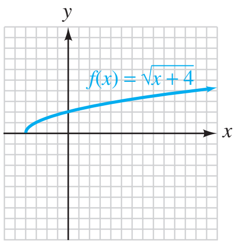
x	$f(x)$
0	1
1	0
4	-1
9	-2
16	-3

D: $[0, \infty)$, R: $(-\infty, 1]$

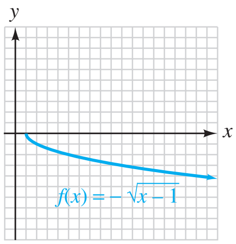


Graph each function and find its domain and range. See Example 5.

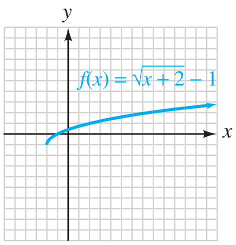
45. $f(x) = \sqrt{x + 4}$
D: $[-4, \infty)$, R: $[0, \infty)$



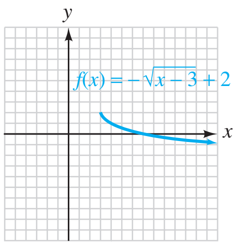
▶ 46. $f(x) = -\sqrt{x - 1}$
D: $[1, \infty)$, R: $(-\infty, 0]$



47. $f(x) = \sqrt{x + 2} - 1$
D: $[-2, \infty)$, R: $[-1, \infty)$



48. $f(x) = -\sqrt{x - 3} + 2$
D: $[3, \infty)$, R: $(-\infty, 2]$



Simplify each cube root. See Example 7.

49. $\sqrt[3]{1}$ 1

▶ 50. $\sqrt[3]{512}$ 8

51. $\sqrt[3]{-125a^3}$ $-5a$

52. $\sqrt[3]{-8y^3}$ $-2y$

53. $\sqrt[3]{-\dfrac{8a^6}{27b^3}}$ $-\dfrac{2a^2}{3b}$

54. $\sqrt[3]{\dfrac{125x^3}{216y^6}}$ $\dfrac{5x}{6y^2}$

55. $\sqrt[3]{0.064a^3b^6}$ $0.4ab^2$

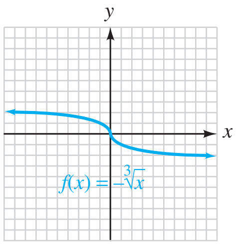
56. $\sqrt[3]{0.001x^6y^6}$ $0.1x^2y^2$

Complete each table, graph the function, and determine its domain and range. See Example 8.

▶ 57. $f(x) = -\sqrt[3]{x}$

x	$f(x)$
-8	2
-1	1
0	0
1	-1
8	-2

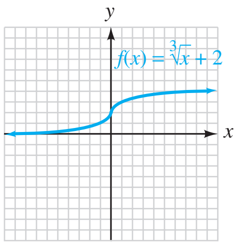
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



58. $f(x) = \sqrt[3]{x} + 2$

x	$f(x)$
-8	0
-1	1
0	2
1	3
8	4

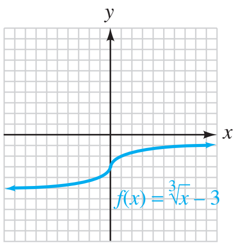
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



59. $f(x) = \sqrt[3]{x} - 3$

x	$f(x)$
-8	-5
-1	-4
0	-3
1	-2
8	-1

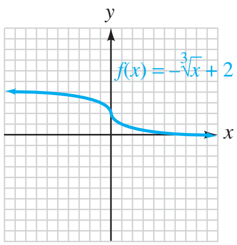
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



60. $f(x) = -\sqrt[3]{x} + 2$

x	$f(x)$
-8	4
-1	3
0	2
1	1
8	0

D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



Evaluate each radical, if possible. See Example 9.

61. $\sqrt[4]{81}$ 3

62. $\sqrt[6]{64}$ 2

63. $-\sqrt[5]{243}$ -3

64. $-\sqrt[4]{625}$ -5

65. $\sqrt[4]{-256}$ not real

66. $\sqrt[6]{-729}$ not real

67. $\sqrt[4]{\dfrac{16}{625}}$ $\dfrac{2}{5}$

▶ 68. $\sqrt[5]{-\dfrac{243}{32}}$ $-\dfrac{3}{2}$

69. $-\sqrt[5]{-\dfrac{1}{32}}$ $\dfrac{1}{2}$

▶ 70. $-\sqrt[4]{\dfrac{81}{256}}$ $-\dfrac{3}{4}$

71. $\sqrt[8]{10^8}$ 10

72. $\sqrt[11]{-9^{11}}$ -9

Simplify each expression. Assume that all variables represent nonzero numbers. See Example 10.

73. $\sqrt[5]{32a^5}$ $2a$

74. $\sqrt[5]{-32x^5}$ $-2x$

75. $\sqrt[4]{16a^4}$ $2|a|$

76. $\sqrt[8]{x^{24}}$ $|x^3|$

77. $\sqrt[4]{k^{12}}$ $|k^3|$

78. $\sqrt[6]{64b^6}$ $2|b|$

79. $\sqrt[4]{\frac{1}{16}m^4}$ $\frac{1}{2}|m|$

80. $\sqrt[4]{\frac{1}{81}x^8}$ $\frac{1}{3}x^2$

81. $\sqrt[4]{(x+2)^4}$ $|x+2|$

82. $\sqrt[5]{(x+1)^5}$ $x+1$

83. $\sqrt[3]{(x+2)^3}$ $x+2$

84. $\sqrt[4]{(x+4)^4}$ $|x+4|$

TRY IT YOURSELF

Evaluate each radical, if possible.

85. $\sqrt{-25}$ not real

86. $-\sqrt{-49}$ not real

87. $\sqrt{(-4)^2}$ 4

88. $\sqrt{(-9)^2}$ 9

89. $\sqrt[3]{8a^3}$ $2a$

90. $\sqrt[3]{-27x^6}$ $-3x^2$

91. $\sqrt[3]{-1{,}000p^3q^3}$ $-10pq$

92. $\sqrt[3]{343a^6b^3}$ $7a^2b$

93. $\sqrt[3]{-\frac{1}{8}m^6n^3}$ $-\frac{1}{2}m^2n$

94. $\sqrt[3]{0.008z^9}$ $0.2z^3$

95. $\sqrt[3]{-0.064s^6t^6}$ $-0.4s^2t^2$

96. $\sqrt[3]{\frac{27}{1{,}000}a^6b^6}$ $\frac{3}{10}a^2b^2$

Use a calculator to find each square root. Give the answer to four decimal places.

97. $\sqrt{12}$ 3.4641

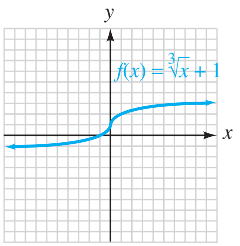
98. $\sqrt{340}$ 18.4391

99. $\sqrt{679.25}$ 26.0624

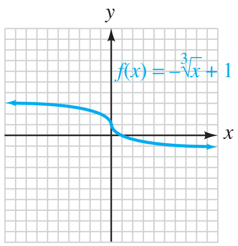
100. $\sqrt{0.0063}$ 0.0794

Graph each function and find its domain and range.

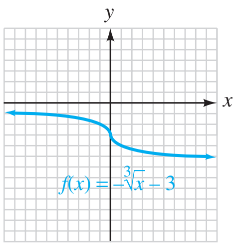
101. $f(x) = \sqrt[3]{x} + 1$
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



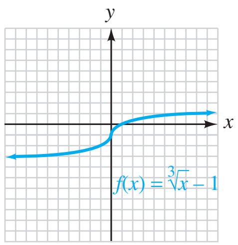
102. $f(x) = -\sqrt[3]{x} + 1$
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



103. $f(x) = -\sqrt[3]{x} - 3$
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



104. $f(x) = \sqrt[3]{x} - 1$
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

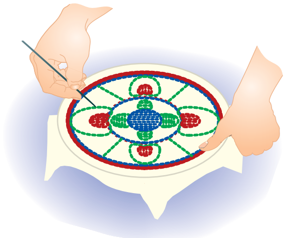


APPLICATIONS

Use a calculator to solve each problem. In each case, round to the nearest tenth.

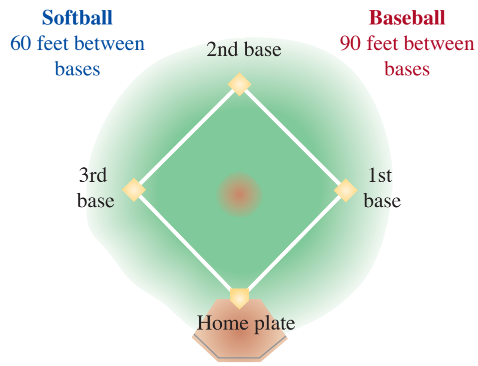
105. EMBROIDERY The radius r of a circle is given by the formula

$$r = \sqrt{\frac{A}{\pi}}$$



where A is its area. Find the diameter of the embroidery hoop shown on the right if there are 38.5 in.^{2} of stretched fabric on which to embroider. 7.0 in.

106. BASEBALL The length of a diagonal of a square is given by the function $d(s) = \sqrt{2s^2}$, where s is the length of a side of the square. Find the distance from home plate to second base on a softball diamond and on a baseball diamond. The illustration gives the dimensions of each type of infield. 84.9 ft, 127.3 ft



107. PULSE RATES The approximate pulse rate (in beats per minute) of an adult who is t inches tall is given by the function

$$p(t) = \frac{590}{\sqrt{t}}$$

The Guinness Book of World Records 1998 lists Ri Myong-hun of North Korea as the tallest living man, at 7 ft $8\frac{1}{2}$ in. Find his approximate pulse rate as predicted by the function. about 61.3 beats/min

108. THE GRAND CANYON The time t (in seconds) that it takes for an object to fall a distance of s feet is given by the formula

$$t = \frac{\sqrt{s}}{4}$$

In some places, the Grand Canyon is one mile (5,280 feet) deep. How long would it take a stone dropped over the edge of the canyon to hit bottom? 18.2 sec

▶ **109.** BIOLOGY Scientists will place five rats inside the controlled environment of a sealed hemisphere to study the rats' behavior. The function

$$d(V) = \sqrt[3]{12\left(\frac{V}{\pi}\right)}$$

gives the diameter of a hemisphere with volume V. Use the function to determine the diameter of the base of the hemisphere if each rat requires 125 cubic feet of living space. 13.4 ft

▶ **110.** AQUARIUMS The function

$$s(g) = \sqrt[3]{\frac{g}{7.5}}$$

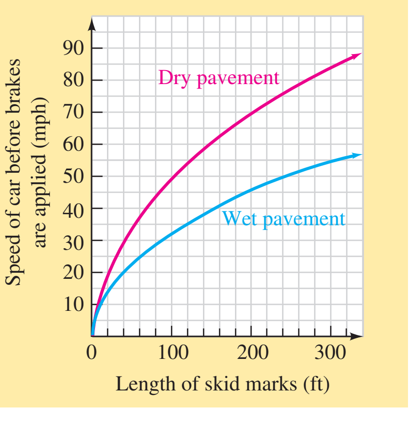
determines how long (in feet) an edge of a cube-shaped tank must be if it is to hold g gallons of water. What dimensions should a cube-shaped aquarium have if it is to hold 1,250 gallons of water? 5.5 ft × 5.5 ft × 5.5 ft

▶ **111.** COLLECTIBLES The *effective rate of interest r* earned by an investment is given by the formula

$$r = \sqrt[n]{\frac{A}{P}} - 1$$

where P is the initial investment that grows to value A after n years. Determine the effective rate of interest earned by a collector on a Lladró porcelain figurine purchased for \$800 and sold for \$950 five years later. 3.5%

▶ **112.** LAW ENFORCEMENT The graphs of the two radical functions shown in the illustration can be used to estimate the speed (in mph) of a car involved in an accident. Suppose a police accident report listed skid marks to be 220 feet long but failed to give the road conditions. Estimate the possible speeds the car was traveling prior to the brakes being applied. dry: about 72 mph, wet: about 47 mph

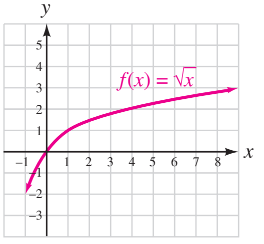


WRITING

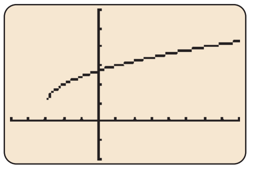
113. Explain why 36 has two square roots, but $\sqrt{36}$ is just 6, not -6.

▶ **114.** If x is any real number, then $\sqrt{x^2} = x$ is not correct. Explain.

115. Explain what is wrong with the graph on the right.



116. Explain how to estimate the domain and range of the radical function that is graphed on the right.



REVIEW

Perform the operations.

117. $\dfrac{x^2 - x - 6}{x^2 - 2x - 3} \cdot \dfrac{x^2 - 1}{x^2 + x - 2}$ 1

▶ **118.** $\dfrac{x^2 - 3x - 4}{x^2 - 5x + 6} \div \dfrac{x^2 - 2x - 3}{x^2 - x - 2}$ $\dfrac{(x-4)(x+1)}{(x-3)^2}$

119. $\dfrac{3}{m+1} + \dfrac{3m}{m-1}$ $\dfrac{3(m^2 + 2m - 1)}{(m+1)(m-1)}$

▶ **120.** $\dfrac{2x+3}{3x-1} - \dfrac{x-4}{2x+1}$ $\dfrac{x^2 + 21x - 1}{(3x-1)(2x+1)}$

SECTION 7.2
Simplifying and Combining Radical Expressions

Objectives

1. Use the product rule to simplify radical expressions.
2. Use prime factorization to simplify radical expressions.
3. Use the quotient rule to simplify radical expressions.
4. Add and subtract radical expressions.

In algebra, it is often helpful to replace an expression with a simpler equivalent expression. This is certainly true when working with radicals. In most cases, radical expressions should be written in simplified form. We use two rules for radicals to do this.

1 Use the product rule to simplify radical expressions.

To introduce the product rule for radicals, we will find $\sqrt{4 \cdot 25}$ and $\sqrt{4}\sqrt{25}$ and compare the results.

Square root of a product	**Product of square roots**	
$\sqrt{4 \cdot 25} = \sqrt{100}$	$\sqrt{4}\sqrt{25} = 2 \cdot 5$	Read as "the square root of 4 times the square root of 25."
$= 10$	$= 10$	

In each case, the answer is 10. Thus, $\sqrt{4 \cdot 25} = \sqrt{4}\sqrt{25}$. Likewise,

$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{216} = 6 \qquad \sqrt[3]{8}\sqrt[3]{27} = 2 \cdot 3 = 6$$

In each case, the answer is 6. Thus, $\sqrt[3]{8 \cdot 27} = \sqrt[3]{8}\sqrt[3]{27}$. These results illustrate the **multiplication property of radicals.**

The Product Rule for Radicals

The nth root of the product of two numbers is equal to the product of their nth roots.

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ represent real numbers, then

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

As long as all radical expressions represent real numbers, *the nth root of the product of two numbers is equal to the product of their nth roots.*

Caution! The multiplication property of radicals applies to the nth root of the product of two numbers. There is no such property for sums or differences. For example,

$$\sqrt{9+4} \neq \sqrt{9} + \sqrt{4} \qquad \sqrt{9-4} \neq \sqrt{9} - \sqrt{4}$$
$$\sqrt{13} \neq 3 + 2 \qquad \sqrt{5} \neq 3 - 2$$
$$\sqrt{13} \neq 5 \qquad \sqrt{5} \neq 1$$

Thus, $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ and $\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$.

The product rule for radicals can be used to simplify radical expressions. When a radical expression is written in **simplified form,** each of the following is true.

Simplified Form of a Radical Expression

1. Each factor in the radicand is to a power that is less than the index of the radical.
2. The radicand contains no fractions or negative numbers.
3. No radicals appear in the denominator of a fraction.

To simplify radical expressions, we must often factor the radicand using two natural-number factors. To simplify square-root, cube-root, and fourth-root radicals, it is helpful to memorize the following lists.

Perfect squares: **1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, . . .**
Perfect cubes: **1, 8, 27, 64, 125, 216, 343, 512, 729, 1,000, . . .**
Perfect-fourth powers: **1, 16, 81, 256, 625, . . .**

Self Check 1

Simplify:

a. $\sqrt{20}$ $2\sqrt{5}$
b. $\sqrt[3]{24}$ $2\sqrt[3]{3}$
c. $\sqrt[5]{128}$ $2\sqrt[5]{4}$

Now Try **Problems 14, 16, and 20**

Teaching Example 1 Simplify:
a. $\sqrt{52}$ **b.** $\sqrt{75}$
c. $\sqrt[3]{72}$ **d.** $-\sqrt[4]{80}$
Answers:
a. $2\sqrt{13}$ **b.** $5\sqrt{3}$
c. $2\sqrt[3]{9}$ **d.** $-2\sqrt[4]{5}$

EXAMPLE 1 Simplify: **a.** $\sqrt{12}$ **b.** $\sqrt{98}$ **c.** $\sqrt[3]{54}$ **d.** $-\sqrt[4]{48}$

Strategy We will factor each radicand into two factors, one of which is a perfect square, perfect cube, or perfect-fourth power, depending on the index of the radical. Then we can use the product rule for radicals to simplify the expression.

WHY Factoring the radicand in this way leads to a square root, cube root, or fourth root of a perfect square, perfect cube, or perfect-fourth power that we can easily simplify.

Solution

a. To simplify $\sqrt{12}$, we first factor 12 so that one factor is the largest perfect square that divides 12. Since 4 is the largest perfect-square factor of 12, we write 12 as $4 \cdot 3$, use the product rule for radicals, and simplify.

$\sqrt{12} = \sqrt{4 \cdot 3}$ Write 12 as $12 = 4 \cdot 3$.
Write the perfect-square factor first.
$= \sqrt{4}\sqrt{3}$ The square root of a product is equal to the product of the square roots.
$= 2\sqrt{3}$ Evaluate $\sqrt{4}$. Read as "2 times the square root of 3" or as "2 radical 3."

We say that $2\sqrt{3}$ is the simplified form of $\sqrt{12}$.

b. The largest perfect-square factor of 98 is 49. Thus,

$\sqrt{98} = \sqrt{49 \cdot 2}$ Write 98 in factored form: $98 = 49 \cdot 2$.
$= \sqrt{49}\sqrt{2}$ The square root of a product is equal to the product of the square roots: $\sqrt{49 \cdot 2} = \sqrt{49}\sqrt{2}$.
$= 7\sqrt{2}$ Evaluate $\sqrt{49}$.

c. Since the largest perfect-cube factor of 54 is 27, we have

$\sqrt[3]{54} = \sqrt[3]{27 \cdot 2}$ Write 54 as $27 \cdot 2$.
$= \sqrt[3]{27}\sqrt[3]{2}$ The cube root of a product is equal to the product of the cube roots: $\sqrt[3]{27 \cdot 2} = \sqrt[3]{27}\sqrt[3]{2}$.
$= 3\sqrt[3]{2}$ Evaluate $\sqrt[3]{27}$.

d. The largest perfect-fourth power factor of 48 is 16. Thus,

$$-\sqrt[4]{48} = -\sqrt[4]{16 \cdot 3} \quad \text{Write 48 as } 16 \cdot 3.$$

$$= -\sqrt[4]{16}\sqrt[4]{3} \quad \text{The fourth root of a product is equal to the product of the fourth roots: } \sqrt[4]{16 \cdot 3} = \sqrt[4]{16} \cdot \sqrt[4]{3}.$$

$$= -2\sqrt[4]{3} \quad \text{Evaluate } \sqrt[4]{16}.$$

Variable expressions can also be perfect squares, perfect cubes, perfect-fourth powers, and so on. For example, x^4 is a perfect square because it is the square of x^2.

Perfect squares: $x^2, x^4, x^6, x^8, x^{10}, \ldots$

Perfect cubes: $x^3, x^6, x^9, x^{12}, x^{15}, \ldots$

Perfect-fourth powers: $x^4, x^8, x^{12}, x^{16}, x^{20}, \ldots$

EXAMPLE 2 Simplify:

a. $\sqrt{m^9}$ **b.** $\sqrt{128a^5}$ **c.** $\sqrt[3]{-24x^5}$ **d.** $\sqrt[5]{a^9b^5}$

All variables represent positive real numbers.

Strategy We will factor each radicand into two factors, one of which is a perfect nth power.

WHY We can then apply the rule *the nth root of a product is the product of the nth roots* to simplify the radical expression.

Solution

a. The largest perfect-square factor of m^9 is m^8.

$$\sqrt{m^9} = \sqrt{m^8 \cdot m} \quad \text{Write } m^9 \text{ in factored form as } m^8 \cdot m.$$

$$= \sqrt{m^8}\sqrt{m} \quad \text{Use the product rule for radicals.}$$

$$= m^4\sqrt{m} \quad \text{Simplify } \sqrt{m^8}.$$

b. Since the largest perfect-square factor of 128 is 64 and the largest perfect-square factor of a^5 is a^4, the largest perfect-square factor of $128a^5$ is $64a^4$. We write $128a^5$ as $64a^4 \cdot 2a$ and proceed as follows:

$$\sqrt{128a^5} = \sqrt{64a^4 \cdot 2a} \quad \text{Write } 128a^5 \text{ in factored form as } 64a^4 \cdot 2a.$$

$$= \sqrt{64a^4}\sqrt{2a} \quad \text{Use the product rule for radicals.}$$

$$= 8a^2\sqrt{2a} \quad \text{Simplify } \sqrt{64a^4}.$$

c. We write $-24x^5$ as $-8x^3 \cdot 3x^2$ and proceed as follows:

$$\sqrt[3]{-24x^5} = \sqrt[3]{-8x^3 \cdot 3x^2} \quad 8x^3 \text{ is the largest perfect-cube factor of } 24x^5. \text{ Since the radicand is negative, we factor it using } -8x^3.$$

$$= \sqrt[3]{-8x^3}\sqrt[3]{3x^2} \quad \text{Use the product rule for radicals.}$$

$$= -2x\sqrt[3]{3x^2} \quad \text{Simplify } \sqrt[3]{-8x^3}.$$

d. The largest perfect-fifth power factor of a^9 is a^5, and b^5 is a perfect-fifth power.

$$\sqrt[5]{a^9b^5} = \sqrt[5]{a^5b^5 \cdot a^4} \quad a^5b^5 \text{ is the largest perfect-fifth power factor of } a^9b^5.$$

$$= \sqrt[5]{a^5b^5}\sqrt[5]{a^4} \quad \text{Use the product rule for radicals.}$$

$$= ab\sqrt[5]{a^4} \quad \text{Simplify } \sqrt[5]{a^5b^5}.$$

Self Check 2

Simplify. All variables represent positive real numbers.

a. $\sqrt{98b^3}$ $7b\sqrt{2b}$

b. $\sqrt[3]{-54y^5}$ $-3y\sqrt[3]{2y^2}$

c. $\sqrt[4]{t^8u^{15}}$ $t^2u^3\sqrt[4]{u^3}$

Now Try **Problems 22, 28, and 32**

Teaching Example 2 Simplify:

a. $\sqrt{b^{11}}$ **b.** $\sqrt{125x^7}$ **c.** $\sqrt[3]{-54x^{11}}$

Answers:

a. $b^5\sqrt{b}$ **b.** $5x^3\sqrt{5x}$ **c.** $-3x^3\sqrt[3]{2x^2}$

2 Use prime factorization to simplify radical expressions.

When simplifying radical expressions, prime factorization can be helpful in determining how to factor the radicand.

Self Check 3

Simplify:

a. $\sqrt{275}$ $5\sqrt{11}$

b. $\sqrt[3]{189c^4d^3}$ $3cd\sqrt[3]{7c}$

Now Try **Problems 34, 36, and 39**

Teaching Example 3 Simplify. All variables represent positive real numbers.

a. $\sqrt{63}$ **b.** $\sqrt[3]{-162x^5}$

c. $\sqrt[4]{162x^5y^{11}}$

Answers:

a. $3\sqrt{7}$ **b.** $-3x\sqrt[3]{6x^2}$

c. $3xy^2\sqrt[4]{2xy^3}$

EXAMPLE 3 Simplify. All variables represent positive real numbers.

a. $\sqrt{150}$ **b.** $\sqrt[3]{297b^4}$ **c.** $\sqrt[4]{224s^8t^7}$

Strategy In each case, the way to factor the radicand is not obvious. Another approach is to prime-factor the coefficient of the radicand and look for groups of like factors.

WHY Identifying groups of like factors of the radicand leads to a factorization of the radicand that can be easily simplified.

Solution

a.

$$\begin{aligned}\sqrt{150} &= \sqrt{2\cdot3\cdot5\cdot5} && \text{Write 150 in prime-factored form.}\\ &= \sqrt{2\cdot3}\sqrt{5\cdot5} && \text{Group the pair of like factors together and use the product rule for radicals.}\\ &= \sqrt{2\cdot3}\sqrt{5^2} && \text{Write } 5\cdot5 \text{ as } 5^2.\\ &= \sqrt{6}\cdot5 && \text{Evaluate } \sqrt{5^2}.\\ &= 5\sqrt{6} && \text{Write the factor 5 first.}\end{aligned}$$

b.

$$\begin{aligned}\sqrt[3]{297b^4} &= \sqrt[3]{3\cdot3\cdot3\cdot11\cdot b^3\cdot b} && \text{Write 297 in prime-factored form. The largest perfect-cube factor of } b^4 \text{ is } b^3.\\ &= \sqrt[3]{3\cdot3\cdot3\cdot b^3}\sqrt[3]{11b} && \text{Group the three like factors of 3 together and use the product rule for radicals.}\\ &= \sqrt[3]{3^3b^3}\sqrt[3]{11b} && \text{Write } 3\cdot3\cdot3 \text{ as } 3^3.\\ &= 3b\sqrt[3]{11b} && \text{Simplify } \sqrt[3]{3^3b^3}.\end{aligned}$$

c.

$$\begin{aligned}\sqrt[4]{224s^8t^7} &= \sqrt[4]{2\cdot2\cdot2\cdot2\cdot2\cdot7\cdot s^8\cdot t^4\cdot t^3} && \text{Write 224 in prime-factored form. The largest perfect-fourth power factor of } t^7 \text{ is } t^4.\\ &= \sqrt[4]{2\cdot2\cdot2\cdot2\cdot s^8\cdot t^4}\sqrt[4]{2\cdot7\cdot t^3} && \text{Group the four like factors of 2 together and use the product rule for radicals.}\\ &= \sqrt[4]{2^4s^8t^4}\sqrt[4]{2\cdot7\cdot t^3} && \text{Write } 2\cdot2\cdot2\cdot2 \text{ as } 2^4.\\ &= 2s^2t\sqrt[4]{14t^3} && \text{Simplify } \sqrt[4]{2^4s^8t^4}.\end{aligned}$$

3 Use the quotient rule to simplify radical expressions.

To introduce the second property of radicals, we consider these examples.

Square root of a quotient	**Quotient of square roots**
$\sqrt{\dfrac{100}{4}} = \sqrt{25}$	$\dfrac{\sqrt{100}}{\sqrt{4}} = \dfrac{10}{2}$
$= 5$	$= 5$

Since the answer is 5 in each case, $\sqrt{\dfrac{100}{4}} = \dfrac{\sqrt{100}}{\sqrt{4}}$. Likewise,

$\sqrt[3]{\dfrac{64}{8}} = \sqrt[3]{8}$	$\dfrac{\sqrt[3]{64}}{\sqrt[3]{8}} = \dfrac{4}{2}$
$= 2$	$= 2$

Since the answer is 2 in each case, $\sqrt[3]{\dfrac{64}{8}} = \dfrac{\sqrt[3]{64}}{\sqrt[3]{8}}$. These results illustrate the **division property of radicals.**

Division Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ represent real numbers, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{where } b \neq 0$$

As long as all radical expressions represent real numbers, *the nth root of the quotient of two numbers is equal to the quotient of their nth roots.*

EXAMPLE 4 Simplify: **a.** $\sqrt{\frac{7}{64}}$ **b.** $\sqrt{\frac{15}{49x^2}}$ **c.** $\sqrt[3]{\frac{10x^2}{27y^6}}$

Assume that the variables represent positive real numbers.

Strategy In each case, the radical is not in simplified form because the radicand contains a fraction. To write each of these expressions in simplified form, we will use the quotient rule for radicals.

WHY Writing these expressions in $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ form leads to square roots of perfect squares and cube roots of perfect cubes that we can easily simplify.

Solution

a. We can write the square root of the quotient as the quotient of two square roots.

$$\sqrt{\frac{7}{64}} = \frac{\sqrt{7}}{\sqrt{64}}$$ The square root of a quotient is equal to the quotient of the square roots.

$$= \frac{\sqrt{7}}{8}$$ Evaluate: $\sqrt{64} = 8$.

b. $$\sqrt{\frac{15}{49x^2}} = \frac{\sqrt{15}}{\sqrt{49x^2}}$$ The square root of a quotient is equal to the quotient of the square roots.

$$= \frac{\sqrt{15}}{7x}$$ Simplify the denominator: $\sqrt{49x^2} = 7x$.

c. We can write the cube root of the quotient as the quotient of two cube roots. Since $y \neq 0$, we have

$$\sqrt[3]{\frac{10x^2}{27y^6}} = \frac{\sqrt[3]{10x^2}}{\sqrt[3]{27y^6}}$$ The cube root of a quotient is equal to the quotient of cube roots.

$$= \frac{\sqrt[3]{10x^2}}{3y^2}$$ Simplify the denominator.

Self Check 4

Simplify:

a. $\sqrt{\frac{11}{36a^2}}$ $\frac{\sqrt{11}}{6a}$

b. $\sqrt[4]{\frac{a^3}{625y^{12}}}$ $\frac{\sqrt[4]{a^3}}{5y^3}$

Assume that all variables represent positive real numbers.

Now Try **Problems 42 and 46**

Teaching Example 4 Simplify: **a.** $\sqrt{\frac{13}{36}}$ **b.** $\sqrt{\frac{22}{81x^2}}$ **c.** $\sqrt[3]{\frac{5a^2}{8b^9}}$.

Assume that the variables represent positive real numbers.

Answers:

a. $\frac{\sqrt{13}}{6}$ **b.** $\frac{\sqrt{22}}{9x}$ **c.** $\frac{\sqrt[3]{5a^2}}{2b^3}$

EXAMPLE 5 Simplify each expression. Assume that all variables represent positive numbers. **a.** $\frac{\sqrt{45xy^2}}{\sqrt{5x}}$ **b.** $\frac{\sqrt[3]{-432x^5}}{\sqrt[3]{8x}}$

Strategy We will use the quotient rule in reverse: $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

WHY When the radicands are written under a single radical symbol, the result is a rational expression. Our hope is that the rational expression can be simplified.

Self Check 5

Simplify each expression. Assume that all variables represent positive numbers.

a. $\frac{\sqrt{50ab^2}}{\sqrt{2a}}$ $5b$

b. $\frac{\sqrt[3]{-2{,}000x^5y^3}}{\sqrt[3]{2x}}$ $-10xy\sqrt[3]{x}$

Now Try **Problems 49 and 53**

Teaching Example 5 Simplify each expression. Assume that all variables represent positive numbers.

a. $\dfrac{\sqrt{98x^2y}}{\sqrt{2y}}$ **b.** $\dfrac{\sqrt[3]{-375x^{11}}}{\sqrt[3]{3x^2}}$

Answers:

a. $7x$ **b.** $-5x^3$

Solution

a. We can write the quotient of the square roots as the square root of a quotient.

$$\begin{aligned} \frac{\sqrt{45xy^2}}{\sqrt{5x}} &= \sqrt{\frac{45xy^2}{5x}} && \text{Use the division property of radicals.} \\ &= \sqrt{9y^2} && \text{Simplify } \tfrac{45xy^2}{5x}. \\ &= 3y && \text{Simplify the radical.} \end{aligned}$$

b. We can write the quotient of the cube roots as the cube root of a quotient.

$$\begin{aligned} \frac{\sqrt[3]{-432x^5}}{\sqrt[3]{8x}} &= \sqrt[3]{\frac{-432x^5}{8x}} && \text{Use the division property of radicals.} \\ &= \sqrt[3]{-54x^4} && \text{Simplify } \tfrac{-432x^5}{8x}. \\ &= \sqrt[3]{-27x^3 \cdot 2x} && -27x^3 \text{ is the largest perfect cube that divides } -54x^4. \\ &= \boldsymbol{\sqrt[3]{-27x^3}}\sqrt[3]{2x} && \text{Use the multiplication property of radicals.} \\ &= \boldsymbol{-3x}\sqrt[3]{2x} && \text{Simplify: } \sqrt[3]{-27x^3} = -3x. \end{aligned}$$

4 Add and subtract radical expressions.

Radical expressions with the same index and the same radicand are called **like** or **similar radicals.** For example, $3\sqrt{2}$ and $2\sqrt{2}$ are like radicals. However,

- $3\sqrt{\mathbf{5}}$ and $4\sqrt{\mathbf{2}}$ are not like radicals, because the radicands are different.
- $3\sqrt[4]{5}$ and $2\sqrt[3]{5}$ are not like radicals, because the indexes are different.

For a given expression containing two or more radical terms, we should attempt to combine like radicals, if possible. For example, to simplify the expression $3\sqrt{2} + 2\sqrt{2}$, we use the distributive property to factor out $\sqrt{2}$ and simplify.

$$\begin{aligned} \mathbf{3}\sqrt{2} + \mathbf{2}\sqrt{2} &= (\mathbf{3} + \mathbf{2})\sqrt{2} \\ &= 5\sqrt{2} \end{aligned}$$

Radicals with the same index but different radicands can often be written as like radicals. For example, to simplify the expression $\sqrt{27} - \sqrt{12}$, we simplify both radicals first, and then we combine the like radicals.

$$\begin{aligned} \sqrt{27} - \sqrt{12} &= \sqrt{9 \cdot 3} - \sqrt{4 \cdot 3} && \text{Write 27 and 12 in factored form.} \\ &= \sqrt{9}\sqrt{3} - \sqrt{4}\sqrt{3} && \text{Use the multiplication property of radicals.} \\ &= \mathbf{3}\sqrt{3} - \mathbf{2}\sqrt{3} && \text{Simplify } \sqrt{9} \text{ and } \sqrt{4}. \\ &= (\mathbf{3} - \mathbf{2})\sqrt{3} && \text{Factor out } \sqrt{3}. \\ &= \sqrt{3} && 1\sqrt{3} = \sqrt{3}. \end{aligned}$$

As the previous examples suggest, we can use the following rule to add or subtract radicals.

Adding and Subtracting Radicals

To add or subtract radicals, simplify each radical expression and combine all like radicals. To add or subtract like radicals, combine the coefficients and keep the common radical.

EXAMPLE 6 Simplify: $2\sqrt{12} - 3\sqrt{48} + 3\sqrt{3}$

Strategy Since the radicals are unlike radicals, we cannot add or subtract them in their current form. However, we will simplify the radicals and hope that like radicals result.

WHY Like radicals can be combined.

Solution

We simplify $2\sqrt{12}$ and $3\sqrt{48}$ separately and then combine like radicals.

$$\begin{aligned} 2\sqrt{12} - 3\sqrt{48} + 3\sqrt{3} &= 2\sqrt{4 \cdot 3} - 3\sqrt{16 \cdot 3} + 3\sqrt{3} \\ &= 2\sqrt{4}\sqrt{3} - 3\sqrt{16}\sqrt{3} + 3\sqrt{3} \\ &= 2(2)\sqrt{3} - 3(4)\sqrt{3} + 3\sqrt{3} \\ &= 4\sqrt{3} - 12\sqrt{3} + 3\sqrt{3} && \text{All three expressions have the same index and radicand.} \\ &= (4 - 12 + 3)\sqrt{3} && \text{Combine the coefficients of these like radicals and keep } \sqrt{3}. \\ &= -5\sqrt{3} \end{aligned}$$

Self Check 6

Simplify:
$3\sqrt{75} - 2\sqrt{12} + 2\sqrt{48}$ $19\sqrt{3}$

Now Try **Problem 59**

Teaching Example 6 Simplify:
$5\sqrt{50} - 3\sqrt{98} - \sqrt{72}$
Answer:
$-2\sqrt{2}$

EXAMPLE 7 Simplify: $\sqrt[3]{16} - \sqrt[3]{54} + \sqrt[3]{24}$

Strategy Since the radicals are unlike radicals, we cannot add or subtract them in their current form. However, we will simplify the radicals and hope that like radicals result.

WHY Like radicals can be combined.

Solution

We begin by simplifying each radical expression separately:

$$\begin{aligned} \sqrt[3]{16} - \sqrt[3]{54} + \sqrt[3]{24} &= \sqrt[3]{8 \cdot 2} - \sqrt[3]{27 \cdot 2} + \sqrt[3]{8 \cdot 3} \\ &= \sqrt[3]{8}\sqrt[3]{2} - \sqrt[3]{27}\sqrt[3]{2} + \sqrt[3]{8}\sqrt[3]{3} \\ &= 2\sqrt[3]{2} - 3\sqrt[3]{2} + 2\sqrt[3]{3} \end{aligned}$$

Now we combine the two radical expressions that have the same index and radicand.

$$\sqrt[3]{16} - \sqrt[3]{54} + \sqrt[3]{24} = -\sqrt[3]{2} + 2\sqrt[3]{3} \qquad 2\sqrt[3]{2} - 3\sqrt[3]{2} = -1\sqrt[3]{2} = -\sqrt[3]{2}.$$

Self Check 7

Simplify: $\sqrt[3]{24} - \sqrt[3]{16} + \sqrt[3]{54}$

Now Try **Problem 70**

Self Check 7 Answer
$2\sqrt[3]{3} + \sqrt[3]{2}$

Teaching Example 7 Simplify:
$\sqrt[3]{40} + 2\sqrt[3]{135} - \sqrt[3]{250}$
Answer:
$8\sqrt[3]{5} - 5\sqrt[3]{2}$

Caution! Even though the radical expressions $-\sqrt[3]{2}$ and $2\sqrt[3]{3}$ in the last line of Example 7 have the same index, we cannot combine them, because their radicands are different. Neither can we combine radical expressions having the same radicand but a different index. For example, the expression $\sqrt[3]{2} + \sqrt[4]{2}$ cannot be simplified.

EXAMPLE 8 Simplify: $\sqrt[3]{16x^4} + \sqrt[3]{54x^4} - \sqrt[3]{-128x^4}$

Strategy Since the radicals are unlike radicals, we cannot add or subtract them in their current form. However, we will simplify the radicals and hope that like radicals result.

Self Check 8

Simplify:
$\sqrt{32x^3} + \sqrt{50x^3} - \sqrt{18x^3}$

Now Try **Problem 77**

Self Check 8 Answer
$6x\sqrt{2x}$

Teaching Example 8 Simplify:
$5\sqrt[4]{48x^7} - 2x\sqrt[4]{3x^3} + \sqrt[4]{243x^7}$
Answer:
$11x\sqrt[4]{3x^3}$

WHY Like radicals can be combined.

Solution

We simplify each radical expression separately, factor out $\sqrt[3]{2x}$, and simplify.

$$\sqrt[3]{16x^4} + \sqrt[3]{54x^4} - \sqrt[3]{-128x^4}$$
$$= \sqrt[3]{8x^3 \cdot 2x} + \sqrt[3]{27x^3 \cdot 2x} - \sqrt[3]{-64x^3 \cdot 2x}$$
$$= \sqrt[3]{8x^3}\sqrt[3]{2x} + \sqrt[3]{27x^3}\sqrt[3]{2x} - \sqrt[3]{-64x^3}\sqrt[3]{2x}$$
$$= 2x\sqrt[3]{2x} + 3x\sqrt[3]{2x} + 4x\sqrt[3]{2x}$$ All three radicals have the same index and radicand.
$$= (2x + 3x + 4x)\sqrt[3]{2x}$$ Combine like radicals.
$$= 9x\sqrt[3]{2x}$$ Within the parentheses, combine like terms.

ANSWERS TO SELF CHECKS

1. a. $2\sqrt{5}$ **b.** $2\sqrt[3]{3}$ **c.** $2\sqrt[5]{4}$ **2. a.** $7b\sqrt{2b}$ **b.** $-3y\sqrt[3]{2y^2}$ **c.** $t^2u^3\sqrt[4]{u^3}$
3. a. $5\sqrt{11}$ **b.** $3cd\sqrt[3]{7c}$ **4. a.** $\frac{\sqrt{11}}{6a}$ **b.** $\frac{\sqrt[4]{a^3}}{5y^3}$ **5. a.** $5b$ **b.** $-10xy\sqrt[3]{x}$
6. $19\sqrt{3}$ **7.** $2\sqrt[3]{3} + \sqrt[3]{2}$ **8.** $6x\sqrt{2x}$

SECTION 7.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. Radical expressions such as $\sqrt[3]{4}$ and $6\sqrt[3]{4}$ with the same index and the same radicand are called _like_ radicals.
2. Numbers such as 1, 4, 9, 16, 25, and 36 are called perfect _squares_. Numbers such as 1, 8, 27, 64, and 125 are called perfect _cubes_.
3. The largest perfect-square _factor_ of 27 is 9.
4. "To _simplify_ $\sqrt{24}$" means to write it as $2\sqrt{6}$.

CONCEPTS

Fill in the blanks.

5. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

   In words, the nth root of the _product_ of two numbers is equal to the product of their nth _roots_.

6. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

   In words, the nth root of the _quotient_ of two numbers is equal to the quotient of their nth _roots_.

7. Consider the expressions $\sqrt{4 \cdot 5}$ and $\sqrt{4}\sqrt{5}$. Which expression is
   a. the square root of a product? $\sqrt{4 \cdot 5}$
   b. the product of square roots? $\sqrt{4}\sqrt{5}$
   c. How are these two expressions related? $\sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5}$

8. Consider the expressions $\frac{\sqrt[3]{a}}{\sqrt[3]{x^2}}$ and $\sqrt[3]{\frac{a}{x^2}}$. Which expression is
   a. the cube root of a quotient? $\sqrt[3]{\frac{a}{x^2}}$
   b. the quotient of cube roots? $\frac{\sqrt[3]{a}}{\sqrt[3]{x^2}}$
   c. How are these two expressions related? $\sqrt[3]{\frac{a}{x^2}} = \frac{\sqrt[3]{a}}{\sqrt[3]{x^2}}$

9. a. Write two radical expressions that have the same radicand but a different index. Can the expressions be added?
   $\sqrt{5}$, $\sqrt[3]{5}$ (answers may vary); no
   b. Write two radical expressions that have the same index but a different radicand. Can the expressions be added?
   $\sqrt{5}$, $\sqrt{6}$ (answers may vary); no

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▶ **10.** Explain the mistake.

$$\begin{aligned}\sqrt[3]{54} &= \sqrt[3]{27+27}\\ &= \sqrt[3]{27}+\sqrt[3]{27}\\ &= 3+3\\ &= 6\end{aligned}$$

The second step is incorrect. The cube root of a sum is not equal to the sum of the cube roots.

NOTATION

Complete each solution.

▶ **11.** $\sqrt[3]{32k^4} = \sqrt[3]{\boxed{8k^3} \cdot 4k}$
$= \sqrt[3]{\boxed{8k^3}}\sqrt[3]{4k}$
$= 2k\sqrt[3]{\boxed{4k}}$

▶ **12.** $\dfrac{\sqrt{80s^2t^4}}{\sqrt{5s^2}} = \sqrt{\dfrac{80s^2t^4}{\boxed{5s^2}}}$
$= \sqrt{\boxed{16t^4}}$
$= 4\boxed{t^2}$

GUIDED PRACTICE

Simplify each expression. See Example 1.

13. $\sqrt{20}$ $2\sqrt{5}$
14. $\sqrt{8}$ $2\sqrt{2}$
15. $-\sqrt{200}$ $-10\sqrt{2}$
16. $-\sqrt{250}$ $-5\sqrt{10}$
17. $\sqrt[3]{80}$ $2\sqrt[3]{10}$
▶ **18.** $\sqrt[3]{270}$ $3\sqrt[3]{10}$
19. $-\sqrt[4]{32}$ $-2\sqrt[4]{2}$
20. $\sqrt[4]{768}$ $4\sqrt[4]{3}$

Simplify each expression. Assume that all variables are positive numbers. See Example 2.

21. $\sqrt{a^7}$ $a^3\sqrt{a}$
22. $\sqrt{b^{11}}$ $b^5\sqrt{b}$
23. $\sqrt{50x^2}$ $5x\sqrt{2}$
▶ **24.** $\sqrt{75a^2}$ $5a\sqrt{3}$
25. $\sqrt{32b}$ $4\sqrt{2b}$
26. $\sqrt{80c}$ $4\sqrt{5c}$
27. $\sqrt[3]{-54x^6}$ $-3x^2\sqrt[3]{2}$
▶ **28.** $-\sqrt[3]{-81a^3}$ $3a\sqrt[3]{3}$
29. $\sqrt[5]{\dfrac{3x^{10}}{32}}$ $\dfrac{x^2\sqrt[5]{3}}{2}$
30. $\sqrt[6]{\dfrac{5x^{18}}{64}}$ $\dfrac{x^3\sqrt[6]{5}}{2}$
31. $\sqrt{175a^2b^3}$ $5ab\sqrt{7b}$
32. $\sqrt{128a^3b^5}$ $8ab^2\sqrt{2ab}$

Simplify each expression. Assume that all variables represent positive numbers. See Example 3.

33. $\sqrt{180}$ $6\sqrt{5}$
34. $\sqrt{112}$ $4\sqrt{7}$
35. $\sqrt[3]{16y^4}$ $2y\sqrt[3]{2y}$
36. $\sqrt[3]{40b^7}$ $2b^2\sqrt[3]{5b}$
37. $-\sqrt{300xy}$ $-10\sqrt{3xy}$
▶ **38.** $\sqrt{200x^2y}$ $10x\sqrt{2y}$
39. $\sqrt[4]{32x^{12}y^5}$ $2x^3y\sqrt[4]{2y}$
40. $\sqrt[5]{64x^{10}y^6}$ $2x^2y\sqrt[5]{2y}$

Simplify each expression. Assume that all variables represent positive numbers. See Example 4.

41. $\sqrt{\dfrac{7}{9}}$ $\dfrac{\sqrt{7}}{3}$
▶ **42.** $\sqrt{\dfrac{3}{4}}$ $\dfrac{\sqrt{3}}{2}$
43. $\sqrt[3]{\dfrac{7}{64}}$ $\dfrac{\sqrt[3]{7}}{4}$
44. $\sqrt[3]{\dfrac{4}{125}}$ $\dfrac{\sqrt[3]{4}}{5}$
45. $\sqrt{\dfrac{z^2}{16x^2}}$ $\dfrac{z}{4x}$
46. $\sqrt{\dfrac{b^4}{64a^8}}$ $\dfrac{b^2}{8a^4}$
47. $\sqrt[4]{\dfrac{5x}{16z^4}}$ $\dfrac{\sqrt[4]{5x}}{2z}$
48. $\sqrt[3]{\dfrac{11a^2}{125b^6}}$ $\dfrac{\sqrt[3]{11a^2}}{5b^2}$

Simplify each expression. Assume that all variables represent positive numbers. See Example 5.

49. $\dfrac{\sqrt{98x^3}}{\sqrt{2x}}$ $7x$
50. $\dfrac{\sqrt{75y^5}}{\sqrt{3y}}$ $5y^2$
51. $\dfrac{\sqrt{180ab^4}}{\sqrt{5ab^2}}$ $6b$
52. $\dfrac{\sqrt{112ab^3}}{\sqrt{7ab}}$ $4b$
53. $\dfrac{\sqrt[3]{96a^5}}{\sqrt[3]{6a}}$ $2a\sqrt[3]{2a}$
▶ **54.** $\dfrac{\sqrt[3]{128y^6}}{\sqrt[3]{8y^2}}$ $2y\sqrt[3]{2y}$
55. $\dfrac{\sqrt[3]{567a^4}}{\sqrt[3]{7a}}$ $3a\sqrt[3]{3}$
56. $\dfrac{\sqrt[3]{972x^7}}{\sqrt[3]{9x}}$ $3x^2\sqrt[3]{4}$

Simplify each expression. See Example 6.

57. $\sqrt{98} - \sqrt{50}$ $2\sqrt{2}$
58. $\sqrt{72} - \sqrt{200}$ $-4\sqrt{2}$
59. $3\sqrt{24} + \sqrt{54}$ $9\sqrt{6}$
▶ **60.** $\sqrt{18} + 2\sqrt{50}$ $13\sqrt{2}$
▶ **61.** $\sqrt{20} + \sqrt{125} - \sqrt{80}$ $3\sqrt{5}$
62. $\sqrt{98} - \sqrt{50} - \sqrt{72}$ $-4\sqrt{2}$
63. $\sqrt{63} + \sqrt{72} - \sqrt{28}$ $\sqrt{7} + 6\sqrt{2}$
64. $\sqrt{80} + \sqrt{45} - \sqrt{27}$ $7\sqrt{5} - 3\sqrt{3}$

Simplify each expression. See Example 7.

65. $\sqrt[3]{32} - \sqrt[3]{108}$ $-\sqrt[3]{4}$
66. $\sqrt[3]{80} - \sqrt[3]{10{,}000}$ $-8\sqrt[3]{10}$
67. $2\sqrt[3]{125} - 5\sqrt[3]{64}$ -10
68. $3\sqrt[3]{27} + 12\sqrt[3]{216}$ 81
69. $2\sqrt[3]{16} - \sqrt[3]{54}$ $\sqrt[3]{2}$
▶ **70.** $2\sqrt[3]{250} - 4\sqrt[3]{5} + \sqrt[3]{16}$ $12\sqrt[3]{2} - 4\sqrt[3]{5}$
71. $\sqrt[4]{48} - \sqrt[4]{243} - \sqrt[4]{768}$ $-5\sqrt[4]{3}$
72. $\sqrt[4]{32} + 5\sqrt[4]{2} - \sqrt[4]{162}$ $4\sqrt[4]{2}$

Simplify each expression. Assume all variables represent positive numbers. See Example 8.

73. $4\sqrt{2x} + 6\sqrt{2x}$ $10\sqrt{2x}$
▶ **74.** $16\sqrt[3]{7y} + 3\sqrt[3]{7y}$ $19\sqrt[3]{7y}$
75. $\sqrt{18t} + \sqrt{300t} - \sqrt{243t}$ $3\sqrt{2t} + \sqrt{3t}$

76. $\sqrt{80m} - \sqrt{128m} + \sqrt{288m}$ $4\sqrt{5m} + 4\sqrt{2m}$

77. $\sqrt[3]{24x} + \sqrt[3]{3x}$ $3\sqrt[3]{3x}$

78. $\sqrt[3]{16y} + \sqrt[3]{128y}$ $6\sqrt[3]{2y}$

79. $\sqrt{50a^2} + 2a\sqrt{8} + 2\sqrt{200a^2}$ $29a\sqrt{2}$

80. $\sqrt[3]{54a^6} - 3\sqrt[3]{16a^6} + 4\sqrt[3]{128a^6}$ $13a^2\sqrt[3]{2}$

TRY IT YOURSELF

Simplify each expression. Assume that all variables represent positive numbers.

81. $\sqrt[7]{m^{19}}$ $m^2\sqrt[7]{m^5}$

82. $\sqrt[5]{a^{14}}$ $a^2\sqrt[5]{a^4}$

83. $\sqrt[3]{a^5b^{16}}$ $ab^5\sqrt[3]{a^2b}$

84. $\sqrt[4]{81x^{12}y^9}$ $3x^3y^2\sqrt[4]{y}$

85. $\sqrt[3]{250}$ $5\sqrt[3]{2}$

86. $\sqrt[5]{96x^{12}}$ $2x^2\sqrt[5]{3x^2}$

87. $\dfrac{\sqrt{500}}{\sqrt{5}}$ 10

88. $\dfrac{\sqrt{128}}{\sqrt{2}}$ 8

89. $\sqrt[3]{-81}$ $-3\sqrt[3]{3}$

90. $\sqrt[3]{-72}$ $-2\sqrt[3]{9}$

91. $\sqrt[5]{96}$ $2\sqrt[5]{3}$

92. $\sqrt[7]{256}$ $2\sqrt[7]{2}$

93. $\sqrt[4]{\dfrac{3}{10{,}000}}$ $\dfrac{\sqrt[4]{3}}{10}$

94. $\sqrt[5]{\dfrac{4}{243}}$ $\dfrac{\sqrt[5]{4}}{3}$

95. $-\sqrt{112a^3}$ $-4a\sqrt{7a}$

96. $\sqrt{147a^5}$ $7a^2\sqrt{3a}$

Simplify and combine like radicals. All variables represent positive numbers.

97. $4\sqrt{2x} + 6\sqrt{2x}$ $10\sqrt{2x}$

98. $6\sqrt[3]{5y} + 3\sqrt[3]{5y}$ $9\sqrt[3]{5y}$

99. $8\sqrt[5]{7a^2} - 7\sqrt[5]{7a^2}$ $\sqrt[5]{7a^2}$

100. $10\sqrt[6]{12xyz} - \sqrt[6]{12xyz}$ $9\sqrt[6]{12xyz}$

101. $\sqrt{2} - \sqrt{8}$ $-\sqrt{2}$

102. $\sqrt{20} - \sqrt{125}$ $-3\sqrt{5}$

103. $14\sqrt[4]{32} - 15\sqrt[4]{162}$ $-17\sqrt[4]{2}$

104. $23\sqrt[4]{768} + \sqrt[4]{48}$ $94\sqrt[4]{3}$

105. $3\sqrt[4]{512} + 2\sqrt[4]{32}$ $16\sqrt[4]{2}$

106. $4\sqrt[4]{243} - \sqrt[4]{48}$ $10\sqrt[4]{3}$

107. $\sqrt{25y^2z} - \sqrt{16y^2z}$ $y\sqrt{z}$

108. $\sqrt{25yz^2} + \sqrt{9yz^2}$ $8z\sqrt{y}$

109. $\sqrt{36xy^2} + \sqrt{49xy^2}$ $13y\sqrt{x}$

110. $3\sqrt{2x} - \sqrt{8x}$ $\sqrt{2x}$

111. $2\sqrt[3]{64a} + 2\sqrt[3]{8a}$ $12\sqrt[3]{a}$

112. $3\sqrt[4]{x^4y} - 2\sqrt[4]{x^4y}$ $x\sqrt[4]{y}$

113. $\sqrt{y^5} - \sqrt{9y^5} - \sqrt{25y^5}$ $-7y^2\sqrt{y}$

114. $\sqrt{8y^7} + \sqrt{32y^7} - \sqrt{2y^7}$ $5y^3\sqrt{2y}$

115. $-2\sqrt[5]{x^6y^2} - \sqrt[5]{32x^6y^2} + \sqrt[5]{x^6y^2}$ $-3x\sqrt[5]{xy^2}$

116. $\sqrt[3]{xy^4} + \sqrt[3]{8xy^4} - \sqrt[3]{27xy^4}$ 0

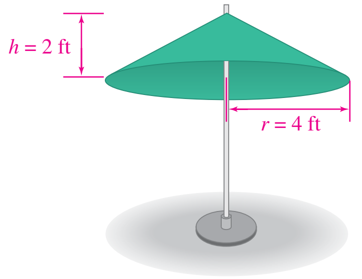
117. $\sqrt{18t} + \sqrt{300t} - \sqrt{243t}$ $3\sqrt{2t} + \sqrt{3t}$

118. $\sqrt{80m} - \sqrt{128m} + \sqrt{288m}$ $4\sqrt{5m} + 4\sqrt{2m}$

APPLICATIONS

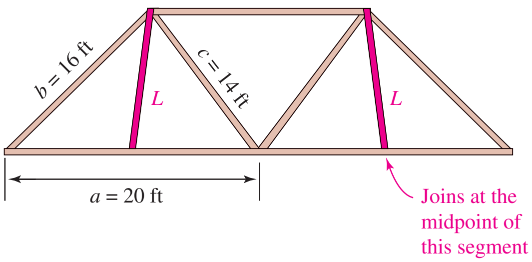
First give the exact answer, expressed as a simplified radical expression. Then give an approximation, rounded to the nearest tenth.

119. UMBRELLAS The surface area of a cone is given by the formula $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is its height. Use this formula to find the number of square feet of waterproof cloth used to make the umbrella. $8\pi\sqrt{5}$ ft^2; 56.2 ft^2



120. STRUCTURAL ENGINEERING Engineers have determined that two additional supports need to be added to strengthen a truss. Find the length L of each new support using the formula

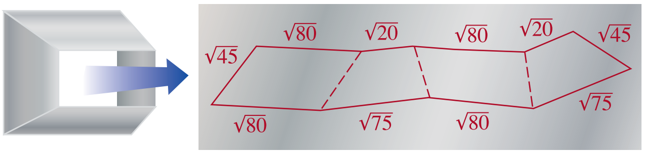
$$L = \sqrt{\frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}}$$ $3\sqrt{14}$ ft; 11.2 ft



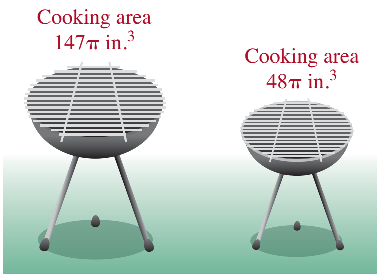
121. BLOW DRYERS The current I (in amps), the power P (in watts), and the resistance R (in ohms) are related by the formula $I = \sqrt{\frac{P}{R}}$. What current is needed for a 1,200-watt hair dryer if the resistance is 16 ohms? $5\sqrt{3}$ amps; 8.7 amps

122. SATELLITES Engineers have determined that a spherical communications satellite needs to have a capacity of 565.2 cubic feet to house all of its operating systems. The volume V of a sphere is related to its radius r by the formula $r = \sqrt[3]{\frac{3V}{4\pi}}$. What radius must the satellite have to meet the engineer's specification? Use 3.14 for π. $3\sqrt[3]{5}$ ft; 5.1 ft

▶ **123.** DUCTWORK The pattern shown below is laid out on a sheet of galvanized tin. Then it is cut out with snips and bent along the dotted lines to make an air conditioning duct connection. Find the total length of the cut that must be made with the tin snips. (All measurements are in inches.) $\left(26\sqrt{5} + 10\sqrt{3}\right)$ in.; 75.5 in.



▶ **124.** OUTDOOR COOKING The diameter of a circle is given by the function $d(A) = 22\sqrt{\frac{A}{\pi}}$, where A is the area of the circle. Find the difference between the diameters of the barbecue grills. $6\sqrt{3}$ in.; 10.4 in.

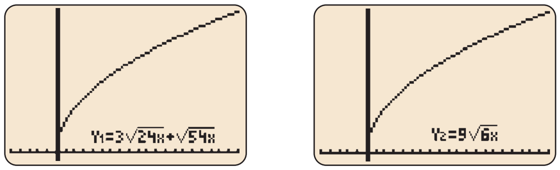


WRITING

125. Explain why $\sqrt[3]{9x^4}$ is not in simplified form.

▶ **126.** How are the procedures used to simplify $3x + 4x$ and $3\sqrt{x} + 4\sqrt{x}$ similar?

127. Explain how the graphs of $Y_1 = 3\sqrt{24x} + \sqrt{54x}$ and $Y_2 = 9\sqrt{6x}$ can be used to verify the simplification $3\sqrt{24x} + \sqrt{54x} = 9\sqrt{6x}$. In each graph, settings of $[-5, 20]$ for x and $[-5, 100]$ for y were used.



(a) (b)

128. Explain how to verify algebraically that $\sqrt{200x^3y^5} = 10xy^2\sqrt{2xy}$.

REVIEW

Perform each operation.

129. $3x^2y^3(-5x^3y^{-4})$ $-\frac{15x^5}{y}$

▶ **130.** $(2x^2 - 9x - 5) \cdot \dfrac{x}{2x^2 + x}$ $x - 5$

131. $2p - 5\overline{)6p^2 - 7p - 25}$ $3p + 4 - \frac{5}{2p - 5}$

132. $\dfrac{xy}{\frac{1}{x} - \frac{1}{y}}$ $\frac{x^2y^2}{y - x}$

SECTION 7.3
Multiplying and Dividing Radical Expressions

Objectives

1. Multiply radical expressions.
2. Find powers of radical expressions.
3. Rationalize denominators.
4. Rationalize denominators that have two terms.
5. Rationalize numerators.

In this section, we will discuss how to multiply and divide radical expressions. These problems often require the use of procedures and properties studied earlier, such as simplifying radical expressions, combining like radicals, the FOIL method, and the distributive property.

1 Multiply radical expressions.

We have used the *product rule for radicals* to write radical expressions in simplified form. We can also use this rule to multiply radical expressions that have the same index.

The Product Rule for Radicals

The product of the nth roots of two nonnegative numbers is equal to the nth root of the product of those numbers.

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

Self Check 1

Multiply and then simplify:

a. $-2\sqrt{14}\left(5\sqrt{2}\right)$

b. $\sqrt[4]{4x^3} \cdot 9\sqrt[4]{8x^2}$

Now Try **Problems 24 and 28**

Self Check 1 Answers

a. $-20\sqrt{7}$ **b.** $18x\sqrt[4]{2x}$

Teaching Example 1 Multiply and then simplify:

a. $\sqrt{5}\sqrt{10}$

b. $3\sqrt[3]{25x} \cdot \sqrt[3]{15x^2}$

Answers:

a. $5\sqrt{2}$

b. $15x\sqrt[3]{3}$

EXAMPLE 1

Multiply and then simplify:

a. $3\sqrt{6}\left(2\sqrt{3}\right)$ **b.** $-2\sqrt[3]{7x} \cdot 6\sqrt[3]{49x^2}$

Strategy In each case, we will multiply the coefficients and then use the product rule for radicals to multiply the factors of the form $\sqrt[n]{a}$ and $\sqrt[n]{b}$.

WHY The product rule for radicals is used to multiply radicals that have the same index.

Solution

We use the commutative and associative properties of multiplication to multiply the coefficients and the radicals separately. Then we simplify any radicals in the product, if possible.

a.

$3\sqrt{6}\left(2\sqrt{3}\right) = 3(2)\sqrt{6}\sqrt{3}$	Write the coefficients together and multiply the radicals together.
$= 6\sqrt{18}$	$3(2) = 6$ and $\sqrt{6}\sqrt{3} = \sqrt{18}$.
$= 6\sqrt{9}\sqrt{2}$	Simplify: $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2}$.
$= 6(3)\sqrt{2}$	Evaluate: $\sqrt{9} = 3$.
$= 18\sqrt{2}$	Multiply.

b.

$-2\sqrt[3]{7x} \cdot 6\sqrt[3]{49x^2} = -2(6)\sqrt[3]{7x}\sqrt[3]{49x^2}$	Write the coefficients together and the radicals together.
$= -12\sqrt[3]{7x \cdot 49x^2}$	Multiply the coefficients and multiply the radicals.
$= -12\sqrt[3]{7x \cdot 7^2x^2}$	Write 49 as 7^2.
$= -12\sqrt[3]{\mathbf{7^3x^3}}$	Write $7x \cdot 7^2x^2$ as 7^3x^3.
$= -12(\mathbf{7x})$	Simplify: $\sqrt[3]{7^3x^3} = 7x$.
$= -84x$	Multiply.

Recall that to multiply a polynomial by a monomial, we use the distributive property. We use the same technique to multiply a radical expression that has two or more terms by a radical expression that has only one term.

Self Check 2

Simplify: $4\sqrt{2}\left(3\sqrt{5} - 2\sqrt{8}\right)$

Now Try **Problem 32**

Self Check 2 Answer

$12\sqrt{10} - 32$

EXAMPLE 2

Simplify: $3\sqrt{3}\left(4\sqrt{8} - 5\sqrt{10}\right)$

Strategy We will use the distributive property and multiply each term within the parentheses by the term outside the parentheses.

WHY The given expression has the form $a(b - c)$.

Solution

$$3\sqrt{3}\left(4\sqrt{8} - 5\sqrt{10}\right)$$

$= 3\sqrt{3} \cdot 4\sqrt{8} - 3\sqrt{3} \cdot 5\sqrt{10}$ — Distribute the multiplication by $3\sqrt{3}$.

$= 12\sqrt{24} - 15\sqrt{30}$ — Multiply the coefficients and multiply the radicals.

$= 12\sqrt{4}\sqrt{6} - 15\sqrt{30}$ — Simplify: $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6}$.

$= 12(2)\sqrt{6} - 15\sqrt{30}$ — $\sqrt{4} = 2$.

$= 24\sqrt{6} - 15\sqrt{30}$

Teaching Example 2 Simplify: $2\sqrt{7}\left(4\sqrt{14} - 5\sqrt{21}\right)$
Answer:
$56\sqrt{2} - 70\sqrt{3}$

Recall that to multiply two binomials, we multiply each term of one binomial by each term of the other binomial and simplify. We multiply two radical expressions, each having two terms, in the same way.

EXAMPLE 3 Multiply: $\left(\sqrt{7} + \sqrt{2}\right)\left(\sqrt{7} - 3\sqrt{2}\right)$

Strategy As with binomials, we will multiply each term within the first set of parentheses by each term within the second set of parentheses.

WHY Within each parenthesis there are two terms. This is an application of the FOIL method for multiplying binomials.

Solution

$$\left(\sqrt{7} + \sqrt{2}\right)\left(\sqrt{7} - 3\sqrt{2}\right)$$

$= \sqrt{7}\sqrt{7} - 3\sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} - 3\sqrt{2}\sqrt{2}$ — Use the FOIL method.

$= 7 - 3\sqrt{14} + \sqrt{14} - 3(2)$ — Perform each multiplication: $\sqrt{7}\sqrt{7} = \sqrt{49} = 7$ and $\sqrt{2}\sqrt{2} = \sqrt{4} = 2$.

$= 7 - 2\sqrt{14} - 6$ — Combine like radicals: $-3\sqrt{14} + \sqrt{14} = -2\sqrt{14}$.

$= 1 - 2\sqrt{14}$ — Combine like terms: $7 - 6 = 1$.

Self Check 3

Multiply: $\left(\sqrt{5} + 2\sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)$

Now Try **Problem 34**

Self Check 3 Answer
$-1 + \sqrt{15}$

Teaching Example 3 Multiply: $\left(\sqrt{11} - 3\sqrt{2}\right)\left(4\sqrt{11} - \sqrt{2}\right)$
Answer:
$50 - 13\sqrt{22}$

EXAMPLE 4 Multiply $\left(\sqrt{3x} - \sqrt{5}\right)\left(\sqrt{2x} + \sqrt{10}\right)$. Assume that $x > 0$.

Strategy As with binomials, we will multiply each term within the first set of parentheses by each term within the second set of parentheses.

WHY Within each parenthesis there are two terms. This is an application of the FOIL method for multiplying binomials.

Solution

$$\left(\sqrt{3x} - \sqrt{5}\right)\left(\sqrt{2x} + \sqrt{10}\right)$$

$= \sqrt{3x}\sqrt{2x} + \sqrt{3x}\sqrt{10} - \sqrt{5}\sqrt{2x} - \sqrt{5}\sqrt{10}$ — Use the FOIL method.

$= \sqrt{6x^2} + \sqrt{30x} - \sqrt{10x} - \sqrt{50}$ — Perform each multiplication.

$= \sqrt{6}\sqrt{x^2} + \sqrt{30x} - \sqrt{10x} - \sqrt{25}\sqrt{2}$ — Simplify $\sqrt{6x^2}$ and $\sqrt{50}$.

$= \sqrt{6}x + \sqrt{30x} - \sqrt{10x} - 5\sqrt{2}$

Self Check 4

Multiply $\left(\sqrt{x} + 1\right)\left(\sqrt{x} - 3\right)$. Assume that $x > 0$.

Now Try **Problem 40**

Self Check 4 Answer
$x - 2\sqrt{x} - 3$

Teaching Example 4 Multiply $\left(\sqrt{7x} + \sqrt{3}\right)\left(\sqrt{x} - \sqrt{5}\right)$. Assume that $x > 0$.
Answer:
$x\sqrt{7} - \sqrt{35x} + \sqrt{3x} - \sqrt{15}$

Success Tip It is important to draw the radical sign carefully so that it completely covers the radicand, but no more than the radicand. To avoid confusion, we often write an expression such as $\sqrt{6x}$ in the form $x\sqrt{6}$.

2 Find powers of radical expressions.

To find the power of a radical expression, such as $\left(\sqrt{5}\right)^2$ or $\left(\sqrt[3]{2}\right)^3$, we can use the definition of exponent and the product rule for radicals.

$$\begin{aligned}\left(\sqrt{5}\right)^2 &= \sqrt{5}\sqrt{5} \\ &= \sqrt{25} \\ &= 5\end{aligned} \qquad \begin{aligned}\left(\sqrt[3]{2}\right)^3 &= \sqrt[3]{2}\cdot\sqrt[3]{2}\cdot\sqrt[3]{2} \\ &= \sqrt[3]{8} \\ &= 2\end{aligned}$$

These results illustrate the following property of radicals.

The *n*th Power of the *n*th Root

If $\sqrt[n]{a}$ is a real number,

$$\left(\sqrt[n]{a}\right)^n = a$$

Self Check 5

Find:

a. $\left(\sqrt{11}\right)^2$

b. $\left(3\sqrt[3]{4y}\right)^3$

c. $\left(\sqrt{x-8}-5\right)^2$

Now Try **Problems 42, 46, and 48**

Self Check 5 Answers

a. 11 **b.** $108y$

c. $x - 10\sqrt{x-8} + 17$

Teaching Example 5 Find:

a. $\left(\sqrt{13}\right)^2$ **b.** $\left(4\sqrt[3]{2x^2}\right)^3$

c. $\left(\sqrt{x}+3\right)^2$

Answers:

a. 13 **b.** $128x^2$ **c.** $x + 6\sqrt{x} + 9$

EXAMPLE 5 Find: **a.** $\left(\sqrt{5}\right)^2$ **b.** $\left(2\sqrt[3]{7x^2}\right)^3$ **c.** $\left(\sqrt{m+1}+2\right)^2$

Strategy In part a, we will use the definition of square root. In part b, we will use a power rule for exponents. In part c, we will use the FOIL method.

WHY Part a is the square of a square root, part b has the form $(xy)^n$, and part c has the form $(x + y)^2$.

Solution

a. $\left(\sqrt{5}\right)^2 = 5$ Because the square of the square root of 5 is 5.

b. We can use the power of a product rule for exponents to find $\left(2\sqrt[3]{7x^2}\right)^3$.

$$\begin{aligned}\left(2\sqrt[3]{7x^2}\right)^3 &= 2^3\left(\sqrt[3]{7x^2}\right)^3 && \text{Raise each factor of } 2\sqrt[3]{7x^2} \text{ to the 3rd power.} \\ &= 8(7x^2) && \text{Evaluate: } 2^3 = 8. \text{ Use } \left(\sqrt[n]{a}\right)^n = a. \\ &= 56x^2\end{aligned}$$

c. We can use the FOIL method to find the product.

$$\begin{aligned}\left(\sqrt{m+1}+2\right)^2 &= \left(\sqrt{m+1}+2\right)\left(\sqrt{m+1}+2\right) \\ &= \left(\sqrt{m+1}\right)^2 + 2\sqrt{m+1} + 2\sqrt{m+1} + 2\cdot 2 \\ &= m + 1 + 2\sqrt{m+1} + 2\sqrt{m+1} + 4 && \text{Use } \left(\sqrt[n]{a}\right)^n = a. \\ &= m + 4\sqrt{m+1} + 5 && \text{Combine like terms.}\end{aligned}$$

3 Rationalize denominators.

In Section 7.2, we saw that a radical expression is in simplified form when each of the following statements is true.

1. Each factor in the radicand is to a power that is less than the index of the radical.
2. The radicand contains no fractions or negative numbers.
3. No radicals appear in the denominator of a fraction.

We now consider radical expressions that do not satisfy requirement 2 and radical expressions that do not satisfy requirement 3 of this list. We will introduce an algebraic technique, called *rationalizing the denominator,* that is used to write such expressions in an equivalent simplified form.

To divide radical expressions, we **rationalize the denominator** of a fraction to replace the denominator with a rational number. For example, to divide $\sqrt{5}$ by $\sqrt{3}$, we write the division as the fraction

$$\frac{\sqrt{5}}{\sqrt{3}}$$ The denominator is the irrational number $\sqrt{3}$. This radical expression is not in simplified form, because a radical appears in the denominator.

To eliminate the radical in the denominator, we multiply the numerator and the denominator by a number that will give a perfect square *under the radical in the denominator.* Because $3 \cdot 3 = 9$ and 9 is a perfect square, $\sqrt{3}$ is such a number.

$$\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ To build an equivalent fraction, multiply by $\frac{\sqrt{3}}{\sqrt{3}} = 1$.

$$= \frac{\sqrt{15}}{\sqrt{9}}$$ Multiply the numerators: $\sqrt{5} \cdot \sqrt{3} = \sqrt{15}$. Multiply the denominators: $\sqrt{3} \cdot \sqrt{3} = \sqrt{9}$.

$$= \frac{\sqrt{15}}{3}$$ Simplify: $\sqrt{9} = 3$. The denominator is now the rational number 3.

Thus, $\frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{15}}{3}$. Since there is no radical in the denominator and $\sqrt{15}$ cannot be simplified, the expression $\frac{\sqrt{15}}{3}$ is in simplest form, and the division is complete.

Success Tip As an informal check, we can use a calculator to evaluate each expression.

$$\frac{\sqrt{5}}{\sqrt{3}} \approx 1.290994449$$

$$\frac{\sqrt{15}}{3} \approx 1.290994449$$

EXAMPLE 6

Rationalize each denominator: **a.** $\sqrt{\frac{20}{7}}$ **b.** $\frac{4}{\sqrt[3]{2}}$

Strategy In part a, we will examine the radicand in the denominator and ask, "By what must we multiply it to obtain a perfect square?" In part b, we will examine the radicand in the denominator and ask, "By what must we multiply it to obtain a perfect cube?"

WHY The answers to those questions will determine what form of 1 we use to rationalize each denominator.

Solution

a. This radical expression is not in simplified form, because the radicand contains a fraction. We begin by writing the square root of the quotient as the quotient of two square roots:

$$\sqrt{\frac{20}{7}} = \frac{\sqrt{20}}{\sqrt{7}}$$ Apply the division property of radicals: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Self Check 6

Rationalize each denominator:

a. $\sqrt{\frac{24}{5}}$ **b.** $\frac{5}{\sqrt[4]{3}}$

Now Try Problems 52 and 56

Self Check 6 Answers

a. $\frac{2\sqrt{30}}{5}$ **b.** $\frac{5\sqrt[4]{27}}{3}$

Teaching Example 6 Rationalize each denominator:

a. $\sqrt{\frac{12}{11}}$ **b.** $\frac{7}{\sqrt[3]{75}}$

Answers:

a. $\frac{2\sqrt{33}}{11}$ **b.** $\frac{7\sqrt[3]{45}}{15}$

To rationalize the denominator, we proceed as follows:

$$\frac{\sqrt{20}}{\sqrt{7}} = \frac{\sqrt{20}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$ Multiply by the form of 1 to rationalize the denominator.

$$= \frac{\sqrt{140}}{\sqrt{49}}$$ Multiply the radicals. This radicand is now a perfect square.

$$= \frac{2\sqrt{35}}{7}$$ Simplify: $\sqrt{140} = \sqrt{4 \cdot 35} = \sqrt{4}\sqrt{35} = 2\sqrt{35}$ and $\sqrt{49} = 7$.

b. Here, we must rationalize a denominator that is a cube root. We multiply the numerator and the denominator by a number that will give a perfect cube under the radical sign. Since $2 \cdot 4 = 8$ is a perfect cube, $\sqrt[3]{4}$ is such a number.

$$\frac{4}{\sqrt[3]{2}} = \frac{4}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$$ Multiply by a form of 1 to rationalize the denominator.

$$= \frac{4\sqrt[3]{4}}{\sqrt[3]{8}}$$ Multiply the radicals in the denominator. This radicand is now a perfect cube.

$$= \frac{4\sqrt[3]{4}}{2}$$ Simplify: $\sqrt[3]{8} = 2$.

$$= 2\sqrt[3]{4}$$ Simplify: $\frac{4\sqrt[3]{4}}{2} = \frac{\overset{1}{\cancel{2}} \cdot 2\sqrt[3]{4}}{\underset{1}{\cancel{2}}} = 2\sqrt[3]{4}$.

Self Check 7

Rationalize the denominator: $\frac{\sqrt{4ab^3}}{\sqrt{2a^2b^2}}$

Assume that $a > 0$ and $b > 0$. $\frac{\sqrt{2ab}}{a}$

Now Try **Problem 62**

Teaching Example 7 Rationalize the denominator: $\frac{\sqrt{6a^2b^5}}{\sqrt{3a^3b}}$

Answer: $\frac{b^2\sqrt{2a}}{a}$

EXAMPLE 7

Rationalize the denominator: $\frac{\sqrt{5xy^2}}{\sqrt{xy^3}}$

Assume that x and y are positive numbers.

Strategy We will begin by using the quotient rule for radicals in reverse $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.

WHY When the radicals are written under a single radical symbol, the result is a rational expression. Our hope is that the rational expression can be simplified, which could possibly make rationalizing the denominator easier.

Solution

There are two methods we can use to rationalize the denominator. In each method, we simplify the expression first.

Method 1

$$\frac{\sqrt{5xy^2}}{\sqrt{xy^3}} = \sqrt{\frac{5xy^2}{xy^3}}$$

$$= \sqrt{\frac{5}{y}}$$

$$= \frac{\sqrt{5}}{\sqrt{y}}$$

$$= \frac{\sqrt{5}}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}}$$ Multiply outside the radical.

$$= \frac{\sqrt{5y}}{y}$$

Method 2

$$\frac{\sqrt{5xy^2}}{\sqrt{xy^3}} = \sqrt{\frac{5xy^2}{xy^3}}$$

$$= \sqrt{\frac{5}{y}}$$

$$= \sqrt{\frac{5}{y} \cdot \frac{y}{y}}$$ Multiply within the radical.

$$= \frac{\sqrt{5y}}{\sqrt{y^2}}$$

$$= \frac{\sqrt{5y}}{y}$$

EXAMPLE 8 Rationalize the denominator and assume $q > 0$: $\sqrt{\dfrac{11}{20q^5}}$

Strategy We will begin by using the quotient rule for radicals $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

WHY When the radicals are written under separate radicals, our hope is that the radical in the denominator can be simplified, which could possibly make rationalizing the denominator easier.

Solution

We write the expression as a quotient of two radicals. Then we simplify the radical in the denominator before rationalizing.

$$\sqrt{\frac{11}{20q^5}} = \frac{\sqrt{11}}{\sqrt{20q^5}}$$ The square root of a quotient is the quotient of the square roots.

$$= \frac{\sqrt{11}}{\sqrt{4q^4 \cdot 5q}}$$ To simplify $\sqrt{20q^5}$, write it as $\sqrt{4q^4 \cdot 5q}$.

$$= \frac{\sqrt{11}}{2q^2\sqrt{5q}}$$ Simplify: $\sqrt{4q^4 \cdot 5q} = \sqrt{4q^4}\sqrt{5q} = 2q^2\sqrt{5q}$.

$$= \frac{\sqrt{11}}{2q^2\sqrt{5q}} \cdot \frac{\sqrt{5q}}{\sqrt{5q}}$$ To rationalize the denominator, multiply by $\dfrac{\sqrt{5q}}{\sqrt{5q}} = 1$.

$$= \frac{\sqrt{55q}}{2q^2(5q)}$$ Multiply the radicals: $\sqrt{5q}\sqrt{5q} = 5q$.

$$= \frac{\sqrt{55q}}{10q^3}$$ Multiply in the denominator.

Self Check 8

Rationalize the denominator: $\sqrt[3]{\dfrac{1}{16h^4}}$ $\dfrac{\sqrt[3]{4h^2}}{4h^2}$

Now Try **Problem 65**

Teaching Example 8 Rationalize the denominator: $\sqrt[3]{\dfrac{7}{18x}}$

Answer: $\dfrac{\sqrt[3]{84x^2}}{6x}$

EXAMPLE 9 Rationalize the denominator: $\dfrac{\sqrt[3]{5}}{\sqrt[3]{9m}}$

Strategy We will examine the radicand in the denominator and ask, "By what must we multiply it to obtain a perfect cube?"

WHY The answers to those questions will determine what form of 1 we use to rationalize each denominator.

Solution

We multiply the numerator and the denominator by $\sqrt[3]{3m^2}$, which will produce a perfect cube, $27m^3$, under the radical sign in the denominator.

$$\frac{\sqrt[3]{5}}{\sqrt[3]{9m}} = \frac{\sqrt[3]{5}}{\sqrt[3]{9m}} \cdot \frac{\sqrt[3]{3m^2}}{\sqrt[3]{3m^2}}$$ Multiply by the form of 1 to rationalize the denominator.

$$= \frac{\sqrt[3]{15m^2}}{\sqrt[3]{27m^3}}$$ Multiply the radicals.
This radicand is now a perfect cube.

$$= \frac{\sqrt[3]{15m^2}}{3m}$$ Simplify the denominator: $\sqrt[3]{27m^3} = 3m$.

Self Check 9

Rationalize the denominator: $\dfrac{\sqrt{5}}{\sqrt{17b}}$ $\dfrac{\sqrt{85b}}{17b}$

Now Try **Problem 70**

Teaching Example 9 Rationalize the denominator: $\dfrac{\sqrt[3]{2}}{\sqrt[3]{25a^2}}$

Answer: $\dfrac{\sqrt[3]{10a}}{5a}$

4 Rationalize denominators that have two terms.

So far, we have rationalized denominators that had only one term. We will now discuss a method to rationalize denominators that have two terms.

One-term denominators	**Two-term denominators**
$\frac{\sqrt{5}}{\sqrt{3}}$, $\frac{11}{\sqrt{20q^5}}$, $\frac{4}{\sqrt[3]{2}}$	$\frac{1}{\sqrt{2}+1}$, $\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$

To rationalize a denominator of $\frac{1}{\sqrt{2}+1}$, for example, we multiply the numerator and denominator by $\sqrt{2}-1$, because the product $(\sqrt{2}+1)(\sqrt{2}-1)$ contains no radicals.

$$\begin{aligned}(\sqrt{2}+1)(\sqrt{2}-1) &= (\sqrt{2})^2-(1)^2 && \text{Use a special product formula.}\\ &= 2-1\\ &= 1\end{aligned}$$

Radical expressions that involve the sum and difference of the same two terms, such as $\sqrt{2}+1$ and $\sqrt{2}-1$, are called **conjugates.**

Self Check 10

Rationalize the denominator:

a. $\frac{2}{\sqrt{3}+1}$ **b.** $\frac{\sqrt{x}-\sqrt{5}}{\sqrt{x}+\sqrt{5}}$

Self Check 10 Answers

a. $\sqrt{3}-1$ **b.** $\frac{x-2\sqrt{5x}+5}{x-5}$

Now Try Problems 74 and 80

Teaching Example 10 Rationalize the denominator:

a. $\frac{3}{\sqrt{5}-1}$ **b.** $\frac{\sqrt{x}-\sqrt{3}}{\sqrt{x}+\sqrt{3}}$

Answers:

a. $\frac{3\sqrt{5}+3}{4}$ **b.** $\frac{x-2\sqrt{3x}+3}{x-3}$

EXAMPLE 10 Rationalize the denominator: **a.** $\frac{1}{\sqrt{2}+1}$ **b.** $\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}}$

Strategy We will rationalize the denominator by multiplying the numerator and the denominator by the conjugate of the denominator.

WHY Multiplying the denominator by its conjugate will produce a new denominator that does not contain radicals.

Solution

a. We multiply the numerator and denominator of the fraction by $\sqrt{2}-1$, which is the conjugate of the denominator.

$$\begin{aligned}\frac{1}{\sqrt{2}+1} &= \frac{1}{(\sqrt{2}+1)}\cdot\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}\\ &= \frac{\sqrt{2}-1}{2-1} && \text{In the denominator, multiply the binomials: } (\sqrt{2}+1)(\sqrt{2}-1) = \sqrt{2}\sqrt{2}-\sqrt{2}+\sqrt{2}-1 = 2-1.\\ &= \sqrt{2}-1 && \text{Simplify: } \frac{\sqrt{2}-1}{2-1} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1.\end{aligned}$$

b. We multiply the numerator and denominator by $\sqrt{x}+\sqrt{2}$, which is the conjugate of $\sqrt{x}-\sqrt{2}$, and simplify.

$$\begin{aligned}\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}-\sqrt{2}} &= \frac{(\sqrt{x}+\sqrt{2})(\sqrt{x}+\sqrt{2})}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}\\ &= \frac{x+\sqrt{2x}+\sqrt{2x}+2}{x-2} && \text{In the numerator and denominator, use the FOIL method.}\\ &= \frac{x+2\sqrt{2x}+2}{x-2} && \text{In the numerator, combine like radicals.}\end{aligned}$$

5 Rationalize numerators.

In calculus, we sometimes have to rationalize a numerator by multiplying the numerator and denominator of the fraction by the conjugate of the numerator.

EXAMPLE 11 Rationalize the numerator: $\dfrac{\sqrt{x} - 3}{\sqrt{x}}$

Strategy To rationalize the numerator, we will multiply the numerator and the denominator by the conjugate of the numerator.

WHY After rationalizing the numerator, we can simplify the expression. Although the result will not be in simplified form, this nonsimplified form is often desirable in calculus.

Solution

We multiply the numerator and denominator by $\sqrt{x} + 3$, which is the conjugate of the numerator.

$$\frac{\sqrt{x} - 3}{\sqrt{x}} = \frac{\sqrt{x} - 3}{\sqrt{x}} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3}$$ Multiply by a form of 1 to rationalize the numerator.

$$= \frac{\left(\sqrt{x}\right)^2 - (3)^2}{x + 3\sqrt{x}}$$ Multiply the numerators using a special-product rule. Multiply the denominators.

$$= \frac{x - 9}{x + 3\sqrt{x}}$$

Self Check 11

Rationalize the numerator: $\dfrac{\sqrt{x} + 3}{\sqrt{x}}$ $\dfrac{x - 9}{x - 3\sqrt{x}}$

Now Try **Problem 82**

Teaching Example 11 Rationalize the numerator: $\dfrac{\sqrt{x} + 5}{\sqrt{x}}$

Answer: $\dfrac{x - 25}{x - 5\sqrt{x}}$

ANSWERS TO SELF CHECKS

1. a. $-20\sqrt{7}$ **b.** $18x\sqrt[4]{2x}$ **2.** $12\sqrt{10} - 32$ **3.** $-1 + \sqrt{15}$ **4.** $x - 2\sqrt{x} - 3$ **5. a.** 11 **b.** $108y$ **c.** $x - 10\sqrt{x - 8} + 17$ **6. a.** $\frac{2\sqrt{30}}{5}$ **b.** $\frac{5\sqrt[4]{27}}{3}$ **7.** $\frac{\sqrt{2ab}}{a}$ **8.** $\frac{\sqrt[3]{4h^2}}{4h^2}$ **9.** $\frac{\sqrt{85b}}{17b}$ **10. a.** $\sqrt{3} - 1$ **b.** $\frac{x - 2\sqrt{5x} + 5}{x - 5}$ **11.** $\frac{x - 9}{x - 3\sqrt{x}}$

SECTION 7.3 STUDY SET

VOCABULARY

Fill in the blanks.

1. To multiply $\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - 2\sqrt{2}\right)$, we can use the FOIL method.

2. To multiply $2\sqrt{5}\left(3\sqrt{8} + \sqrt{3}\right)$, use the distributive property to remove parentheses.

3. The denominator of the fraction $\dfrac{4}{\sqrt{5}}$ is an irrational number.

▶ **4.** The conjugate of $\sqrt{x} + 1$ is $\sqrt{x} - 1$.

5. To rationalize the denominator of $\dfrac{4}{\sqrt{5}}$, we multiply the numerator and denominator by $\sqrt{5}$.

6. To obtain a perfect cube under the radical in the denominator of $\dfrac{\sqrt[3]{7}}{\sqrt[3]{5n}}$, we multiply the numerator and denominator by $\sqrt[3]{25n^2}$.

▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

Perform each operation, if possible.

7. $4\sqrt{6} + 2\sqrt{6}$
$6\sqrt{6}$

8. $4\sqrt{6}\left(2\sqrt{6}\right)$
48

9. $3\sqrt{2} - 2\sqrt{3}$
can't be simplified

10. $3\sqrt{2}\left(-2\sqrt{3}\right)$
$-6\sqrt{6}$

Perform each operation, if possible.

▶ **11.** $5 + 6\sqrt[3]{6}$
can't be simplified

▶ **12.** $5\left(6\sqrt[3]{6}\right)$
$30\sqrt[3]{6}$

▶ **13.** $\dfrac{30\sqrt[3]{15}}{5}$
$6\sqrt[3]{15}$

▶ **14.** $\dfrac{\sqrt[3]{15}}{5}$
can't be simplified

15. Consider $\dfrac{\sqrt{3}}{\sqrt{7}} = \dfrac{\sqrt{3}\sqrt{7}}{\sqrt{7}\sqrt{7}}$. Explain why the expressions on the left-hand side and the right-side side of the equation are equal.
When the numerator and the denominator of a fraction are multiplied by the same nonzero number, the value of the fraction is not changed.

16. To rationalize the denominator of $\frac{\sqrt[3]{2}}{\sqrt[3]{3}}$, why wouldn't we multiply the numerator and denominator by $\frac{\sqrt[3]{3}}{\sqrt[3]{3}}$?
We don't get a perfect-cube factor under the radical in the denominator.

17. Explain why $\frac{\sqrt[3]{12}}{\sqrt[3]{5}}$ is not in simplified form.
A radical appears in the denominator.

18. Explain why $\sqrt{\frac{3a}{11k}}$ is not in simplified form.
The radicand contains a fraction.

NOTATION

Fill in the blanks.

19. Multiply: $5\sqrt{8} \cdot 7\sqrt{6}$

$$\begin{aligned} 5\sqrt{8} \cdot 7\sqrt{6} &= 5(7)\sqrt{8}\boxed{\sqrt{6}} \\ &= 35\sqrt{\boxed{48}} \\ &= 35\sqrt{\boxed{16} \cdot 3} \\ &= 35(\boxed{4})\sqrt{3} \\ &= 140\sqrt{3} \end{aligned}$$

20. Rationalize the denominator: $\frac{9}{\sqrt[3]{4a^2}}$

$$\begin{aligned} \frac{9}{\sqrt[3]{4a^2}} &= \frac{9 \cdot \sqrt[3]{2a}}{\sqrt[3]{4a^2} \cdot \sqrt[3]{\boxed{2a}}} \\ &= \frac{9\sqrt[3]{2a}}{\sqrt[3]{\boxed{8a^3}}} \\ &= \frac{9\sqrt[3]{2a}}{\boxed{2a}} \end{aligned}$$

GUIDED PRACTICE

Perform each multiplication and simplify. All variables represent positive numbers. See Example 1.

21. $\sqrt{5}\sqrt{10}$
$5\sqrt{2}$

22. $\sqrt{7}\sqrt{35}$
$7\sqrt{5}$

23. $4\sqrt{5a}(3\sqrt{8a})$
$24a\sqrt{10}$

24. $-5\sqrt{8b}(4\sqrt{3b})$
$-40b\sqrt{6}$

25. $3\sqrt[3]{2}\sqrt[3]{12}$
$6\sqrt[3]{3}$

26. $\sqrt[3]{3}(4\sqrt[3]{18})$
$12\sqrt[3]{2}$

27. $(3\sqrt[3]{9a^2})(2\sqrt[3]{3a})$
$18a$

28. $(2\sqrt[3]{16b^2})(-\sqrt[3]{4b^4})$
$-8b^2$

Perform each multiplication and simplify. See Example 2.

29. $3\sqrt{5}(4 - \sqrt{5})$
$12\sqrt{5} - 15$

30. $2\sqrt{7}(3\sqrt{7} - 1)$
$42 - 2\sqrt{7}$

31. $3\sqrt{2}(4\sqrt{6} + 2\sqrt{7})$
$24\sqrt{3} + 6\sqrt{14}$

32. $-\sqrt{3}(\sqrt{7} - \sqrt{15})$
$-\sqrt{21} + 3\sqrt{5}$

Perform each multiplication and simplify. See Example 3.

33. $(\sqrt{2} + 1)(\sqrt{2} - 3)$
$-1 - 2\sqrt{2}$

34. $(2\sqrt{3} + 1)(\sqrt{3} - 1)$
$5 - \sqrt{3}$

35. $(\sqrt{5} + \sqrt{3})(\sqrt{5} - 2\sqrt{3})$
$-1 - \sqrt{15}$

36. $(\sqrt{6} + 3\sqrt{5})(\sqrt{6} - 4\sqrt{5})$
$-54 - \sqrt{30}$

Perform each multiplication and simplify. All variables represent positive numbers. See Example 4.

37. $(\sqrt{5z} + \sqrt{3})(\sqrt{5z} + \sqrt{3})$
$5z + 2\sqrt{15z} + 3$

38. $(\sqrt{3p} - \sqrt{2})(\sqrt{3p} + \sqrt{2})$
$3p - 2$

39. $(\sqrt{5b} - \sqrt{3})(\sqrt{2b} + \sqrt{6})$
$b\sqrt{10} + \sqrt{30b} - \sqrt{6b} - 3\sqrt{2}$

40. $(\sqrt{3y} - \sqrt{2})(\sqrt{2y} - \sqrt{3})$
$y\sqrt{6} - 5\sqrt{y} + \sqrt{6}$

Perform each multiplication and simplify. All variables represent positive numbers. See Example 5.

41. $(\sqrt{7})^2$
7

42. $(\sqrt{23})^2$
23

43. $(3\sqrt{2x})^2$
$18x$

44. $(2x\sqrt{5})^2$
$20x^2$

45. $(3\sqrt[3]{5x^2})^3$
$135x^2$

46. $(-2\sqrt[3]{9y^2})^3$
$-72y^2$

47. $(\sqrt{x - 2} + 3)^2$
$x + 6\sqrt{x - 2} + 7$

48. $(\sqrt{x + 1} - 2)^2$
$x - 4\sqrt{x + 1} + 5$

Rationalize each denominator. See Example 6.

49. $\sqrt{\frac{1}{7}}$ $\frac{\sqrt{7}}{7}$

50. $\sqrt{\frac{5}{3}}$ $\frac{\sqrt{15}}{3}$

51. $\sqrt{\frac{36}{30}}$ $\frac{\sqrt{30}}{5}$

52. $\sqrt{\frac{64}{10}}$ $\frac{4\sqrt{10}}{5}$

53. $\frac{1}{\sqrt[3]{2}}$ $\frac{\sqrt[3]{4}}{2}$

54. $\frac{2}{\sqrt[3]{6}}$ $\frac{\sqrt[3]{36}}{3}$

55. $\frac{3}{\sqrt[3]{9}}$ $\sqrt[3]{3}$

56. $\frac{-5}{\sqrt[3]{10}}$ $-\frac{\sqrt[3]{100}}{2}$

57. $\frac{\sqrt[3]{2}}{\sqrt[3]{9}}$ $\frac{\sqrt[3]{6}}{3}$

58. $\frac{\sqrt[3]{9}}{\sqrt[3]{54}}$ $\frac{\sqrt[3]{36}}{6}$

59. $\frac{1}{\sqrt[4]{8}}$ $\frac{\sqrt[4]{2}}{2}$

60. $\frac{1}{\sqrt[5]{2}}$ $\frac{\sqrt[5]{16}}{2}$

Rationalize each denominator. Assume all variables represent positive numbers. See Example 7.

61. $\frac{\sqrt{8}}{\sqrt{xy}}$ $\frac{2\sqrt{2xy}}{xy}$

62. $\frac{\sqrt{9xy}}{\sqrt{3x^2y}}$ $\frac{\sqrt{3x}}{x}$

63. $\dfrac{\sqrt{10xy^2}}{\sqrt{2xy^3}}$ $\dfrac{\sqrt{5y}}{y}$

▶ **64.** $\dfrac{\sqrt{5ab^2c}}{\sqrt{10abc}}$ $\dfrac{\sqrt{2b}}{2}$

Rationalize each denominator. Assume all variables represent positive numbers. **See Example 8.**

65. $\sqrt{\dfrac{7}{8a^3}}$ $\dfrac{\sqrt{14a}}{4a^2}$

66. $\sqrt{\dfrac{5}{2b^5}}$ $\dfrac{\sqrt{10b}}{2b^3}$

67. $-\sqrt{\dfrac{13}{5p^7}}$ $-\dfrac{\sqrt{65p}}{5p^4}$

▶ **68.** $-\sqrt{\dfrac{2}{3t}}$ $-\dfrac{\sqrt{6t}}{3t}$

Rationalize each denominator. Assume all variables represent positive numbers. **See Example 9.**

69. $\dfrac{\sqrt[3]{4a^2}}{\sqrt[3]{2ab}}$ $\dfrac{\sqrt[3]{2ab^2}}{b}$

▶ **70.** $\dfrac{\sqrt[3]{9x}}{\sqrt[3]{3xy}}$ $\dfrac{\sqrt[3]{3y^2}}{y}$

71. $\dfrac{\sqrt[3]{7}}{\sqrt[3]{4p}}$ $\dfrac{\sqrt[3]{14p^2}}{2p}$

72. $\dfrac{\sqrt[3]{5p^2}}{\sqrt[3]{9q}}$ $\dfrac{\sqrt[3]{15p^2q^2}}{3q}$

Rationalize each denominator. Assume all variables represent positive numbers. **See Example 10.**

73. $\dfrac{1}{\sqrt{2}-1}$ $\sqrt{2}+1$

74. $\dfrac{3}{\sqrt{3}-1}$ $\dfrac{3\left(\sqrt{3}+1\right)}{2}$

75. $\dfrac{\sqrt{2}}{\sqrt{5}+3}$ $\dfrac{3\sqrt{2}-\sqrt{10}}{4}$

76. $\dfrac{\sqrt{3}}{\sqrt{3}-2}$ $-3-2\sqrt{3}$

77. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ $2+\sqrt{3}$

78. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ $3-2\sqrt{2}$

79. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ $\dfrac{x-2\sqrt{xy}+y}{x-y}$

▶ **80.** $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$ $\dfrac{x+2\sqrt{xy}+y}{x-y}$

Rationalize each numerator. Assume all variables represent positive numbers. **See Example 11.**

81. $\dfrac{\sqrt{a}-2}{\sqrt{a}}$ $\dfrac{a-4}{a+2\sqrt{a}}$

82. $\dfrac{\sqrt{p}+3}{\sqrt{p}}$ $\dfrac{p-9}{p-3\sqrt{p}}$

83. $\dfrac{3+\sqrt{b}}{\sqrt{b}}$ $\dfrac{9-b}{3\sqrt{b}-b}$

▶ **84.** $\dfrac{2-\sqrt{p}}{\sqrt{p}}$ $\dfrac{4-p}{2\sqrt{p}+p}$

TRY IT YOURSELF

Simplify each radical expression. Assume that all variables represent positive numbers.

85. $\sqrt{11}\sqrt{11}$ 11

86. $\sqrt{35}\sqrt{35}$ 35

87. $\sqrt{2}\sqrt{8}$ 4

88. $\sqrt{3}\sqrt{27}$ 9

89. $2\sqrt{3}\sqrt{6}$ $6\sqrt{2}$

90. $-3\sqrt{11}\sqrt{33}$ $-33\sqrt{3}$

91. $\sqrt[3]{5}\sqrt[3]{25}$ 5

92. $-\sqrt[3]{7}\sqrt[3]{49}$ -7

93. $\left(-2\sqrt{2}\right)^2$ 8

94. $\left(-3\sqrt{10}\right)^2$ 90

95. $\sqrt{ab^3}\sqrt{ab}$ ab^2

96. $\sqrt{8x}\sqrt{2x^3y}$ $4x^2\sqrt{y}$

97. $\sqrt{5ab}\sqrt{5a}$ $5a\sqrt{b}$

▶ **98.** $\sqrt{15rs^2}\sqrt{10r}$ $5rs\sqrt{6}$

99. $-4\sqrt[3]{5r^2s}\left(5\sqrt[3]{2r}\right)$ $-20r\sqrt[3]{10s}$

100. $-\sqrt[3]{3xy^2}\left(-\sqrt[3]{9x^3}\right)$ $3x\sqrt[3]{xy^2}$

▶ **101.** $\sqrt{x(x+3)}\sqrt{x^3(x+3)}$ $x^2(x+3)$

102. $\sqrt{y^2(x+y)}\sqrt{(x+y)^3}$ $y(x+y)^2$

▶ **103.** $-2\sqrt{5x}\left(4\sqrt{2x}-3\sqrt{3}\right)$ $-8x\sqrt{10}+6\sqrt{15x}$

104. $3\sqrt{7t}\left(2\sqrt{7t}+3\sqrt{3t^2}\right)$ $42t+9t\sqrt{21t}$

105. $\left(\sqrt{3x}-\sqrt{2y}\right)\left(\sqrt{3x}+\sqrt{2y}\right)$ $3x-2y$

▶ **106.** $\left(\sqrt{3m}+\sqrt{2n}\right)\left(\sqrt{3m}+\sqrt{2n}\right)$ $3m+2\sqrt{6mn}+2n$

107. $\left(2\sqrt{3a}-\sqrt{b}\right)\left(\sqrt{3a}+3\sqrt{b}\right)$ $6a+5\sqrt{3ab}-3b$

▶ **108.** $\left(5\sqrt{p}-\sqrt{3q}\right)\left(\sqrt{p}+2\sqrt{3q}\right)$ $5p+9\sqrt{3pq}-6q$

▶ **109.** $\left(3\sqrt{2r}-2\right)^2$ $18r-12\sqrt{2r}+4$

110. $\left(2\sqrt{3t}+5\right)^2$ $12t+20\sqrt{3t}+25$

111. $-2\left(\sqrt{3x}+\sqrt{3}\right)^2$ $-6x-12\sqrt{x}-6$

112. $3\left(\sqrt{5x}-\sqrt{3}\right)^2$ $15x-6\sqrt{15x}+9$

Simplify each radical expression by rationalizing the denominator. All variables represent positive real numbers.

113. $\dfrac{\sqrt{5}}{\sqrt{8}}$ $\dfrac{\sqrt{10}}{4}$

114. $\dfrac{\sqrt{3}}{\sqrt{50}}$ $\dfrac{\sqrt{6}}{10}$

115. $\dfrac{\sqrt{8}}{\sqrt{2}}$ 2

116. $\dfrac{\sqrt{27}}{\sqrt{3}}$ 3

117. $\dfrac{1}{\sqrt[5]{16}}$ $\dfrac{\sqrt[5]{2}}{2}$

118. $\dfrac{4}{\sqrt[4]{32}}$ $\sqrt[4]{8}$

119. $\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{2}+\sqrt{7}}$ $\dfrac{9-2\sqrt{14}}{5}$

▶ **120.** $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$ $5+2\sqrt{6}$

121. $\dfrac{2}{\sqrt{x}+1}$ $\dfrac{2\left(\sqrt{x}-1\right)}{x-1}$

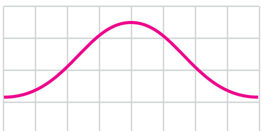
▶ **122.** $\dfrac{3}{\sqrt{x}-2}$ $\dfrac{3\sqrt{x}+6}{x-4}$

123. $\dfrac{2z-1}{\sqrt{2z}-1}$ $\sqrt{2z}+1$

124. $\dfrac{3t-1}{\sqrt{3t}+1}$ $\sqrt{3t}-1$

APPLICATIONS

▶ **125.** STATISTICS An example of a normal distribution curve, or *bell-shaped* curve, is shown in the illustration. A fraction that is part of the equation that models this curve is



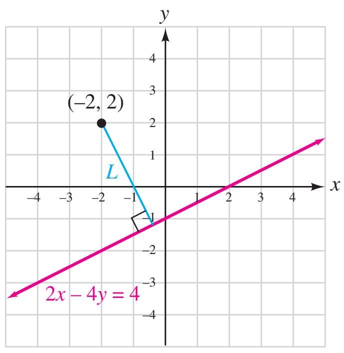
$$\frac{1}{\sigma\sqrt{2\pi}}$$

where σ is a letter from the Greek alphabet. Rationalize the denominator of the fraction. $\dfrac{\sqrt{2\pi}}{2\pi\sigma}$

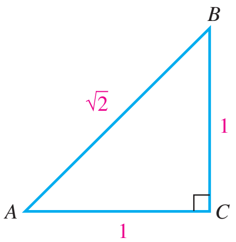
126. ANALYTIC GEOMETRY The length of the perpendicular segment drawn from $(-2, 2)$ to the line with equation $2x - 4y = 4$ is given by

$$L = \frac{|2(-2) + (-4)(2) + (-4)|}{\sqrt{(2)^2 + (-4)^2}}$$

Find L. Express the result in simplified radical form. Then give an approximation to the nearest tenth. $\frac{8\sqrt{5}}{5} \approx 3.6$



127. TRIGONOMETRY In trigonometry, we must often find the ratio of the lengths of two sides of right triangles. Use the information in the illustration to find the ratio



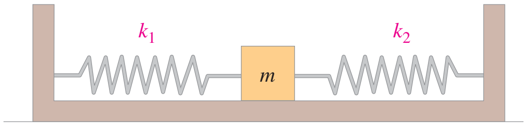
$$\frac{\text{length of side } AC}{\text{length of side } AB}$$

Write the result in simplified radical form. $\frac{\sqrt{2}}{2}$

128. MECHANICAL ENGINEERING A measure of how fast the block shown in the next column will oscillate when the system is set in motion is given by the formula

$$\omega = \sqrt{\frac{k_1 + k_2}{m}}$$

where k_1 and k_2 indicate the stiffness of the springs and m is the mass of the block. Rationalize the right-hand side and restate the formula. $\omega = \frac{\sqrt{(k_1 + k_2)m}}{m}$



129. The period of a pendulum is the time required for the pendulum to swing back and forth to complete one cycle. The period (in seconds) is a function given by

$$f(L) = 2\pi\sqrt{\frac{L}{32}}$$

Rationalize the right-hand side and restate the formula. $f(L) = \frac{\pi\sqrt{2L}}{4}$

130. ELECTRONICS A formula that is used when designing AC (alternating current) circuits is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

Rationalize the right-hand side and restate the formula. $f_0 = \frac{\sqrt{LC}}{2\pi LC}$

WRITING

131. Explain why $\sqrt{m} \cdot \sqrt{m} = m$ but $\sqrt[3]{m} \cdot \sqrt[3]{m} \neq m$. (Assume that $m > 0$.)

132. Explain why the product of $\sqrt{m} + 3$ and $\sqrt{m} - 3$ does not contain a radical.

REVIEW

Solve each equation.

133. $\frac{2}{3 - a} = 1$ 1

134. $5(s - 4) = -5(s - 4)$ 4

135. $\frac{8}{b - 2} + \frac{3}{2 - b} = -\frac{1}{b}$ $\frac{1}{3}$

136. $\frac{2}{x - 2} + \frac{1}{x + 1} = \frac{1}{(x + 1)(x - 2)}$ $\frac{1}{3}$

SECTION 7.4
Solving Radical Equations

Objectives

1. Solve equations containing one radical.
2. Solve equations containing two radicals.
3. Solve formulas containing radicals.

Many situations can be modeled by equations that contain radicals. In this section, we will develop techniques to solve such equations.

1 Solve equations containing one radical.

Radical equations contain a radical expression with a variable radicand. Some examples are

$$\sqrt{x + 3} = 4 \qquad \sqrt[3]{x} = 2 \qquad \sqrt{x} - \sqrt{x + 1} = -1$$

To solve equations containing radicals, we will use the **power rule.**

The Power Rule

If we raise two equal quantities to the same power, the results are equal quantities.

If x, y, and n represent real numbers and $x = y$, then

$$x^n = y^n$$

If we raise both sides of an equation to the same power, the resulting equation might not be equivalent to the original equation. For example, if we square both sides of the equation

(1) $x = 3$

with a solution set of $\{3\}$, we obtain the equation

(2) $x^2 = 9$

with a solution set of $\{3, -3\}$.

Equations 1 and 2 are not equivalent, because they have different solution sets, and the solution -3 of Equation 2 does not satisfy Equation 1. Since raising both sides of an equation to the same power can produce an equation with solutions that don't satisfy the original equation, we must always check each proposed solution in the original equation and discard any **extraneous solutions.**

When we use the power rule to solve square root radical equations, it produces expressions of the form $\left(\sqrt{a}\right)^2$. We have seen that when this expression is simplified, the radical symbol is removed.

The Square of a Square Root

For any nonnegative real number a,

$$\left(\sqrt{a}\right)^2 = a$$

EXAMPLE 1 Solve: $\sqrt{x + 3} = 4$

Strategy We will use the power rule and square both sides of the equation.

WHY Squaring both sides will produce, on the left side, the expression $\left(\sqrt{x + 3}\right)^2$ that simplifies to $x + 3$. This step clears the equation of the radical.

Solution

To eliminate the radical, we apply the power rule by squaring both sides of the equation and proceed as follows:

$$\sqrt{x + 3} = 4 \quad \text{This is the equation to solve.}$$

$$\left(\sqrt{x + 3}\right)^2 = (4)^2 \quad \text{To clear the equation of the square root, square both sides.}$$

$$x + 3 = 16 \quad \text{Perform the operations on each side.}$$

$$x = 13 \quad \text{Subtract 3 from both sides.}$$

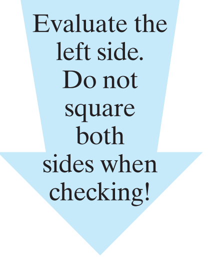
We must check the proposed solution 13 to see whether it satisfies the original equation.

Self Check 1

Solve: $\sqrt{a - 2} = 3$ 11

Now Try Problem 22

Teaching Example 1 Solve: $\sqrt{x - 6} = 5$
Answer:
31



Check:

$$\sqrt{x+3} = 4 \quad \text{This is the original equation.}$$

$$\sqrt{13+3} \stackrel{?}{=} 4 \quad \text{Substitute 13 for } x.$$

$$\sqrt{16} \stackrel{?}{=} 4$$

$$4 = 4 \quad \text{True.}$$

Since 13 satisfies the original equation, it is the solution.

The method used in Example 1 to solve a radical equation containing a radical can be generalized as follows.

Solving an Equation Containing Radicals

1. Isolate one radical expression on one side of the equation.
2. Raise both sides of the equation to the power that is the same as the index of the radical.
3. Solve the resulting equation. If it still contains a radical, go back to step 1.
4. Check the results to eliminate extraneous solutions.

Self Check 2

AMUSEMENT PARK RIDES In Example 2, how long a vertical drop is needed if the riders are to free fall for 3.5 seconds? 196 ft

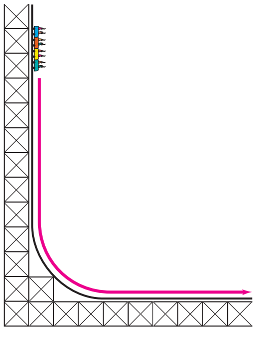
Now Try **Problem 28**

Teaching Example 2 AMUSEMENT PARK RIDES In Example 2, how long a vertical drop is needed if the riders are to free fall for 4 seconds?
Answer:
256 ft

EXAMPLE 2 ***Amusement Park Rides*** The distance d in feet that an object will fall in t seconds is given by the formula

$$t = \sqrt{\frac{d}{16}}$$

If the designers of the amusement park attraction shown to the right want the riders to experience 3 seconds of vertical free fall, what length of vertical drop is needed?



Strategy We will begin by substituting 3 for the time t in the formula.

WHY We can then solve the resulting radical equation in one variable to find the unknown distance d.

Solution

We substitute 3 for t in the formula and solve for d.

$$t = \sqrt{\frac{d}{16}}$$

$$3 = \sqrt{\frac{d}{16}} \quad \text{Here the radical is isolated on the right-hand side.}$$

$$(3)^2 = \left(\sqrt{\frac{d}{16}}\right)^2 \quad \text{Raise both sides to the second power.}$$

$$9 = \frac{d}{16} \quad \text{Simplify.}$$

$$144 = d \quad \text{Solve the resulting equation by multiplying both sides by 16.}$$

The amount of vertical drop needs to be 144 feet.

EXAMPLE 3 Solve: $\sqrt{3x+1}+1=x$

Strategy Since 1 is added outside the square root symbol, there are two terms on the left side of the equation. To isolate the radical, we will subtract 1 from both sides.

WHY This will put the equation in the form where we can square both sides to clear the radical.

Solution

We first subtract 1 from both sides to isolate the radical. Then, to eliminate the radical, we square both sides of the equation and proceed as follows:

$$\sqrt{3x+1}+1=x$$

$$\sqrt{3x+1}=x-1 \qquad \text{Subtract 1 from both sides.}$$

$$\left(\sqrt{3x+1}\right)^2=(x-1)^2 \qquad \text{Square both sides to eliminate the square root.}$$

$$3x+1=x^2-2x+1 \qquad \text{On the right-hand side, } (x-1)^2=(x-1)(x-1)=x^2-x-x+1=x^2-2x+1.$$

$$0=x^2-5x \qquad \text{Subtract 3x and 1 from both sides. This is a quadratic equation. Use factoring to solve it.}$$

$$0=x(x-5) \qquad \text{Factor } x^2-5x.$$

$$x=0 \quad \text{or} \quad x-5=0 \qquad \text{Set each factor each to 0.}$$

$$x=0 \quad \Big| \quad x=5$$

We must check each proposed solution to see whether it satisfies the original equation.

Check:

This is the check for 0:

$$\sqrt{3x+1}+1=x$$
$$\sqrt{3(0)+1}+1\stackrel{?}{=}0$$
$$\sqrt{1}+1\stackrel{?}{=}0$$
$$2\neq 0$$

This is the check for 5:

$$\sqrt{3x+1}+1=x$$
$$\sqrt{3(5)+1}+1\stackrel{?}{=}5$$
$$\sqrt{16}+1\stackrel{?}{=}5$$
$$5=5$$

Since 0 does not check, it must be discarded. The only solution of the original equation is 5.

Self Check 3

Solve: $\sqrt{4x+1}+1=x$

Now Try **Problem 29**

Self Check 3 Answer

6, 0 is extraneous

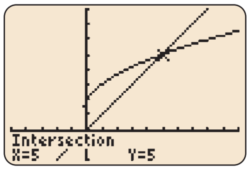
Teaching Example 3 Solve: $\sqrt{2x+4}+2=x$

Answer:

6, 0 is extraneous

Using Your CALCULATOR Solving Equations Containing Radicals

To find approximate solutions for $\sqrt{3x+1}+1=x$ with a graphing calculator, we use window settings of $[-5, 10]$ for x and $[-2, 8]$ for y and graph the functions $f(x)=\sqrt{3x+1}+1$ and $g(x)=x$. We then use the INTERSECT feature to approximate the point of intersection of the graphs. See the figure to the right. The intersection point of (5, 5) implies that 5 is a solution of the radical equation. Check this result.



EXAMPLE 4 Solve: $\sqrt[3]{x^3+7}=x+1$

Strategy Note that the index of the radical is 3. We will use the power rule and cube both sides of the equation.

WHY Cubing both sides will produce, on the left side, the expression $\left(\sqrt[3]{x^3+7}\right)^3$ that simplifies to x^3+7. This step clears the equation of the radical.

Self Check 4

Solve: $\sqrt[3]{x^3+8}=x+2$ 0, -2

Now Try **Problem 34**

Teaching Example 4 Solve: $\sqrt[3]{x^3+27}=x+3$

Answer:

0, -3

Solution

$$\sqrt[3]{x^3 + 7} = x + 1 \quad \text{This is the equation to solve.}$$

$$\left(\sqrt[3]{x^3 + 7}\right)^3 = (x + 1)^3 \quad \text{Cube both sides to eliminate the cube root.}$$

$$x^3 + 7 = x^3 + 3x^2 + 3x + 1 \quad (x + 1)^3 = (x + 1)(x + 1)(x + 1).$$

$$0 = 3x^2 + 3x - 6 \quad \text{Subtract } x^3 \text{ and 7 from both sides.}$$

$$0 = x^2 + x - 2 \quad \text{Divide both sides by 3. To solve this quadratic equation, use factoring.}$$

$$0 = (x + 2)(x - 1) \quad \text{Factor the trinomial.}$$

$$x + 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -2 \qquad\qquad x = 1$$

We check each proposed solution to see whether it satisfies the original equation.

Check:

$$\sqrt[3]{x^3 + 7} = x + 1 \qquad \sqrt[3]{x^3 + 7} = x + 1$$

$$\sqrt[3]{(-2)^3 + 7} \stackrel{?}{=} -2 + 1 \qquad \sqrt[3]{1^3 + 7} \stackrel{?}{=} 1 + 1$$

$$\sqrt[3]{-8 + 7} \stackrel{?}{=} -1 \qquad \sqrt[3]{1 + 7} \stackrel{?}{=} 2$$

$$\sqrt[3]{-1} \stackrel{?}{=} -1 \qquad \sqrt[3]{8} \stackrel{?}{=} 2$$

$$-1 = -1 \qquad 2 = 2$$

Both -2 and 1 are solutions of the original equation.

Self Check 5

Let $g(x) = \sqrt[5]{10x + 1}$. For what value(s) of x is $g(x) = 1$? 0

Now Try **Problem 39**

Teaching Example 5 Let $f(x) = \sqrt[3]{4x - 4}$. For what value(s) of x is $f(x) = 2$?
Answer:
3

EXAMPLE 5

Let $f(x) = \sqrt[4]{2x + 1}$. For what value(s) of x is $f(x) = 5$?

Strategy We will substitute 5 for $f(x)$ and solve the equation $5 = \sqrt[4]{2x + 1}$. To do so, we will raise both sides of the equation to the fourth power.

WHY Raising both sides to the fourth power will produce, on the right side, the expression $\left(\sqrt[4]{2x + 1}\right)^4$ that simplifies to $2x + 1$. This step clears the equation of the radical.

Solution

To find the value(s) where $f(x) = 5$, we substitute 5 for $f(x)$ and solve for x.

$$f(x) = \sqrt[4]{2x + 1}$$

$$5 = \sqrt[4]{2x + 1} \quad \text{This is the equation to solve.}$$

Since the equation contains a fourth root, we raise both sides to the fourth power to solve for x.

$$(5)^4 = \left(\sqrt[4]{2x + 1}\right)^4 \quad \text{Use the power rule to eliminate the radical.}$$

$$625 = 2x + 1 \quad \text{Perform the operations on each side.}$$

$$624 = 2x \quad \text{To solve the resulting equation, subtract 1 from both sides.}$$

$$312 = x \quad \text{Divide both sides by 2.}$$

If $x = 312$, then $f(x) = 5$. Verify this by evaluating $f(312)$.

2 Solve equations containing two radicals.

Self Check 6

Solve: $\sqrt{x - 4} = 2\sqrt{x - 16}$ 20

Now Try **Problem 43**

EXAMPLE 6

Solve: $\sqrt{5x + 9} = 2\sqrt{3x + 4}$

Strategy We will square both sides to clear the equation of both radicals.

WHY We can immediately square both sides since each radical is isolated on one side of the equation.

Solution

$\sqrt{5x+9} = 2\sqrt{3x+4}$	This is the equation to solve.
$\left(\sqrt{5x+9}\right)^2 = \left(2\sqrt{3x+4}\right)^2$	Square both sides.
$5x + 9 = 4(3x+4)$	On the right-hand side: $\left(2\sqrt{3x+4}\right)^2 = 2^2\left(\sqrt{3x+4}\right)^2 = 4(3x+4)$.
$5x + 9 = 12x + 16$	Distribute the multiplication by 4.
$-7 = 7x$	Subtract 5x and 16 from both sides.
$-1 = x$	Divide both sides by 7.

We check the proposed solution by substituting -1 for x in the original equation.

$\sqrt{5x+9} = 2\sqrt{3x+4}$	This is the original equation.
$\sqrt{5(-1)+9} \stackrel{?}{=} 2\sqrt{3(-1)+4}$	Substitute -1 for x.
$\sqrt{4} \stackrel{?}{=} 2\sqrt{1}$	
$2 = 2$	True

The solution is -1.

Teaching Example 6 Solve: $\sqrt{3x+21} = 3\sqrt{x-5}$
Answer: 11

When more than one radical appears in an equation, it is often necessary to apply the power rule more than once.

EXAMPLE 7 Solve: $\sqrt{x} + \sqrt{x+2} = 2$

Strategy We will isolate $\sqrt{x+2}$ on the left side of the equation and square both sides to eliminate it. After simplifying the resulting equation, we will isolate the remaining radical term and square both sides a second time to eliminate it.

WHY Each time that we square both sides, we are able to clear the equation of one radical.

Solution

To remove the radicals, we must square both sides of the equation. This is easier to do if one radical is on each side of the equation. So we subtract $\sqrt{x}$ from both sides to isolate $\sqrt{x+2}$ on the left-hand side of the equation.

$$\sqrt{x} + \sqrt{x+2} = 2$$

$\sqrt{x+2} = 2 - \sqrt{x}$	Subtract $\sqrt{x}$ from both sides.
$\left(\sqrt{x+2}\right)^2 = \left(2-\sqrt{x}\right)^2$	Square both sides to eliminate the square root.
$x + 2 = 4 - 2\sqrt{x} - 2\sqrt{x} + x$	$\left(2-\sqrt{x}\right)^2 = \left(2-\sqrt{x}\right)\left(2-\sqrt{x}\right) = 4 - 2\sqrt{x} - 2\sqrt{x} + x$.
$x + 2 = 4 - 4\sqrt{x} + x$	Combine like radicals: $-2\sqrt{x} - 2\sqrt{x} = -4\sqrt{x}$.
$2 = 4 - 4\sqrt{x}$	Subtract x from both sides.
$-2 = -4\sqrt{x}$	Subtract 4 from both sides.
$\frac{1}{2} = \sqrt{x}$	Divide both sides by -4 and simplify.
$\frac{1}{4} = x$	Square both sides.

Self Check 7

Solve: $\sqrt{a} + \sqrt{a+3} = 3$ 1

Now Try **Problem 46**

Teaching Example 7 Solve: $\sqrt{9x+16} + \sqrt{9x} = 8$
Answer: 1

$$\textbf{Check:}\quad \sqrt{x} + \sqrt{x+2} = 2 \qquad \text{This is the original equation.}$$

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{1}{4} + 2} \stackrel{?}{=} 2 \qquad \text{Substitute } \tfrac{1}{4} \text{ for } x.$$

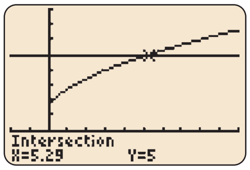
$$\frac{1}{2} + \sqrt{\frac{9}{4}} \stackrel{?}{=} 2 \qquad \text{Think of 2 as } \tfrac{8}{4} \text{ and add: } \tfrac{1}{4} + \tfrac{8}{4} = \tfrac{9}{4}.$$

$$\frac{1}{2} + \frac{3}{2} \stackrel{?}{=} 2 \qquad \text{Evaluate } \sqrt{\tfrac{9}{4}}.$$

$$2 = 2 \qquad \text{True}$$

The solution is $\frac{1}{4}$.

Using Your CALCULATOR **Solving Equations Containing Radicals**



To find approximate solutions for $\sqrt{x} + \sqrt{x+2} = 5$ (an equation similar to that in Example 7) with a graphing calculator, we can use window settings of $[-2, 10]$ for x and $[-2, 8]$ for y and graph the functions $f(x) = \sqrt{x} + \sqrt{x+2}$ and $g(x) = 5$.

The figure shows that the INTERSECT feature gives the approximate coordinates of the point of intersection of the two graphs as (5.29, 5). Therefore, an approximate solution of the radical equation is 5.29. Check its reasonableness.

3 Solving formulas containing radicals.

To *solve a formula for a variable* means to isolate that variable on one side of the equation, with all other quantities on the other side.

Self Check 8

STATISTICS A formula used in statistics to determine the necessary size of a sample to obtain the desired degree of accuracy is

$$e = z_0\sqrt{\frac{pq}{n}}$$

Solve the formula for n. $n = \dfrac{z_0^{\,2}pq}{e^2}$

Now Try **Problem 54**

Teaching Example 8 STATISTICS
A formula used in statistics to determine the necessary size of a sample to obtain the desired degree of accuracy is

$$E = \frac{Z_c\sigma}{\sqrt{n}}$$

Solve the formula for n.
Answer:
$$n = \left(\frac{Z_c\sigma}{E}\right)^2$$

EXAMPLE 8 ***Depreciation Rates*** A piece of office equipment that is now worth V dollars originally cost C dollars 3 years ago. The rate r at which it has depreciated (lost value) is given by

$$r = 1 - \sqrt[3]{\frac{V}{C}}$$

Solve the formula for C.

Strategy To isolate the radical, we will subtract 1 from both sides. We can then eliminate the radical by cubing both sides.

WHY Cubing both sides will produce, on the right, the expression $\left(-\sqrt[3]{\frac{V}{C}}\right)^3$ that simplifies to $-\frac{V}{C}$. This step clears the equation of the radical.

Solution
We begin by isolating the cube root on the right-hand side of the equation.

$$r = 1 - \sqrt[3]{\frac{V}{C}}$$

$$r - 1 = -\sqrt[3]{\frac{V}{C}} \qquad \text{Subtract 1 from both sides.}$$

$$(r-1)^3 = \left[-\sqrt[3]{\frac{V}{C}}\right]^3 \quad \text{Cube both sides.}$$

$$(r-1)^3 = -\frac{V}{C} \quad \text{Simplify the right-hand side.}$$

$$C(r-1)^3 = -V \quad \text{Multiply both sides by } C.$$

$$C = -\frac{V}{(r-1)^3} \quad \text{Divide both sides by } (r-1)^3.$$

ANSWERS TO SELF CHECKS

1. 11 **2.** 196 ft **3.** 6, 0 is extraneous **4.** 0, −2 **5.** 0 **6.** 20 **7.** 1 **8.** $n = \frac{z_0^2 pq}{e^2}$

SECTION 7.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. Equations such as $\sqrt{x+4} - 4 = 5$ and $\sqrt[3]{x+1} = 12$ are called radical equations.

2. When solving equations containing radicals, try to isolate one radical expression on one side of the equation.

3. Squaring both sides of an equation can introduce extraneous solutions.

▶ **4.** To check a proposed solution means to substitute it into the original equation and see whether a true statement results.

CONCEPTS

What is the first step in solving each equation?

5. $\sqrt{x+4} = 5$
Square both sides.

6. $\sqrt[3]{x+4} = 2$
Cube both sides.

Simplify each expression. Assume all variables represent positive numbers.

7. $\left(\sqrt{x}\right)^2$ x

8. $\left(\sqrt{x-5}\right)^2$ $x - 5$

9. $\left(4\sqrt{2x}\right)^2$ $32x$

10. $\left(-\sqrt{x+3}\right)^2$ $x + 3$

Simplify each expression. Assume all variables represent positive numbers.

▶ **11.** $\left(\sqrt[3]{x}\right)^3$ x

▶ **12.** $\left(\sqrt[4]{x}\right)^4$ x

▶ **13.** $\left(-\sqrt[3]{2x}\right)^3$ $-2x$

▶ **14.** $\left(2\sqrt[3]{x+3}\right)^3$ $8x + 24$

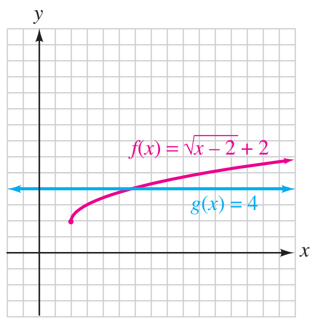
▶ **15.** Fill in the blank to make a true statement. $\sqrt{x}$ represents the number that, when squared, gives x.

16. Find the error.

$$\sqrt{x+1} - 3 = 8$$
$$\sqrt{x+1} = 11$$
$$\left(\sqrt{x+1}\right)^2 = 11$$
$$x + 1 = 11$$
$$x = 10$$

Only one side of the equation was squared.

▶ **17.** Solve $\sqrt{x-2} + 2 = 4$ graphically, using the graphs at the right. 6



18. Use your own words to restate the power rule: If x, y, and n represent real numbers and $x = y$ then $x^n = y^n$.
If we raise two equal quantities to the same power, the results are equal quantities.

NOTATION

Complete each solution to solve the equation.

19.
$$2\sqrt{x-2} = 4$$
$$\left(2\sqrt{x-2}\right)^2 = 4^2$$
$$4(x-2) = 16$$
$$4x - 8 = 16$$
$$4x = 24$$
$$x = 6$$

▶ Selected exercises available online at **www.webassign.net/brookscole**

▶ **20.**
$$\begin{aligned}\sqrt{1-2x} &= \sqrt{x+10}\\ \left(\sqrt{1-2x}\right)^2 &= \left(\sqrt{x+10}\right)^2\\ 1-2x &= x+10\\ -3x &= 9\\ x &= -3\end{aligned}$$

GUIDED PRACTICE

Solve each equation. See Example 1.

21. $\sqrt{5x-6}=2$ 2

22. $\sqrt{7x-10}=12$ 22

23. $\sqrt{6x+1}=5$ 4

▶ **24.** $\sqrt{6x+13}=7$ 6

Use the formula $t=\sqrt{\frac{d}{16}}$ *to find the indicated value.* See Example 2.

25. If $d=16$, find t. 1

26. If $d=32$, find t. $\sqrt{2}$

27. If $t=2$, find d. 64

▶ **28.** If $t=4$, find d. 256

Solve each equation. Cross out any extraneous solutions. See Example 3.

29. $\sqrt{2r-3}+9=r$ $14, \cancel{6}$

30. $-s-3=2\sqrt{5-s}$ $-11, \cancel{1}$

31. $\sqrt{y+2}=4-y$ $2, \cancel{7}$

▶ **32.** $\sqrt{-x+2}+2=x$ $2, \cancel{1}$

Solve each equation. See Example 4.

33. $\sqrt[3]{x^3-7}=x-1$ $2, -1$

34. $\sqrt[3]{x^3+56}=x+2$ $2, -4$

35. $\sqrt[3]{b^3-63}=b-3$ $4, -1$

▶ **36.** $\sqrt[3]{m^3+26}=m+2$ $1, -3$

See Example 5.

37. Let $f(x)=\sqrt[3]{3x-6}$. For what value(s) of x is $f(x)=-3$? -7

▶ **38.** Let $f(x)=\sqrt{2x^2-7x}$. For what value(s) of x is $f(x)=2$? $-\frac{1}{2}, 4$

39. Let $f(x)=\sqrt[4]{3x+1}$. For what value(s) of x is $f(x)=4$? 85

40. Let $f(x)=\sqrt[5]{4x-4}$. For what value(s) of x is $f(x)=-2$? -7

Solve each equation. See Example 6.

41. $2\sqrt{x}=\sqrt{5x-16}$ 16

42. $3\sqrt{x}=\sqrt{3x+54}$ 9

43. $2\sqrt{4x+1}=\sqrt{x+4}$ 0

▶ **44.** $\sqrt{3(x+4)}=\sqrt{5x-12}$ 12

Solve each equation. See Example 7.

45. $\sqrt{x-5}+\sqrt{x}=5$ 9

46. $\sqrt{x-7}+\sqrt{x}=7$ 16

47. $\sqrt{z+3}-\sqrt{z}=1$ 1

▶ **48.** $\sqrt{x+12}+\sqrt{x}=6$ 4

Solve each equation for the indicated variable. See Example 8.

49. $v=\sqrt{2gh}$ for h $h=\frac{v^2}{2g}$

▶ **50.** $d=1.4\sqrt{h}$ for h $h=\frac{d^2}{1.96}$

51. $T=2\pi\sqrt{\frac{l}{32}}$ for l $l=\frac{8T^2}{\pi^2}$

52. $d=\sqrt[3]{\frac{12V}{\pi}}$ for V $V=\frac{\pi d^3}{12}$

53. $r=\sqrt[3]{\frac{A}{P}}-1$ for A $A=P(r+1)^3$

▶ **54.** $r=\sqrt[3]{\frac{A}{P}}-1$ for P $P=\frac{A}{(r+1)^3}$

55. $L_A=L_B\sqrt{1-\frac{v^2}{c^2}}$ for v^2 $v^2=c^2\left(1-\frac{L_A^2}{L_B^2}\right)$

56. $R_1=\sqrt{\frac{A}{\pi}-R_2^2}$ for A $A=\pi R_1^2+\pi R_2^2$

TRY IT YOURSELF

Solve each equation. Cross out all extraneous solutions.

57. $\sqrt[3]{7n-1}=3$ 4

58. $\sqrt[3]{12m+4}=4$ 5

59. $\sqrt[4]{10p+1}=\sqrt[4]{11p-7}$ 8

60. $\sqrt[4]{10y+6}=2\sqrt[4]{y}$ 1

61. $x=\frac{\sqrt{12x-5}}{2}$ $\frac{5}{2}, \frac{1}{2}$

62. $x=\frac{\sqrt{16x-12}}{2}$ $1, 3$

63. $\sqrt{x+2}-\sqrt{4-x}=0$ 1

64. $\sqrt{6-x}-\sqrt{2x+3}=0$ 1

65. $\sqrt{-5x+24}=6-x$ $4, 3$

▶ **66.** $\sqrt{22y+86}=y+9$ $5, -1$

67. $\sqrt[4]{x^4+4x^2-4}=-x$ $-1, \cancel{1}$

68. $\sqrt[4]{8x+8}+2=0$ $\cancel{3}$

69. $\sqrt[4]{12t+4}+2=0$ $\cancel{1}$

▶ **70.** $u=\sqrt[4]{u^4-6u^2+24}$ 2

71. $\sqrt{x+5}+\sqrt{x-3}=4$ 4

72. $\sqrt{b+7}-\sqrt{b-5}=2$ 9

73. $\sqrt{2y+1}=1-2\sqrt{y}$ $0, \cancel{4}$

74. $\sqrt{u}+3=\sqrt{u-3}$ $\cancel{4}$

75. $\sqrt{y+7}+3=\sqrt{y+4}$ $\cancel{-3}$

▶ **76.** $1+\sqrt{z}=\sqrt{z+3}$ 1

77. $2+\sqrt{u}=\sqrt{2u+7}$ $1, 9$

78. $5r + 4 = \sqrt{5r + 20} + 4r$ 1, −4

79. $\sqrt{6t + 1} - 3\sqrt{t} = -1$ 4, $\cancel{0}$

80. $\sqrt{4s + 1} - \sqrt{6s} = -1$ 6, $\cancel{0}$

81. $\sqrt{2x + 5} + \sqrt{x + 2} = 5$ 2, $\cancel{142}$

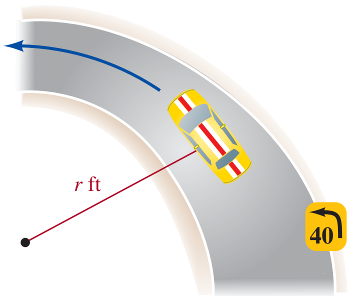
82. $\sqrt{2x + 5} + \sqrt{2x + 1} + 4 = 0$ $\frac{5}{8}$

83. $\sqrt{x - 5} - \sqrt{x + 3} = 4$ $\cancel{6}$

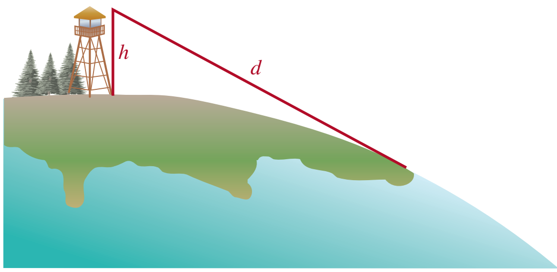
84. $\sqrt{x + 8} - \sqrt{x - 4} = -2$ 8

APPLICATIONS

85. HIGHWAY DESIGNS A curved concrete road will accommodate traffic traveling s mph if the radius of the curve is r feet, according to the formula $s = 3\sqrt{r}$. If engineers expect 40-mph traffic, what radius should they specify? Give the result to the nearest foot. 178 ft



86. FORESTRY The higher a lookout tower is built, the farther an observer can see. That distance d (called the *horizon distance,* measured in miles) is related to the height h of the observer (measured in feet) by the formula $d = 1.4\sqrt{h}$. How tall must a lookout tower be to see the edge of the forest, 25 miles away? (Round to the nearest foot.) 319 ft



87. WIND POWER The power generated by a certain windmill is related to the velocity of the wind by the formula

$$v = \sqrt[3]{\frac{P}{0.02}}$$

where P is the power (in watts) and v is the velocity of the wind (in mph). Find how much power the windmill is generating when the wind is 29 mph. about 488 watts

88. DIAMONDS The *effective rate of interest* r earned by an investment is given by the formula

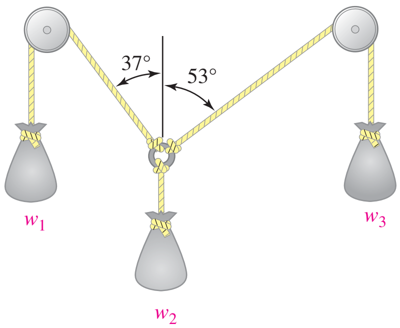
$$r = \sqrt[n]{\frac{A}{P}} - 1$$

where P is the initial investment that grows to value A after n years. If a diamond buyer got \$4,000 for a 1.73-carat diamond that he had purchased 4 years earlier, and earned an annual rate of return of 6.5% on the investment, what did he originally pay for the diamond? \$3,109

89. THEATER PRODUCTIONS The ropes, pulleys, and sandbags shown are part of a mechanical system used to raise and lower scenery for a stage play. For the scenery to be in the proper position, the following formula must apply:

$$w_2 = \sqrt{w_1^{\,2} + w_3^{\,2}}$$

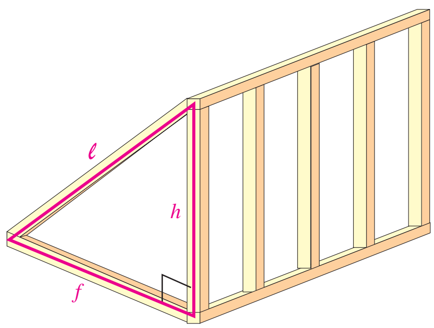
If $w_2 = 12.5$ lb and $w_3 = 7.5$ lb, find w_1. 10 lb



90. CARPENTRY During construction, carpenters often brace walls as shown, where the length of the brace is given by the formula

$$\ell = \sqrt{f^2 + h^2}$$

If a carpenter nails a 10-ft brace to the wall 6 feet above the floor, how far from the base of the wall should he nail the brace to the floor? 8 ft



91. SUPPLY AND DEMAND The number of wrenches that will be produced at a given price can be predicted by the formula $s = \sqrt{5x}$, where s is the supply (in thousands) and x is the price (in dollars). The demand d for wrenches can be predicted by the formula $d = \sqrt{100 - 3x^2}$. Find the equilibrium price; that is, find the price at which supply will equal demand. \$5

▶ **92.** SUPPLY AND DEMAND The number of mirrors that will be produced at a given price can be predicted by the formula $s = \sqrt{23x}$, where s is the supply (in thousands) and x is the price (in dollars). The demand d for mirrors can be predicted by the formula $d = \sqrt{312 - 2x^2}$. Find the equilibrium price—that is, find the price at which supply will equal demand. \$8

WRITING

93. If both sides of an equation are raised to the same power, the resulting equation might not be equivalent to the original equation. Explain.

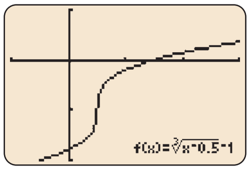
▶ **94.** Explain how the radical equation $\sqrt{2x - 1} = x$ can be solved graphically.

95. Explain how the table on the right can be used to solve $\sqrt{4x - 3} - 2 = \sqrt{2x - 5}$ if $Y_1 = \sqrt{4x - 3} - 2$ and $Y_2 = \sqrt{2x - 5}$.

X	Y1	Y2
2	.23607	ERROR
3	1	1
4	1.6056	1.7321
5	2.1231	2.2361
6	2.5826	2.6458
7	3	3
8	3.3852	3.3166

X=2

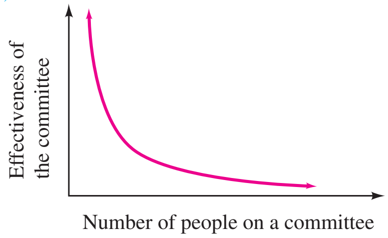
96. Explain how to use the graph shown on the right to approximate the solution of $\sqrt[3]{x - 0.5} = 1$.



REVIEW

97. LIGHTING The intensity of the light reaching you from a light bulb varies inversely as the square of your distance from the bulb. If you are 5 feet away from a light bulb and the intensity is 40 foot-candles, what will the intensity be if you move 20 feet away from the bulb? 2.5 foot-candles

98. COMMITTEES What type of variation is shown below? As the number of people on this committee increased, what happened to its effectiveness? inverse; it decreased



99. TYPESETTING If 12-point type is 0.166044 inch tall, how tall is 30-point type? 0.41511 in.

▶ **100.** GUITAR STRINGS The frequency of vibration of a string varies directly as the square root of the tension and inversely as the length of the string. Suppose a string 2.5 feet long, under a tension of 16 pounds, vibrates 25 times per second. Find k, the constant of variation. 15.625

Objectives

1. Simplify expressions of the form $a^{1/n}$.
2. Convert between radicals and rational exponents.
3. Simplify exponential expressions with variables in their bases.
4. Simplify expressions of the form $a^{m/n}$.
5. Simplify expressions with negative rational exponents.
6. Use rules for exponents to simplify expressions.
7. Simplify radical expressions.

SECTION 7.5 Rational Exponents

We have worked with exponential expressions containing natural-number exponents, such as 5^3 and x^2. In Chapter 5, the definition of exponent was extended to include zero and negative integers, which gave meaning to expressions such as 8^{-3} and $(-9xy)^0$. In this section, we will again extend the definition of exponent—this time to include rational (fractional) exponents. We will see how expressions such as $9^{1/2}$, $\left(\frac{1}{16}\right)^{3/4}$, and $(-32x^5)^{-2/5}$ can be simplified by writing them in an equivalent radical form or by using the rules for exponents.

1 Simplify expressions of the form $a^{1/n}$.

We have seen that positive-integer exponents indicate the number of times that a base is to be used as a factor in a product. For example, x^5 means that x is to be used as a factor five times.

$$x^5 = \overbrace{x \cdot x \cdot x \cdot x \cdot x}^{\text{5 factors of } x}$$

Furthermore, we recall the following rules for exponents.

Rules for Exponents

If there are no divisions by 0, then for all integers m and n,

1. $x^m x^n = x^{m+n}$ **2.** $(x^m)^n = x^{mn}$ **3.** $(xy)^n = x^n y^n$ **4.** $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

5. $x^0 = 1 \quad (x \neq 0)$ **6.** $x^{-n} = \frac{1}{x^n}$ **7.** $\frac{x^m}{x^n} = x^{m-n}$ **8.** $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

It is possible to raise many bases to fractional powers. Since we want fractional exponents to obey the same rules as integer exponents, the square of $10^{1/2}$ must be 10, because

$$\begin{aligned} (10^{1/2})^2 &= 10^{(1/2)\cdot 2} && \text{Keep the base and multiply the exponents.} \\ &= 10^1 && \tfrac{1}{2} \cdot 2 = 1. \\ &= 10 && 10^1 = 10 \end{aligned}$$

However, we have seen that

$$\left(\sqrt{10}\right)^2 = 10$$

Since $(10^{1/2})^2$ and $\left(\sqrt{10}\right)^2$ both equal 10, we define $10^{1/2}$ to be $\sqrt{10}$. Likewise, we define

$$10^{1/3} \text{ to be } \sqrt[3]{10} \quad \text{and} \quad 10^{1/4} \text{ to be } \sqrt[4]{10}$$

Rational Exponents

A **rational exponent** of $\frac{1}{n}$ indicates the nth root of its base.

If n represents a natural number greater than 1 and $\sqrt[n]{x}$ represents a real number, then

$$x^{1/n} = \sqrt[n]{x}$$

We can use this definition to simplify exponential expressions that have rational exponents with a numerator of 1. For example, to simplify $8^{1/3}$, we write it as an equivalent expression in radical form and proceed as follows:

Index

$$8^{1/3} = \sqrt[3]{8} = 2$$

Radicand

The base of the exponential expression, 8, is the radicand. The denominator of the fractional exponent, 3, is the index of the radical.

Thus, $8^{1/3} = 2$.

EXAMPLE 1 Write each expression in radical form and simplify, if possible.

a. $9^{1/2}$ **b.** $-\left(\frac{16}{9}\right)^{1/2}$ **c.** $(-64)^{1/3}$ **d.** $16^{1/4}$ **e.** $\left(\frac{1}{32}\right)^{1/5}$ **f.** $0^{1/8}$

g. $y^{1/4}$ **h.** $-(2x^2)^{1/5}$

Self Check 1

Write each expression in radical form and simplify, if possible:

a. $16^{1/2}$ **b.** $\left(-\frac{27}{8}\right)^{1/3}$

c. $-(6x^3)^{1/4}$

Now Try **Problems 20, 22, 23, 27, and 33**

Self Check 1 Answer

a. 4 **b.** $-\frac{3}{2}$ **c.** $-\sqrt[4]{6x^3}$

Teaching Example 1 Write each expression in radical form and simplify, if possible:

a. $25^{1/2}$ **b.** $\left(\frac{81}{16}\right)^{1/4}$ **c.** $-(3x^4)^{1/5}$

Answers:

a. 5 **b.** $\frac{3}{2}$ **c.** $-\sqrt[5]{3x^4}$

Strategy First, we will identify the base and the exponent of the exponential expression. Then we will write the expression in an equivalent radical form using the rule for rational exponents $x^{1/n} = \sqrt[n]{x}$.

WHY We can use the methods from Section 7.1 to evaluate the resulting square root, cube root, fourth root, and fifth root.

Solution

a. $9^{1/2} = \sqrt{9} = 3$

b. $-\left(\frac{16}{9}\right)^{1/2} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$

c. $(-64)^{1/3} = \sqrt[3]{-64} = -4$

d. $16^{1/4} = \sqrt[4]{16} = 2$

e. $\left(\frac{1}{32}\right)^{1/5} = \sqrt[5]{\frac{1}{32}} = \frac{1}{2}$

f. $0^{1/8} = \sqrt[8]{0} = 0$

g. $y^{1/4} = \sqrt[4]{y}$

h. $-(2x^2)^{1/5} = -\sqrt[5]{2x^2}$

2 Convert between radicals and rational exponents.

We can use the rules for rational exponents to convert expressions from radical form to exponential form, and visa versa.

Self Check 2

Write the radical with a fractional exponent: $\sqrt[6]{7ab}$

Now Try **Problem 38**

Self Check 2 Answer

$(7ab)^{1/6}$

Teaching Example 2 Write the radical with a fractional exponent: $\sqrt[11]{9xy^2z}$

Answer:

$(9xy^2z)^{1/11}$

EXAMPLE 2 Write $\sqrt{5xyz}$ as an exponential expression with a rational exponent.

Strategy We will use the rule for rational exponents in reverse: $\sqrt[n]{x} = x^{1/n}$.

WHY We are given the radical expression and we want to write an equivalent exponential expression.

Solution

The radicand is $5xyz$, so the base of the exponential expression is $5xyz$. The index of the radical is an understood 2, so the denominator of the fractional exponent is 2.

$\sqrt{5xyz} = (5xyz)^{1/2}$ Recall: $\sqrt{5xyz} = \sqrt[2]{5xyz}$

Rational exponents appear in formulas used in many disciplines, such as science and engineering.

Self Check 3

STATISTICS The formula

$$\sigma = \left(\frac{\Sigma(x-\mu)^2}{N}\right)^{1/2}$$

gives the population standard deviation. Write the formula using a radical.

Now Try **Problem 44**

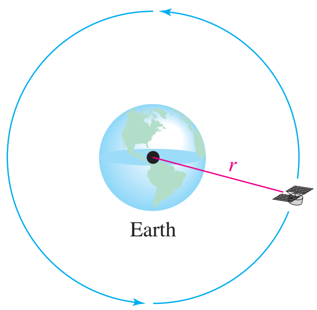
Self Check 3 Answer

$\sigma = \sqrt{\frac{\Sigma(x-\mu)^2}{N}}$

EXAMPLE 3 ***Satellites*** The formula

$$r = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3}$$

gives the orbital radius (in meters) of a satellite circling Earth, where G and M are constants and P is the time in seconds for the satellite to make one complete revolution. Write the formula using a radical.



Strategy We will write the exponential expression in an equivalent radical form using the rule for rational exponents $x^{1/n} = \sqrt[n]{x}$.

WHY We are given an exponential expression involving a rational exponent with a numerator of 1 and we want to write an equivalent radical expression.

Solution

The fractional exponent $\frac{1}{3}$ has a denominator of 3, which indicates that we are to find the cube root of the base of the exponential expression. So we have

$$r = \sqrt[3]{\frac{GMP^2}{4\pi^2}}$$

Teaching Example 3 STATISTICS The formula $s = \left(\frac{\Sigma(x - \bar{x})^2}{n - 1}\right)^{1/2}$ gives the sample standard deviation of a set of data. Write the formula using a radical.

Answer:

$$s = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

3 Simplify exponential expressions with variables in their bases.

As with radicals, when n represents an *odd natural number* in the expression $x^{1/n}$ where $n > 1$, there is exactly one real nth root, and we don't have to worry about absolute value symbols.

When n represents an *even natural number,* there are two nth roots. Since we want the expression $x^{1/n}$ to represent the positive nth root, we must often use absolute value symbols to guarantee that the simplified result is positive. Thus, if n is even,

$$(x^n)^{1/n} = |x|$$

When n is even and x is negative, the expression $x^{1/n}$ is not a real number.

EXAMPLE 4 Simplify each exponential expression. Assume that the variables can be any real number.

a. $(-8x^3)^{1/3}$ **b.** $(256a^8)^{1/8}$ **c.** $[(y + 4)^2]^{1/2}$ **d.** $(25b^4)^{1/2}$ **e.** $(-256x^4)^{1/4}$

Strategy We will write each exponential expression in an equivalent radical form using the rule for rational exponents $x^{1/n} = \sqrt[n]{x}$.

WHY We can use the methods from Section 7.1 to evaluate the resulting cube root, eighth root, square root, and fourth root.

Solution

a. $(-8x^3)^{1/3} = -2x$ Because $(-2x)^3 = -8x^3$. Since n is odd, no absolute value symbols are needed.

b. $(256a^8)^{1/8} = 2|a|$ Because $(2|a|)^8 = 256a^8$. Since n is even and a can be any real number, $2a$ can be negative. Thus, absolute value symbols are needed.

c. $[(y + 4)^2]^{1/2} = |y + 4|$ Because $|y + 4|^2 = (y + 4)^2$. Since n is even and y can be any real number, $y + 4$ can be negative. Thus, absolute value symbols are needed.

d. $(25b^4)^{1/2} = 5b^2$ Because $(5b^2)^2 = 25b^4$. Since $b^2 \geq 0$, no absolute value symbols are needed.

e. $(-256x^4)^{1/4}$ is not a real number. Because no real number raised to the 4th power is $-256x^4$.

Self Check 4

Simplify each expression. Assume that the variables can be any real number.

a. $(625a^4)^{1/4}$ $5|a|$

b. $(b^4)^{1/2}$ b^2

Now Try **Problems 48, 49, and 53**

Teaching Example 4 Simplify each exponential expression. Assume that the variables can be any real number.

a. $(-32x^5)^{1/5}$

b. $(81x^4)^{1/4}$

c. $(25x^4)^{1/2}$

d. $(-16x^4)^{1/4}$

Answers:

a. $-2x$ **b.** $3|x|$ **c.** $5x^2$

d. not a real number

If we are told that the variables represent positive real numbers in parts b and c of Example 4, the absolute value symbols in the answers are not needed.

$(256a^8)^{1/8} = 2a$ If a represents a positive number, then $2a$ is positive.

$[(y + 4)^2]^{1/2} = y + 4$ If y represents a positive number, then $y + 4$ is positive.

We summarize the cases as follows.

Summary of the Definitions of $x^{1/n}$

If n represents a natural number greater than 1 and x represents a real number,

If $x > 0$, then $x^{1/n}$ is the positive number such that $(x^{1/n})^n = x$.

If $x = 0$, then $x^{1/n} = 0$.

$$\text{If } x < 0 \begin{cases} \text{and } n \text{ is odd, then } x^{1/n} \text{ is the real number such that } (x^{1/n})^n = x. \\ \text{and } n \text{ is even, then } x^{1/n} \text{ is not a real number.} \end{cases}$$

4 Simplify expressions of the form $a^{m/n}$.

We can extend the definition of $x^{1/n}$ to include fractional exponents with numerators other than 1. For example, since $8^{2/3}$ can be written as $(8^{1/3})^2$, we have

$$\begin{aligned} 8^{2/3} &= (8^{1/3})^2 \\ &= \left(\sqrt[3]{8}\right)^2 && \text{Write } 8^{1/3} \text{ in radical form.} \\ &= 2^2 && \text{Find the cube root first: } \sqrt[3]{8} = 2. \\ &= 4 && \text{Then find the power.} \end{aligned}$$

Thus, we can simplify $8^{2/3}$ by finding the second power of the cube root of 8.

The numerator of the rational exponent is the power.

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2$$

The base of the exponential expression is the radicand.

The denominator of the exponent is the index of the radical.

We can also simplify $8^{2/3}$ by taking the cube root of 8 squared.

$$\begin{aligned} 8^{2/3} &= (8^2)^{1/3} \\ &= 64^{1/3} && \text{Find the power first: } 8^2 = 64. \\ &= \sqrt[3]{64} && \text{Write } 64^{1/3} \text{ in radical form.} \\ &= 4 && \text{Now find the cube root.} \end{aligned}$$

In general, we have the following rule.

The Definition of $x^{m/n}$

If m and n represent positive integers $(n \neq 1)$ and $\sqrt[n]{x}$ represents a real number,

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m \quad \text{and} \quad x^{m/n} = \sqrt[n]{x^m}$$

Because of the previous definition, we can interpret $x^{m/n}$ in two ways:

1. $x^{m/n}$ means the mth power of the nth root of x.
2. $x^{m/n}$ means the nth root of the mth power of x.

To avoid large numbers, it is usually better to find the root of the base first, and then calculate the power using the rule $x^{m/n} = \left(\sqrt[n]{x}\right)^m$.

EXAMPLE 5 Simplify: **a.** $9^{3/2}$ **b.** $\left(\frac{1}{16}\right)^{3/4}$ **c.** $(-8x^3)^{4/3}$

Strategy First, we will identify the base and the exponent of the exponential expression. Then we will write the expression in an equivalent radical form using the rule for rational exponents $x^{m/n} = \left(\sqrt[n]{x}\right)^m$.

WHY We can use the methods from Section 7.1 to evaluate the resulting square root, fourth root, and cube root.

Solution

Power
Root

a. $9^{3/2} = \left(\sqrt[2]{9}\right)^3$ Because the exponent is 3/2, find the square root of the base, 9, to get 3. Then find the third power of 3.

$= 3^3$

$= 27$

Power
Root

b. $\left(\frac{1}{16}\right)^{3/4} = \left(\sqrt[4]{\frac{1}{16}}\right)^3$ Because the exponent is 3/4, find the fourth root of the base, $\frac{1}{16}$, to get $\frac{1}{2}$. Then find the third power of $\frac{1}{2}$ to get $\frac{1}{8}$.

$= \left(\frac{1}{2}\right)^3$

$= \frac{1}{8}$

Power
Root

c. $(-8x^3)^{4/3} = \left(\sqrt[3]{-8x^3}\right)^4$ Because the exponent is 4/3, find the cube root of the base, $-8x^3$, to get $-2x$. Then find the fourth power of $-2x$ to get $16x^4$.

$= (-2x)^4$

$= 16x^4$

Self Check 5

Simplify:

a. $16^{3/2}$ 64

b. $(-27x^6)^{2/3}$ $9x^4$

Now Try **Problems 57, 62, and 66**

Teaching Example 5 Simplify:

a. $25^{3/2}$ **b.** $\left(\frac{1}{8}\right)^{4/3}$ **c.** $(-27x^3)^{4/3}$

Answers:

a. 125 **b.** $\frac{1}{16}$ **c.** $81x^4$

Success Tip We can also evaluate $x^{m/n}$ using $\sqrt[n]{x^m}$, however the resulting radicand is often extremely large and more difficult to work with.

$$(-8x^3)^{4/3} = \sqrt[3]{(-8x^3)^4}$$
$$= \sqrt[3]{4{,}096x^{12}}$$
$$= 16x^4$$

Using Your CALCULATOR **Rational Exponents**

We can evaluate exponential expressions containing rational exponents using the exponential key $\boxed{y^x}$ or $\boxed{x^y}$ on a scientific calculator. For example, to evaluate $10^{2/3}$, we enter these numbers and press these keys:

10 $\boxed{y^x}$ $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{=}$ 4.641588834

Note that parentheses were used when entering the power. Without them, the calculator would interpret the entry as $10^2 \div 3$.

To evaluate the exponential expression using a direct entry or graphing calculator, we use the $\boxed{\wedge}$ key, which raises a base to a power. Again, we use parentheses when entering the power.

10 $\boxed{\wedge}$ $\boxed{(}$ 2 $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{\text{ENTER}}$

```
10^(2/3)
        4.641588834
```

To the nearest hundredth, $10^{2/3} \approx 4.64$.

5 Simplify expressions with negative rational exponents.

To be consistent with the definition of negative-integer exponents, we define $x^{-m/n}$ as follows.

Definition of $x^{-m/n}$

If m and n represent positive integers, $\frac{m}{n}$ is in simplified form, and $x^{1/n}$ represents a real number, then

$$x^{-m/n} = \frac{1}{x^{m/n}} \quad \text{and} \quad \frac{1}{x^{-m/n}} = x^{m/n} \quad \text{where } x \neq 0$$

Self Check 6

Write using positive exponents only and simplify:

a. $16^{-1/4}$ $\frac{1}{2}$

b. $(-27a^3)^{-2/3}$ $\frac{1}{9a^2}$

Now Try **Problems 71, 73, and 75**

Teaching Example 6 Write each expression using positive exponents only and simplify if possible:
a. $8^{-2/3}$ **b.** $(-8)^{2/3}$ **c.** $(-8)^{-2/3}$
d. $-8^{-2/3}$ **e.** $\frac{1}{8^{-2/3}}$

Answers:
a. $\frac{1}{4}$ **b.** 4 **c.** $\frac{1}{4}$ **d.** $-\frac{1}{4}$ **e.** 4

EXAMPLE 6 Write each expression using positive exponents only and simplify, if possible: **a.** $64^{-1/2}$ **b.** $(-16)^{-3/4}$ **c.** $(-32x^5)^{-2/5}$ **d.** $\frac{1}{16^{-3/2}}$

Strategy We will use one of the rules $x^{-m/n} = \frac{1}{x^{m/n}}$ or $\frac{1}{x^{-m/n}} = x^{m/n}$ to write the reciprocal of each exponential expression and change the exponent's sign to positive.

WHY If we can produce an equivalent expression having a positive rational exponent, we can use the method of this section to simplify it.

Solution

a.
$$64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}$$

b. $(-16)^{-3/4}$ is not a real number, because $(-16)^{1/4}$ is not a real number.

c.
$$(-32x^5)^{-2/5} = \frac{1}{(-32x^5)^{2/5}} = \frac{1}{[(-32x^5)^{1/5}]^2} = \frac{1}{\left(\sqrt[5]{-32x^5}\right)^2} = \frac{1}{(-2x)^2} = \frac{1}{4x^2}$$

d.
$$\frac{1}{16^{-3/2}} = 16^{3/2} = (16^{1/2})^3 = \left(\sqrt{16}\right)^3 = 4^3 = 64$$

Caution! By definition, 0^0 is undefined. A base of 0 raised to a negative power is also undefined. For example, 0^{-2} would equal $\frac{1}{0^2}$, which is undefined because we cannot divide by 0.

6 Use rules for exponents to simplify expressions.

We can use the rules for exponents to simplify many expressions with fractional exponents. If all variables represent positive numbers, absolute value symbols are not needed.

EXAMPLE 7

Simplify each expression. All variables represent positive real numbers. Write all answers using positive exponents only.

a. $5^{2/7}5^{3/7}$ **b.** $(5^{2/7})^3$ **c.** $(a^{2/3}b^{1/2})^6$ **d.** $\dfrac{a^{8/3}a^{1/3}}{a^2}$

Strategy We will use the product, power, and quotient rules for exponents to simplify each expression.

WHY The familiar rules for exponents discussed in Chapter 5 are valid for rational exponents.

Solution

a. $5^{2/7}5^{3/7} = 5^{2/7+3/7}$ Use the rule $x^m x^n = x^{m+n}$.

$= 5^{5/7}$ Add: $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$.

b. $(5^{2/7})^3 = 5^{(2/7)(3)}$ Use the rule $(x^m)^n = x^{mn}$.

$= 5^{6/7}$ Multiply: $\frac{2}{7}(3) = \frac{6}{7}$.

c. $(a^{2/3}b^{1/2})^6 = (a^{2/3})^6(b^{1/2})^6$ Use the rule $(xy)^n = x^n y^n$.

$= a^{12/3}b^{6/2}$ Use the rule $(x^m)^n = x^{mn}$ twice.

$= a^4b^3$ Simplify the exponents.

d. $\dfrac{a^{8/3}a^{1/3}}{a^2} = a^{8/3+1/3-2}$ Use the rules $x^m x^n = x^{m+n}$ and $\frac{x^m}{x^n} = x^{m-n}$.

$= a^{8/3+1/3-6/3}$ $2 = \frac{6}{3}$.

$= a^{3/3}$ $\frac{8}{3} + \frac{1}{3} - \frac{6}{3} = \frac{3}{3}$.

$= a$ $\frac{3}{3} = 1$.

Self Check 7

Simplify. All variables represent positive numbers.

a. $(x^{1/3}y^{3/2})^6$ x^2y^9

b. $\dfrac{x^{5/3}x^{2/3}}{x^{1/3}}$ x^2

Now Try **Problems 80, 81, 84, and 86**

Teaching Example 7 Simplify. All variables represent positive numbers.

a. $7^{2/9}7^{5/9}$ **b.** $(7^{1/5})^4$

c. $(x^{1/8}y^{2/3})^{24}$ **d.** $\dfrac{x^{1/5}x^{7/5}}{x^{3/5}}$

Answers:

a. $7^{7/9}$ **b.** $7^{4/5}$

c. x^3y^{16} **d.** x

EXAMPLE 8

Assume that all variables represent positive numbers, and perform the operations. Write all answers using positive exponents only.

a. $a^{4/5}(a^{1/5} + a^{3/5})$ **b.** $x^{1/2}(x^{-1/2} + x^{1/2})$

Strategy We will use the distributive property and multiply each term within the parentheses by the term outside the parentheses.

WHY Then each expression has the form $a(b + c)$.

Solution

a. $a^{4/5}(a^{1/5} + a^{3/5}) = a^{4/5}a^{1/5} + a^{4/5}a^{3/5}$ Use the distributive property.

$= a^{4/5+1/5} + a^{4/5+3/5}$ Use the rule $x^m x^n = x^{m+n}$.

$= a^{5/5} + a^{7/5}$ Simplify the exponents.

$= a + a^{7/5}$ We cannot add these terms because they are not like terms.

Self Check 8

Simplify. All variables represent positive numbers. Write all answers using positive exponents only.

a. $x^{2/7}(x^{4/7} + x^{5/7})$ $x^{6/7} + x$

b. $x^{1/3}(x^{-1/3} - 2x^{1/3})$ $1 - 2x^{2/3}$

Now Try **Problems 88 and 90**

Teaching Example 8 Simplify. All variables represent positive numbers. Write all answers using positive exponents only.

a. $x^{1/4}(x^{1/4} - x^{3/4})$

b. $x^{5/8}(x^{3/8} + x^{-5/8})$

Answers:

a. $x^{1/2} - x$ **b.** $x + 1$

b. $x^{1/2}(x^{-1/2} + x^{1/2}) = x^{1/2}x^{-1/2} + x^{1/2}x^{1/2}$ Use the distributive property.

$= x^{1/2+(-1/2)} + x^{1/2+1/2}$ Use the rule $x^m x^n = x^{m+n}$.

$= x^0 + x^1$ Simplify each exponent.

$= 1 + x$ $x^0 = 1$.

7 Simplify radical expressions.

We can simplify many radical expressions by using the following steps.

Using Rational Exponents to Simplify Radicals

1. Change the radical expression into an exponential expression.
2. Simplify the rational exponents.
3. Change the exponential expression back into a radical.

Self Check 9

Simplify:

a. $\sqrt[6]{3^3}$ $\sqrt{3}$

b. $\sqrt[4]{64x^2y^2}$ $\sqrt{8xy}$

c. $\sqrt[3]{\sqrt[4]{m}}$ $\sqrt[12]{m}$

Now Try Problems 92 and 94

Teaching Example 9 Simplify:

a. $\sqrt[15]{2^3}$ **b.** $\sqrt[21]{x^7y^{14}}$ **c.** $\sqrt[6]{\sqrt[3]{x}}$

Answers:

a. $\sqrt[5]{2}$ **b.** $\sqrt[3]{xy^2}$ **c.** $\sqrt[18]{x}$

EXAMPLE 9

Simplify: **a.** $\sqrt[4]{3^2}$ **b.** $\sqrt[8]{x^6}$ **c.** $\sqrt[9]{27x^6y^3}$ **d.** $\sqrt[5]{\sqrt[3]{t}}$

Strategy We will write each radical expression as an equivalent exponential expression and use rules for exponents to simplify it. Then we will change that result back into a radical.

WHY When the given expression is written in an equivalent exponential form, we can use rules for exponents and our arithmetic skills with fractions to simplify the exponents.

Solution

a. $\sqrt[4]{3^2} = (3^2)^{1/4}$ Change the radical to an exponential expression.

$= 3^{2/4}$ Use the rule $(x^m)^n = x^{mn}$.

$= 3^{1/2}$ Simplify the fractional exponent: $\frac{2}{4} = \frac{1}{2}$.

$= \sqrt{3}$ Change back to radical form.

b. $\sqrt[8]{x^6} = (x^6)^{1/8}$ Change the radical to an exponential expression.

$= x^{6/8}$ Use the rule $(x^m)^n = x^{mn}$.

$= x^{3/4}$ Simplify the fractional exponent: $\frac{6}{8} = \frac{3}{4}$.

$= (x^3)^{1/4}$ Write $\frac{3}{4}$ as $3\left(\frac{1}{4}\right)$.

$= \sqrt[4]{x^3}$ Change back to radical form.

c. $\sqrt[9]{27x^6y^3} = (3^3x^6y^3)^{1/9}$ Write 27 as 3^3 and change the radical to an exponential expression.

$= 3^{3/9}x^{6/9}y^{3/9}$ Raise each factor to the $\frac{1}{9}$ power by multiplying the fractional exponents.

$= 3^{1/3}x^{2/3}y^{1/3}$ Simplify each fractional exponent.

$= (3x^2y)^{1/3}$ Use the rule $(xy)^n = x^n y^n$.

$= \sqrt[3]{3x^2y}$ Change back to radical form.

d. $\sqrt[5]{\sqrt[3]{t}} = \sqrt[5]{t^{1/3}}$ Change the radical $\sqrt[3]{t}$ to exponential notation.

$= (t^{1/3})^{1/5}$ Change the radical $\sqrt[5]{t^{1/3}}$ to exponential notation.

$= t^{1/15}$ Use the rule $(x^m)^n = x^{mn}$. Multiply: $\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$.

$= \sqrt[15]{t}$ Change back to radical form.

ANSWERS TO SELF CHECKS

1. a. 4 **b.** $-\frac{3}{2}$ **c.** $-\sqrt[4]{6x^3}$ **2.** $(7ab)^{1/6}$ **3.** $\sigma = \sqrt{\frac{\Sigma(x-\mu)^2}{N}}$ **4. a.** $5|a|$ **b.** b^2
5. a. 64 **b.** $9x^4$ **6. a.** $\frac{1}{2}$ **b.** $\frac{1}{9a^2}$ **7. a.** x^2y^9 **b.** x^2 **8. a.** $x^{6/7} + x$ **b.** $1 - 2x^{2/3}$
9. a. $\sqrt{3}$ **b.** $\sqrt{8xy}$ **c.** $\sqrt[12]{m}$

SECTION 7.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. The expressions $4^{1/2}$ and $(-8)^{-2/3}$ have rational (or fractional) exponents.

2. In the exponential expression $27^{4/3}$, 27 is the base, and 4/3 is the exponent.

3. In the radical expression $\sqrt[3]{4{,}096x^{12}}$, 3 is the index, and $4{,}096x^{12}$ is the radicand.

▶ **4.** $32^{4/5}$ means the fourth power of the fifth root of 32.

CONCEPTS

5. Complete the table by writing the given expression in the alternate form.

Radical form	Exponential form
$\sqrt[5]{25}$	$25^{1/5}$
$(\sqrt[3]{-27})^2$	$(-27)^{2/3}$
$(\sqrt[4]{16})^{-3}$	$16^{-3/4}$
$(\sqrt{81})^3$	$81^{3/2}$
$-\sqrt{\frac{9}{64}}$	$-\left(\frac{9}{64}\right)^{1/2}$

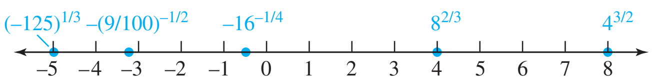
6. Explain the two rules for rational exponents illustrated in the diagrams below.

a. $(-32)^{1/5} = \sqrt[5]{-32}$
The denominator of the rational exponent indicates that we are to find the 5th root of the base.

b. $125^{4/3} = (\sqrt[3]{125})^4$
The denominator of the rational exponent indicates that we are to find the cube root of the base. The numerator indicates that we are to find the 4th power of the cube root.

▶ **7.** Graph each number on the number line.

$$\left\{8^{2/3}, (-125)^{1/3}, -16^{-1/4}, 4^{3/2}, -\left(\frac{9}{100}\right)^{-1/2}\right\}$$



▶ **8.** Evaluate $25^{3/2}$ in two ways. Which way is easier?
125. The easier way is to compute the root first, and then find the power. We can also find the power first, and then find the root.

Complete each rule for exponents.

9. $x^m x^n = x^{m+n}$

10. $(x^m)^n = x^{mn}$

11. $\frac{x^m}{x^n} = x^{m-n}$

▶ **12.** $x^{-n} = \frac{1}{x^n}$

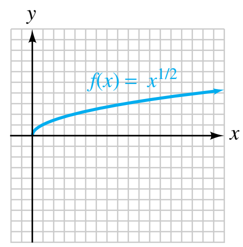
13. $x^{1/n} = \sqrt[n]{x}$

14. $x^{m/n} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$

Complete each table and graph the function.

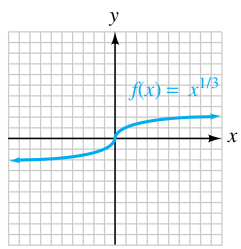
15. $f(x) = x^{1/2}$

x	$f(x)$
0	0
1	1
4	2
9	3
16	4



16. $f(x) = x^{1/3}$

x	$f(x)$
−8	−2
−1	−1
0	0
1	1
8	2



NOTATION

Complete each solution.

17.
$$\begin{aligned}(100a^4)^{3/2} &= (\sqrt{100a^4})^3\\ &= (10a^2)^3\\ &= 1{,}000a^6\end{aligned}$$

▶ Selected exercises available online at **www.webassign.net/brookscole**

18. $(m^{1/3}n^{1/2})^6 = (m^{1/3})^6(n^{1/2})^6$
$= m^{6/3}n^{6/2}$
$= m^2n^3$

GUIDED PRACTICE

Write each expression in radical form and simplify. Assume all variables represent positive numbers. See Example 1.

19. $4^{1/2}$ 2
20. $25^{1/2}$ 5
21. $\left(\frac{1}{16}\right)^{1/2}$ $\frac{1}{4}$
22. $-\left(\frac{1}{4}\right)^{1/2}$ $-\frac{1}{2}$
23. $(-27)^{1/3}$ -3
24. $(-125)^{1/3}$ -5
25. $81^{1/4}$ 3
26. $625^{1/4}$ 5
27. $\left(\frac{1}{16}\right)^{1/4}$ $\frac{1}{2}$
28. $\left(\frac{1}{10{,}000}\right)^{1/4}$ $\frac{1}{10}$
29. $32^{1/5}$ 2
30. $\left(\frac{32}{243}\right)^{1/5}$ $\frac{2}{3}$
31. $0^{1/7}$ 0
32. $b^{1/2}$ $\sqrt{b}$
33. $-(3p^2)^{1/3}$ $-\sqrt[3]{3p^2}$
34. $-(5q^3)^{1/4}$ $-\sqrt[4]{5q^3}$

Write each radical as an exponential expression with a rational exponent. See Example 2.

35. $\sqrt{m}$ $m^{1/2}$
36. $\sqrt[3]{r}$ $r^{1/3}$
37. $\sqrt[4]{3a}$ $(3a)^{1/4}$
38. $3\sqrt[5]{a}$ $3a^{1/5}$
39. $\sqrt[6]{\frac{1}{7}abc}$ $\left(\frac{1}{7}abc\right)^{1/6}$
40. $\sqrt[7]{\frac{3}{8}p^2q}$ $\left(\frac{3}{8}p^2q\right)^{1/7}$
41. $\sqrt[3]{a^2 - b^2}$ $(a^2 - b^2)^{1/3}$
42. $\sqrt{x^2 + y^2}$ $(x^2 + y^2)^{1/2}$

Write each expression using a radical. See Example 3.

43. $c = (a^2 + b^2)^{1/2}$ $c = \sqrt{a^2 + b^2}$
44. $v = (2gh)^{1/2}$ $v = \sqrt{2gh}$
45. $d = \left(\frac{12V}{\pi}\right)^{1/3}$ $d = \sqrt[3]{\frac{12V}{\pi}}$
46. $r = \left(\frac{A}{P}\right)^{1/3} - 1$ $r = \sqrt[3]{\frac{A}{P}} - 1$

Simplify each expression, if possible. Assume all variables are unrestricted and use absolute value symbols when necessary. See Example 4.

47. $(25y^2)^{1/2}$ $5|y|$
48. $(-27x^3)^{1/3}$ $-3x$
49. $(16x^4)^{1/4}$ $2|x|$
50. $(-16x^4)^{1/2}$ not real
51. $(243x^5)^{1/5}$ $3x$
52. $[(x + 1)^4]^{1/4}$ $|x + 1|$
53. $(-64x^8)^{1/4}$ not real
54. $[(x + 5)^3]^{1/3}$ $x + 5$

Simplify each expression. Assume all variables represent positive numbers. See Example 5.

55. $36^{3/2}$ 216
56. $27^{2/3}$ 9
57. $81^{3/4}$ 27
58. $100^{3/2}$ 1,000
59. $144^{3/2}$ 1,728
60. $1{,}000^{2/3}$ 100
61. $\left(\frac{1}{8}\right)^{2/3}$ $\frac{1}{4}$
62. $\left(\frac{4}{9}\right)^{3/2}$ $\frac{8}{27}$
63. $(25x^4)^{3/2}$ $125x^6$
64. $(27a^3b^3)^{2/3}$ $9a^2b^2$
65. $\left(\frac{8x^3}{27}\right)^{2/3}$ $\frac{4x^2}{9}$
66. $\left(-\frac{27}{64y^6}\right)^{2/3}$ $\frac{9}{16y^4}$

Write each expression without using negative exponents, if possible. Assume all variables represent positive numbers. See Example 6.

67. $4^{-1/2}$ $\frac{1}{2}$
68. $8^{-1/3}$ $\frac{1}{2}$
69. $(-81)^{-3/4}$ not a real number
70. $(-8x^3)^{-2/3}$ $\frac{1}{4x^2}$
71. $4^{-3/2}$ $\frac{1}{8}$
72. $25^{-5/2}$ $\frac{1}{3{,}125}$
73. $(16x^2)^{-3/2}$ $\frac{1}{64x^3}$
74. $(81c^4)^{-3/2}$ $\frac{1}{729c^6}$
75. $\frac{1}{8^{-2/3}}$ 4
76. $\frac{1}{32^{-3/5}}$ 8
77. $\frac{1}{(16x^2)^{-3/2}}$ $64x^3$
78. $\frac{1}{(27y^3)^{-2/3}}$ $9y^2$

Perform the operations. Write the answers without negative exponents. Assume all variables represent positive numbers. See Example 7.

79. $5^{3/7}5^{2/7}$ $5^{5/7}$
80. $4^{2/5}4^{2/5}$ $4^{4/5}$
81. $(4^{1/5})^3$ $4^{3/5}$
82. $(3^{1/3})^5$ $3^{5/3}$
83. $(a^{1/2}b^{1/3})^{3/2}$ $a^{3/4}b^{1/2}$
84. $(mn^{-2/3})^{-3/5}$ $\frac{n^{2/5}}{m^{3/5}}$
85. $\frac{b^{4/3}b^{1/3}}{b^{2/3}}$ b
86. $\frac{c^{5/6}c^{1/3}}{c^{1/2}}$ $c^{2/3}$

Perform the multiplications. Assume all variables represent positive numbers. See Example 8.

87. $y^{1/3}(y^{2/3} + y^{5/3})$ $y + y^2$
88. $y^{2/5}(y^{-2/5} + y^{3/5})$ $1 + y$
89. $x^{3/5}(x^{7/5} - x^{2/5} + 1)$ $x^2 - x + x^{3/5}$
90. $x^{4/3}(x^{2/3} + 3x^{5/3} - 4)$ $x^2 + 3x^3 - 4x^{4/3}$

Use rational exponents to simplify each radical. Assume all variables represent positive numbers. See Example 9.

91. $\sqrt[6]{p^3}$ $\sqrt{p}$
92. $\sqrt[8]{q^2}$ $\sqrt[4]{q}$
93. $\sqrt[4]{25b^2}$ $\sqrt{5b}$
94. $\sqrt[9]{-8x^6}$ $-\sqrt[3]{2x^2}$

TRY IT YOURSELF

Simplify each expression, if possible. Assume all variables represent positive numbers.

95. $125^{1/3}$ 5

96. $8^{1/3}$ 2

97. $32^{1/5}$ 2

98. $0^{1/5}$ 0

99. $-16^{1/4}$ -2

100. $-125^{1/3}$ -5

101. $(-64)^{1/2}$ not real

102. $(-216)^{1/2}$ not real

103. $(-27y^3)^{-2/3}$ $\frac{1}{9y^2}$

▶ **104.** $(-8z^9)^{-2/3}$ $\frac{1}{4z^6}$

105. $\left(\frac{27}{8}\right)^{-4/3}$ $\frac{16}{81}$

106. $\left(\frac{25}{49}\right)^{-3/2}$ $\frac{343}{125}$

107. $\left(-\frac{8x^3}{27}\right)^{-1/3}$ $-\frac{3}{2x}$

108. $\left(\frac{16}{81y^4}\right)^{-3/4}$ $\frac{27y^3}{8}$

Write each expression using a radical.

109. $(3x)^{1/4}$ $\sqrt[4]{3x}$

110. $(5ab)^{1/6}$ $\sqrt[6]{5ab}$

111. $(17x^3y)^{1/4}$ $\sqrt[4]{17x^3y}$

112. $(34a^2b^2)^{1/5}$ $\sqrt[5]{34a^2b^2}$

113. $(x^2 + y^2)^{1/2}$ $\sqrt{x^2 + y^2}$

114. $(x^3 + y^3)^{1/3}$ $\sqrt[3]{x^3 + y^3}$

Use a calculator to evaluate each expression. Round to the nearest hundredth.

115. $\sqrt[3]{15}$ 2.47

116. $\sqrt[4]{50.5}$ 2.67

117. $\sqrt[5]{1.045}$ 1.01

118. $\sqrt[5]{-1{,}000}$ -3.98

Simplify each expression. Assume all variables represent positive numbers.

119. $\frac{9^{4/5}}{9^{3/5}}$ $9^{1/5}$

120. $\frac{7^{2/3}}{7^{1/2}}$ $7^{1/6}$

121. $6^{-2/3}6^{-4/3}$ $\frac{1}{36}$

122. $5^{1/3}5^{-5/3}$ $\frac{1}{5^{4/3}}$

123. $a^{2/3}a^{1/3}$ a

▶ **124.** $b^{3/5}b^{1/5}$ $b^{4/5}$

125. $(a^{2/3})^{1/3}$ $a^{2/9}$

126. $(t^{4/5})^{10}$ t^8

127. $\frac{(4x^3y)^{1/2}}{(9xy)^{1/2}}$ $\frac{2x}{3}$

128. $\frac{(27x^3y)^{1/3}}{(8xy^2)^{2/3}}$ $\frac{3x^{1/3}}{4y}$

129. $(27x^{-3})^{-1/3}$ $\frac{1}{3}x$

130. $(16a^{-2})^{-1/2}$ $\frac{1}{4}a$

131. $(2x^2y^{-1/4})^3(8y^{-2})^{2/3}$ $\frac{32x^6}{y^{25/12}}$

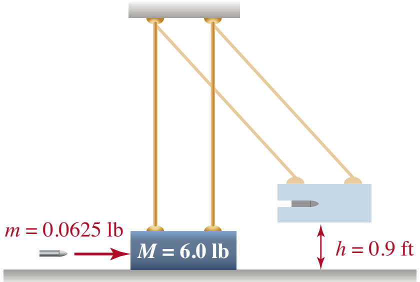
132. $(27a^3b)^{-1/3}(9a^{-2}b^2)^{-1/2}$ $\frac{1}{9b^{4/3}}$

APPLICATIONS

▶ **133.** BALLISTIC PENDULUMS See the illustration in the next column. The formula

$$v = \frac{m + M}{m}(2gh)^{1/2}$$

gives the velocity (in ft/sec) of a bullet with weight m fired into a block with weight M, that raises the height of the block h feet after the collision. The letter g represents the constant, 32. Find the velocity of the bullet to the nearest ft/sec. 736 ft/sec



▶ **134.** GEOGRAPHY The formula

$$A = [s(s - a)(s - b)(s - c)]^{1/2}$$

gives the area of a triangle with sides of length a, b, and c, where s is one-half of the perimeter. Estimate the area of Virginia (to the nearest square mile) using the data given below. 40,700 mi²

▶ **135.** RELATIVITY One of the concepts of relativity theory is that an object moving past an observer at a speed near the speed of light appears to have a larger mass because of its motion. If the mass of the object is m_0 when the object is at rest relative to the observer, its mass m will be given by the formula

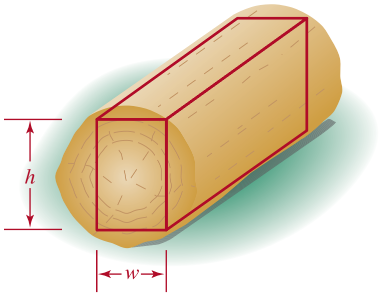
$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

when it is moving with speed v (in miles per second) past the observer. The letter c is the speed of light, 186,000 mi/sec. If a proton with a rest mass of 1 unit is accelerated by a nuclear accelerator to a speed of 160,000 mi/sec, what mass will the technicians observe it to have? Round to the nearest hundredth. 1.96 units

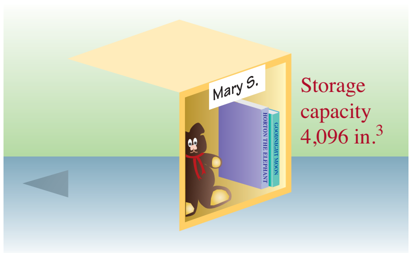
136. LOGGING The width w and height h of the strongest rectangular beam that can be cut from a cylindrical log of radius a are given by

$$w = \frac{2a}{3}(3^{1/2}) \qquad h = a\left(\frac{8}{3}\right)^{1/2}$$

Find the width, height, and cross-sectional area of the strongest beam that can be cut from a log with *diameter* 4 feet. Round to the nearest hundredth. 2.31 ft, 3.27 ft, 7.55 ft^2



137. CUBICLES The area of the base of a cube is given by the function $A(V) = V^{2/3}$, where V is the volume of the cube. In a preschool room, 18 children's cubicles like that shown are placed on the floor around the room. Estimate how much floor space is lost to the cubicles. Give your answer in square inches and in square feet. 4,608 in.^{2}, 32 ft^2



138. The length L of the longest board that can be carried horizontally around the right-angle corner of two intersecting hallways is given by the formula

$$L = (a^{2/3} + b^{2/3})^{3/2}$$

where a and b represent the widths of the hallways. Find the longest shelf that a carpenter can carry around the corner if $a = 40$ in. and $b = 64$ in. Give your result in inches and in feet. In each case, round to the nearest tenth. 145.8 in. or 12.1 ft

from Campus to Careers
General Contractor

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WRITING

139. What is a rational exponent? Give some examples.

140. Explain how the root key $\boxed{\sqrt[x]{y}}$ on a scientific calculator can be used in combination with other keys to evaluate the expression $16^{3/4}$.

REVIEW

Solve each inequality. Write the solution set using interval notation.

141. $5x - 4 < 11$
$(-\infty, 3)$

142. $-2(3t - 5) \geq 8$
$\left(-\infty, \frac{1}{3}\right]$

143. $\frac{4}{5}(r - 3) > \frac{2}{3}(r + 2)$
$(28, \infty)$

144. $-4 < 2x - 4 \leq 8$
$(0, 6]$

SECTION 7.6
Geometric Applications of Radicals

Objectives

1. Use the Pythagorean theorem to solve problems.
2. Solve problems involving 45°–45°–90° triangles.
3. Solve problems involving 30°–60°–90° triangles.
4. Use the distance formula to solve problems.

We will now consider applications of square roots that occur in geometry. Then we will find the distance between two points on a rectangular coordinate system, using a formula that contains a square root. We begin by considering an important theorem (mathematical statement) about right triangles.

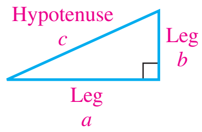
1 Use the Pythagorean theorem to solve problems.

If we know the lengths of two legs of a right triangle, we can find the length of the hypotenuse (the side opposite the 90° angle) by using the **Pythagorean theorem.**

Pythagorean Theorem

If a and b represent the lengths of the legs of a right triangle and c represents the length of the hypotenuse,

$$a^2 + b^2 = c^2$$

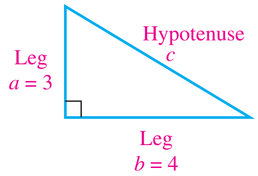


In words, the Pythagorean theorem is expressed as follows:

In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the two legs.

Suppose the right triangle shown in the figure has legs of length 3 and 4 units. To find the length of the hypotenuse, we use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = c^2 \quad \text{Substitute 3 for } a \text{ and 4 for } b.$$
$$9 + 16 = c^2$$
$$25 = c^2$$



To solve for c, we ask, "What number, when squared, is equal to 25?" There are two such numbers: the positive square root of 25 and the negative square root of 25. Since c represents the length of the hypotenuse, and it cannot be negative, it follows that c is the positive square root of 25.

$$\sqrt{25} = \sqrt{c^2} \quad \text{Since } c \text{ represents the length of a side of a triangle, } c > 0.$$
$$5 = c$$

The length of the hypotenuse is 5 units.

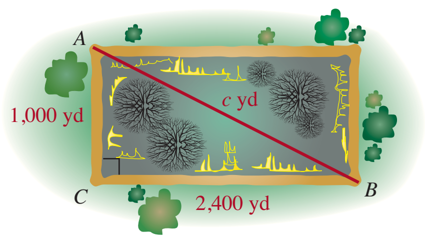
EXAMPLE 1 ***Firefighting*** To fight a forest fire, the forestry department plans to clear a rectangular fire break around the fire, as shown in the figure below. Crews are equipped with mobile communications that have a 3,000-yard range. Can crews at points A and B remain in radio contact?

Strategy We will use the Pythagorean theorem to find the distance between points A and B.

WHY If this distance is less than 3,000 yards, they can communicate. If it is greater than 3,000 yards, they cannot communicate.

Solution

Points A, B, and C form a right triangle. To find the distance c from point A to point B, we can use the Pythagorean theorem, substituting 2,400 for a and 1,000 for b and solving for c.



$$a^2 + b^2 = c^2$$
$$2{,}400^2 + 1{,}000^2 = c^2$$
$$5{,}760{,}000 + 1{,}000{,}000 = c^2$$
$$6{,}760{,}000 = c^2$$
$$\sqrt{6{,}760{,}000} = \sqrt{c^2} \quad \text{Take the positive square root of both sides.}$$
$$2{,}600 = c \quad \text{Use a calculator to find the square root.}$$

The two crews are 2,600 yards apart. Because this distance is less than the range of the radios, they can communicate by radio.

Self Check 1

FIREFIGHTING In Example 1, can the crews communicate if $b = 1{,}500$ yards? yes

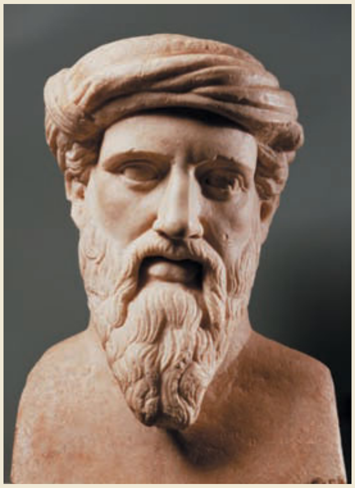
Now Try **Problem 18**

Teaching Example 1 FIREFIGHTING In Example 1, can the crews communicate if $b = 2{,}000$ yards?
Answer:
no

THINK IT THROUGH *Pythagorean Triples*

"Fraternity and sorority membership nationwide is declining, down about 30% in the last decade."

Chronicle of Higher Education, 2002



SEF/Art Resource, NY

The first college social fraternity, Phi Beta Kappa, was founded in 1776 on the campus of The College of William and Mary. However, secret societies have existed since ancient times, and from these roots the essence of today's fraternities and sororities have their foundation.

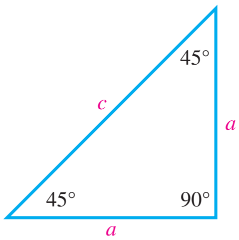
Pythagoras, the Greek mathematician of the 6th century B.C., was the leader of a secret fraternity/sorority called the Pythagoreans. They were a community of men and women that studied mathematics, and in particular, the "magic 3-4-5 triangle." This right triangle is special because the sum of the squares of the lengths of its legs is equal to the square of the length of its hypotenuse: $3^2 + 4^2 = 5^2$ or $9 + 16 = 25$. Today, we call a set of three natural numbers a, b, and c that satisfy $a^2 + b^2 = c^2$ a **Pythagorean triple.** Show that each list of numbers is a Pythagorean triple.

1. 5, 12, 13 **2.** 7, 24, 25 **3.** 8, 15, 17

4. 9, 40, 41 **5.** 28, 45, 53 **6.** 15, 112, 113

2 Solve problems involving 45°–45°–90° triangles.

An **isosceles right triangle** is a right triangle with two legs of equal length. Isosceles right triangles have angle measures of 45°, 45°, and 90°. If we know the length of one leg of an isosceles right triangle, we can use the Pythagorean theorem to find the length of the hypotenuse. Since the triangle shown is a right triangle, we have



$$c^2 = a^2 + b^2$$

$$c^2 = a^2 + a^2 \quad \text{Both legs are } a \text{ units long, so replace } b \text{ with } a.$$

$$c^2 = 2a^2 \quad \text{Combine like terms.}$$

$$c = \sqrt{2a^2} \quad \text{Take the positive square root of both sides.}$$

$$c = a\sqrt{2} \quad \text{Simplify the radical: } \sqrt{2a^2} = \sqrt{2}\sqrt{a^2} = \sqrt{2}a = a\sqrt{2}.$$

Thus, *in an isosceles right triangle, the length of the hypotenuse is the length of one leg times* $\sqrt{2}$.

Self Check 2

Find the length of the hypotenuse of an isosceles right triangle if one leg is 12 meters long. $12\sqrt{2}$ m

Now Try **Problem 21**

EXAMPLE 2 If one leg of an isosceles right triangle is 10 feet long, find the length of the hypotenuse. Then approximate the length to two decimal places.

Strategy We will multiply the length of the known leg by $\sqrt{2}$.

WHY The length of the hypotenuse of an isosceles right triangle is $\sqrt{2}$ times the length of one leg.

Solution

Since the length of the hypotenuse is the length of a leg times $\sqrt{2}$, we have

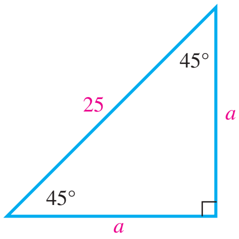
$$c = 10\sqrt{2}$$

The length of the hypotenuse is $10\sqrt{2}$ units. To two decimal places, the length is 14.14 units.

Teaching Example 2 Find the length of the hypotenuse of an isosceles right triangle if one leg is 14 cm.
Answer:
$14\sqrt{2}$ cm

If the length of the hypotenuse of an isosceles right triangle is known, we can use the Pythagorean theorem to find the length of each leg.

EXAMPLE 3 ***Isosceles Right Triangle*** Find the exact length of each leg of the isosceles right triangle shown to the right. Then approximate the length to two decimal places.



Strategy We will use the Pythagorean theorem to form an equation that we can solve to find the unknown length of a leg of the triangle.

WHY We use the Pythagorean theorem because the triangle is a right triangle and we are given the length of its hypotenuse.

Solution

We use the Pythagorean theorem.

$$c^2 = a^2 + b^2$$

$$25^2 = a^2 + a^2$$ Since both legs are a units long, substitute a for b. The hypotenuse is 25 units long. Substitute 25 for c.

$$25^2 = 2a^2$$ Combine like terms.

$$\frac{625}{2} = a^2$$ To isolate a^2, square 25 and divide both sides by 2.

$$\sqrt{\frac{625}{2}} = a$$ To solve for a, take the positive square root of both sides: $\sqrt{a^2} = a$.

$$\frac{\sqrt{625}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = a$$ Write $\sqrt{\frac{625}{2}}$ as $\frac{\sqrt{625}}{\sqrt{2}}$. Then rationalize the denominator.

$$\frac{25\sqrt{2}}{2} = a$$ In the numerator, simplify the radical: $\sqrt{625} = 25$. In the denominator, do the multiplication: $\sqrt{2} \cdot \sqrt{2} = 2$.

The exact length of each leg is $\frac{25\sqrt{2}}{2}$ units. To two decimal places, the length is 17.68 units.

Self Check 3

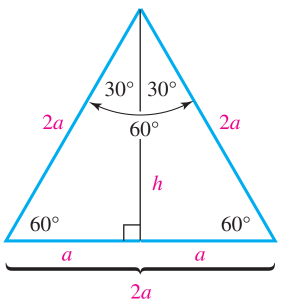
Find the length of each leg of an isosceles right triangle if the hypotenuse is 5 feet. $\frac{5\sqrt{2}}{2}$ ft

Now Try **Problem 23**

Teaching Example 3 Find the length of each leg of an isosceles right triangle if the hypotenuse is 18 meters.
Answer:
$9\sqrt{2}$ m

3 Solve problems involving 30°–60°–90° triangles.

From geometry, we know that an **equilateral triangle** is a triangle with three sides of equal length and three 60° angles. Each side of the equilateral triangle shown at the right is $2a$ units long. If an **altitude** is drawn to its base, the altitude bisects the base and divides the equilateral triangle into two 30°–60°–90° triangles. We can see that the shorter leg of each 30°–60°–90° triangle (the side *opposite* the 30° angle) is a units long. Thus,



The length of the shorter leg of a 30°–60°–90° right triangle is half as long as the hypotenuse.

We can discover another important relationship between the legs of a 30°–60°–90° triangle if we find the length of the altitude h in the figure. We begin by applying the Pythagorean theorem to one of the 30°–60°–90° triangles.

$$a^2 + b^2 = c^2$$

$$a^2 + h^2 = (2a)^2 \quad \text{One leg is } h \text{ units long, so replace } b \text{ with } h. \text{ The hypotenuse is } 2a \text{ units long, so replace } c \text{ with } 2a.$$

$$a^2 + h^2 = 4a^2 \quad (2a)^2 = (2a)(2a) = 4a^2.$$

$$h^2 = 3a^2 \quad \text{Subtract } a^2 \text{ from both sides.}$$

$$h = \sqrt{3a^2} \quad \text{Take the positive square root of both sides.}$$

$$h = a\sqrt{3} \quad \text{Simplify the radical: } \sqrt{3a^2} = \sqrt{3}\sqrt{a^2} = a\sqrt{3}.$$

We see that the altitude (the longer side of the 30°–60°–90° triangle) is $\sqrt{3}$ times as long as the shorter leg. Thus,

The length of the longer leg of a 30°–60°–90° *triangle is the length of the shorter leg times* $\sqrt{3}$.

Self Check 4

Find the length of the hypotenuse and the longer leg of a 30°–60°–90° triangle if the shorter leg is 8 centimeters long.

Now Try **Problem 27**

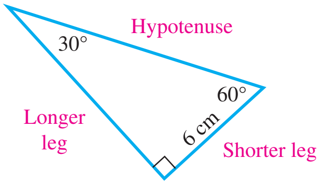
Self Check 4 Answer

16 cm, $8\sqrt{3}$ cm

Teaching Example 4 Find the length of the hypotenuse and the longer leg of a 30°–60°–90° triangle if the shorter leg is 10 cm long.
Answer:
20 cm, $10\sqrt{3}$ cm

EXAMPLE 4 ***30°–60°–90° Triangles*** Find the length of the hypotenuse and the longer leg of the right triangle shown.

Strategy To find the length of the hypotenuse, we will multiply the length of the shorter side by 2. To find the length of the longer leg, we will multiply the length of the shorter leg by $\sqrt{3}$.



WHY These side-length relationships are true for any 30°–60°–90° triangle.

Solution

Since the shorter leg of a 30°–60°–90° triangle is half as long as the hypotenuse, the hypotenuse is 12 centimeters long.

Since the length of the longer leg is the length of the shorter leg times $\sqrt{3}$, the longer leg is $6\sqrt{3}$ (about 10.39) centimeters long.

Self Check 5

Find the length of the hypotenuse and the shorter leg of a 30°–60°–90° triangle if the longer leg is 15 feet long. Then approximate the lengths to two decimal places.

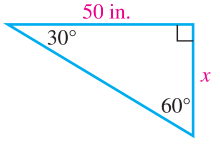
Now Try **Problem 28**

Self Check 5 Answer

$10\sqrt{3}$ in. ≈ 17.32 in., $5\sqrt{3}$ in. ≈ 8.66 in.

Teaching Example 5 Find the length of the shorter leg and the length of the hypotenuse of the 30°–60°–90° triangle if the longer leg is 16 cm long.
Answer:
$\frac{16\sqrt{3}}{3}$ cm, $\frac{32\sqrt{3}}{3}$ cm

EXAMPLE 5 Find the length of the hypotenuse and the length of the shorter leg of the 30°–60°–90° triangle shown. Then approximate the lengths to two decimal places.



Strategy We will find the length of the shorter leg first.

WHY Once we know the length of the shorter leg, we can multiply it by 2 to find the length of the hypotenuse.

Solution

If we let x = the length in inches of the shorter leg of the triangle, we can form an equation by translating the following statement:

The length of the longer leg of a 30°–60°–90° triangle	is	$\sqrt{3}$	times	the length of the shorter leg.
50	=	$\sqrt{3}$	·	x

To find the length of the shorter leg, we solve the equation for x.

$$50 = \sqrt{3}x$$

$$\frac{50}{\sqrt{3}} = \frac{\sqrt{3}x}{\sqrt{3}} \qquad \text{To isolate } x\text{, divide both sides by } \sqrt{3}.$$

$$\frac{50}{\sqrt{3}} = x$$

The length of the shorter leg is exactly $\frac{50}{\sqrt{3}}$ inches. To write this number in simplified radical form, we rationalize the denominator.

$$\frac{50}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{50\sqrt{3}}{3}$$

Thus, the length of the shorter leg is exactly $\frac{50\sqrt{3}}{3}$ inches (about 28.87 inches).

Since the length of the hypotenuse of a 30°–60°–90° triangle is twice as long as the shorter leg, the hypotenuse is $2 \cdot \frac{50\sqrt{3}}{3} = \frac{100\sqrt{3}}{3}$ inches (about 57.74 inches).

Success Tip In a 30°–60°–90° triangle, the side opposite the 30° angle is the shorter leg, the side opposite the 60° angle is the longer leg, and the hypotenuse is opposite the 90° angle.

EXAMPLE 6 ***Stretching Exercises*** A doctor prescribed the back-strengthening exercise shown in figure (a) for a patient. The patient was instructed to raise his leg to an angle of 60° and hold the position for 10 seconds. If the patient's leg is 36 inches long, how high off the floor will his foot be when his leg is held at the proper angle?

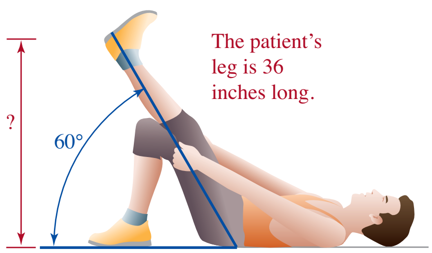
Strategy This situation is modeled by a 30°–60°–90° triangle. We will begin by finding the length of the shorter leg.

WHY Once we find the length of the shorter leg, we can easily find the length of the longer leg, which represents the distance the patient's foot is off the ground.

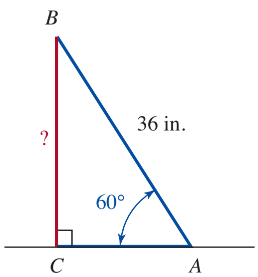
Solution

In figure (b), we see that a 30°–60°–90° triangle, which we will call triangle ABC, models the situation. Since the side opposite the 30° angle of a 30°–60°–90° triangle is half as long as the hypotenuse, side AC is 18 inches long.

Since the length of the side opposite the 60° angle is the length of the side opposite the 30° angle times $\sqrt{3}$, side BC is $18\sqrt{3}$, or about 31 inches long. So the patient's foot will be about 31 inches from the floor when his leg is in the proper stretching position.



(a)



(b)

Self Check 6

STRETCHING EXERCISES Refer to Example 6. The doctor prescribed the same exercise for a patient whose leg is 29 inches long. Find how high off the floor this patient's foot is when her leg is held at the proper angle.

Now Try **Problem 22**

Self Check 6 Answer
$\frac{29\sqrt{3}}{2}$ in. $\approx$ 25 in.

Teaching Example 6 STRETCHING EXERCISES Repeat Example 6 for a patient whose leg is 30 inches long.
Answer:
$15\sqrt{3}$ in. $\approx$ 26 in.

4 Use the distance formula to solve problems.

With the *distance formula,* we can find the distance between any two points that are graphed on a rectangular coordinate system.

To find the distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ shown in the figure below, we construct the right triangle PRQ. The distance between P and R is $|x_2 - x_1|$, and the distance between R and Q is $|y_2 - y_1|$. We apply the Pythagorean theorem to the right triangle PRQ to get

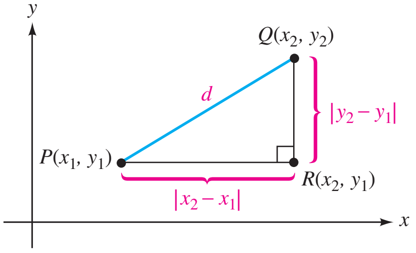
$$\begin{aligned} d^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$

Because $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$.

Because d represents the distance between two points, we take the positive square root of both sides:

(1) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Equation 1 is called the **distance formula.**



Distance Formula

The distance d between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Self Check 7

Find the distance between $(-2, -2)$ and $(3, 10)$. 13

Now Try **Problem 34**

Teaching Example 7 Find the distance between $(-5, 2)$ and $(4, 14)$.
Answer:
15

EXAMPLE 7 Find the distance between points $(-2, 3)$ and $(4, -5)$.

Strategy We will use the distance formula.

WHY We know the x- and y-coordinates of both points.

Solution

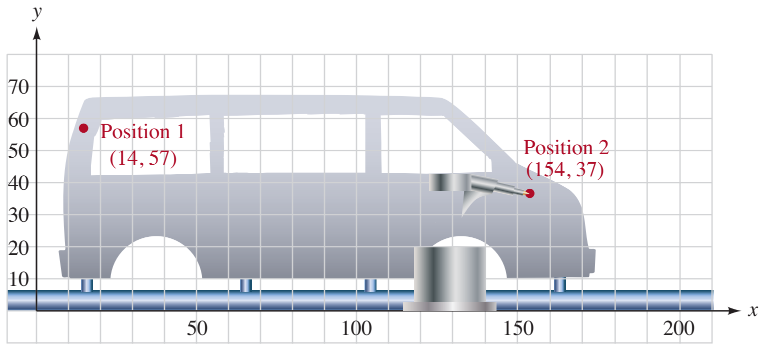
To find the distance, we can use the distance formula by substituting 4 for x_2, -2 for x_1, -5 for y_2, and 3 for y_1.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[4 - (-2)]^2 + (-5 - 3)^2} \\ &= \sqrt{(4 + 2)^2 + (-5 - 3)^2} \\ &= \sqrt{6^2 + (-8)^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

The distance between the points is 10 units.

EXAMPLE 8 ***Robotics*** Computerized robots are used to weld the parts of an automobile chassis together on an automated production line. To do this, an imaginary coordinate system is superimposed on the side of the vehicle, and the robot is programmed to move to specific positions to make each weld. See the

figure below, which is scaled in inches. If the welder unit moves from point to point at an average rate of speed of 48 in. per sec, how long will it take it to move from position 1 to position 2?



Strategy We will use the distance formula to find how far the tip of the welder unit moves.

WHY We know the x- and y-coordinates of position 1 and position 2.

Solution

This is a uniform motion problem. We can use the formula $t = \frac{d}{r}$ to find the time it takes for the welder to move from position 1 coordinates (14, 57) to position 2 with coordinates (154, 37).

We can use the distance formula to find the distance d that the welder unit moves.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(154 - 14)^2 + (37 - 57)^2}$$ Substitute 154 for x_2, 14 for x_1, 37 for y_2, and 57 for y_1.

$$= \sqrt{140^2 + (-20)^2}$$

$$= \sqrt{20{,}000}$$ $140^2 + (-20)^2 = 19{,}600 + 400 = 20{,}000$.

$$= 100\sqrt{2}$$ Simplify: $\sqrt{20{,}000} = \sqrt{100 \cdot 100 \cdot 2} = \sqrt{100^2 \cdot 2} = 100\sqrt{2}$.

The welder travels $100\sqrt{2}$ inches as it moves from position 1 to position 2. To find the time this will take, we divide the distance by the average rate of speed, 48 in. per sec.

$$t = \frac{d}{r}$$

$$t = \frac{100\sqrt{2}}{48}$$ Substitute $100\sqrt{2}$ for d and 48 for r.

$$t \approx 2.9$$ Use a calculator to find an approximation to the nearest tenth.

It will take the welder about 2.9 seconds to travel from position 1 to position 2.

Self Check 8

ROBOTICS Refer to Example 8. The welder unit moves from position 2 to a new position at (100, 35). How long will it take the welder unit to move from position 2 to the new position? about 1.1 sec

Now Try **Problem 49**

Teaching Example 8 ROBOTICS The welder unit in Example 8 is to move from position 2 to a new position to make a weld at (160, 17). How long will it take the welder to move to this new position?
Answer:
≈ 0.4 min

ANSWERS TO SELF CHECKS

1. yes **2.** $12\sqrt{2}$ m **3.** $\frac{5\sqrt{2}}{2}$ ft **4.** 16 cm, $8\sqrt{3}$ cm **5.** $10\sqrt{3}$ in. ≈ 17.32 in., $5\sqrt{3}$ in. ≈ 8.66 in. **6.** $\frac{29\sqrt{3}}{2}$ in. ≈ 25 in. **7.** 13 **8.** about 1.1 sec

SECTION 7.6 STUDY SET

VOCABULARY

Fill in the blanks.

1. In a right triangle, the side opposite the 90° angle is called the hypotenuse.

2. An isosceles right triangle is a right triangle with two legs of equal length.

3. The Pythagorean theorem states that in any right triangle, the square of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

4. An equilateral triangle has three sides of equal length and three 60° angles.

CONCEPTS

Fill in the blanks.

5. If a and b are the lengths of two legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.

6. In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

7. In an isosceles right triangle, the length of the hypotenuse is the length of one leg times $\sqrt{2}$.

8. The shorter leg of a 30°–60°–90° triangle is half as long as the hypotenuse.

9. The length of the longer leg of a 30°–60°–90° triangle is the length of the shorter leg times $\sqrt{3}$.

10. The formula to find the distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

11. In a right triangle, the shorter leg is opposite the 30° angle, and the longer leg is opposite the 60° angle.

12. An isosceles triangle has two sides of equal length.

13. To solve the equation $c^2 = 20$, where c represents the length of the hypotenuse of a right triangle, how do we undo the operation performed on c? Take the positive square root of both sides.

14. What is the first step when solving the equation $25 + b^2 = 81$? Subtract 25 from both sides.

NOTATION

Complete each solution.

15. Evaluate: $\sqrt{(-1-3)^2 + [2-(-4)]^2}$

$$\begin{aligned}\sqrt{(-1-3)^2 + [2-(-4)]^2} &= \sqrt{(-4)^2 + [6]^2}\\ &= \sqrt{52}\\ &= \sqrt{4 \cdot 13}\\ &= 2\sqrt{13}\\ &\approx 7.21\end{aligned}$$

16. Solve: $8^2 + 4^2 = c^2$

$$\begin{aligned}64 + 16 &= c^2\\ 80 &= c^2\\ \sqrt{80} &= \sqrt{c^2}\\ \sqrt{16 \cdot 5} &= c\\ 4\sqrt{5} &= c\\ c &\approx 8.94\end{aligned}$$

GUIDED PRACTICE

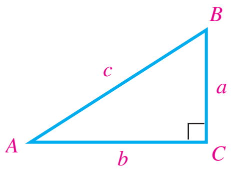
The lengths of two sides of the right triangle ABC are given. Find the length of the missing side. See Example 1.

17. $a = 6$ ft and $b = 8$ ft 10 ft

18. $a = 10$ cm and $c = 26$ cm 24 cm

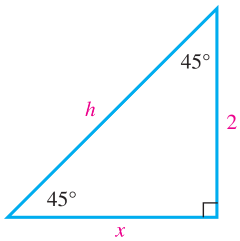
19. $b = 18$ m and $c = 82$ m 80 m

20. $a = 14$ in. and $c = 50$ in. 48 in.

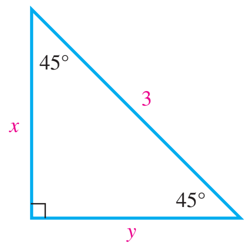


Find the missing lengths in each triangle. Give the exact answer and then as an approximation to two decimal places, when applicable. See Examples 2–3.

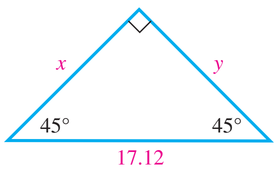
21. $h = 2\sqrt{2} \approx 2.83$, $x = 2$



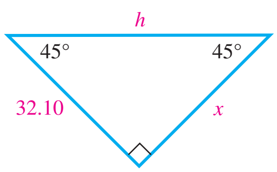
22. $x = \frac{3\sqrt{2}}{2} \approx 2.12$, $y = \frac{3\sqrt{2}}{2} \approx 2.12$



23. $x = 12.11$, $y = 12.11$

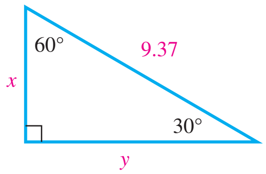


24. $x = 32.10$, $h = 45.40$

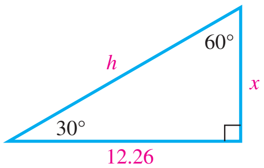


Find the missing lengths in each triangle. Give the answer to two decimal places. **See Examples 4–6.**

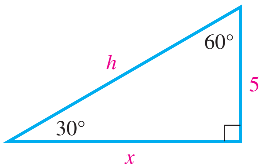
25. $x = 4.69, y = 8.11$



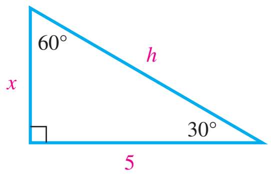
26. $x = 7.08, h = 14.16$



27. $x = 5\sqrt{3} \approx 8.66, h = 10$



28. $x = \frac{7\sqrt{3}}{3} \approx 4.04, h = \frac{14\sqrt{3}}{3} \approx 8.08$



Find the distance between each pair of points. **See Example 7.**

29. $(0, 0), (3, -4)$ 5

30. $(0, 0), (-12, 16)$ 20

31. $(-2, -8), (3, 4)$ 13

32. $(-5, -2), (7, 3)$ 13

33. $(6, 8), (12, 16)$ 10

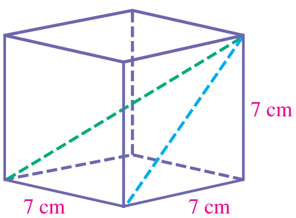
34. $(10, 4), (2, -2)$ 10

35. $(-3, 5), (-5, -5)$ $2\sqrt{26}$

36. $(2, -3), (4, -8)$ $\sqrt{29}$

APPLICATIONS

37. GEOMETRY Find the exact length of the diagonal (in blue) of one of the *faces* of the cube shown. $7\sqrt{2}$ cm



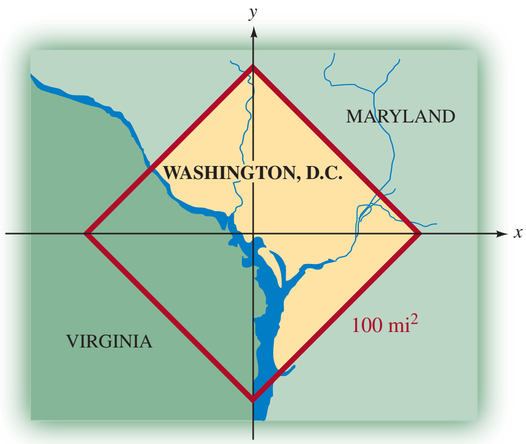
38. GEOMETRY Find the exact length of the diagonal (in green) of the cube shown at the right. $7\sqrt{3}$ cm

39. ISOSCELES TRIANGLES Use the distance formula to show that a triangle with vertices $(-2, 4)$, $(2, 8)$, and $(6, 4)$ is isosceles.

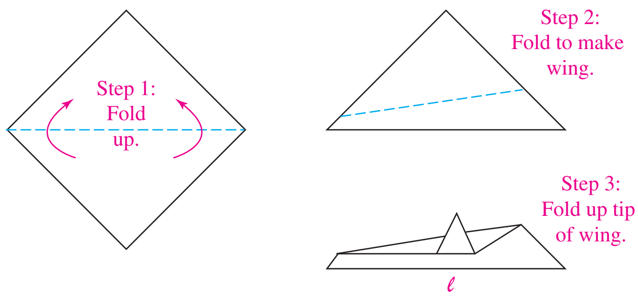
40. RIGHT TRIANGLES Use the distance formula and the Pythagorean theorem to show that a triangle with vertices $(2, 3)$, $(-3, 4)$, and $(1, -2)$ is a right triangle.

Give the exact answer and round it to two decimal places.

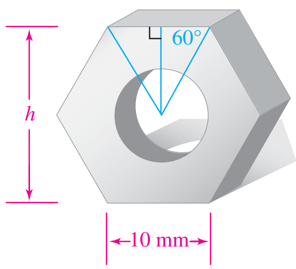
41. WASHINGTON, D.C. The square shows the 100-square-mile site selected by George Washington in 1790 to serve as a permanent capital for the United States. In 1847, the part of the district lying on the west bank of the Potomac was returned to Virginia. Find the coordinates of each corner of the original square that outlined the District of Columbia.
$(5\sqrt{2}, 0), (0, 5\sqrt{2}), (-5\sqrt{2}, 0), (0, -5\sqrt{2})$; $(7.07, 0), (0, 7.07), (-7.07, 0), (0, -7.07)$



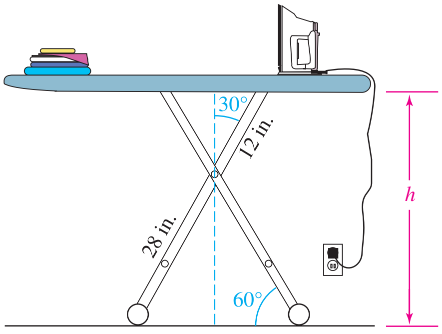
42. PAPER AIRPLANES The illustration below gives the directions for making a paper airplane from a square piece of paper with sides 8 inches long. Find the length l of the plane when it is completed. $8\sqrt{2}$ in., 11.31 in.



43. HARDWARE The sides of the regular hexagonal nut shown at the right are 10 millimeters long. Find the height h of the nut. $10\sqrt{3}$ mm, 17.32 mm

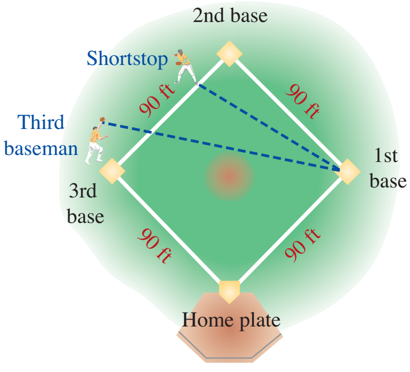


44. IRONING BOARDS Find the height h of the ironing board shown below. $20\sqrt{3}$ in., 34.64 in.

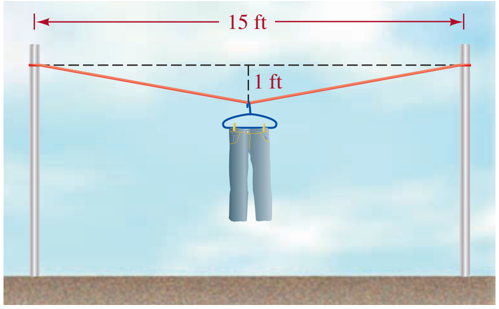


45. BASEBALL The baseball diamond shown below is a square, 90 feet on a side. If the third baseman fields a ground ball 10 feet directly behind third base, how far must he throw the ball to throw a runner out at first base? $10\sqrt{181}$ ft, 134.54 ft

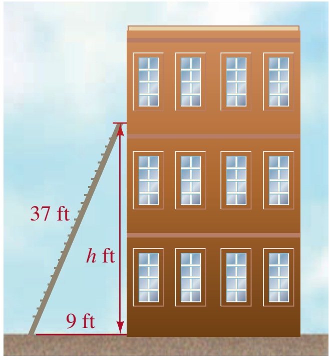
46. BASEBALL A shortstop fields a grounder at a point one-third of the way from second base to third base. (See the illustration below.) How far will he have to throw the ball to make an out at first base? $30\sqrt{10}$ ft, 94.87 ft



47. CLOTHESLINES A pair of damp jeans are hung on a clothesline to dry. They pull the center down 1 foot. By how much is the line stretched? Give the answer to the nearest hundredth. about 0.13 ft



48. FIREFIGHTING The base of the 37-foot ladder shown below is 9 feet from the wall. Will the top reach a window ledge that is 35 feet above the ground? Explain how you arrived at your answer. yes

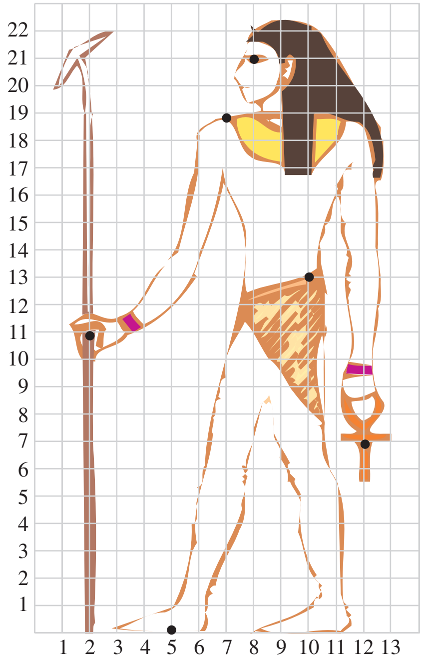


49. ART HISTORY A figure displaying some of the basic characteristics of Egyptian art is shown below. Use the distance formula to find the following dimensions of the drawing. Round your answers to two decimal places.

a. From the foot to the eye 21.21 units

b. From the belt to the hand holding the staff 8.25 units

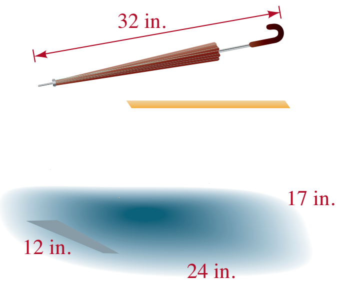
c. From the shoulder to the symbol held in the hand 13.00 units



50. PACKAGING The diagonal d of a rectangular box with dimensions $a \times b \times c$ is given by

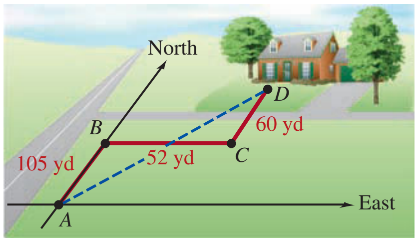
$$d = \sqrt{a^2 + b^2 + c^2}$$

Will the umbrella fit in the shipping carton? Explain how you arrived at your answer. not quite



51. PACKAGING An archaeologist wants to ship a 34-inch femur bone. Will it fit in a 4-inch-tall box that has a 24-inch-square base? (See Exercise 50.) Explain how you arrived at your answer. yes

52. TELEPHONE SERVICE A telephone cable runs from A to B to C to D. How much cable is required to run from A to D directly? 173 yd



WRITING

53. State the Pythagorean theorem in words.

54. List the facts that you learned about special right triangles in this section.

REVIEW

55. DISCOUNT BUYING A repairman purchased some washing machine motors for a total of \$224. When the unit cost decreased by \$4, he was able to buy one additional motor for the same total price. How many motors did he buy originally? 7

56. AVIATION An airplane can fly 650 miles with the wind in the same amount of time as it can fly 475 miles against the wind. If the wind speed is 40 mph, find the speed of the plane in still air. about 257 mph

57. Find the mean of 16, 6, 10, 4, 5, 13. 9

58. Find the median of 16, 6, 10, 4, 5. 6

SECTION 7.7
Complex Numbers

Objectives

1. Express square roots of negative numbers in terms of *i*.
2. Write complex numbers in the form $a + bi$.
3. Add and subtract complex numbers.
4. Multiply complex numbers.
5. Divide complex numbers.
6. Perform operations involving powers of *i*.

Recall that the square root of a negative number is not a real number. However, an expanded number system, called the *complex number system,* gives meaning to square roots of negative numbers, such as $\sqrt{-9}$ and $\sqrt{-25}$. To define complex numbers, we use a number that is denoted by the letter i.

1 Express square roots of negative numbers in terms of *i*.

Some equations do not have real-number solutions. For example, $x^2 = -1$ has no real-number solutions because the square of a real number is never negative. To provide a solution to this equation, mathematicians have defined the number i so that $i^2 = -1$.

The Number *i*

The **imaginary number *i*** is defined as

$$i = \sqrt{-1}$$

From the definition, it follows that $i^2 = -1$.

This definition enables us to write the square root of any negative number in terms of i.

We can use extensions of the product and quotient rules for radicals to write the square root of a negative number as the product of a real number and i.

Self Check 1

Write each expression in terms of i:

a. $\sqrt{-25}$ $5i$
b. $-\sqrt{-19}$ $-i\sqrt{19}$
c. $\sqrt{-45}$ $3i\sqrt{5}$
d. $\sqrt{-\frac{50}{81}}$ $\frac{5\sqrt{2}}{9}i$

Now Try **Problems 19, 21, and 27**

Teaching Example 1 Write each expression in terms of i:
a. $\sqrt{-49}$ **b.** $\sqrt{-19}$
c. $-\sqrt{-12}$ **d.** $\sqrt{-\frac{18}{121}}$

Answers:
a. $7i$ **b.** $i\sqrt{19}$
c. $-2i\sqrt{3}$ **d.** $\frac{3\sqrt{2}}{11}i$

EXAMPLE 1 Write each expression in terms of i:

a. $\sqrt{-9}$ **b.** $\sqrt{-7}$ **c.** $-\sqrt{-18}$ **d.** $\sqrt{-\frac{24}{49}}$

Strategy We will write each radicand as the product of -1 and a positive number. Then we will apply the appropriate rules for radicals.

WHY We want our work to produce a factor of $\sqrt{-1}$ so that we can replace it with i.

Solution

After factoring the radicand, we use an extension of the product rule for radicals.

a. $\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1}\sqrt{9} = i \cdot 3 = 3i$ Replace $\sqrt{-1}$ with i.

b. $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$ or $\sqrt{7}i$ Replace $\sqrt{-1}$ with i.

c. $-\sqrt{-18} = -\sqrt{-1 \cdot 9 \cdot 2} = -\sqrt{-1}\sqrt{9}\sqrt{2} = -i \cdot 3 \cdot \sqrt{2} = -3i\sqrt{2}$ or $-3\sqrt{2}i$

d. After factoring the radicand, use an extension of the product and quotient rules for radicals.

$$\sqrt{-\frac{24}{49}} = \sqrt{-1 \cdot \frac{24}{49}} = \frac{\sqrt{-1 \cdot 24}}{\sqrt{49}} = \frac{\sqrt{-1}\sqrt{4}\sqrt{6}}{\sqrt{49}} = \frac{2i\sqrt{6}}{7} \text{ or } \frac{2\sqrt{6}}{7}i$$

The Language of Algebra For years, mathematicians thought numbers like $\sqrt{-9}$ and $\sqrt{-25}$ were useless. In the 17th century, French mathematician René Descartes (1596–1650) called them *imaginary numbers*. Today they have important uses, such as describing alternating electric current.

The results from Example 1 illustrate a rule for simplifying square roots of negative numbers.

Square Root of a Negative Number

For any positive real number b,

$$\sqrt{-b} = i\sqrt{b}$$

To justify this rule, we use the fact that $\sqrt{-1} = i$.

$$\begin{aligned}\sqrt{-b} &= \sqrt{-1 \cdot b}\\ &= \sqrt{-1}\sqrt{b}\\ &= i\sqrt{b}\end{aligned}$$

Success Tip Since it is easy to confuse $\sqrt{b}i$ with $\sqrt{bi}$, we usually write i first so that it is clear that the i is not under the radical symbol.

However, both $i\sqrt{b}$ and $\sqrt{b}i$ are correct.

2 Write complex numbers in the form $a + bi$.

The imaginary number i is used to define *complex numbers.*

Complex Numbers

A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

Complex numbers of the form $a + bi$, where $b \neq 0$, are also called **imaginary numbers.***

For a complex number written in the **standard form** $a + bi$, we call a the **real part** and b the **imaginary part.** Some examples of complex numbers written in standard form are

$$2 + 11i \qquad 6 - 9i \qquad -\frac{1}{2} + 0i \qquad 0 + i\sqrt{3}$$

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$. Thus, $0.5 + 0.9i = \frac{1}{2} + \frac{9}{10}i$ because $0.5 = \frac{1}{2}$ and $0.9 = \frac{9}{10}$.

Success Tip It is acceptable to use $a - bi$ as a substitute for the form $a + bi$. For example:

$$6 - 9i = 6 + (-9)i$$

EXAMPLE 2 Write each number in the form $a + bi$:

a. 6 **b.** $\sqrt{-64}$ **c.** $-2 + \sqrt{-63}$

Strategy We will determine a, the real part, and we will simplify the radical (if necessary) to determine the bi part.

WHY We can put the two parts together to produce the desired $a + bi$ form.

Solution

a. $6 = 6 + 0i$ The real part is 6. The imaginary part is 0.

b. $\sqrt{-64} = 0 + 8i$ The real part is 0. Simplify: $\sqrt{-64} = \sqrt{-1}\sqrt{64} = 8i$.

c. $-2 + \sqrt{-63} = -2 + 3i\sqrt{7}$ The real part is -2. Simplify: $\sqrt{-63} = \sqrt{-1}\sqrt{63} = \sqrt{-1}\sqrt{9}\sqrt{7} = 3i\sqrt{7}$.

Self Check 2

Write each number in the form $a + bi$:

a. -18 $-18 + 0i$

b. $\sqrt{-36}$ $0 + 6i$

c. $1 + \sqrt{-24}$ $1 + 2i\sqrt{6}$

Now Try **Problems 29 and 33**

Teaching Example 2 Write each number in the form $a + bi$:
a. 23 **b.** $\sqrt{-144}$ **c.** $5 + \sqrt{-18}$
Answers:
a. $23 + 0i$ **b.** $0 + 12i$ **c.** $5 + 3\sqrt{2}i$

Success Tip Just as real numbers are either rational or irrational, but not both, complex numbers are either real or imaginary, but not both.

The following illustration shows the relationship between the real numbers, the imaginary numbers, and the complex numbers.

*Some textbooks define imaginary numbers as complex numbers with $a = 0$ and $b \neq 0$.

Complex numbers						
Real numbers				Imaginary numbers		
-6	$\frac{5}{16}$	-1.75	π	$9 + 7i$	$-2i$	$\frac{1}{4} - \frac{3}{4}i$
$48 + 0i$	0	$-\sqrt{10}$	$-\frac{7}{2}$	$0.56i$	$\sqrt{-10}$	$6 + i\sqrt{3}$

3 Add and subtract complex numbers.

Adding and subtracting complex numbers is similar to adding and subtracting polynomials.

Addition and Subtraction of Complex Numbers

1. To add complex numbers, add their real parts and add their imaginary parts.
2. To subtract complex numbers, add the opposite of the complex number being subtracted.

Self Check 3

Perform the operations. Write the answers in the form $a + bi$.

a. $(3 - 5i) + (-2 + 7i)$

b. $\left(3 - \sqrt{-25}\right) - \left(-2 + \sqrt{-49}\right)$

Now Try **Problems 37 and 43**

Self Check 3 Answers

a. $1 + 2i$ **b.** $5 - 12i$

Teaching Example 3 Perform each operation. Write the answers in the form $a + bi$.

a. $(4 - 14i) + (12 + 56i)$

b. $(22 - 11i) - (5 + 2i)$

c. $\left(9 - \sqrt{-49}\right) + \left(2 - \sqrt{-64}\right)$

Answers:

a. $16 + 42i$

b. $17 - 13i$

c. $11 - 15i$

EXAMPLE 3 Perform each operation. Write the answers in the form $a + bi$.

a. $(8 + 4i) + (12 + 8i)$ **b.** $(-6 + 4i) - (3 + 2i)$

c. $\left(7 - \sqrt{-16}\right) + \left(9 + \sqrt{-4}\right)$

Strategy To add the complex numbers, we will add their real parts and add their imaginary parts. To subtract the complex numbers, we will add the opposite of the complex number to be subtracted.

WHY We perform the indicated operations as if the complex numbers were polynomials with i as a variable.

Solution

a. $(8 + 4i) + (12 + 8i) = (8 + 12) + (4 + 8)i$

The sum of the imaginary parts

The sum of the real parts

$= 20 + 12i$

b. $(-6 + 4i) - (3 + 2i) = (-6 + 4i) + (-3 - 2i)$ — Add; the opposite

To find the opposite, change the sign of each term of $3 + 2i$.

$= [-6 + (-3)] + [4 + (-2)]i$ Add the real parts. Add the imaginary parts.

$= -9 + 2i$

c. $\left(7 - \sqrt{-16}\right) + \left(9 + \sqrt{-4}\right)$

$= (7 - 4i) + (9 + 2i)$ Write $\sqrt{-16}$ and $\sqrt{-4}$ in terms of i.

$= (7 + 9) + (-4 + 2)i$ Add the real parts. Add the imaginary parts.

$= 16 - 2i$ Write $16 + (-2i)$ in the form $16 - 2i$.

Success Tip Always change complex numbers to $a + bi$ form before performing any arithmetic.

4 Multiply complex numbers.

Since imaginary numbers are not real numbers, some properties of real numbers do not apply to imaginary numbers. For example, we cannot use the product rule for radicals to multiply two imaginary numbers.

Caution! If a and b are both negative, then $\sqrt{a}\sqrt{b} \neq \sqrt{ab}$. For example, if $a = -4$ and $b = -9$,

$$\cancel{\sqrt{-4}\sqrt{-9} = \sqrt{-4(-9)} = \sqrt{36} = 6}$$

$$\sqrt{-4}\sqrt{-9} = 2i(3i) = 6i^2 = 6(-1) = -6$$

EXAMPLE 4 Multiply: $\sqrt{-2}\sqrt{-20}$

Strategy To multiply the imaginary numbers, we will first write $\sqrt{-2}$ and $\sqrt{-20}$ in $i\sqrt{b}$ form. Then we will use the product rule for radicals.

WHY We cannot immediately use the product rule for radicals because it does not apply when both radicands are negative.

Solution

$$\begin{aligned}\sqrt{-2}\sqrt{-20} &= \left(i\sqrt{2}\right)\left(2i\sqrt{5}\right) && \text{Simplify: } \sqrt{-20} = i\sqrt{20} = 2i\sqrt{5}.\\ &= 2i^2\sqrt{2\cdot 5} && \text{Multiply: } i\cdot 2i = 2i^2. \text{ Use the product rule for radicals.}\\ &= 2i^2\sqrt{10} &&\\ &= 2(-1)\sqrt{10} && \text{Replace } i^2 \text{ with } -1.\\ &= -2\sqrt{10} && \text{Multiply.}\end{aligned}$$

Self Check 4

Multiply: $\sqrt{-3}\sqrt{-32}$ $-4\sqrt{6}$

Now Try **Problem 47**

Teaching Example 4 Multiply: $\sqrt{-5}\sqrt{-45}$
Answer:
-15

Multiplying complex numbers is similar to multiplying polynomials.

EXAMPLE 5 Multiply. Write the answers in the form $a + bi$.

a. $6(2 + 9i)$ **b.** $-5i(4 - 8i)$

Strategy We will use the distributive property to find the products.

WHY We perform the indicated operations as if the complex numbers were polynomials with i as a variable.

Solution

a.
$$\begin{aligned}6(2 + 9i) &= 6(2) + 6(9i) && \text{Use the distributive property.}\\ &= 12 + 54i && \text{Perform each multiplication.}\end{aligned}$$

b.
$$\begin{aligned}-5i(4 - 8i) &= -5i(4) - (-5i)8i && \text{Use the distributive property.}\\ &= -20i + 40i^2 && \text{Perform each multiplication.}\\ &= -20i + 40(-1) && \text{Replace } i^2 \text{ with } -1.\\ &= -20i - 40 && \text{Multiply.}\\ &= -40 - 20i && \text{Write the real part, } -40\text{, as the first term.}\end{aligned}$$

Self Check 5

Multiply. Write the answers in the form $a + bi$.

a. $-2(-9 - i)$ $18 + 2i$

b. $10i(7 + 4i)$ $-40 + 70i$

Now Try **Problems 49 and 55**

Teaching Example 5 Multiply. Write the answers in the form $a + bi$.
a. $-3(4 - 5i)$ **b.** $6i(2 - 5i)$
Answers:
a. $-12 + 15i$ **b.** $30 + 12i$

Self Check 6

Multiply. Write the answers in the form $a + bi$:
$(-2 + 3i)(3 - 2i)$ $0 + 13i$

Now Try **Problem 59**

Teaching Example 6 Multiply. Write the answers in the form $a + bi$.
a. $(3 - 5i)(-2 + 6i)$
b. $(4 + \sqrt{-25})(1 - \sqrt{-4})$
Answers:
a. $24 + 28i$
b. $14 - 3i$

EXAMPLE 6 Multiply. Write the answers in the form $a + bi$.

a. $(2 + 3i)(3 - 2i)$ **b.** $(-4 + 2i)(2 + i)$

Strategy We will use the FOIL method to multiply the two complex numbers.

WHY We perform the indicated operations as if the complex numbers were binomials with i as a variable.

Solution

a.
$$\begin{aligned}
(2 + 3i)(3 - 2i) &= \overset{F}{6} - \overset{O}{4i} + \overset{I}{9i} - \overset{L}{6i^2} && \text{Use the FOIL method.} \\
&= 6 + 5i - 6(-1) && \text{Combine the imaginary terms: } -4i + 9i = 5i. \text{ Replace } i^2 \text{ with } -1. \\
&= 6 + 5i + 6 && \text{Simplify the last term.} \\
&= 12 + 5i && \text{Combine like terms.}
\end{aligned}$$

b.
$$\begin{aligned}
(-4 + 2i)(2 + i) &= -8 - 4i + 4i + 2i^2 && \text{Use the FOIL method.} \\
&= -8 + 0i + 2(-1) && \text{Combine like terms: } -4i + 4i = 0i. \text{ Replace } i^2 \text{ with } -1. \\
&= -8 + 0i - 2 && \text{Multiply.} \\
&= -10 + 0i && \text{Combine like terms.}
\end{aligned}$$

Success Tip i is not a variable, but you can think of it as one when adding, subtracting, and multiplying. For example:

$$\begin{aligned}
-4i + 9i &= 5i \\
6i - 2i &= 4i \\
i \cdot i &= i^2
\end{aligned}$$

Remember that the expression i^2 simplifies to -1.

5 Divide complex numbers.

Before we can discuss division of complex numbers, we must introduce an important fact about *complex conjugates*.

Complex Conjugates

The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates.**

For example,

- $7 + 4i$ and $7 - 4i$ are complex conjugates.
- $5 - i$ and $5 + i$ are complex conjugates.
- $-6i$ and $6i$ are complex conjugates, because $-6i = 0 - 6i$ and $6i = 0 + 6i$.

In general, the product of the complex number $a + bi$ and its complex conjugate $a - bi$ is the real number $a^2 + b^2$, as the following work shows:

$$\begin{aligned}
(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 && \text{Use the FOIL method.} \\
&= a^2 - b^2(-1) && -abi + abi = 0. \text{ Replace } i^2 \text{ with } -1. \\
&= a^2 + b^2
\end{aligned}$$

The Language of Algebra Recall that the word *conjugate* was used earlier when we rationalized the denominators of radical expressions such as

$$\frac{5}{\sqrt{6} - 1}$$

EXAMPLE 7

Find the product of $3 + 5i$ and its complex conjugate.

Strategy The complex conjugate of $3 + 5i$ is $3 - 5i$. We will find their product by using the FOIL method.

WHY We perform the indicated operations as if the complex numbers were binomials with i as a variable.

Solution

We can find the product as follows:

$$\begin{aligned}(3 + 5i)(3 - 5i) &= 9 - 15i + 15i - 25i^2 && \text{Use the FOIL method.}\\ &= 9 - 25i^2 && \text{Combine like terms: } -15i + 15i = 0.\\ &= 9 - 25(-1) && \text{Replace } i^2 \text{ with } -1.\\ &= 9 + 25\\ &= 34\end{aligned}$$

The product of $3 + 5i$ and its conjugate $3 - 5i$ is the real number 34.

Self Check 7

Multiply: $(2 + 3i)(2 - 3i)$ 13

Now Try Problem 65

Teaching Example 7 Find the product of $-2 + 7i$ and its complex conjugate.
Answer:
53

Recall that to divide *radical expressions,* we rationalized the denominator. We will use a similar approach to divide complex numbers. To divide two complex numbers when the divisor has two terms, we use the following strategy.

Division of Complex Numbers

To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator.

EXAMPLE 8

Divide. Write the answers in the form $a + bi$.

a. $\dfrac{3}{6 + i}$ **b.** $\dfrac{1 + 2i}{3 - 4i}$

Strategy We will build each fraction by multiplying it by a form of 1 that uses the conjugate of the denominator.

WHY This step produces a *real number* in the denominator so that the result can then be written in the form $a + bi$.

Solution

a. We want to build a fraction equivalent to $\frac{3}{6+i}$ that does not have i in the denominator. To make the denominator, $6 + i$, a real number, we need to multiply it by its complex conjugate, $6 - i$. It follows that $\frac{6-i}{6-i}$ should be the form of 1 that is used to build $\frac{3}{6+i}$.

Self Check 8

Divide. Write the answers in the form $a + bi$.

a. $\dfrac{6}{5 + 2i}$ $\frac{30}{29} - \frac{12}{29}i$

b. $\dfrac{2 - 4i}{5 - 3i}$ $\frac{11}{17} - \frac{7}{17}i$

Now Try Problems 71 and 77

Teaching Example 8 Divide. Write the answer in the form $a + bi$.

a. $\dfrac{5}{3 + 2i}$ **b.** $\dfrac{3 - 2i}{2 - i}$

Answers:

a. $\dfrac{15}{13} - \dfrac{10}{13}i$ **b.** $\dfrac{8}{5} - \dfrac{1}{5}i$

$$\frac{3}{6+i} = \frac{3}{6+i} \cdot \frac{\mathbf{6-i}}{\mathbf{6-i}}$$ To build an equivalent fraction, multiply by $\frac{6-i}{6-i} = 1$.

$$= \frac{18-3i}{36-6i+6i-\mathbf{i^2}}$$ To multiply the numerators, distribute the multiplication by 3. Use the FOIL method to multiply the denominators.

$$= \frac{18-3i}{36-(\mathbf{-1})}$$ Combine like terms: $-6i + 6i = 0$. Replace i^2 with -1. Note that the denominator no longer contains i.

$$= \frac{18-3i}{37}$$ Simplify the denominator. This notation represents the difference of two fractions that have the common denominator 37: $\frac{18}{37}$ and $\frac{3i}{37}$.

$$= \frac{18}{37} - \frac{3}{37}i$$ Write the complex number in the form $a + bi$.

b. We can make the denominator of $\frac{1+2i}{3-4i}$ a real number by multiplying it by the complex conjugate of $3 - 4i$, which is $3 + 4i$. It follows that $\frac{3+4i}{3+4i}$ should be the form of 1 that is used to build $\frac{1+2i}{3-4i}$.

$$\frac{1+2i}{3-4i} = \frac{1+2i}{3-4i} \cdot \frac{\mathbf{3+4i}}{\mathbf{3+4i}}$$ To build an equivalent fraction, multiply by $\frac{3+4i}{3+4i} = 1$.

$$= \frac{3+4i+6i+8\mathbf{i^2}}{9+12i-12i-16\mathbf{i^2}}$$ Use the FOIL method to multiply the numerators and the denominators.

$$= \frac{3+10i+8(\mathbf{-1})}{9-16(\mathbf{-1})}$$ Combine like terms in the numerator and denominator. Replace i^2 with -1. The denominator is now a real number.

$$= \frac{3+10i-8}{9+16}$$ Simplify the numerator and denominator.

$$= \frac{-5+10i}{25}$$ Combine like terms in the numerator and denominator.

$$= \frac{\overset{1}{\cancel{5}}(-1+2i)}{\underset{1}{\cancel{5}} \cdot 5}$$ Factor out 5 in the numerator and remove the common factor of 5.

$$= \frac{-1+2i}{5}$$ Simplify. This notation represents the sum of two fractions that have the common denominator 5.

$$= -\frac{1}{5} + \frac{2}{5}i$$ Write the complex number in the form $a + bi$.

Self Check 9

Divide and write the answer in the form $a + bi$: $\frac{21 - \sqrt{-49}}{3 - \sqrt{-1}}$

Now Try **Problem 81**

Self Check 9 Answers
$7 + 0i$

Teaching Example 9 Divide and write the answer in the form $a + bi$:
$\frac{2 + \sqrt{-25}}{6 + \sqrt{-225}}$
Answer:
$\frac{1}{3} + 0i$

EXAMPLE 9

Divide and write the answer in the form $a + bi$: $\frac{4 + \sqrt{-16}}{2 + \sqrt{-4}}$

Strategy We will begin by writing $\sqrt{-16}$ and $\sqrt{-4}$ in $i\sqrt{b}$ form.

WHY To perform any computations, the numerator and denominator should be written in the form $a + bi$.

Solution

$$\frac{4 + \sqrt{-16}}{2 + \sqrt{-4}} = \frac{4+4i}{2+2i}$$ Simplify: $\sqrt{-16} = \sqrt{-1}\sqrt{16} = 4i$ and $\sqrt{-4} = \sqrt{-1}\sqrt{4} = 2i$.

$$= \frac{2\overset{1}{\cancel{(2+2i)}}}{\underset{1}{\cancel{2+2i}}}$$ Factor out 2 in the numerator and remove the common factor of $2 + 2i$.

$$= 2$$

$$= 2 + 0i$$ Write 2 in the form $a + bi$.

EXAMPLE 10 Divide and write the result in the form $a + bi$: $\dfrac{7}{2i}$

Strategy We will use $\frac{-2i}{-2i}$ as the form of 1 to build $\frac{7}{2i}$.

WHY Since the denominator $2i$ can be expressed as $0 + 2i$, its conjugate is $0 - 2i$. However, instead of building with $\frac{0-2i}{0-2i}$ we will drop the zeros and just use $\frac{-2i}{-2i}$.

Solution

$$\frac{7}{2i} = \frac{7}{2i} \cdot \frac{-2i}{-2i}$$ To build an equivalent fraction, multiply by $\frac{-2i}{-2i} = 1$.

$$= \frac{-14i}{-4i^2}$$ Multiply the numerators and multiply the denominators.

$$= \frac{-14i}{-4(-1)}$$ Replace i^2 with -1. The denominator is now a real number.

$$= \frac{-14i}{4}$$ Simplify the denominator.

$$= -\frac{7i}{2}$$ Simplify the fraction: $-\frac{\overset{1}{\cancel{2}} \cdot 7i}{\underset{1}{\cancel{2}} \cdot 2}$.

$$= 0 - \frac{7}{2}i$$ Write in the form $a + bi$.

Self Check 10 Divide and write the answer in the form $a + bi$: $\dfrac{3}{4i}$ $\quad 0 - \frac{3}{4}i$

Now Try **Problem 85**

Teaching Example 10 Divide and write the result in the form $a + bi$: $\dfrac{11}{6i}$

Answer:

$0 - \dfrac{11}{6}i$

6 Perform operations involving powers of *i*.

The powers of i produce an interesting pattern:

$$\begin{aligned} i &= \sqrt{-1} = \boldsymbol{i} & i^5 &= i^4 i = 1i = \boldsymbol{i} \\ i^2 &= \left(\sqrt{-1}\right)^2 = \mathbf{-1} & i^6 &= i^4 i^2 = 1(-1) = \mathbf{-1} \\ i^3 &= i^2 i = -1i = \boldsymbol{-i} & i^7 &= i^4 i^3 = 1(-i) = \boldsymbol{-i} \\ i^4 &= i^2 i^2 = (-1)(-1) = \mathbf{1} & i^8 &= i^4 i^4 = (1)(1) = \mathbf{1} \end{aligned}$$

The pattern continues: $i, -1, -i, 1, \ldots$.

Larger powers of i can be simplified by using the fact that $i^4 = 1$. For example, to simplify i^{29}, we note that 29 divided by 4 gives a quotient of 7 and a remainder of 1. Thus, $29 = 4 \cdot 7 + 1$ and

$$i^{29} = i^{4\cdot7+1}$$ $4 \cdot 7 = 28$.

$$= (i^4)^7 \cdot i^1$$ Use the rules for exponents $x^{m \cdot n} = (x^m)^n$ and $x^{m+n} = x^m \cdot x^n$.

$$= 1^7 \cdot i$$ Simplify: $i^4 = 1$.

$$= i$$ Simplify: $1 \cdot i = i$.

The result of this example illustrates the following fact.

Powers of *i*

If n is a natural number that has a remainder of r when divided by 4, then

$$i^n = i^r$$

Self Check 11

Simplify: i^{62} -1

Now Try **Problem 91**

Teaching Example 11 Simplify:
a. i^{33} **b.** i^{52} **c.** i^{63}
Answers:
a. i **b.** 1 **c.** $-i$

EXAMPLE 11 Simplify: i^{55}

Strategy We will examine the remainder when we divide the exponent 55 by 4.

WHY The remainder determines the power to which i is raised in the simplified form.

Solution

We divide 55 by 4 and get a remainder of 3. Therefore,

$$i^{55} = i^3 = -i$$

$$\begin{array}{r} 13R\,3 \\ 4\overline{)55} \\ \underline{-4} \\ 15 \\ \underline{-12} \\ 3 \end{array}$$

Success Tip If we divide the natural number exponent n of a power of i by 4, the remainder indicates the simplified form of i^n:

$$R = 1: i$$
$$R = 2: -1$$
$$R = 3: -i$$
$$R = 0: 1$$

ANSWERS TO SELF CHECKS

1. a. $5i$ **b.** $-i\sqrt{19}$ **c.** $3i\sqrt{5}$ **d.** $\frac{5\sqrt{2}}{9}i$ **2. a.** $-18 + 0i$ **b.** $0 + 6i$ **c.** $1 + 2i\sqrt{6}$ **3. a.** $1 + 2i$ **b.** $5 - 12i$ **4.** $-4\sqrt{6}$ **5. a.** $18 + 2i$ **b.** $-40 + 70i$ **6.** $0 + 13i$ **7.** 13 **8. a.** $\frac{30}{29} - \frac{12}{29}i$ **b.** $\frac{11}{17} - \frac{7}{17}i$ **9.** $7 + 0i$ **10.** $0 - \frac{3}{4}i$ **11.** -1

SECTION 7.7 STUDY SET

VOCABULARY

Fill in the blanks.

1. The imaginary number i is defined as $i = \sqrt{-1}$. We call i^{25} a power of i.

2. A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

▶ **3.** For the complex number $2 + 5i$, we call 2 the real part and 5 the imaginary part.

4. $6 + 3i$ and $6 - 3i$ are called complex conjugates.

CONCEPTS

Fill in the blanks.

5. a. $i = \sqrt{-1}$ **b.** $i^2 = -1$

c. $i^3 = -i$ **d.** $i^4 = 1$

e. In general, the powers of i cycle through four possible outcomes.

6. Simplify:

$$\sqrt{-36} = \sqrt{-1 \cdot 36} = \sqrt{-1}\sqrt{36} = 6i$$

7. To add (or subtract) complex numbers, add (or subtract) their real parts and add (or subtract) their imaginary parts.

8. To multiply two complex numbers, such as $(2 + 3i)(3 + 5i)$, we can use the FOIL method.

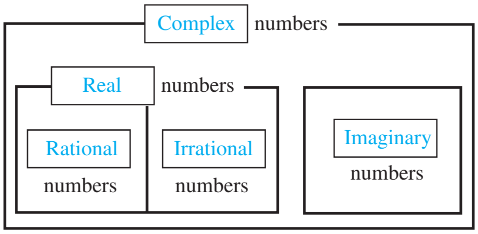
9. To divide $6 + 7i$ by $1 - 8i$, we multiply $\frac{6 + 7i}{1 - 8i}$ by a 1 in the form of $\frac{1 + 8i}{1 + 8i}$.

10. Give the complex conjugate of each number.

a. $2 - 3i$ $2 + 3i$ **b.** 2 $2 - 0i$ **c.** $-3i$ $0 + 3i$

▶ Selected exercises available online at **www.webassign.net/brookscole**

11. Complete the illustration. Label the real numbers, the imaginary numbers, the complex numbers, the rational numbers, and the irrational numbers.



12. Determine whether each statement is true or false.

a. Every complex number is a real number. false

b. Every real number is a complex number. true

c. i is a real number. false

d. The square root of a negative number is an imaginary number. true

NOTATION

Complete each solution.

13. $(3 + 2i)(3 - i) = \boxed{9} - 3i + \boxed{6i} - 2i^2$
$= 9 + 3i + \boxed{2}$
$= \boxed{11} + 3i$

14. $\dfrac{3}{2 - i} = \dfrac{3}{2 - i} \cdot \dfrac{\boxed{2 + i}}{\boxed{2 + i}}$
$= \dfrac{6 + \boxed{3i}}{4 - \boxed{i^2}}$
$= \dfrac{6 + 3i}{\boxed{5}}$
$= \dfrac{\boxed{6}}{\boxed{5}} + \dfrac{3}{5}i$

15. Determine whether each statement is true or false.

a. $\sqrt{6}i = i\sqrt{6}$ true **b.** $\sqrt{8i} = \sqrt{8}i$ false

c. $\sqrt{-25} = -\sqrt{25}$ false **d.** $-i = i$ false

16. Write each number in the form $a + bi$.

a. $\dfrac{9 + 11i}{4}$ $\frac{9}{4} + \frac{11}{4}i$ **b.** $\dfrac{1 - i}{18}$ $\frac{1}{18} - \frac{1}{18}i$

GUIDED PRACTICE

Express each number in terms of i. See Example 1.

17. $\sqrt{-9}$ $3i$

18. $\sqrt{-4}$ $2i$

19. $\sqrt{-7}$ $\sqrt{7}i$ or $i\sqrt{7}$

20. $\sqrt{-11}$ $\sqrt{11}i$ or $i\sqrt{11}$

21. $\sqrt{-24}$ $2\sqrt{6}i$ or $2i\sqrt{6}$

▶ **22.** $\sqrt{-28}$ $2\sqrt{7}i$ or $2i\sqrt{7}$

23. $-\sqrt{-72}$ $-6\sqrt{2}i$ or $-6i\sqrt{2}$

24. $-\sqrt{-24}$ $-2\sqrt{6}i$ or $-2i\sqrt{6}$

25. $5\sqrt{-81}$ $45i$

26. $6\sqrt{-49}$ $42i$

27. $\sqrt{-\dfrac{25}{9}}$ $\frac{5}{3}i$

28. $-\sqrt{-\dfrac{121}{144}}$ $-\frac{11}{12}i$

Write each number in the form a + bi. See Example 2.

29. **a.** 5 $5 + 0i$
b. $\sqrt{-49}$ $0 + 7i$

30. **a.** -43 $-43 + 0i$
b. $\sqrt{-169}$ $0 + 13i$

31. **a.** $1 + \sqrt{-25}$ $1 + 5i$
b. $-3 + \sqrt{-8}$ $-3 + 2i\sqrt{2}$

▶ **32.** **a.** $21 + \sqrt{-16}$ $21 + 4i$
b. $-9 + \sqrt{-12}$ $-9 + 2i\sqrt{3}$

33. **a.** $76 - \sqrt{-54}$ $76 - 3i\sqrt{6}$
b. $-7 + \sqrt{-19}$ $-7 + i\sqrt{19}$

34. **a.** $88 - \sqrt{-98}$ $88 - 7i\sqrt{2}$
b. $-2 + \sqrt{-35}$ $-2 + i\sqrt{35}$

35. **a.** $-6 - \sqrt{-9}$ $-6 - 3i$
b. $3 + \sqrt{-6}$ $3 + i\sqrt{6}$

36. **a.** $-45 - \sqrt{-81}$ $-45 - 9i$
b. $8 + \sqrt{-7}$ $8 + i\sqrt{7}$

Perform the operations. Write all answers in the form a + bi. See Example 3.

37. $(3 + 4i) + (5 - 6i)$ $8 - 2i$

38. $(8 + 3i) + (-7 - 2i)$ $1 + i$

39. $(6 - i) + (9 + 3i)$ $15 + 2i$

40. $(5 + 3i) - (6 - 9i)$ $-1 + 12i$

41. $(7 - 3i) - (4 + 2i)$ $3 - 5i$

▶ **42.** $(5 - 4i) - (3 + 2i)$ $2 - 6i$

43. $\left(8 + \sqrt{-25}\right) - \left(7 + \sqrt{-4}\right)$ $1 + 3i$

44. $\left(-7 + \sqrt{-81}\right) - \left(-2 - \sqrt{-64}\right)$ $-5 + 17i$

Multiply. See Example 4.

45. $\sqrt{-1}\sqrt{-36}$ -6

▶ **46.** $\sqrt{-9}\sqrt{-100}$ -30

47. $\sqrt{-2}\sqrt{-12}$ $-2\sqrt{6}$

48. $\sqrt{-3}\sqrt{-45}$ $-3\sqrt{15}$

Multiply. Write all answers in the form a + bi. See Example 5.

49. $3(2 - 9i)$ $6 - 27i$

▶ **50.** $-4(3 + 4i)$ $-12 - 16i$

51. $7(5 - 4i)$ $35 - 28i$

52. $-5(3 + 2i)$ $-15 - 10i$

53. $2i(7 - 3i)$ $6 + 14i$

54. $i(8 + 2i)$ $-2 + 8i$

55. $-5i(5 - 5i)$ $-25 - 25i$

56. $2i(7 + 2i)$ $-4 + 14i$

Multiply. Write all answers in the form a + bi. See Example 6.

57. $(2 + i)(3 - i)$ $7 + i$

▶ **58.** $(4 - i)(2 + i)$ $9 + 2i$

59. $(3 - 2i)(2 + 3i)$ $12 + 5i$

60. $(3 - i)(2 + 3i)$ $9 + 7i$

61. $(4 + i)(3 - i)$ $13 - i$

62. $(1 - 5i)(1 - 4i)$ $-19 - 9i$

63. $(2 + i)^2$ $3 + 4i$

64. $(3 - 2i)^2$ $5 - 12i$

Find the product of the given complex number and its conjugate. See Example 7.

65. $2 + 6i$ 40

66. $5 + 2i$ 29

67. $-4 - 7i$ 65

68. $-10 - 9i$ 181

Divide. Write all answers in the form $a + bi$. See Example 8.

69. $\dfrac{9}{5 + i}$ $\frac{45}{26} - \frac{9}{26}i$

70. $\dfrac{4}{2 - i}$ $\frac{8}{5} + \frac{4}{5}i$

71. $\dfrac{11i}{4 - 7i}$ $-\frac{77}{65} + \frac{44}{65}i$

72. $\dfrac{2i}{3 + 8i}$ $\frac{16}{73} + \frac{6}{73}i$

73. $\dfrac{3 - 2i}{4 - i}$ $\frac{14}{17} - \frac{5}{17}i$

74. $\dfrac{6 - i}{2 + i}$ $\frac{11}{5} - \frac{8}{5}i$

75. $\dfrac{7 + 4i}{2 - 5i}$ $-\frac{6}{29} + \frac{43}{29}i$

76. $\dfrac{2 + 3i}{2 - 3i}$ $-\frac{5}{13} + \frac{12}{13}i$

77. $\dfrac{7 + 3i}{4 - 2i}$ $\frac{11}{10} + \frac{13}{10}i$

78. $\dfrac{5 - 3i}{4 + 2i}$ $\frac{7}{10} - \frac{11}{10}i$

79. $\dfrac{1 - 3i}{3 + i}$ $0 - i$

80. $\dfrac{3 + 5i}{1 - i}$ $-1 + 4i$

Divide. Write all answers in the form $a + bi$. See Example 9.

81. $\dfrac{8 + \sqrt{-144}}{2 + \sqrt{-9}}$ $4 + 0i$

82. $\dfrac{3 + \sqrt{-36}}{1 + \sqrt{-4}}$ $3 + 0i$

83. $\dfrac{-4 - \sqrt{-4}}{2 + \sqrt{-1}}$ $-2 + 0i$

84. $\dfrac{-5 - \sqrt{-25}}{1 + \sqrt{-1}}$ $-5 + 0i$

Divide. Write all answers in the form $a + bi$. See Example 10.

85. $\dfrac{5}{3i}$ $0 - \frac{5}{3}i$

86. $\dfrac{3}{8i}$ $0 - \frac{3}{8}i$

87. $-\dfrac{2}{7i}$ $0 + \frac{2}{7}i$

88. $-\dfrac{8}{5i}$ $0 + \frac{8}{5}i$

Simplify each expression. See Example 11.

89. i^{21} i

90. i^{19} $-i$

91. i^{27} $-i$

92. i^{22} -1

93. i^{100} 1

94. i^{97} i

95. i^{42} -1

96. i^{200} 1

TRY IT YOURSELF

Perform the operations. Write all answers in the form $a + bi$.

97. $(3 - i) - (-1 + 10i)$ $4 - 11i$

98. $(14 + 4i) - (-9 - i)$ $23 + 5i$

99. $(2 - \sqrt{-16})(3 + \sqrt{-4})$ $14 - 8i$

100. $(3 - \sqrt{-4})(4 - \sqrt{-9})$ $6 - 17i$

101. $(-6 - 9i) + (4 + 3i)$ $-2 - 6i$

102. $(-3 + 11i) + (-1 - 6i)$ $-4 + 5i$

103. $\dfrac{-2i}{3 + 2i}$ $-\frac{4}{13} - \frac{6}{13}i$

104. $\dfrac{-4i}{2 - 6i}$ $\frac{3}{5} - \frac{1}{5}i$

105. $6i(2 - 3i)$ $18 + 12i$

106. $-9i(4 - 6i)$ $-54 - 36i$

107. $\dfrac{4}{5i^{35}}$ $0 + \frac{4}{5}i$

108. $\dfrac{3}{2i^{17}}$ $0 - \frac{3}{2}i$

109. $(2 + i\sqrt{2})(3 - i\sqrt{2})$ $8 + \sqrt{2}i$ or $8 + i\sqrt{2}$

110. $(5 + i\sqrt{3})(2 - i\sqrt{3})$ $13 - 3\sqrt{3}i$ or $13 - 3i\sqrt{3}$

111. $\dfrac{5 + 9i}{1 - i}$ $-2 + 7i$

112. $\dfrac{5 - i}{3 + 2i}$ $1 - i$

113. $(4 - 8i)^2$ $-48 - 64i$

114. $(7 - 3i)^2$ $40 - 42i$

115. $\dfrac{\sqrt{5} - i\sqrt{3}}{\sqrt{5} + i\sqrt{3}}$ $\frac{1}{4} - \frac{\sqrt{15}}{4}i$

116. $\dfrac{\sqrt{3} + i\sqrt{2}}{\sqrt{3} - i\sqrt{2}}$ $\frac{1}{5} + \frac{2\sqrt{6}}{5}i$

APPLICATIONS

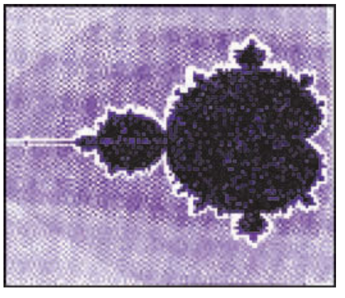
117. FRACTALS Complex numbers are fundamental in the creation of the intricate geometric shape shown below, called a fractal. The process of creating this image is based on the following sequence of steps, which begins by picking any complex number, which we will call z.

1. Square z, and then add that result to z.

2. Square the result from step 1, and then add it to z.

3. Square the result from step 2, and then add it to z.

If we begin with the complex number i, what is the result after performing steps 1, 2, and 3? $-1 + i$



118. ELECTRONICS The impedance Z in an AC (alternating current) circuit is a measure of how much the circuit impedes (hinders) the flow of current through it. The impedance is related to the voltage V and the current I by the formula

$$V = IZ$$

If a circuit has a current of $(0.5 + 2.0i)$ amps and an impedance of $(0.4 - 3.0i)$ ohms, find the voltage. $6.2 - 0.7i$

WRITING

119. What is an imaginary number? What is a complex number?

120. The method used to divide complex numbers is similar to the method used to divide radical expressions. Explain why. Give an example.

121. Explain the error. Then find the correct result.

a. Add: $\sqrt{-16} + \sqrt{-9} = \sqrt{-25}$

b. Multiply: $\sqrt{-2}\sqrt{-3} = \sqrt{-2(-3)} = \sqrt{6}$

122. Determine whether the pair of complex numbers are equal. Explain your reasoning.

a. $4 - \frac{2}{5}i, \frac{8}{2} - 0.4i$

b. $0.25 + 0.7i, \frac{1}{4} + \frac{7}{10}i$

REVIEW

123. WIND SPEEDS A plane that can fly 200 mph in still air makes a 330-mile flight with a tail wind and returns, flying into the same wind. Find the speed of the wind if the total flying time is $3\frac{1}{3}$ hours. 20 mph

124. FINDING RATES A student drove a distance of 135 miles at an average speed of 50 mph. How much faster would she have to drive on the return trip to save 30 minutes of driving time? about 11.4 mph faster

STUDY SKILLS CHECKLIST

Preparing for the Chapter 7 Test

There are several common mistakes that students make when working with the topics of Chapter 7. To make sure you are prepared for the test over this material, read the list below to help you avoid these mistakes.

☐ Although $-\sqrt{25}$ and $\sqrt{-25}$ look similar, these radical expressions have very different meanings: $-\sqrt{25} = -5$ and $\sqrt{-25} = 5i$

☐ With an odd index root, when the radicand is negative, the number is a real number. $\sqrt[3]{-27} = -3$ because $(-3)^3 = -27$

☐ When adding or subtracting radicals expressions, you must have *like radicals.* Like radicals are radical expressions in which the index and the radicand are the same.

$5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$ Both expressions have the same index and radicand. Add the coefficients of the terms and keep the same radical.

$5\sqrt{3} + 2\sqrt{7}$ Cannot be combined. The operation is addition but the radicals are not like radicals. The expression is in simplest form.

$5\sqrt{18} + 2\sqrt{50} = 5\sqrt{9 \cdot 2} + 2\sqrt{25 \cdot 2}$	Write 18 as $18 = 9 \cdot 2$. Write 50 as $50 = 25 \cdot 2$.
$= 5\sqrt{9} \cdot \sqrt{2} + 2\sqrt{25} \cdot \sqrt{2}$	The square root of a product is equal to the product of the square roots.
$= 5 \cdot 3 \cdot \sqrt{2} + 2 \cdot 5 \cdot \sqrt{2}$	Evaluate $\sqrt{9} = 3$ and $\sqrt{25} = 5$.
$= 15\sqrt{2} + 10\sqrt{2}$	Multiply $5 \cdot 3 = 15$ and $2 \cdot 5 = 10$.
$= 25\sqrt{2}$	Both radicals have the same index and the same radicand. To combine them, add the coefficients and keep the radical.

☐ To multiply radicals, only the index has to be the same. Use the product rule for radicals to carry out the multiplication. Be sure to simplify all answers.

$3\sqrt{15}\left(2\sqrt{10}\right) = 3 \cdot 2\sqrt{15} \cdot \sqrt{10}$	Multiply the integer factors, 3 and 2, and multiply the radicals.
$= 6\sqrt{150}$	Use the product rule for radicals.
$= 6\sqrt{25}\sqrt{6}$	
$= 6(5)\sqrt{6}$	
$= 30\sqrt{6}$	

☐ To multiply radical expressions with more than one term, we use the distributive property.

$$\begin{aligned} 3\sqrt{5}\left(2\sqrt{15} - 6\sqrt{10}\right) &= 3\sqrt{5} \cdot 2\sqrt{15} - 3\sqrt{5}\,6\sqrt{10} \\ &= 6\sqrt{75} - 18\sqrt{50} \\ &= 6\sqrt{3 \cdot 5 \cdot 5} - 18\sqrt{2 \cdot 5 \cdot 5} \\ &= 6 \cdot 5\sqrt{3} - 18 \cdot 5\sqrt{2} \\ &= 30\sqrt{3} - 90\sqrt{2} \end{aligned}$$

☐ When solving radical equations, isolate the radical on one side of the equation before raising both sides to the power that matches the index.

☐ Even if you are certain that no algebraic mistakes were made when solving a radical equation, you must still check your solutions. Raising both sides to a power can introduce extraneous solutions that must be discarded.

☐ i is not a variable, but you can think of it as one when adding, subtracting, and multiplying complex numbers. For example:

$-6i + 9i = 3i$

$5i \cdot 3i = 15i^2$ Remember, $i^2 = -1$ to simplify the answer: $15i^2 = 15(-1) = -15$

CHAPTER 7 SUMMARY AND REVIEW

SECTION 7.1 Radical Expressions and Radical Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
The number b is a **square root of a** if $b^2 = a$.	7 is a square root of 49 because $7^2 = 49$. -7 is a square root of 49 because $(-7)^2 = 49$.
A **radical symbol** $\sqrt{}$ represents the **positive** or **principal square root** of a number. For any real number x, $\sqrt{x^2} = \lvert x \rvert$ The symbol $-\sqrt{}$ represents the **negative square root** of a number.	Simplify: $\sqrt{25} = 5$ because $5^2 = 25$. $\sqrt{36x^2} = \lvert 6x \rvert = 6\lvert x \rvert$ because $(\lvert 6x \rvert)^2 = 36x^2$. $\sqrt{\frac{r^8}{100}} = \frac{r^4}{10}$ because $\left(\frac{r^4}{10}\right)^2 = \frac{r^8}{100}$. Since $\frac{r^4}{10} \geq 0$, no absolute value symbols are needed. $-\sqrt{81} = -9$ because $(-9)^2 = 81$.
A function of the form $f(x) = \sqrt{x}$ is called a **square root function.**	Find the domain of $f(x) = \sqrt{x - 2}$. Since the expression $\sqrt{x - 2}$ is not a real number when $x - 2$ is negative, we must require that $x - 2 \geq 0$. It follows that x must be greater than or equal to 2. Thus, the domain of $f(x)$ is $[2, \infty)$.
The **cube root** of x is denoted as $\sqrt[3]{x}$ and is defined as $\sqrt[3]{x} = y$ if $y^3 = x$ A function of the form $f(x) = \sqrt[3]{x}$ is called a **cube root function.**	Simplify: $\sqrt[3]{8} = 2$ because $2^3 = 8$. $\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$.
The ***n*th root of *x*** is denoted as $\sqrt[n]{x}$. If x is a real number and $n > 1$, then: $\begin{cases} \text{If } n \text{ is an odd natural number, } \sqrt[n]{x^n} = x. \\ \text{If } n \text{ is an even natural number, } \sqrt[n]{x^n} = \lvert x \rvert. \end{cases}$	Simplify: $\sqrt[4]{81x^4} = \lvert 3x \rvert = 3\lvert x \rvert$ because $(\lvert 3x \rvert)^4 = 81x^4$. $\sqrt[5]{32a^{10}} = 2a^2$ because $(2a^2)^5 = 32a^{10}$. $\sqrt[6]{(m-4)^6} = \lvert m - 4 \rvert$ because $(\lvert m - 4 \rvert)^6 = (m-4)^6$.

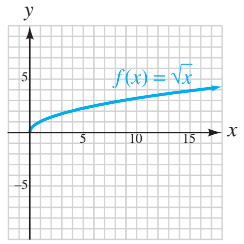
REVIEW EXERCISES

Simplify each expression, if possible. Assume that x and y can be any real number.

1. $\sqrt{49}$ 7
2. $-\sqrt{121}$ -11
3. $\sqrt{\frac{225}{49}}$ $\frac{15}{7}$
4. $\sqrt{-4}$ not a real number
5. $\sqrt{100a^{12}}$ $10a^6$
6. $\sqrt{25x^2}$ $5\lvert x \rvert$
7. $\sqrt{x^8}$ x^4
8. $\sqrt{x^2 + 4x + 4}$ $\lvert x + 2 \rvert$
9. $\sqrt[3]{-27}$ -3
10. $-\sqrt[3]{216}$ -6
11. $\sqrt[3]{64x^6y^3}$ $4x^2y$
12. $\sqrt[3]{\frac{x^9}{125}}$ $\frac{x^3}{5}$
13. $\sqrt[6]{64}$ 2
14. $\sqrt[5]{-32}$ -2
15. $\sqrt[4]{256x^8y^4}$ $4x^2\lvert y \rvert$
16. $\sqrt[15]{(x+1)^{15}}$ $x + 1$
17. $-\sqrt[4]{\frac{1}{16}}$ $-\frac{1}{2}$
18. $\sqrt[4]{-16}$ not a real number
19. $\sqrt[6]{-1}$ not a real number
20. $\sqrt[3]{0}$ 0
21. GEOMETRY The side of a square with area A square feet is given by the function $s(A) = \sqrt{A}$. Find the length of one side of a square that has an area of 169 ft^2. 13 ft
22. SURFACE AREA OF A CUBE The total surface area of a cube is related to its volume V by the function $A(V) = 6\sqrt[3]{V^2}$. Find the surface area of a cube with a volume of 8 cm^3. 24 cm^2

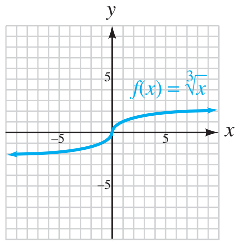
Graph each function. Find the domain and range.

23. $f(x) = \sqrt{x}$



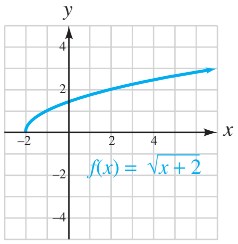
D: $[0, \infty)$, R: $[0, \infty)$

24. $f(x) = \sqrt[3]{x}$



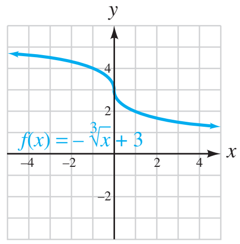
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

25. $f(x) = \sqrt{x + 2}$



D: $[-2, \infty)$, R: $[0, \infty)$

26. $f(x) = -\sqrt[3]{x} + 3$



D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

SECTION 7.2 Simplifying and Combining Radical Expressions

DEFINITIONS AND CONCEPTS	EXAMPLES
Product rule for radicals: $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ The product rule for radicals can be used to simplify radical expressions. **Simplified form** of a radical: **1.** Except for 1, the radicand has no perfect-square factors. **2.** No fraction appears in the radicand. **3.** No radical appears in the denominator.	Simplify: $\sqrt{98} = \sqrt{49 \cdot 2}$ Write 98 as the product of it greatest perfect-square factor and one other factor. $= \sqrt{49}\sqrt{2}$ The square root of a product is equal to the product of the square roots. $= 7\sqrt{2}$ Evaluate $\sqrt{49}$. Simplify: $\sqrt[3]{16x^4} = \sqrt[3]{8x^3 \cdot 2x}$ Write $16x^4$ as the product of it greatest perfect-cube factor and one other factor. $= \sqrt[3]{8x^3}\sqrt[3]{2x}$ The cube root of a product is equal to the product of the cube roots. $= 2x\sqrt[3]{2x}$ Simplify $\sqrt[3]{8x^3}$.
Quotient rule for radicals: $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	Simplify: $\sqrt{\dfrac{10}{25x^4}} = \dfrac{\sqrt{10}}{\sqrt{25x^4}} = \dfrac{\sqrt{10}}{5x^2}$ Simplify: $\sqrt[3]{\dfrac{16y^3}{125a^3}} = \dfrac{\sqrt[3]{16y^3}}{\sqrt[3]{125a^3}} = \dfrac{\sqrt[3]{8y^3}\sqrt[3]{2}}{5a} = \dfrac{2y\sqrt[3]{2}}{5a}$
Radical expressions with the same index and radicand are called **like radicals.** Like radicals can be combined by addition and subtraction. To **combine like radicals** we use the distributive property in reverse.	Add: $3\sqrt{6} + 5\sqrt{6} = (3 + 5)\sqrt{6} = 8\sqrt{6}$ Subtract: $8\sqrt[4]{2y} - 9\sqrt[4]{2y} = (8 - 9)\sqrt[4]{2y} = -\sqrt[4]{2y}$

If a sum or difference involves unlike radicals, make sure that each one is written in simplified form. After doing so, like radicals may result that can be combined.	Simplify: $\sqrt[3]{54a^4} - \sqrt[3]{16a^4} = \sqrt[3]{27a^3 \cdot 2a} - \sqrt[3]{8a^3 \cdot 2a}$ $= \sqrt[3]{27a^3}\sqrt[3]{2a} - \sqrt[3]{8a^3}\sqrt[3]{2a}$ $= \mathbf{3a}\sqrt[3]{2a} - \mathbf{2a}\sqrt[3]{2a}$ $= (\mathbf{3a} - \mathbf{2a})\sqrt[3]{2a}$ $= a\sqrt[3]{2a}$

REVIEW EXERCISES

Simplify each expression. All variables represent positive real numbers.

27. $\sqrt{80}$ $4\sqrt{5}$

28. $\sqrt[3]{54}$ $3\sqrt[3]{2}$

29. $\sqrt[4]{160}$ $2\sqrt[4]{10}$

30. $\sqrt[5]{-96}$ $-2\sqrt[5]{3}$

31. $\sqrt{8x^5}$ $2x^2\sqrt{2x}$

32. $\sqrt[4]{r^{17}}$ $r^4\sqrt[4]{r}$

33. $\sqrt[3]{-27j^7k}$ $-3j^2\sqrt[3]{jk}$

34. $\sqrt[3]{-16x^5y^4}$ $-2xy\sqrt[3]{2x^2y}$

35. $\sqrt{\dfrac{m}{144n^{12}}}$ $\dfrac{\sqrt{m}}{12n^6}$

36. $\sqrt{\dfrac{17xy}{64a^4}}$ $\dfrac{\sqrt{17xy}}{8a^2}$

37. $\dfrac{\sqrt[5]{64x^8}}{\sqrt[5]{2x^3}}$ $2x$

38. $\dfrac{\sqrt[5]{243x^{16}}}{\sqrt[5]{x}}$ $3x^3$

Simplify and combine like radicals. All variables represent positive real numbers.

39. $\sqrt{2} + 2\sqrt{2}$
$3\sqrt{2}$

40. $6\sqrt{20} - \sqrt{5}$
$11\sqrt{5}$

41. $2\sqrt[3]{3} - \sqrt[3]{24}$
0

42. $-\sqrt[4]{32a^5} - 2\sqrt[4]{162a^5}$
$-8a\sqrt[4]{2a}$

43. $2x\sqrt{8} + 2\sqrt{200x^2} + \sqrt{50x^2}$
$29x\sqrt{2}$

44. $\sqrt[3]{54x^3} - 3\sqrt[3]{16x^3} + 4\sqrt[3]{128x^3}$
$13x\sqrt[3]{2}$

45. $2\sqrt[4]{32t^3} - 8\sqrt[4]{6t^3} + 5\sqrt[4]{2t^3}$
$9\sqrt[4]{2t^3} - 8\sqrt[4]{6t^3}$

46. $10\sqrt[4]{16x^9} - 8x^2\sqrt[4]{x} + 5\sqrt[4]{x^5}$
$12x^2\sqrt[4]{x} + 5x\sqrt[4]{x}$

47. Explain the error in each simplification.

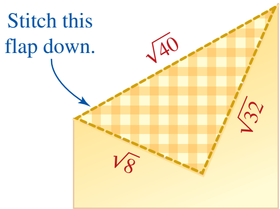
a. $2\sqrt{5x} + 3\sqrt{5x} = \cancel{5\sqrt{10x}}$

b. $30 + 30\sqrt[4]{2} = \cancel{60\sqrt[4]{2}}$

c. $7\sqrt[3]{y^2} - 5\sqrt[3]{y^2} = \cancel{2}$

d. $6\sqrt{11ab} - 3\sqrt{5ab} = \cancel{3\sqrt{6ab}}$

48. SEWING A corner of fabric is folded over to form a collar and stitched down as shown. From the dimensions given in the figure, determine the exact number of inches of stitching that must be made. Then give an approximation to one decimal place. (All measurements are in inches.) $(6\sqrt{2} + 2\sqrt{10})$ in., 14.8 in.



SECTION 7.3 Multiplying and Dividing Radical Expressions

DEFINITIONS AND CONCEPTS	EXAMPLES
We can use the product rule for radicals to **multiply radical expressions** that have the same index: $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$	Multiply and then simplify, if possible: $\sqrt{\mathbf{6}}\sqrt{\mathbf{8}} = \sqrt{\mathbf{6 \cdot 8}} = \sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16}\sqrt{3} = 4\sqrt{3}$ $\sqrt[3]{\mathbf{9x^4}}\sqrt[3]{\mathbf{3x^2}} = \sqrt[3]{\mathbf{9x^4 \cdot 3x^2}} = \sqrt[3]{27x^6} = 3x^2$
We can use the **distributive property** to multiply a radical expression with two or more terms by a radical expression with one term.	Multiply and then simplify, if possible: $\mathbf{2\sqrt{3}}(4\sqrt{5} - 5\sqrt{2}) = \mathbf{2\sqrt{3}} \cdot 4\sqrt{5} - \mathbf{2\sqrt{3}} \cdot 5\sqrt{2}$ $= 2 \cdot 4\sqrt{3 \cdot 5} - 2 \cdot 5\sqrt{3 \cdot 2}$ $= 8\sqrt{15} - 10\sqrt{6}$

We can use the **FOIL method** to multiply a radical expression with two terms by another radical expression with two terms.	Multiply and then simplify, if possible: $\left(\sqrt[3]{x}-\sqrt[3]{3}\right)\left(\sqrt[3]{x}+\sqrt[3]{9}\right)=\overset{F}{\sqrt[3]{x}\sqrt[3]{x}}+\overset{O}{\sqrt[3]{x}\sqrt[3]{9}}-\overset{I}{\sqrt[3]{3}\sqrt[3]{x}}-\overset{L}{\sqrt[3]{3}\sqrt[3]{9}}$ $=\sqrt[3]{x^2}+\sqrt[3]{9x}-\sqrt[3]{3x}-\sqrt[3]{27}$ $=\sqrt[3]{x^2}+\sqrt[3]{9x}-\sqrt[3]{3x}-3$
If a radical appears in a denominator of a fraction, or if a radicand contains a fraction, we can write the radical in simplest form by **rationalizing the denominator.** To rationalize a denominator, we multiply the given expression by a carefully chosen form of 1.	Rationalize the denominator: $\sqrt{\frac{10}{3}}=\frac{\sqrt{10}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$ $=\frac{\sqrt{30}}{3}$ $\quad\sqrt[3]{\frac{5}{2p^2}}=\frac{\sqrt[3]{5}}{\sqrt[3]{2p^2}}\cdot\frac{\sqrt[3]{4p}}{\sqrt[3]{4p}}$ $=\frac{\sqrt[3]{20p}}{\sqrt[3]{8p^3}}$ $=\frac{\sqrt[3]{20p}}{2p}$
Radical expressions that involve the sum and difference of the same two terms are called **conjugates.**	Conjugates: $\sqrt{2x}+3$ and $\sqrt{2x}-3$
To **rationalize a two-termed denominator** of a fraction, multiply the numerator and the denominator by the **conjugate** of the denominator.	Rationalize the denominator: $\frac{\sqrt{x}-2}{\sqrt{x}+2}=\frac{\sqrt{x}-2}{\sqrt{x}+2}\cdot\frac{\sqrt{x}-2}{\sqrt{x}-2}$ $=\frac{\sqrt{x}\sqrt{x}-2\sqrt{x}-2\sqrt{x}+4}{\sqrt{x}\sqrt{x}-2\sqrt{x}+2\sqrt{x}-4}$ $=\frac{x-4\sqrt{x}+4}{x-4}$

REVIEW EXERCISES

Simplify each expression. All variables represent positive real numbers.

49. $\sqrt{7}\sqrt{7}$
7

50. $\left(2\sqrt{5}\right)\left(3\sqrt{2}\right)$
$6\sqrt{10}$

51. $\left(-2\sqrt{8}\right)^2$
32

52. $2\sqrt{6}\sqrt{15}$
$6\sqrt{10}$

53. $\sqrt{9x}\sqrt{x}$
$3x$

54. $\left(\sqrt[3]{x+1}\right)^3$
$x+1$

55. $-\sqrt[3]{2x^2}\sqrt[3]{4x^8}$
$-2x^3\sqrt[3]{x}$

56. $\sqrt[5]{9}\cdot\sqrt[5]{27}$
3

57. $3\sqrt{7t}\left(2\sqrt{7t}+3\sqrt{3t^2}\right)$
$42t+9t\sqrt{21t}$

58. $-\sqrt[4]{4x^5y^{11}}\sqrt[4]{8x^9y^3}$
$-2x^3y^3\sqrt[4]{2x^2y^2}$

59. $\left(\sqrt{3b}+\sqrt{3}\right)^2$
$3b+6\sqrt{b}+3$

60. $\left(\sqrt[3]{3p}-2\sqrt[3]{2}\right)\left(\sqrt[3]{3p}+\sqrt[3]{2}\right)$
$\sqrt[3]{9p^2}-\sqrt[3]{6p}-2\sqrt[3]{4}$

Rationalize each denominator. All variables represent positive real numbers.

61. $\frac{10}{\sqrt{3}}$ $\quad\frac{10\sqrt{3}}{3}$

62. $\sqrt{\frac{3}{5xy}}$ $\quad\frac{\sqrt{15xy}}{5xy}$

63. $\frac{\sqrt[3]{6u}}{\sqrt[3]{u^5}}$ $\quad\frac{\sqrt[3]{6u^2}}{u^2}$

64. $\frac{\sqrt[4]{a}}{\sqrt[4]{3b^2}}$ $\quad\frac{\sqrt[4]{27ab^2}}{3b}$

65. $\frac{2}{\sqrt{2}-1}$
$2\left(\sqrt{2}+1\right)$ or $2\sqrt{2}+2$

66. $\frac{4\sqrt{x}-2\sqrt{z}}{\sqrt{z}+4\sqrt{x}}$
$\frac{12\sqrt{xz}-16x-2z}{z-16x}$

67. Rationalize the numerator: $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}}$ $\quad\frac{a-b}{a+\sqrt{ab}}$

68. VOLUME The formula relating the radius r of a sphere and its volume V is $r=\sqrt[3]{\frac{3V}{4\pi}}$. Write the radical in simplest form. $r=\frac{\sqrt[3]{6\pi^2V}}{2\pi}$

SECTION 7.4 Solving Radical Equations

DEFINITIONS AND CONCEPTS	EXAMPLES
We can use the **power rule** to solve equations containing radicals. If $x = y$, then $x^n = y^n$. To **solve equations containing radicals:** **1.** Isolate one radical expression on one side of the equation. **2.** Raise both sides of the equation to the power that is the same as the index. **3.** Solve the resulting equation. If it still contains a radical, go back to step 1. **4.** Check the solutions to eliminate extraneous solutions.	Solve each equation. $\sqrt{2x-2}+1=x$ $\sqrt{2x-2}=x-1$ $\left(\sqrt{2x-2}\right)^2=(x-1)^2$ $2x-2=x^2-2x+1$ $0=x^2-4x+3$ $0=(x-3)(x-1)$ $x-3=0$ or $x-1=0$ $x=3$ | $x=1$ The solutions are 3 and 1. Verify that each satisfies the original equation. $\sqrt[3]{x+2}=3$ $\left(\sqrt[3]{x+2}\right)^3=3^3$ $x+2=27$ $x=25$ The solution is 25. Verify that it satisfies the original equation.
When **more than one radical** appears in an equation, we must often use the power rule more than once.	Solve: $\sqrt{x}+\sqrt{x+5}=5$ $\sqrt{x+5}=5-\sqrt{x}$ — To isolate each radical, subtract $\sqrt{x}$ from both sides. $\left(\sqrt{x+5}\right)^2=\left(5-\sqrt{x}\right)^2$ — To eliminate one radical, square both sides. $x+5=25-10\sqrt{x}+x$ — Perform the operations on each side. $-20=-10\sqrt{x}$ — To isolate the radical term, subtract 25 and x from both sides. $2=\sqrt{x}$ — To isolate the radical, divide both sides by -10. $(2)^2=\left(\sqrt{x}\right)^2$ — To eliminate the radical, square both sides again. $4=x$ The solution is 4. Verify that it satisfies the original equation.

REVIEW EXERCISES

Solve each equation. Write all proposed solutions. Cross out those that are extraneous.

69. $\sqrt{7x-10}-1=11$
22

70. $u=\sqrt{25u-144}$
16, 9

71. $2\sqrt{y-3}=\sqrt{2y+1}$
$\frac{13}{2}$

72. $\sqrt{z+1}+\sqrt{z}=2$
$\frac{9}{16}$

73. $\sqrt[3]{x^3+56}-2=x$
2, -4

74. $a=\sqrt{a^2+5a-35}$
7

75. $(x+2)^{1/2}-(4-x)^{1/2}=0$
1

76. $\sqrt{b^2+b}=\sqrt{3-b^2}$
$-\frac{3}{2}$, 1

77. $\sqrt[4]{8x-8}+2=0$
~~3~~, no solution

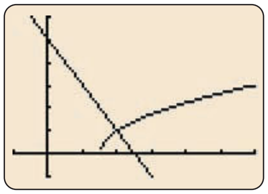
78. $\sqrt{2m+4}-\sqrt{m+3}=1$
6, -2

79. Let $f(x)=\sqrt{2x^2-7x}$. For what value(s) of x is $f(x)=2$?
$-\frac{1}{2}$, 4

80. Using the graphs of $f(x) = \sqrt{2x - 3}$ and $g(x) = -2x + 5$, estimate the solution of

$$\sqrt{2x - 3} = -2x + 5$$

Check the result. 2



Solve each equation for the specified variable.

81. $r = \sqrt{\frac{A}{P}} - 1$ for P $\quad P = \frac{A}{(r+1)^2}$

82. $h = \sqrt[3]{\frac{12I}{b}}$ for I $\quad I = \frac{h^3 b}{12}$

SECTION 7.5 Rational Exponents

DEFINITIONS AND CONCEPTS	EXAMPLES
To simplify exponential expressions involving **rational (fractional) exponents,** use the following rules to write the expressions in an equivalent radical form. $x^{1/n} = \sqrt[n]{x}$ $\quad x^{m/n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$	Simplify. All variables represent positive real numbers. $25^{1/2} = \sqrt{25} = 5$ $\quad \left(\frac{256}{d^4}\right)^{1/4} = \sqrt[4]{\frac{256}{d^4}} = \frac{4}{d}$ $\quad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$ $\quad (t^{10})^{6/5} = \left(\sqrt[5]{t^{10}}\right)^6 = (t^2)^6 = t^{12}$
To be consistent with the definition of negative integer exponents, we define $x^{-m/n}$ as follows. $x^{-m/n} = \frac{1}{x^{m/n}}$ $\quad \frac{1}{x^{-m/n}} = x^{m/n}$	Simplify. All variables represent positive real numbers. $(125)^{-2/3} = \frac{1}{(125)^{2/3}} = \frac{1}{\left(\sqrt[3]{125}\right)^2} = \frac{1}{25}$ $\quad \frac{1}{(-32x^5)^{-3/5}} = (-32x^5)^{3/5} = \left(\sqrt[5]{-32x^5}\right)^3 = -8x^3$
The **rules for exponents** can be used to simplify expressions with rational (fractional) exponents.	Simplify: $\frac{p^{5/3}p^{8/3}}{p^4} = p^{5/3+8/3-4} = p^{5/3+8/3-12/3} = p^{1/3}$
We can write certain radical expressions as an equivalent exponential expression and use rules for exponents to simplify it. Then we can change that result back into a radical.	Simplify: $\sqrt[4]{9} = \sqrt[4]{3^2} = 3^{2/4} = 3^{1/2} = \sqrt{3}$

REVIEW EXERCISES

Write each expression in radical form.

83. $t^{1/2}$ $\quad \sqrt{t}$

84. $(5xy^3)^{1/4}$ $\quad \sqrt[4]{5xy^3}$

Simplify each expression, if possible. Assume that all variables represent positive real numbers.

85. $25^{1/2}$ $\quad 5$

86. $-36^{1/2}$ $\quad -6$

87. $(-36)^{1/2}$ not a real number

88. $1^{1/5}$ $\quad 1$

89. $\left(\frac{9}{x^2}\right)^{1/2}$ $\quad \frac{3}{x}$

90. $(-8)^{1/3}$ $\quad -2$

91. $625^{1/4}$ $\quad 5$

92. $(81c^4d^4)^{1/4}$ $\quad 3cd$

93. $9^{3/2}$ $\quad 27$

94. $8^{-2/3}$ $\quad \frac{1}{4}$

95. $-49^{5/2}$ $\quad -16{,}807$

96. $\frac{1}{100^{-1/2}}$ $\quad 10$

97. $\left(\frac{4}{9}\right)^{-3/2}$ $\quad \frac{27}{8}$

98. $\frac{1}{25^{5/2}}$ $\quad \frac{1}{3{,}125}$

99. $(25x^2y^4)^{3/2}$ $\quad 125x^3y^6$

100. $(8u^6v^3)^{-2/3}$ $\quad \frac{1}{4u^4v^2}$

Perform the operations. Write answers using positive exponents only. Assume that all variables represent positive real numbers.

101. $5^{1/4}5^{1/2}$ $5^{3/4}$

102. $a^{3/7}a^{-2/7}$ $a^{1/7}$

103. $(k^{4/5})^{10}$ k^8

104. $\dfrac{3^{5/6}3^{1/3}}{3^{1/2}}$ $3^{2/3}$

Perform the multiplications. Assume all variables represent positive real numbers.

105. $u^{1/2}(u^{1/2} - u^{-1/2})$ $u - 1$

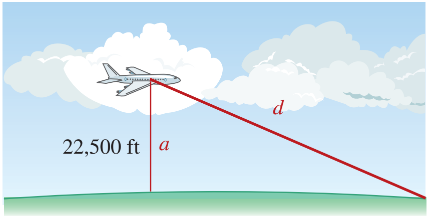
106. $v^{2/3}(v^{1/3} + v^{4/3})$ $v + v^2$

Use rational exponents to simplify each radical. All variables represent positive real numbers.

107. $\sqrt[4]{a^2}$ $\sqrt{a}$

108. $\sqrt[3]{\sqrt{c}}$ $\sqrt[6]{c}$

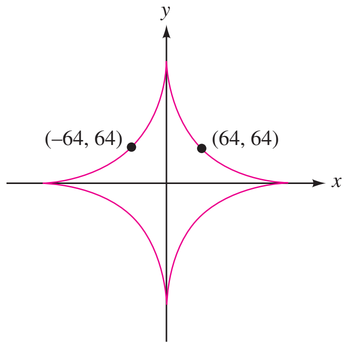
109. VISIBILITY See the illustration in the next column. The distance d in miles a person in an airplane can see to the horizon on a clear day is given by the formula $d = 1.22a^{1/2}$, where a is the altitude of the plane in feet. Find d. 183 mi



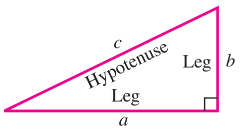
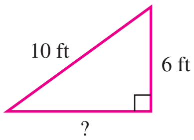
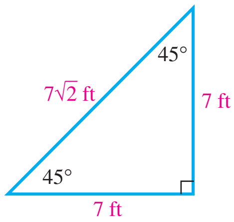
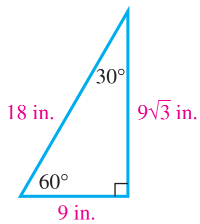
110. Substitute the x- and y-coordinates of each point labeled in the graph into the equation

$$x^{2/3} + y^{2/3} = 32$$

Show that each one satisfies the equation.
Two true statements result: $32 = 32$.



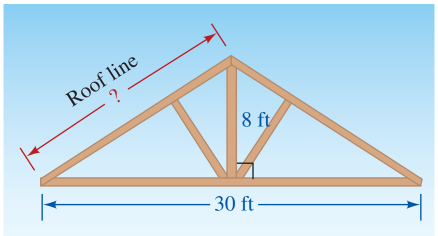
SECTION 7.6 Geometric Applications of Radicals

DEFINITIONS AND CONCEPTS	EXAMPLES
The Pythagorean theorem: If a and b are the lengths of the **legs** of a right triangle and c is the length of the **hypotenuse,** then $a^2 + b^2 = c^2$.  	Find the length of the third side of the right triangle.  $a^2 + b^2 = c^2$ This is the Pythagorean theorem. $6^2 + b^2 = 10^2$ Substitute 6 for a and 10 for c. $36 + b^2 = 100$ $b^2 = 64$ To isolate b^2, subtract 36 from both sides. $b = \sqrt{64}$ Since b must be positive, find the positive square root of 64. $b = 8$ The length of the third side of the triangle is 8 ft.
In an **isosceles right triangle,** the length of the hypotenuse is $\sqrt{2}$ times the length of one leg.	If the length of one leg of an isosceles right triangle is 7 feet, the length of the hypotenuse is $7\sqrt{2}$ feet.  Isosceles right triangle
The hypotenuse of a **30°–60°–90° triangle** is twice as long as the shorter leg (the leg opposite the 30° angle.) The length of the longer leg (the leg opposite the 60° angle) is $\sqrt{3}$ times the length of the shorter leg.	If the shorter leg of a 30°–60°–90° triangle is 9 inches long: • The length of the hypotenuse is $2 \cdot 9 = 18$ inches. • The length of the longer leg is $\sqrt{3} \cdot 9 = 9\sqrt{3}$ inches.  30°–60°–90° Triangle

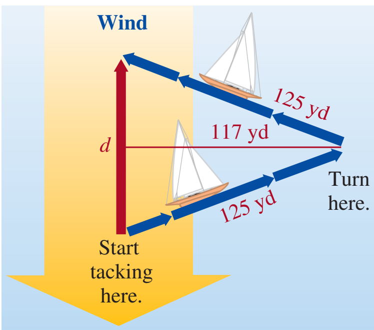
The distance d between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by the **distance formula:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	The distance between points $(-2, 3)$ and $(1, 7)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{[1 - (-2)]^2 + (7 - 3)^2}$ $= \sqrt{3^2 + 4^2}$ $= \sqrt{9 + 16}$ $= \sqrt{25}$ $= 5$

REVIEW EXERCISES

111. CARPENTRY The gable end of the roof shown below is divided in half by a vertical brace, 8 feet in height. Find the length of the roof line. 17 ft



112. SAILING A technique called *tacking* allows a sailboat to make progress into the wind. A sailboat follows the course shown below. Find d, the distance the boat advances into the wind after tacking. 88 yd



For problems 113–118, give the exact answer and then an approximation to two decimal places, when appropriate.

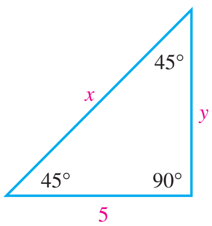
113. Find the length of the hypotenuse of an isosceles right triangle if the length of one leg is 7 meters.
$7\sqrt{2}$ m ≈ 9.90 m

114. The length of the hypotenuse of an isosceles right triangle is 15 yards. Find the length of one leg of the triangle.
$\frac{15\sqrt{2}}{2}$ yd ≈ 10.61 yd

115. The length of the hypotenuse of a 30°–60°–90° triangle is 12 centimeters. Find the length of each leg.
shorter leg: 6 cm, longer leg: $6\sqrt{3}$ cm ≈ 10.39 cm

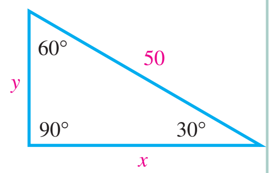
116. In a 30°–60°–90° triangle, the length of the longer leg is 60 feet. Find the length of the hypotenuse and the length of the shorter leg.
$40\sqrt{3}$ ft ≈ 69.28 ft, $20\sqrt{3}$ ft ≈ 34.64 ft

117. Find x and y.



$x = 5\sqrt{2} \approx 7.07$, $y = 5$

118. Find x and y.



$x = 25\sqrt{3} \approx 43.30$, $y = 25$

Find the distance between the points.

119. $(1, 3)$ and $(6, -9)$
13

120. $(-4, 6)$ and $(-2, 8)$
$2\sqrt{2}$

SECTION 7.7 Complex Numbers

DEFINITIONS AND CONCEPTS	EXAMPLES
The **imaginary number** i is defined as $i = \sqrt{-1}$ From the definition, it follows that $i^2 = -1$.	Write each expression in terms of i: $\sqrt{-81} = \sqrt{-1 \cdot 81}$ $= \sqrt{-1}\sqrt{81}$ $= i \cdot 9$ $= 9i$ $\sqrt{-24} = \sqrt{-1 \cdot 24}$ $= \sqrt{-1}\sqrt{24}$ $= i\sqrt{4}\sqrt{6}$ $= 2i\sqrt{6}$ or $2\sqrt{6}i$
A **complex number** is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. We call a the **real part** and b the **imaginary part.**	Complex numbers: $5 + 3i$ 5 is the real part and 3 is the imaginary part. $16 = 16 + 0i$ 16 is the real part and 0 is the imaginary part. $9i = 0 + 9i$ 0 is the real part and 9 is the imaginary part.
Adding and subtracting complex numbers is similar to adding and subtracting polynomials. To **add two complex numbers,** add their real parts and add their imaginary parts. To **subtract two complex numbers,** add the opposite of the complex number being subtracted.	Add. Write the answer in the form $a + bi$. $(7 - 5i) + (3 + 9i) = (7 + 3) + (-5 + 9)i$ Add the real parts. Add the imaginary parts. $= 10 + 4i$ Subtract. Write the answer in the form $a + bi$. $(8 - i) - (-1 + 6i) = (8 - i) + (1 - 6i)$ Add the opposite of $-1 + 6i$. $= (8 + 1) + [-1 + (-6)]i$ Add the real parts. Add the imaginary parts. $= 9 - 7i$
Multiplying complex numbers is similar to multiplying polynomials.	Multiply. Write the answers in the form $a + bi$. $3i(6 - 4i) = 18i - 12i^2$ $= 18i - 12(-1)$ $= 18i + 12$ $= 12 + 18i$ $(4 + 7i)(2 - i) = 8 - 4i + 14i - 7i^2$ $= 8 + 10i - 7(-1)$ $= 8 + 10i + 7$ $= 15 + 10i$
The complex numbers $a + bi$ and $a - bi$ are called **complex conjugates.**	The complex numbers $7 - 2i$ and $7 + 2i$ are complex conjugates.
To **divide complex numbers,** multiply the numerator and denominator by the complex conjugate of the denominator. The process is similar to rationalizing denominators.	Divide. Write the answers in the form $a + bi$. $\frac{3}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{3(1 - i)}{1 - i + i - i^2}$ $= \frac{3(1 - i)}{1 - (-1)}$ $= \frac{3 - 3i}{2}$ $= \frac{3}{2} - \frac{3}{2}i$ $\frac{6 + i}{2 - i} \cdot \frac{2 + i}{2 + i} = \frac{12 + 6i + 2i + i^2}{4 + 2i - 2i - i^2}$ $= \frac{12 + 8i + (-1)}{4 - (-1)}$ $= \frac{11 + 8i}{5}$ $= \frac{11}{5} + \frac{8}{5}i$

The **powers of i** cycle through four possible outcomes: i, -1, $-i$, and 1.

$$\begin{aligned} i^1 &= \mathbf{i} = i^5 = i^9 = \ldots \\ i^2 &= \mathbf{-1} = i^6 = i^{10} = \ldots \\ i^3 &= \mathbf{-i} = i^7 = i^{11} = \ldots \\ i^4 &= \mathbf{1} = i^8 = i^{12} = \ldots \end{aligned}$$

Simplify: i^{66}

We divide 66 by 4 to get a remainder of 2. Thus, $i^{66} = i^{\mathbf{2}} = -1$.

$$\begin{array}{r} 16 \text{ R}2 \\ 4\overline{)66} \\ \underline{-4} \\ 26 \\ \underline{-24} \\ \mathbf{2} \end{array}$$

REVIEW EXERCISES

Write each expression in terms of ***i.***

121. $\sqrt{-25}$ $5i$

122. $\sqrt{-18}$ $3i\sqrt{2}$

123. $-\sqrt{-6}$ $-i\sqrt{6}$

124. $\sqrt{-\frac{9}{64}}$ $\frac{3}{8}i$

125. Complete the diagram.

Complex numbers

Real numbers	Imaginary numbers

126. Determine whether each statement is true or false.

a. Every real number is a complex number. true

b. $3 - 4i$ is a complex number. true

c. $\sqrt{-4}$ is a real number. false

d. i is a real number. false

Give the complex conjugate of each number.

127. a. $3 + 6i$ $3 - 6i$

b. $19i$ $0 - 19i$

128. a. $-1 - 7i$ $-1 + 7i$

b. $-i$ $0 + i$

Perform the operations. Write all answers in the form ***a + bi.***

129. $(3 + 4i) + (5 - 6i)$ $8 - 2i$

130. $\left(7 - \sqrt{-9}\right) - \left(4 + \sqrt{-4}\right)$ $3 - 5i$

131. $3i(2 - i)$ $3 + 6i$

132. $(2 - 7i)(-3 + 4i)$ $22 + 29i$

133. $\sqrt{-3} \cdot \sqrt{-9}$ $-3\sqrt{3} + 0i$

134. $(9i)^2$ $-81 + 0i$

135. $\dfrac{5 + 14i}{2 + 3i}$ $4 + i$

136. $\dfrac{3}{11i}$ $0 - \frac{3}{11}i$

Simplify each expression.

137. i^{42} -1

138. i^{97} i

CHAPTER 7 TEST

1. Fill in the blanks.

a. The symbol $\sqrt{}$ is called a radical symbol.

b. The imaginary number i is defined as $i = \sqrt{-1}$.

c. Squaring both sides of an equation can introduce extraneous solutions.

d. An isosceles right triangle is a right triangle with two legs of equal length.

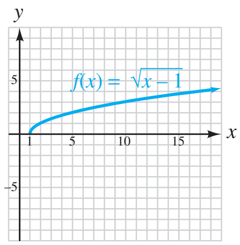
e. To rationalize the denominator of $\frac{4}{\sqrt{5}}$, we multiply the fraction by $\frac{\sqrt{5}}{\sqrt{5}}$.

f. A complex number is any number that can be written in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

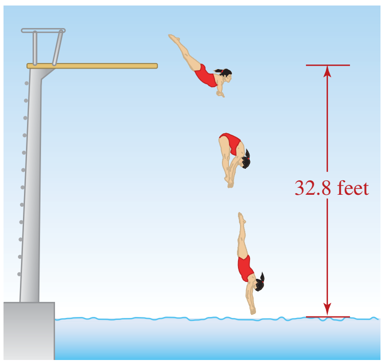
2. a. State the product rule for radicals.
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.

b. State the quotient rule for radicals.
If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$. $(b \neq 0)$

3. Graph $f(x) = \sqrt{x - 1}$. Find the domain and range of the function. D: $[1, \infty)$, R: $[0, \infty)$



4. DIVING Refer to the illustration below. The velocity v of an object in feet per second after it has fallen a distance of d feet is approximated by the function $v(d) = \sqrt{64.4d}$. Olympic diving platforms are 10 meters tall (approximately 32.8 feet). Estimate the velocity at which a diver hits the water from this height. Round to the nearest foot per second. 46 ft/sec



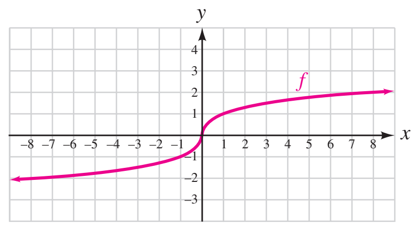
5. Use the graph to find each of the following.

a. $f(-1)$ -1

b. $f(8)$ 2

c. The value(s) of x for which $f(x) = 1$ 1

d. The domain and range of f D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



6. Explain why $\sqrt[4]{-16}$ is not a real number.
No real number raised to the fourth power is -16.

Simplify each expression. The variables are unrestricted.

7. $\sqrt{x^2}$ $|x|$

8. $\sqrt{y^2 - 10y + 25}$ $|y - 5|$

Simplify each expression. All variables represent positive real numbers.

9. $\sqrt[3]{-64x^3y^6}$ $-4xy^2$

10. $\sqrt{\frac{4a^2}{9}}$ $\frac{2}{3}a$

11. $\sqrt[5]{(t + 8)^5}$ $t + 8$

12. $\sqrt{540x^3y^5}$ $6xy^2\sqrt{15xy}$

13. $\frac{\sqrt[3]{24x^{15}y^4}}{\sqrt[3]{y}}$ $2x^5y\sqrt[3]{3}$

14. $\sqrt[4]{32}$ $2\sqrt[4]{2}$

Perform the operations and simplify. All variables represent positive real numbers.

15. $2\sqrt{48y^5} - 3y\sqrt{12y^3}$ $2y^2\sqrt{3y}$

16. $2\sqrt[3]{40} - \sqrt[3]{5{,}000} + 4\sqrt[3]{625}$ $14\sqrt[3]{5}$

17. $\sqrt[4]{243z^{13}} + z\sqrt[4]{48z^9}$ $5z^3\sqrt[4]{3z}$

18. $-2\sqrt{xy}\left(3\sqrt{x} + \sqrt{xy^3}\right)$ $-6x\sqrt{y} - 2xy^2$

19. $\left(3\sqrt{2}+\sqrt{3}\right)\left(2\sqrt{2}-3\sqrt{3}\right)$ $3-7\sqrt{6}$

20. $\left(\sqrt[3]{2a}+9\right)^2$ $\sqrt[3]{4a^2}+18\sqrt[3]{2a}+81$

21. $\dfrac{8}{\sqrt{10}}$ $\frac{4\sqrt{10}}{5}$

22. $\dfrac{3t-1}{\sqrt{3t}-1}$ $\sqrt{3t}+1$

23. $\sqrt[3]{\dfrac{9}{4a}}$ $\frac{\sqrt[3]{18a^2}}{2a}$

24. Rationalize the numerator: $\dfrac{\sqrt{5}+3}{-4\sqrt{2}}$

$\frac{1}{\sqrt{2}\left(\sqrt{5}-3\right)}=\frac{1}{\sqrt{10}-3\sqrt{2}}$

Solve each equation. Write all proposed solutions. Cross out those that are extraneous.

25. $4\sqrt{x}=\sqrt{x+1}$ $\frac{1}{15}$

26. $\sqrt[3]{6n+4}-4=0$ 10

27. $1=\sqrt{u-3}+\sqrt{u}$ $\cancel{4}$, no solution

28. $(2m^2-9)^{1/2}=m$ $3, \cancel{-3}$

29. $\sqrt{t-2}-t+2=0$ $2, 3$

30. $\sqrt{x-8}+10=0$ $\cancel{108}$, no solution

31. $\sqrt[4]{15-a}=\sqrt[4]{13-2a}$ -2

32. Solve $r=\sqrt[3]{\dfrac{GMt^2}{4\pi^2}}$ for G. $G=\frac{4\pi^2r^3}{Mt^2}$

Simplify each expression. All variables represent positive real numbers. Write answers using positive exponents only.

33. $(49x^4)^{1/2}$ $7x^2$

34. $-27^{2/3}$ -9

35. $36^{-3/2}$ $\frac{1}{216}$

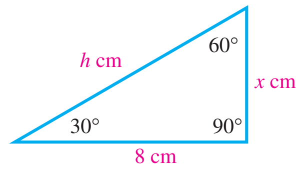
36. $\left(-\dfrac{8}{125n^6}\right)^{-2/3}$ $\frac{25n^4}{4}$

37. $\dfrac{2^{5/3}2^{1/6}}{2^{1/2}}$ $2^{4/3}$

38. $(a^{2/3})^{1/6}$ $a^{1/9}$

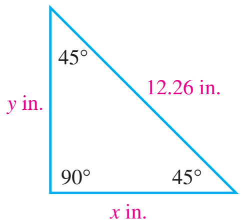
Find the missing side lengths in each triangle. Give the exact answer and then an approximation to two decimal places, when appropriate.

39.



$x=\frac{8\sqrt{3}}{3}$ cm ≈ 4.62 cm,
$h=\frac{16\sqrt{3}}{3}$ cm ≈ 9.24 cm

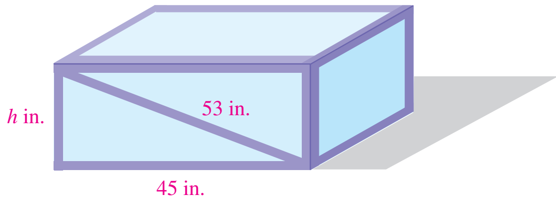
40.



$x=\frac{(12.26)\sqrt{2}}{2}$ in. ≈ 8.67 in.,
$y=\frac{(12.26)\sqrt{2}}{2}$ in. ≈ 8.67 in.

41. Find the distance between $(-2, 5)$ and $(22, 12)$. 25

42. SHIPPING CRATES The diagonal brace on the shipping crate in the illustration is 53 inches. Find the height h of the crate. 28 in.



43. Express $\sqrt{-45}$ in terms of i. $3i\sqrt{5}$

44. Simplify: i^{106} -1

Perform the operations. Write all answers in the form a + bi.

45. $(9+4i)+(-13+7i)$ $-4+11i$

46. $\left(3-\sqrt{-9}\right)-\left(-1+\sqrt{-16}\right)$ $4-7i$

47. $15i(3-5i)$ $75+45i$

48. $(8+10i)(-7-i)$ $-46-78i$

49. $\dfrac{1}{i\sqrt{2}}$ $0-\frac{\sqrt{2}}{2}i$

50. $\dfrac{2+i}{3-i}$ $\frac{1}{2}+\frac{1}{2}i$

CHAPTERS 1–7 CUMULATIVE REVIEW

1. a. What is a rational number? [Section 1.2]
A rational number is any number that can be written as a fraction with an integer numerator and a nonzero integer denominator.

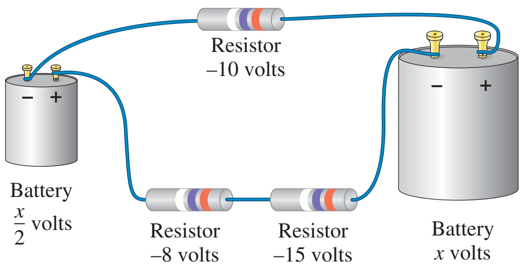
b. What is an irrational number?
An irrational number is a nonterminating, nonrepeating decimal.

c. What is a real number?
A real number is any number that is either a rational number or an irrational number.

2. Evaluate $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$ for $A = 6, B = -8,$ $C = -5$, and $x_0 = y_0 = -2$. [Section 1.3] $\frac{1}{10}$

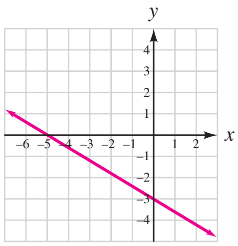
3. Solve $S = \dfrac{a - \ell r}{1 - r}$ for ℓ. [Section 1.5] $\ell = \dfrac{a - S + Sr}{r}$

4. ELECTRONICS The illustration shows a closed circuit with two voltage sources and three resistors. The sum of the voltages in the loop must be 0. Find x. [Section 1.6] 22



5. SALAD DRESSING A caterer is going to combine 10% vinegar-oil dressing with 18% vinegar-oil dressing to make 10 cups of a 15% vinegar-oil dressing. How many cups of each type of dressing will she need? [Section 1.7] 10%: $3\frac{3}{4}$ cups, 18%: $6\frac{1}{4}$ cups

6. Refer to the illustration.

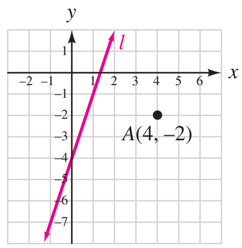


a. What is the slope of the line? [Section 2.4] $-\frac{3}{5}$

b. What is the y-intercept of the line? $(0, -3)$

c. What is the equation of the line? $y = -\frac{3}{5}x - 3$

7. Refer to the illustration.

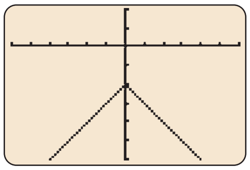


a. What is the slope of line l? [Section 2.4] 3

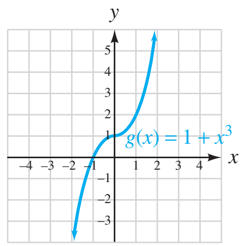
b. Write the equation of the line that passes through point A and is perpendicular to line l. Answer in slope–intercept form. $y = -\frac{1}{3}x - \frac{2}{3}$

8. If $f(x) = 1.25 - x^5$, find $f(2)$. [Section 2.5] -30.75

9. Determine the domain and range of the function $f(x) = -|x| - 2$, which is graphed to the right. (The x- and y-axes are scaled in units of 1.) [Section 2.6]
D: $(-\infty, \infty)$, R: $(-\infty, -2)$

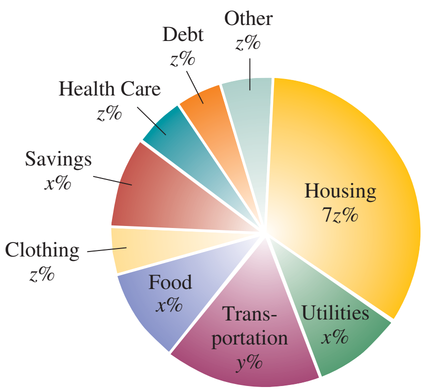


10. Graph: $g(x) = 1 + x^3$ [Section 2.6]



11. Solve, if possible: $\begin{cases} x = \frac{3}{2}y + 5 \\ 2x - 3y = 8 \end{cases}$ [Section 3.2] no solution

12. BUDGETS See the family budget guidelines below. It is recommended that housing, utilities, and transportation costs should be 60% of the budget; and food, clothing, and savings should be 25% of the budget. Find x, y, and z. (*Hint:* The sum of the percents from all categories in a circle graph is what number?) [Section 3.4] 10, 15, 5

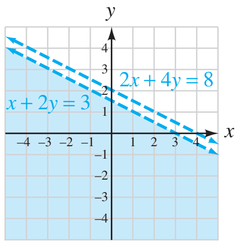


Solve each inequality or compound inequality and graph the solution set. Then express the solution set using interval notation.

13. $5(x + 1) \le 4(x + 3)$ and $x + 12 < -3$ [Section 4.2]
$(-\infty, -15)$ 

14. $|-1 - 2x| > 5$ [Section 4.3]
$(-\infty, -3) \cup (2, \infty)$ 

15. Graph: $\begin{cases} x + 2y < 3 \\ 2x + 4y < 8 \end{cases}$ [Section 4.5]



16. Simplify: $\left(\dfrac{4a^{-2}b}{3ab^{-3}}\right)^3$ [Section 5.1] $\dfrac{64b^{12}}{27a^9}$

17. Find $(6.1 \times 10^8)(3.9 \times 10^5)$. Give the answer in scientific notation. [Section 5.2] 2.379×10^{14}

18. Subtract:
$(-2x^2y^3 + 6xy + 5y^2) - (-4x^2y^3 - 7xy + 2y^2)$
[Section 5.3] $2x^2y^3 + 13xy + 3y^2$

19. Multiply: $(3y + 1)(2y^2 + 3y + 2)$
[Section 5.4] $6y^3 + 11y^2 + 9y + 2$

20. Simplify: $(x + 3)(x - 3) + (2x - 1)(x + 2)$
[Section 5.4] $3x^2 + 3x - 11$

Factor each polynomial completely.

21. $3c - cd + 3d - c^2$ [Section 5.5] $(3 - c)(c + d)$

22. $x^3 - 8y^3$ [Section 5.6] $(x - 2y)(x^2 + 2xy + 4y^2)$

23. $(a + b)^2 - 2(a + b) + 1$ [Section 5.7] $(a + b - 1)^2$

24. $x^4 - 17x^2 + 16$ [Section 5.8] $(x + 1)(x - 1)(x + 4)(x - 4)$

Solve each equation and check the result.

25. $2z^3 - 200z = 0$ [Section 5.9] $0, 10, -10$

26. $3m^2 + 10m = -3$ [Section 5.9] $-\frac{1}{3}, -3$

27. Simplify: $\dfrac{3x^2 - 10xy - 8y^2}{4y^2 - xy}$ [Section 6.1] $\dfrac{-3x - 2y}{y}$

28. For what values of x is $\dfrac{2}{x^2 - x - 56}$ undefined?
[Section 6.1] $-7, 8$

Perform the operations.

29. $(2x^2 - 9x - 5) \cdot \dfrac{x}{2x^2 + x}$ [Section 6.2] $x - 5$

30. $\dfrac{2x}{x^2 - 4} - \dfrac{1}{x^2 - 3x + 2} + \dfrac{x + 1}{x^2 + x - 2}$
[Section 6.3] $\dfrac{3x + 2}{(x + 2)(x - 1)}$

31. SHARED WORK One pipe fills a tank in 4 hours, and another fills it in 6 hours. How long will it take to fill the tank using both pipes? [Section 6.8] $2\frac{2}{5}$ hr

32. Explain the difference between direct variation and inverse variation. Assume a positive constant of variation. [Section 6.9]
Direct variation: As one quantity increases, the other increases, in a predictable way. Inverse variation: As one quantity increases, the other decreases, in a predictable way.

Simplify each expression.

33. $\sqrt{200x^4y^3z}$ [Section 7.2] $10x^2y\sqrt{2yz}$

34. $\sqrt[3]{16} + \sqrt[3]{128}$ [Section 7.2] $6\sqrt[3]{2}$

35. $\left(\sqrt{5z} + \sqrt{3}\right)\left(\sqrt{5z} + \sqrt{3}\right)$
[Section 7.3] $5z + 2\sqrt{15z} + 3$

36. Rationalize the denominator: $\dfrac{\sqrt{3}}{\sqrt{50}}$ [Section 7.3] $\dfrac{\sqrt{6}}{10}$

37. Solve: $\sqrt{-5x + 24} = 6 - x$ [Section 7.4] $4, 3$

38. $\left(-\dfrac{8x^3}{27}\right)^{-1/3}$ [Section 7.5] $-\frac{3}{2x}$

Quadratic Equations, Functions, and Inequalities

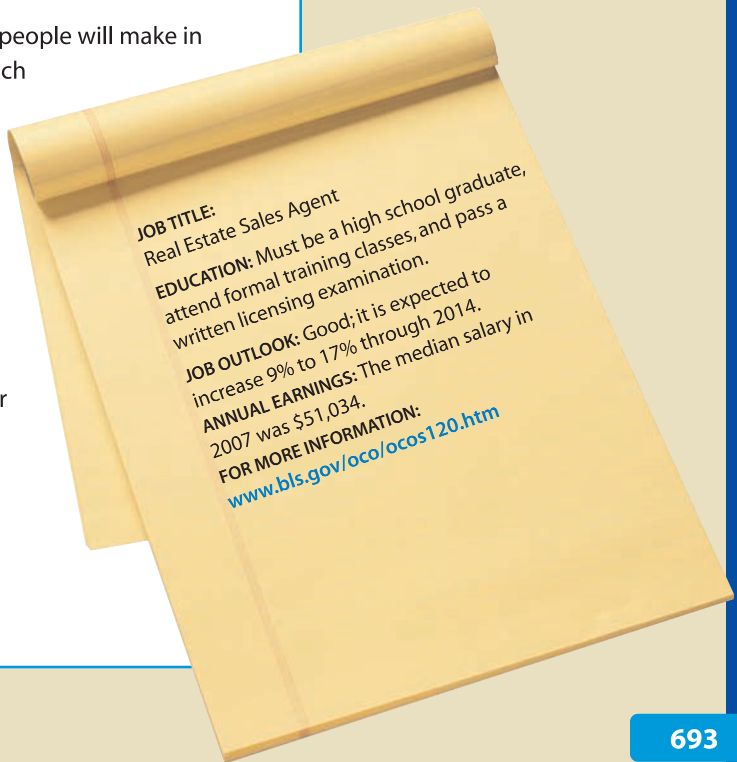
8

from Campus to Careers

Real Estate Sales Agent

Buying a house is probably the biggest purchase that most people will make in their lives. The complex process of purchasing a home is much easier with the help of a real estate agent. Real estate agents use their mathematical skills in many ways. They compute square footage, appraise property, calculate commissions, and write offer sheets. Technology is widely used in the real estate industry. Most sales agents use computers to locate and list available properties and identify sources of financing.

In **Problem 96** of **Study Set 8.1,** you will find the rate of appreciation of a house purchased for $150,000 that sold for $200,000 in two years.



Objectives

1. Review solving quadratic equations by factoring.
2. Use the square root property to solve quadratic equations.
3. Solve quadratic equations by completing the square.
4. Use quadratic equations to solve application problems.

SECTION 8.1

The Square Root Property and Completing the Square

We have seen that equations involving first-degree polynomials, such as $12x - 4 = 0$, are called *linear equations.* We have also seen that equations involving second-degree polynomials, such as $12x^2 - 4x = 0$, are called *quadratic equations.* In Chapter 5, we learned how to solve quadratic equations by factoring. However, as we shall see, the factoring method has its limitations. In this section, we will introduce a more general method that enables us to solve any quadratic equation.

1 Review solving quadratic equations by factoring.

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. We have discussed how to solve quadratic equations by factoring and the zero-factor property.

Self Check 1

Solve: $15x^2 - 17x - 4 = 0$

Now Try **Problem 25**

Self Check 1 Answer

$\frac{4}{3}, -\frac{1}{5}$

Teaching Example 1 Solve: $7x^2 + 6x - 1 = 0$

Answer:

$\frac{1}{7}, -1$

EXAMPLE 1 Solve: $6x^2 - 7x - 3 = 0$

Strategy We will factor the trinomial on the left side of the equation and use the zero-factor property to solve for x.

WHY To use the zero-factor property, we need one side of the equation to be factored completely and the other side to be 0.

Solution

$$6x^2 - 7x - 3 = 0$$

$$(2x - 3)(3x + 1) = 0 \qquad \text{Factor the trinomial.}$$

$$2x - 3 = 0 \quad \text{or} \quad 3x + 1 = 0 \qquad \text{Set each factor equal to 0.}$$

$$x = \frac{3}{2} \quad \Big| \quad x = -\frac{1}{3} \qquad \text{Solve each linear equation.}$$

Many expressions do not factor as easily as $6x^2 - 7x - 3$. For example, it would be difficult to solve $2x^2 + 4x + 1 = 0$ by factoring, because $2x^2 + 4x + 1$ cannot be factored by using only integers. With this in mind, we will now develop another method of solving quadratic equations. It is based on the *square root property.*

2 Use the square root property to solve quadratic equations.

To develop general methods for solving all quadratic equations, we first consider the equation $x^2 = c$. If $c \geq 0$, we can find the real solutions of $x^2 = c$ as follows:

$$x^2 = c$$

$$x^2 - c = 0 \qquad \text{Subtract } c \text{ from both sides.}$$

$$x^2 - \left(\sqrt{c}\right)^2 = 0 \qquad \text{Replace } c \text{ with } \left(\sqrt{c}\right)^2\text{, since } c = \left(\sqrt{c}\right)^2.$$

$$\left(x + \sqrt{c}\right)\left(x - \sqrt{c}\right) = 0 \qquad \text{Factor the difference of two squares.}$$

$$x + \sqrt{c} = 0 \quad \text{or} \quad x - \sqrt{c} = 0 \qquad \text{Set each factor equal to 0.}$$

$$x = -\sqrt{c} \quad \Big| \quad x = \sqrt{c} \qquad \text{Solve each linear equation.}$$

The two solutions of $x^2 = c$ are $x = \sqrt{c}$ and $x = -\sqrt{c}$.

Square Root Property

For any nonnegative real number c, if $x^2 = c$, then

$$x = \sqrt{c} \qquad \text{or} \qquad x = -\sqrt{c}$$

EXAMPLE 2 Solve: $x^2 - 12 = 0$

Strategy We will add 12 to both sides of the equation and use the square root property to solve for x.

WHY After adding 12 to both sides, the resulting equivalent equation will have the desired form $x^2 = c$.

Solution

$x^2 - 12 = 0$ This is the equation to solve.

$x^2 = 12$ To isolate x^2 on the left side, add 12 to both sides.

$x = \sqrt{12}$ or $x = -\sqrt{12}$ Use the square root property.

$x = 2\sqrt{3}$ | $x = -2\sqrt{3}$ Simplify: $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$.

Verify that $2\sqrt{3}$ and $-2\sqrt{3}$ satisfy the original equation.

We can use **double-sign notation** $\pm$ to write the solutions in more compact form as $\pm 2\sqrt{3}$. Read $\pm$ as "positive or negative." We can use a calculator to approximate the solutions. To the nearest hundredth, they are ± 3.46.

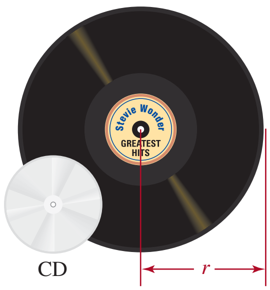
Self Check 2

Solve: $x^2 - 18 = 0$ $\pm 3\sqrt{2}$

Now Try **Problem 34**

Teaching Example 2 Solve: $x^2 - 75 = 0$
Answer:
$\pm 5\sqrt{3}$

EXAMPLE 3 ***Phonograph Records*** Before compact disc (CD) technology, one way of recording music was by engraving grooves on thin vinyl discs called records. The vinyl discs used for long-playing records had a surface area of about 111 square inches per side and were played at $33\frac{1}{3}$ revolutions per minute on a turntable. What is the radius of a long-playing record?



Strategy The area A of a circle with radius r is given by the formula $A = \pi r^2$. We will find the radius of a record by substituting 111 for A and dividing both sides by π. Then we will use the square root property to solve for r.

WHY After substituting 111 for A and dividing both sides by π, the resulting equivalent equation will have the desired form $r^2 = c$.

Solution

$A = \pi r^2$ This is the formula for the area of a circle.

$111 = \pi r^2$ Substitute 111 for A.

$\frac{111}{\pi} = r^2$ To undo the multiplication by π, divide both sides by π.

$r = \sqrt{\frac{111}{\pi}}$ or $r = -\sqrt{\frac{111}{\pi}}$ Use the square root property. Since the radius of the record cannot be negative, discard the second solution.

The radius of a record is $\sqrt{\frac{111}{\pi}}$ inches—to the nearest tenth, 5.9 inches.

Self Check 3

RESTAURANT SEATING A restaurant requires 1,257 square inches of area for their round tables. Find the radius of the tables. $\sqrt{\frac{1257}{\pi}}$ in.

Now Try **Problem 40**

Teaching Example 3 COMPACT DISCS The surface area of a compact disc (CD) is approximately 15.9 square inches. What is the radius of a CD?
Answer:
2.25 in.

Self Check 4

Solve: $(x + 2)^2 = 9$ $1, -5$

Now Try **Problem 41**

Teaching Example 4 Solve: $(x + 7)^2 = 25$
Answer: $-2, -12$

EXAMPLE 4 Solve: $(x - 3)^2 = 16$

Strategy Instead of a variable squared on the left side, we have a quantity squared. We can still use the square root property to solve the equation.

WHY We want to eliminate the square on the binomial, so that we can eventually isolate the variable on one side of the equation.

Solution

$$(x - 3)^2 = 16$$

$x - 3 = \sqrt{16}$	or	$x - 3 = -\sqrt{16}$	Use the square root property.
$x - 3 = 4$		$x - 3 = -4$	Simplify: $\sqrt{16} = 4$.
$x = 3 + 4$		$x = 3 - 4$	Add 3 to both sides.
$x = 7$		$x = -1$	Simplify.

Verify that 7 and -1 satisfy the original equation.

Some quadratic equations have solutions that are not real numbers.

Self Check 5

Solve: $16x^2 + 81 = 0$ $\frac{9}{4}i, -\frac{9}{4}i$

Now Try **Problem 48**

Teaching Example 5 Solve: $9x^2 + 49 = 0$
Answer: $\frac{7}{3}i, -\frac{7}{3}i$

EXAMPLE 5 Solve: $4x^2 + 25 = 0$

Strategy We will subtract 25 from both sides of the equation and divide both sides by 4. Then we will use the square root property to solve for x.

WHY After subtracting 25 from both sides and dividing both sides by 4, the resulting equivalent equation will have the desired form $x^2 = c$.

Solution

$4x^2 + 25 = 0$ This is the equation to solve.

$x^2 = -\frac{25}{4}$ To isolate x^2, subtract 25 from both sides and divide both sides by 4.

$x = \pm\sqrt{-\frac{25}{4}}$ Use the square root property.

Since

$$\sqrt{-\frac{25}{4}} = \sqrt{-1 \cdot \frac{25}{4}} = \sqrt{-1}\frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2}i$$

we have

$$x = \pm\frac{5}{2}i$$

Since the solutions are $\frac{5}{2}i$ and $-\frac{5}{2}i$, the solution set is $\left\{\frac{5}{2}i, -\frac{5}{2}i\right\}$.

Check:

$$4x^2 + 25 = 0 \qquad\qquad 4x^2 + 25 = 0$$
$$4\left(\frac{5}{2}i\right)^2 + 25 \stackrel{?}{=} 0 \qquad\qquad 4\left(-\frac{5}{2}i\right)^2 + 25 \stackrel{?}{=} 0$$
$$4\left(\frac{25}{4}\right)i^2 + 25 \stackrel{?}{=} 0 \qquad\qquad 4\left(\frac{25}{4}\right)i^2 + 25 \stackrel{?}{=} 0$$
$$25(-1) + 25 \stackrel{?}{=} 0 \qquad\qquad 25(-1) + 25 \stackrel{?}{=} 0$$
$$0 = 0 \quad \text{True} \qquad\qquad 0 = 0 \quad \text{True}$$

The Language of Algebra The $\pm$ symbol is often seen in political polls. A candidate with 48% ($\pm$4%) support could be between $48 + 4 = 52\%$ and $48 - 4 = 44\%$.

3 Solve quadratic equations by completing the square.

All quadratic equations can be solved by **completing the square.** This method involves the special products

$$x^2 + 2ax + a^2 = (x + a)^2 \qquad \text{and} \qquad x^2 - 2ax + a^2 = (x - a)^2$$

The trinomials $x^2 + \mathbf{2a}x + a^2$ and $x^2 - \mathbf{2a}x + a^2$ are both perfect-square trinomials, because both factor as the square of a binomial. In each case, the coefficient of the first term is 1, and if we take one-half of the coefficient of x in the middle term and square it, we obtain the third term.

$$\left[\frac{1}{2}(\mathbf{2a})\right]^2 = a^2 \qquad \left[\frac{1}{2}(\mathbf{-2a})\right]^2 = (-a)^2 = a^2$$

EXAMPLE 6

Add a number to make each binomial a perfect-square trinomial:

a. $x^2 + 10x$ **b.** $x^2 - 6x$ **c.** $x^2 - 11x$

Strategy We will add the square of one-half of the coefficient of x to the given binomial.

WHY Adding such a term will change the binomial into a perfect-square trinomial that will factor.

Solution

a. To make $x^2 + 10x$ a perfect-square trinomial, we find one-half of 10, square it, and add that result to $x^2 + 10x$.

$$x^2 + \mathbf{10}x + \left[\frac{1}{2}(\mathbf{10})\right]^2 = x^2 + 10x + (5)^2 \qquad \text{Simplify: } \tfrac{1}{2}(10) = 5.$$

$$= x^2 + 10x + 25 \qquad \text{Note that } x^2 + 10x + 25 = (x + 5)^2.$$

b. To make $x^2 - 6x$ a perfect-square trinomial, we find one-half of -6, square it, and add that result to $x^2 - 6x$.

$$x^2 - \mathbf{6}x + \left[\frac{1}{2}(\mathbf{-6})\right]^2 = x^2 - 6x + (-3)^2 \qquad \text{Simplify: } \tfrac{1}{2}(-6) = -3.$$

$$= x^2 - 6x + 9 \qquad \text{Note that } x^2 - 6x + 9 = (x - 3)^2.$$

c. To make $x^2 - 11x$ a perfect-square trinomial, we find one-half of -11, square it, and add that result to $x^2 - 11x$.

$$x^2 - \mathbf{11}x + \left[\frac{1}{2}(\mathbf{-11})\right]^2$$

$$= x^2 - 11x + \left(-\frac{11}{2}\right)^2 \qquad \text{Simplify: } \tfrac{1}{2}(-11) = -\tfrac{11}{2}.$$

$$= x^2 - 11x + \frac{121}{4} \qquad \text{Note that } x^2 - 11x + \tfrac{121}{4} = \left(x - \tfrac{11}{2}\right)^2.$$

Self Check 6

Add a number to $a^2 - 5a$ to make it a perfect-square trinomial. $a^2 - 5a + \frac{25}{4}$

Now Try **Problem 49**

Teaching Example 6 Add a number to $x^2 + 7x$ to make it a perfect-square trinomial.
Answer:
$x^2 + 7x + \dfrac{49}{4}$

To solve an equation of the form $ax^2 + bx + c = 0$ by completing the square, we use the following steps.

Completing the Square to Solve a Quadratic Equation in x

1. If the coefficient of x^2 is 1, go to step 2. If it is not, make it 1 by dividing both sides of the equation by the coefficient of x^2.
2. Get all variable terms on one side of the equation and constants on the other side.
3. Complete the square:
   a. Find one-half of the coefficient of x and square it.
   b. Add that square to both sides of the equation.
4. Factor the perfect-square trinomial on one side of the equation. Combine like terms on the other side.
5. Solve the resulting equation using the square root property.
6. Check your answers in the original equation.

Self Check 7

Use completing the square to solve: $x^2 + 6x - 16 = 0$ $-8, 2$

Now Try **Problem 54**

Teaching Example 7 Use completing the square to solve: $x^2 - 10x + 9 = 0$
Answer:
1, 9

EXAMPLE 7 Use completing the square to solve $x^2 + 8x + 7 = 0$.

Strategy We will begin by subtracting 7 from both sides of the equation. Then we will proceed to complete the square to solve for x.

WHY We subtract 7 from both sides to isolate the variable terms, x^2 and $8x$, on the left side of the equation and the constant term on the right side.

Solution

Step 1 In this example, the coefficient of x^2 is understood to be 1.

Step 2 We subtract 7 from both sides of the equation so that only terms with variables are on the left-hand side.

$$x^2 + 8x + 7 = 0 \quad \text{This is the equation to solve.}$$
$$x^2 + 8x = -7$$

Step 3 The coefficient of x is 8, one-half of 8 is 4, and $4^2 = 16$. To complete the square, we add 16 to both sides.

$$x^2 + 8x + 16 = 16 - 7$$
$$\text{(1)} \quad x^2 + 8x + 16 = 9 \quad \text{Simplify: } 16 - 7 = 9.$$

Step 4 Since the left-hand side of Equation 1 is a perfect-square trinomial, we can factor it to get $(x + 4)^2$.

$$x^2 + 8x + 16 = 9$$
$$\text{(2)} \quad (x + 4)^2 = 9$$

Step 5 We then solve Equation 2 by using the square root property.

$$x + 4 = \pm\sqrt{9}$$
$$x + 4 = 3 \quad \text{or} \quad x + 4 = -3 \quad \text{Write as two equations. Simplify: } \sqrt{9} = 3.$$
$$x = -1 \quad \Big| \quad x = -7 \quad \text{To isolate } x\text{, subtract 4 from both sides.}$$

Step 6 Verify that -1 and -7 satisfy the original equation.

Caution! When using the square root property to solve an equation, always write the $\pm$ symbol. If you forget, you will lose one of the solutions.

EXAMPLE 8 Solve: $6x^2 + 5x - 6 = 0$

Strategy We will begin by dividing both sides of the equation by 6.

WHY This will create a leading coefficient that is 1 so that we can proceed to complete the square to solve the equation.

Solution

Step 1 To make the coefficient of x^2 equal to 1, we divide both sides of the equation by 6.

$$6x^2 + 5x - 6 = 0 \quad \text{This is the equation to solve.}$$

$$\frac{6x^2}{6} + \frac{5}{6}x - \frac{6}{6} = \frac{0}{6} \quad \text{Divide both sides by 6, term by term.}$$

$$x^2 + \frac{5}{6}x - 1 = 0 \quad \text{Simplify.}$$

Step 2 We add 1 to both sides so that only terms with variables are on the left-hand side of the equation.

$$x^2 + \frac{5}{6}x = 1$$

Step 3 The coefficient of x is $\frac{5}{6}$, one-half of $\frac{5}{6}$ is $\frac{5}{12}$, and $\left(\frac{5}{12}\right)^2 = \frac{25}{144}$. To complete the square, we add $\frac{25}{144}$ to both sides.

$$x^2 + \frac{5}{6}x + \frac{25}{144} = 1 + \frac{25}{144}$$

$$(3) \quad x^2 + \frac{5}{6}x + \frac{25}{144} = \frac{169}{144} \quad \text{Simplify the right side: } 1 + \frac{25}{144} = \frac{144}{144} + \frac{25}{144} = \frac{169}{144}.$$

Step 4 Since the left-hand side of Equation 3 is a perfect-square trinomial, we can factor it to get $\left(x + \frac{5}{12}\right)^2$.

$$(4) \quad \left(x + \frac{5}{12}\right)^2 = \frac{169}{144} \quad x^2 + \frac{5}{6}x + \frac{25}{144} \text{ is a perfect-square trinomial.}$$

Step 5 We can solve Equation 4 using the square root property.

$$x + \frac{5}{12} = \pm\sqrt{\frac{169}{144}}$$

$$x + \frac{5}{12} = \frac{13}{12} \quad \text{or} \quad x + \frac{5}{12} = -\frac{13}{12} \quad \text{Simplify: } \pm\sqrt{\frac{169}{144}} = \pm\frac{13}{12}.$$

$$x = -\frac{5}{12} + \frac{13}{12} \quad\Big|\quad x = -\frac{5}{12} - \frac{13}{12} \quad \text{Subtract } \frac{5}{12} \text{ from both sides.}$$

$$x = \frac{8}{12} \quad\Big|\quad x = -\frac{18}{12} \quad \text{Simplify.}$$

$$x = \frac{2}{3} \quad\Big|\quad x = -\frac{3}{2} \quad \text{Simplify each fraction.}$$

Step 6 Verify that $\frac{2}{3}$ and $-\frac{3}{2}$ satisfy the original equation.

Self Check 8

Solve: $4x^2 + 5x - 6 = 0$ $-2, \frac{3}{4}$

Now Try Problem 63

Teaching Example 8 Solve: $3a^2 + 2a - 1 = 0$

Answer: $-1, \frac{1}{3}$

Caution! A common error is to add a constant to one side of an equation to complete the square and forget to add it to the other side.

Self Check 9

Solve: $3x^2 + 6x + 1 = 0$

Now Try **Problem 65**

Self Check 9 Answer

$\frac{-3 \pm \sqrt{6}}{3}$

Teaching Example 9 Solve: $2x^2 - 12x + 3 = 0$

Answer:

$\frac{6 \pm \sqrt{30}}{2}$

EXAMPLE 9 Solve: $2x^2 + 4x + 1 = 0$

Strategy We will follow the steps for solving a quadratic equation by completing the square.

WHY Since the trinomial $2x^2 + 4x + 1$ cannot be factored using only integers, solving the equation by completing the square is our only option at this time.

Solution

$$2x^2 + 4x + 1 = 0 \quad \text{This is the equation to solve.}$$

$$x^2 + 2x + \frac{1}{2} = 0 \quad \text{Divide both sides by 2 to make the coefficient of } x^2 \text{ equal to 1.}$$

$$x^2 + 2x = -\frac{1}{2} \quad \text{Subtract } \tfrac{1}{2} \text{ from both sides.}$$

$$x^2 + 2x + \mathbf{1} = \mathbf{1} - \frac{1}{2} \quad \text{Square one-half of the coefficient of } x \text{ and add it to both sides.}$$

$$(x + 1)^2 = \frac{1}{2} \quad \text{Factor and combine like terms.}$$

$$x + 1 = \pm\sqrt{\frac{1}{2}} \quad \text{Use the square root property.}$$

To write $\sqrt{\frac{1}{2}}$ in simplified radical form, we write it as a quotient of square roots and then rationalize the denominator.

$$x + 1 = \frac{\sqrt{2}}{2} \quad \Big| \quad x + 1 = -\frac{\sqrt{2}}{2} \qquad \sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$x = -1 + \frac{\sqrt{2}}{2} \quad \Big| \quad x = -1 - \frac{\sqrt{2}}{2} \qquad \text{Subtract 1 from both sides.}$$

We can express each solution in an alternate form if we write -1 as a fraction with a denominator of 2.

$$x = -\frac{2}{2} + \frac{\sqrt{2}}{2} \quad \Big| \quad x = -\frac{2}{2} - \frac{\sqrt{2}}{2} \qquad \text{Write } -1 \text{ as } -\tfrac{2}{2}.$$

$$x = \frac{-2 + \sqrt{2}}{2} \quad \Big| \quad x = \frac{-2 - \sqrt{2}}{2} \qquad \text{Add (subtract) the numerators and keep the common denominator of 2.}$$

The exact solutions are $\frac{-2 + \sqrt{2}}{2}$ and $\frac{-2 - \sqrt{2}}{2}$, or more concisely, $x = \frac{-2 \pm \sqrt{2}}{2}$. (Read $\pm$ as "plus or minus.") We can use a calculator to approximate them. To the nearest hundredth, they are -0.29 and -1.71.

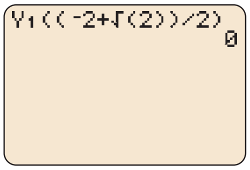
Caution! Recall that to simplify a fraction, we remove common *factors* of the numerator and denominator. In Example 9, since -2 is a *term* of the numerator of $\frac{-2 \pm \sqrt{2}}{2}$, no further simplification of this expression can be made.

Using Your CALCULATOR **Checking Solutions of Quadratic Equations**

We can use a graphing calculator to check the solutions of the quadratic equation $2x^2 + 4x + 1 = 0$ found in Example 9. After entering $Y_1 = 2x^2 + 4x + 1$, we call up the home screen by pressing $\boxed{\text{2nd}}$ QUIT. Then we press the $\boxed{\text{VARS}}$ key, arrow $\boxed{\blacktriangleright}$ to Y-VARS, and enter 1 and enter 1 again to get the display shown in figure (a). We evaluate $2x^2 + 4x + 1$

for $x = \frac{-2 + \sqrt{2}}{2}$ by inputting the solution using function notation, as shown in figure (b). When $\boxed{\text{ENTER}}$ is pressed, the result of 0 is confirmation that $x = \frac{-2 + \sqrt{2}}{2}$ is a solution of the equation.

(a)



(b)

In the next example, the solutions of the equation are two complex numbers that contain i.

EXAMPLE 10 Solve: $3x^2 + 2x + 2 = 0$

Strategy We will follow the steps for solving a quadratic equation by completing the square.

WHY Since the trinomial $3x^2 + 2x + 2$ cannot be factored using only integers, solving the equation by completing the square is our only option at this time.

Solution

$$3x^2 + 2x + 2 = 0$$ This is the equation to solve.

$$x^2 + \frac{2}{3}x + \frac{2}{3} = \frac{0}{3}$$ Divide both sides by 3 to make the coefficient of x^2 equal to 1.

$$x^2 + \frac{2}{3}x = -\frac{2}{3}$$ Subtract $\frac{2}{3}$ from both sides.

$$x^2 + \frac{2}{3}x + \mathbf{\frac{1}{9}} = \mathbf{\frac{1}{9}} - \frac{2}{3}$$ $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ and $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$. Add $\frac{1}{9}$ to both sides.

$$\left(x + \frac{1}{3}\right)^2 = -\frac{5}{9}$$ Factor the left side and combine terms: $\frac{1}{9} - \frac{2}{3} = \frac{1}{9} - \frac{6}{9} = -\frac{5}{9}$.

$$x + \frac{1}{3} = \pm\sqrt{-\frac{5}{9}}$$ Use the square root property.

$$x = -\frac{1}{3} \pm \sqrt{-\frac{5}{9}}$$ To isolate x, subtract $\frac{1}{3}$ from both sides.

Since

$$\sqrt{-\frac{5}{9}} = \sqrt{-1 \cdot \frac{5}{9}} = \sqrt{-1}\frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}i$$

we have

$$x = -\frac{1}{3} \pm \frac{\sqrt{5}}{3}i$$

The solutions are $-\frac{1}{3} + \frac{\sqrt{5}}{3}i$ and $-\frac{1}{3} - \frac{\sqrt{5}}{3}i$.

Self Check 10

Solve: $x^2 + 4x + 6 = 0$

Now Try **Problem 71**

Self Check 10 Answer

$-2 \pm i\sqrt{2}$

Teaching Example 10 Solve: $2x^2 - x + 3 = 0$

Answer:

$\frac{1}{4} \pm \frac{\sqrt{23}}{4}i$

4 Use quadratic equations to solve application problems.

Self Check 11

PLANNING A GARDEN A memorial garden is to be constructed so that the area of the garden is 911 square feet. The garden's length is to be 2 feet more than its width. Find the dimensions of the garden. Round to the nearest tenth of a foot.

Now Try **Problem 97**

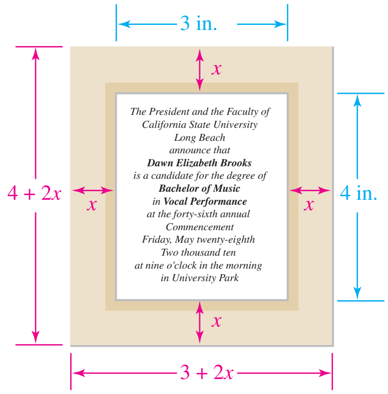
Self Check 11 Answer
29.2 ft by 31.2 ft

Teaching Example 11 THE U.S. FLAG
In 1912, an order by President Taft fixed the width and length of the U.S. flag in the ratio of 1 to 1.9. If 17.1 square feet of cloth are to be used to make a U.S. flag, estimate its dimensions to the nearest tenth of a foot.
Answer:
3.0 ft by 5.7 ft

EXAMPLE 11 ***Graduation Announcements*** In creating the announcement shown in the figure below, the graphic artist wants to follow two design criteria:

- A border of uniform width should surround the text.
- Equal areas should be devoted to the text and to the border.

To meet these requirements, how wide should the border be?



Analyze The text occupies $4 \cdot 3 = 12$ in.^{2} of space. The border must also have an area of 12 in.^{2}.

Form If we let $x =$ the width of the border, the length of the announcement is $(4 + 2x)$ inches and the width is $(3 + 2x)$ inches. We can now form the equation.

The area of the announcement	minus	the area of the text	equals	the area of the border.
$(4 + 2x)(3 + 2x)$	$-$	12	$=$	12

Solve

$$(4 + 2x)(3 + 2x) - 12 = 12$$

$$12 + 8x + 6x + 4x^2 - 12 = 12 \quad \text{On the left-hand side, multiply the binomials.}$$

$$4x^2 + 14x = 12 \quad \text{Combine like terms.}$$

$$4x^2 + 14x - 12 = 0 \quad \text{Subtract 12 from both sides.}$$

$$2x^2 + 7x - 6 = 0 \quad \text{Divide both sides by 2.}$$

Since the trinomial on the left-hand side does not factor, we will solve the equation by completing the square.

$$x^2 + \frac{7}{2}x - 3 = 0 \quad \text{Divide both sides by 2 so that the coefficient of } x^2 \text{ is 1.}$$

$$x^2 + \frac{7}{2}x = 3 \quad \text{Add 3 to both sides.}$$

$$x^2 + \frac{7}{2}x + \frac{49}{16} = 3 + \frac{49}{16} \quad \text{One-half of } \tfrac{7}{2} \text{ is } \tfrac{7}{4}\text{. Square } \tfrac{7}{4}\text{, which is } \tfrac{49}{16}\text{, and add it to both sides.}$$

$$\left(x + \frac{7}{4}\right)^2 = \frac{97}{16} \quad \text{On the left-hand side, factor the trinomial. On the right-hand side, } 3 = \tfrac{3 \cdot 16}{1 \cdot 16} = \tfrac{48}{16} \text{ and } \tfrac{48}{16} + \tfrac{49}{16} = \tfrac{97}{16}.$$

$$x + \frac{7}{4} = \pm\frac{\sqrt{97}}{4}$$ Apply the square root property. On the right-hand side, $\sqrt{\frac{97}{16}} = \frac{\sqrt{97}}{\sqrt{16}} = \frac{\sqrt{97}}{4}$.

$$x = -\frac{7}{4} + \frac{\sqrt{97}}{4} \quad \text{or} \quad x = -\frac{7}{4} - \frac{\sqrt{97}}{4}$$ Subtract $\frac{7}{4}$ from both sides.

$$x = \frac{-7 + \sqrt{97}}{4} \quad\quad x = \frac{-7 - \sqrt{97}}{4}$$ Write each expression as a single fraction.

State The width of the border should be $\frac{-7+\sqrt{97}}{4} \approx 0.71$ inch. (We discard the solution $\frac{-7-\sqrt{97}}{4}$, since it is negative.)

Check If the border is 0.71 inch wide, the announcement has an area of about $5.42 \cdot 4.42 \approx 23.96$ in.^{2}. If we subtract the area of the text from the area of the announcement, we get $23.96 - 12 = 11.96$ in.^{2}. This represents the area of the border, which was to be 12 in.^{2}. The answer seems reasonable.

ANSWERS TO SELF CHECKS

1. $\frac{4}{3}, -\frac{1}{5}$ **2.** $\pm 3\sqrt{2}$ **3.** $\sqrt{\frac{1257}{\pi}}$ in. **4.** 1, −5 **5.** $\frac{9}{4}i, -\frac{9}{4}i$ **6.** $a^2 - 5a + \frac{25}{4}$ **7.** −8, 2 **8.** $-2, \frac{3}{4}$ **9.** $\frac{-3 \pm \sqrt{6}}{3}$ **10.** $-2 \pm i\sqrt{2}$ **11.** 29.2 ft by 31.2 ft

SECTION 8.1 STUDY SET

VOCABULARY

Fill in the blanks.

1. An equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ is called a quadratic equation.
2. In the expression $2 \pm \sqrt{3}$, the symbol $\pm$ is read as "plus or minus."
3. $x^2 + 6x + 9$ is called a perfect-square trinomial because it factors as $(x+3)^2$.
4. The coefficient of x^2 in $x^2 - 12x + 36 = 0$ is 1, and the constant term is 36.

CONCEPTS

Fill in the blanks.

5. The solutions of $x^2 = c$, where $c > 0$, are $x = \sqrt{c}$ and $x = -\sqrt{c}$.
6. To complete the square on x in $x^2 + 6x$, find one-half of 6, square it to get 9, and add 9 to get $x^2 + 6x + 9$.
7. Is $3\sqrt{2}$ a solution of $x^2 - 18 = 0$? yes
8. Is $-2 + \sqrt{2}$ a solution of $x^2 + 4x + 2 = 0$? yes
9. Find one-half of the coefficient of x and then square it. What is the result?

   a. $x^2 + 12x$ 36 **b.** $x^2 - 5x$ $\frac{25}{4}$ **c.** $x^2 - \frac{x}{2}$ $\frac{1}{16}$

10. Add a number to make each binomial a perfect-square trinomial and factor the result.

    a. $x^2 + 8x$ $x^2 + 8x + 16 = (x+4)^2$

    b. $x^2 - 8x$ $x^2 - 8x + 16 = (x-4)^2$

    c. $x^2 - x$ $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

11. What is the first step in solving the equation $x^2 + 12x = 35$

    a. by the factoring method? Subtract 35 from both sides.

    b. by completing the square? Add 36 to both sides.

12. Solve the equation: $x^2 = 16$

    a. by the factoring method. 4, −4

    b. by the square root method. 4, −4

13. Write the expression on the right-hand side of the equation in simplified form by applying the quotient rule for radicals and then rationalizing the denominator.

$$x - 1 = \pm\sqrt{\frac{5}{2}} \quad \pm\frac{\sqrt{10}}{2}$$

14. Write the expression on the right-hand side of the equation as a single fraction by first expressing 1 as a fraction with a denominator of 2.

$$x = 1 \pm \frac{\sqrt{10}}{2} \quad \frac{2 \pm \sqrt{10}}{2}$$

15. Explain the error in the work shown below.

$$\frac{4 \pm \sqrt{3}}{8} = \frac{\overset{1}{\cancel{4}} \pm \sqrt{3}}{\underset{1}{\cancel{4}} \cdot 2} = \frac{1 \pm \sqrt{3}}{2}$$

4 is not a factor of the numerator. It cannot be divided out.

16. Explain the error in the work shown below.

$$\frac{1 \pm \sqrt{5}}{5} = \frac{1 \pm \sqrt{\cancel{5}^{1}}}{\underset{1}{\cancel{5}}} = \frac{1 \pm 1}{1}$$

5 is not a factor of the numerator. It cannot be divided out.

NOTATION

17. Explain why the following is true:

$$3 \pm \frac{\sqrt{10}}{5} \neq \frac{3 \pm \sqrt{10}}{5}$$

$\frac{3 \pm \sqrt{10}}{5} = \frac{3}{5} \pm \frac{\sqrt{10}}{5}$

▶ **18.** Isolate x on the left-hand side of the equation $x + 3 = \pm\sqrt{5}$ by subtracting 3 from both sides of the equation. $x = -3 \pm \sqrt{5}$

19. In solving a quadratic equation, a student obtains $x = \pm 2\sqrt{5}$.

a. How many solutions are represented by this notation? List them. $2; 2\sqrt{5}, -2\sqrt{5}$

b. Approximate the solutions to the nearest hundredth. ± 4.47

▶ **20.** In solving a quadratic equation, a student obtains $x = \frac{-5 \pm \sqrt{7}}{3}$.

a. How many solutions are represented by this notation? List them. $2; \frac{-5 + \sqrt{7}}{3}, \frac{-5 - \sqrt{7}}{3}$

b. Approximate the solutions to the nearest hundredth. $-0.78, -2.55$

GUIDED PRACTICE

Use factoring to solve each equation. See Example 1.

21. $6x^2 + 12x = 0$ $0, -2$

22. $5x^2 + 11x = 0$ $0, -\frac{11}{5}$

23. $2y^2 - 50 = 0$ $5, -5$

24. $4y^2 - 64 = 0$ $4, -4$

25. $r^2 + 6r + 8 = 0$ $-2, -4$

▶ **26.** $x^2 + 9x + 20 = 0$ $-5, -4$

27. $2z^2 = -2 + 5z$ $2, \frac{1}{2}$

28. $3x^2 = 8 - 10x$ $-4, \frac{2}{3}$

Use the square root property to solve each equation. See Example 2.

29. $t^2 - 81 = 0$ ± 9

30. $w^2 - 49 = 0$ ± 7

31. $x^2 = 36$ ± 6

32. $x^2 = 144$ ± 12

33. $z^2 - 50 = 0$ $\pm 5\sqrt{2}$

34. $u^2 - 24 = 0$ $\pm 2\sqrt{6}$

▶ **35.** $3x^2 - 16 = 0$ $\pm \frac{4\sqrt{3}}{3}$

36. $5x^2 - 49 = 0$ $\pm \frac{7\sqrt{5}}{5}$

In each formula, solve for the unknown variable. Assume that all variables represent positive numbers. Express all radicals in simplified form. See Example 3.

37. $2d^2 = 3h, h = 3$ $d = \frac{3\sqrt{2}}{2}$

38. $2x^2 = d^2, x = 5$ $d = 5\sqrt{2}$

39. $d = 16t^2, d = 48$ $t = \sqrt{3}$

▶ **40.** $A = \pi r^2, A = 27$ $r = \frac{3\sqrt{3\pi}}{\pi}$

Use the square root property to solve each equation. See Example 4.

41. $(x + 1)^2 = 1$ $0, -2$

▶ **42.** $(x - 1)^2 = 4$ $3, -1$

43. $(s - 7)^2 - 9 = 0$ $4, 10$

▶ **44.** $(t + 4)^2 = 16$ $0, -8$

Use the square root property to solve each equation. See Example 5.

45. $p^2 + 16 = 0$ $\pm 4i$

46. $q^2 + 25 = 0$ $\pm 5i$

47. $4m^2 + 81 = 0$ $\pm \frac{9}{2}i$

▶ **48.** $9n^2 + 121 = 0$ $\pm \frac{11}{3}i$

Complete the square and factor the resulting perfect-square trinomial. See Example 6.

49. $x^2 + 24x$ $x^2 + 24x + 144 = (x + 12)^2$

50. $y^2 - 18y$ $y^2 - 18y + 81 = (y - 9)^2$

▶ **51.** $a^2 - 7a$ $a^2 - 7a + \frac{49}{4} = \left(a - \frac{7}{2}\right)^2$

52. $b^2 + 11b$ $b^2 + 11b + \frac{121}{4} = \left(b + \frac{11}{2}\right)^2$

Use completing the square to solve each equation. See Example 7.

53. $x^2 + 2x - 8 = 0$ $2, -4$

▶ **54.** $x^2 + 6x + 5 = 0$ $-1, -5$

55. $k^2 - 8k + 12 = 0$ $2, 6$

56. $p^2 - 4p + 3 = 0$ $3, 1$

57. $g^2 + 5g - 6 = 0$ $1, -6$

▶ **58.** $s^2 + 5s - 14 = 0$ $2, -7$

▶ **59.** $x^2 - 3x - 4 = 0$ $-1, 4$

60. $x^2 - 7x + 12 = 0$ $3, 4$

Use completing the square to solve each equation. See Example 8.

61. $2x^2 - x - 1 = 0$
$1, -\frac{1}{2}$

62. $2x^2 - 5x + 2 = 0$
$2, \frac{1}{2}$

63. $6x^2 + x - 2 = 0$
$\frac{1}{2}, -\frac{2}{3}$

64. $9 - 6r = 8r^2$
$\frac{3}{4}, -\frac{3}{2}$

Use completing the square to solve each equation. See Example 9.

65. $b^2 - 3b = 5$
$\frac{3 \pm \sqrt{29}}{2}$

66. $a^2 - a = 3$
$\frac{1 \pm \sqrt{13}}{2}$

67. $3x^2 - 6x = 1$
$\frac{3 \pm 2\sqrt{3}}{3}$

68. $2x^2 - 6x = -3$
$\frac{3 \pm \sqrt{3}}{2}$

Use completing the square to solve each equation. See Example 10.

69. $p^2 + 2p + 2 = 0$
$-1 \pm i$

70. $x^2 - 6x + 10 = 0$
$3 \pm i$

71. $y^2 + 8y + 18 = 0$
$-4 \pm i\sqrt{2}$

72. $n^2 + 10n + 28 = 0$
$-5 \pm i\sqrt{3}$

TRY IT YOURSELF

Solve each equation.

73. $(x + 5)^2 - 3 = 0$
$-5 \pm \sqrt{3}$

74. $(x + 3)^2 - 7 = 0$
$-3 \pm \sqrt{7}$

75. $x^2 + 8x + 6 = 0$
$-4 \pm \sqrt{10}$

76. $x^2 + 6x + 4 = 0$
$-3 \pm \sqrt{5}$

77. $x^2 - 2x - 17 = 0$
$1 \pm 3\sqrt{2}$

78. $x^2 + 10x - 7 = 0$
$-5 \pm 4\sqrt{2}$

79. $4x^2 - 4x = 7$
$\frac{1 \pm 2\sqrt{2}}{2}$

80. $2x^2 - 8x = -5$
$\frac{4 \pm \sqrt{6}}{2}$

81. $2x^2 + 5x - 2 = 0$
$\frac{-5 \pm \sqrt{41}}{4}$

82. $4x^2 - 4x - 1 = 0$
$\frac{1 \pm \sqrt{2}}{2}$

83. $\frac{7x + 1}{5} = -x^2$
$\frac{-7 \pm \sqrt{29}}{10}$

84. $\frac{3x^2}{8} = \frac{1}{8} - x$
$\frac{-4 \pm \sqrt{19}}{3}$

85. $3a^2 - 1 = 6a$
$\frac{3 \pm 2\sqrt{3}}{3}$

86. $2a^2 + 3 = 6a$
$\frac{3 \pm \sqrt{3}}{2}$

87. $t^2 + t + 3 = 0$
$-\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$

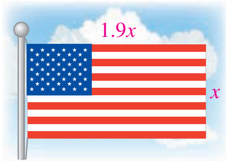
88. $b^2 - b + 5 = 0$
$\frac{1}{2} \pm \frac{\sqrt{19}}{2}i$

89. $4q^2 + 2q + 3 = 0$
$-\frac{1}{4} \pm \frac{\sqrt{11}}{4}i$

90. $3p^2 - 2p + 3 = 0$
$\frac{1}{3} \pm \frac{2\sqrt{2}}{3}i$

APPLICATIONS

91. FLAGS In 1912, an executive order by President Taft fixed the overall width and length of the U.S. flag in the ratio 1 to 1.9. If 100 square feet of cloth are to be used to make a U.S. flag, estimate its dimensions to the nearest $\frac{1}{4}$ foot. width: $7\frac{1}{4}$ ft, length: $13\frac{3}{4}$ ft

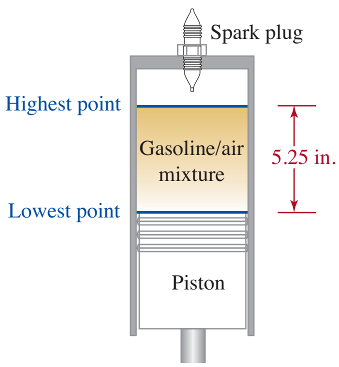


92. MOVIE STUNTS According to the *Guinness Book of World Records, 1998,* stuntman Dan Koko fell a distance of 312 feet into an airbag after jumping from the Vegas World Hotel and Casino. The distance d in feet traveled by a free-falling object in t seconds is given by the formula $d = 16t^2$. To the nearest tenth of a second, how long did the stuntman's free fall last? 4.4 sec

93. ACCIDENTS The height h (in feet) of an object that is dropped from a height of s feet is given by the formula $h = s - 16t^2$, where t is the time the object has been falling. A 5-foot-tall woman on a sidewalk looks directly overhead and sees a window washer drop a bottle from 4 stories up. How long does she have to get out of the way? Round to the nearest tenth. (A story is 12 feet.) 1.6 sec

94. GEOGRAPHY The surface area S of a sphere is given by the formula $S = 4\pi r^2$, where r is the radius of the sphere. An almanac lists the surface area of Earth as 196,938,800 square miles. Assuming Earth to be spherical, what is its radius to the nearest mile? 3,959 mi

95. AUTOMOBILE ENGINES As the piston shown in the illustration below moves upward, it pushes a cylinder of a gasoline/air mixture that is ignited by the spark plug. The formula that gives the volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h the height. Find the radius of the piston (to the nearest hundredth of an inch) if it displaces 47.75 cubic inches of gasoline/air mixture as it moves from its lowest to its highest point. 1.70 in.



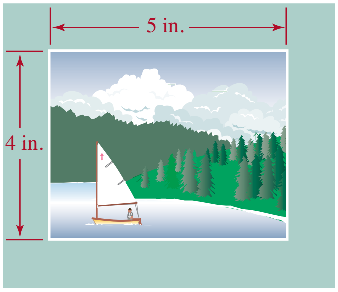
96. A house, purchased for \$150,000, was expected to appreciate according to the formula $V = P(1 + r)^n$, where V is the future value of the house after n years, P is the purchase price, and r is the rate of appreciation. If the house sold for \$200,000 in two years, find the rate of appreciation to the nearest percent. 15%

from Campus to Careers

Real Estate Sales Agent

© Image Source Black/Getty Images

97. PICTURE FRAMING The matting around the picture below has a uniform width. How wide is the matting if its area equals the area of the picture? Round to the nearest hundredth of an inch. 0.92 in.



98. SWIMMING POOLS See the advertisement in the next column. How wide will the free concrete decking be if a uniform width is constructed around the perimeter of the pool? Round to the nearest hundredth of a yard. (*Hint:* Note the difference in units.) 0.80 yd

99. DIMENSIONS OF A RECTANGLE A rectangle is 4 feet longer than it is wide, and its area is 20 square feet. Find its dimensions to the nearest tenth of a foot. 2.9 ft, 6.9 ft

100. DIMENSIONS OF A TRIANGLE The height of a triangle is 4 meters longer than twice its base. Find the base and height if the area of the triangle is 10 square meters. Round to the nearest hundredth of a meter. 2.32 m, 8.63 m

WRITING

101. Explain how to complete the square.

102. Explain why a cannot be 0 in the quadratic equation $ax^2 + bx + c = 0$.

REVIEW

Simplify each expression. All variables represent positive real numbers.

103. $\sqrt[3]{40a^3b^6}$ $2ab^2\sqrt[3]{5}$

104. $\sqrt[3]{-27x^6}$ $-3x^2$

105. $\sqrt[8]{x^{24}}$ x^3

106. $\sqrt[4]{\dfrac{16}{625}}$ $\frac{2}{5}$

107. $\sqrt{175a^2b^3}$ $5ab\sqrt{7b}$

108. $\sqrt{\dfrac{z^2}{16x^2}}$ $\frac{z}{4x}$

Objectives

1. Derive the quadratic formula.
2. Solve quadratic equations using the quadratic formula.
3. Write equivalent equations to make quadratic formula computations easier.
4. Use the quadratic formula to solve application problems.

SECTION 8.2 The Quadratic Formula

We can solve any quadratic equation by completing the square, but the work is often tedious. In this section, we will develop a formula, called the *quadratic formula,* that lets us solve quadratic equations with much less effort.

1 Derive the quadratic formula.

To develop a formula we can use to solve quadratic equations, we start with the **general quadratic equation** $ax^2 + bx + c = 0$ for x, where $a > 0$, and solve it for x by completing the square.

$$ax^2 + bx + c = 0$$

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$ Divide both sides by a so the coefficient of x^2 is 1.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ Simplify $\frac{ax^2}{a} = x^2$. Write $\frac{bx}{a}$ as $\frac{b}{a}x$.

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$ Subtract $\frac{c}{a}$ from both sides so that only the terms involving x are on the left side of the equation.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$ Complete the square on x. One-half of $\frac{b}{a}$ is $\frac{b}{2a}$. Add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4aa}$$ Find both powers. Get a common denominator of $4a^2$ on the right-hand side.

(1) $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$ Factor the left-hand side and add the fractions on the right-hand side.

We can solve Equation 1 using the square root property.

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \qquad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If $a < 0$, it can be shown that the same two solutions are obtained. The result is called the **quadratic formula.**

Quadratic Formula

The solutions of $ax^2 + bx + c = 0$, with $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Read the symbol ± as "plus or minus."

Caution! Be sure to draw the fraction bar under both parts of the numerator, and be sure to draw the radical symbol only over $b^2 - 4ac$. Do not write the quadratic formula as

$$\cancel{x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}} \quad \text{or as} \quad \cancel{x = -b \pm \sqrt{\frac{b^2 - 4ac}{2a}}}$$

2 Solve quadratic equations using the quadratic formula.

EXAMPLE 1 Solve $6x^2 - x - 5 = 0$ by using the quadratic formula.

Strategy We will begin by comparing $6x^2 - x - 5 = 0$ to the general quadratic equation $ax^2 + bx + c = 0$.

WHY To use the quadratic formula, we need to identify the values of a, b, and c.

Solution

$$6x^2 - 1x - 5 = 0$$
$$ax^2 + bx + c = 0$$

Self Check 1

Solve $3x^2 - 5x - 2 = 0$ by using the quadratic formula. $2, -\frac{1}{3}$

Now Try Problem 17

Teaching Example 1 Solve $4x^2 - 7x - 2 = 0$ by using the quadratic formula.
Answer:
$2, -\frac{1}{4}$

We see that $a = 6$, $b = -1$, and $c = -5$. To find the solutions of the equation, we substitute these values into the quadratic formula and evaluate the right side.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ This is the quadratic formula.

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-5)}}{2(6)}$$ Substitute 6 for a, -1 for b, and -5 for c.

$$x = \frac{1 \pm \sqrt{1 + 120}}{12}$$ Simplify: $-(-1) = 1$. Do the operations within the radical.

$$x = \frac{1 \pm \sqrt{121}}{12}$$ Simplify within the radical.

$$x = \frac{1 \pm 11}{12}$$ Simplify: $\sqrt{121} = 11$.

This result represents two solutions. To find the first solution, we evaluate the expression using the + symbol. To find the second solution, we evaluate the expression using the − symbol.

$$x = \frac{1 + 11}{12} \quad \text{or} \quad x = \frac{1 - 11}{12}$$

$$x = \frac{12}{12} \qquad\qquad x = \frac{-10}{12}$$

$$x = 1 \qquad\qquad x = -\frac{5}{6}$$

Verify that 1 and $-\frac{5}{6}$ satisfy the original equation.

Success Tip You may have noticed in Example 1 that $6x^2 - x - 5$ factors as $(6x + 5)(x - 1)$. Therefore, we could have solved $6x^2 - x - 5 = 0$ using the factoring method.

Solving a Quadratic Equation in x Using the Quadratic Formula

To solve a quadratic equation in x using the quadratic formula, we follow these steps.

1. Write the equation in general quadratic form: $ax^2 + bx + c = 0$.
2. Identify a, b, and c.
3. Substitute the values for a, b, and c in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and evaluate the right side to obtain the solutions.

Self Check 2

Solve: $3x^2 = 2x + 3$
Approximate the solutions to two decimal places.

Now Try **Problem 23**

Self Check 2 Answer

$\frac{1 \pm \sqrt{10}}{3}$; $-0.72, 1.39$

EXAMPLE 2 Solve: $2x^2 = -4x - 1$

Strategy We will write the equation in general quadratic form, $ax^2 + bx + c = 0$. Then we will identify the values of a, b, and c, and substitute these values into the quadratic formula.

WHY The quadratic equation must be in general quadratic form to identify the values of a, b, and c.

Solution

To write the equation in general quadratic form, we need to have all nonzero terms on the left side and 0 on the right side.

$$2x^2 = -4x - 1$$ This is the equation to solve.

$$2x^2 + 4x + 1 = 0$$ Add 4x and 1 to both sides.

In this equation, $a = 2$, $b = 4$, and $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ This is the quadratic formula.

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)}$$ Substitute 2 for a, 4 for b, and 1 for c.

$$x = \frac{-4 \pm \sqrt{16 - 8}}{4}$$ Simplify within the radical.

$$x = \frac{-4 \pm \sqrt{8}}{4}$$

$$x = \frac{-4 \pm 2\sqrt{2}}{4}$$ Simplify: $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$.

We can write the solutions in simpler form by factoring out 2 from the terms in the numerator and then dividing out the common factor.

$$x = \frac{-4 \pm 2\sqrt{2}}{4} = \frac{2(-2 \pm \sqrt{2})}{4} = \frac{\overset{1}{\cancel{2}}(-2 \pm \sqrt{2})}{\underset{1}{\cancel{2}} \cdot 2} = \frac{-2 \pm \sqrt{2}}{2}$$

The solutions are $\frac{-2 + \sqrt{2}}{2}$ and $\frac{-2 - \sqrt{2}}{2}$. We can approximate the solutions using a calculator. To two decimal places, they are -0.29 and -1.71.

Teaching Example 2 Solve $4x^2 = 2x + 1$. Approximate the solutions to two decimal places.
Answer:
$\frac{1 \pm \sqrt{5}}{4}$; 0.81, −0.31

The solutions to the next example are imaginary numbers.

EXAMPLE 3 Solve: $x^2 + x = -1$

Strategy We will write the equation in standard form $ax^2 + bx + c = 0$. Then we will identify the values of a, b, and c, and substitute these values into the quadratic formula.

WHY The quadratic equation must be in standard form to identify the values of a, b, and c.

Solution
To write $x^2 + x = -1$ in standard form, we add 1 to both sides, to get

$$x^2 + x + 1 = 0$$ This is the equation to solve.

In this equation, $a = 1$, $b = 1$, and $c = 1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$ Substitute 1 for a, 1 for b, and 1 for c.

$$x = \frac{-1 \pm \sqrt{1 - 4}}{2}$$ Evaluate the expression within the radical.

$$x = \frac{-1 \pm \sqrt{-3}}{2}$$

$$x = \frac{-1 \pm i\sqrt{3}}{2}$$ $\sqrt{-3} = \sqrt{-1 \cdot 3} = \sqrt{-1}\sqrt{3} = i\sqrt{3}$.

The solutions are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Self Check 3
Solve: $a^2 + 3a + 5 = 0$

Now Try Problem 26

Self Check 3 Answer
$-\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$

Teaching Example 3 Solve: $x^2 - x = -2$
Answer:
$\frac{1}{2} + \frac{\sqrt{7}}{2}i, \frac{1}{2} - \frac{\sqrt{7}}{2}i$

Notation The solutions are written in complex number form $a + bi$. They could also be written as

$$\frac{-1 \pm i\sqrt{3}}{2}$$

3 Write equivalent equations to make quadratic formula computations easier.

When solving a quadratic equation by the quadratic formula, we can often simplify the computations by solving an equivalent equation.

Self Check 4

For each equation, write an equivalent equation so that the quadratic formula computations will be simpler.

a. $-6x^2 + 7x - 9 = 0$

b. $\frac{1}{3}x^2 - \frac{2}{3}x - \frac{5}{6} = 0$

c. $44x^2 + 66x - 55 = 0$

Now Try **Problem 31**

Self Check 4 Answers

a. $6x^2 - 7x + 9 = 0$
b. $2x^2 - 4x - 5 = 0$
a. $4x^2 + 6x - 5 = 0$

Teaching Example 4 For each equation, write an equivalent equation so that the quadratic formula computations will be simpler.

a. $\frac{1}{2}x^2 - \frac{3}{5}x + \frac{7}{10} = 0$
b. $-5x^2 + 3x - 9 = 0$
c. $75x^2 - 25x + 50 = 0$

Answers:
a. $5x^2 - 6x + 7 = 0$
b. $5x^2 - 3x + 9 = 0$
c. $3x^2 - x + 2 = 0$

EXAMPLE 4 For each equation, write an equivalent equation so that the quadratic formula computations will be easier.

a. $-2x^2 + 4x - 1 = 0$ **b.** $x^2 + \frac{4}{5}x - \frac{1}{3} = 0$ **c.** $20x^2 - 60x - 40 = 0$

Strategy We will multiply both sides of each equation by a carefully chosen number.

WHY In each case, the objective is to find an equivalent equation whose values of a, b, and c are easier to work with than those of the given equation.

Solution

a. It is often easier to solve a quadratic equation using the quadratic formula if a is positive. If we multiply (or divide) both sides of $-2x^2 + 4x - 1 = 0$ by -1, we obtain an equivalent equation with $a > 0$.

$$\begin{aligned} -2x^2 + 4x - 1 &= 0 && \text{Here, } a = -2. \\ (-1)(-2x^2 + 4x - 1) &= (-1)(0) \\ 2x^2 - 4x + 1 &= 0 && \text{Now } a = 2. \end{aligned}$$

b. For $x^2 + \frac{4}{5}x - \frac{1}{3} = 0$, two coefficients are fractions: $b = \frac{4}{5}$ and $c = -\frac{1}{3}$. We can multiply both sides of the equation by their least common denominator, 15, to obtain an equivalent equation having coefficients that are integers.

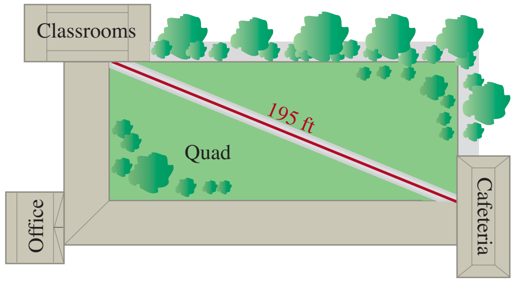
$$\begin{aligned} x^2 + \frac{4}{5}x - \frac{1}{3} &= 0 && \text{Here, } a = 1, b = \tfrac{4}{5}, \text{ and } c = -\tfrac{1}{3}. \\ 15\left(x^2 + \frac{4}{5}x - \frac{1}{3}\right) &= 15(0) \\ 15x^2 + 12x - 5 &= 0 && \text{Now } a = 15, b = 12, \text{ and } c = -5. \end{aligned}$$

c. For $20x^2 - 60x - 40 = 0$, the coefficients 20, -60, and -40 have a common factor of 20. If we divide both sides of the equation by their GCF, we obtain an equivalent equation having smaller coefficients.

$$\begin{aligned} 20x^2 - 60x - 40 &= 0 && \text{Here, } a = 20, b = -60, \text{ and } c = -40. \\ \frac{20x^2}{20} - \frac{60x}{20} - \frac{40}{20} &= \frac{0}{20} \\ x^2 - 3x - 2 &= 0 && \text{Now } a = 1, b = -3, \text{ and } c = -2. \end{aligned}$$

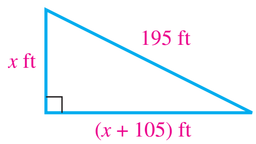
4 Use the quadratic formula to solve application problems.

EXAMPLE 5 ***Taking Shortcuts*** Instead of using the existing hallways, students are wearing a path through a planted quad area to walk 195 feet directly from the classrooms to the cafeteria. If the length of the hallway from the office to the cafeteria is 105 feet longer than the hallway from the office to the classrooms, how much walking are the students saving by taking the shortcut?



Analyze The two hallways and the shortcut form a right triangle with a hypotenuse 195 feet long. We will use the Pythagorean theorem to solve this problem.

Form If we let $x =$ the length (in feet) of the hallway from the classrooms to the office, then the length of the hallway from the office to the cafeteria is $(x + 105)$ feet. Substituting these lengths into the Pythagorean theorem, we have



$$a^2 + b^2 = c^2 \quad \text{This is the Pythagorean theorem.}$$

$$x^2 + (x + 105)^2 = 195^2 \quad \text{Substitute } x \text{ for } a, (x + 105) \text{ for } b, \text{ and } 195 \text{ for } c.$$

$$x^2 + x^2 + 105x + 105x + 11{,}025 = 38{,}025 \quad \text{Find } (x + 105)^2.$$

$$2x^2 + 210x + 11{,}025 = 38{,}025 \quad \text{Combine like terms.}$$

$$2x^2 + 210x - 27{,}000 = 0 \quad \text{Subtract 38,025 from both sides.}$$

$$x^2 + 105x - 13{,}500 = 0 \quad \text{Divide both sides by 2.}$$

Solve To solve $x^2 + 105x - 13{,}500 = 0$, we will use the quadratic formula with $a = 1$, $b = 105$, and $c = -13{,}500$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-105 \pm \sqrt{(105)^2 - 4(1)(-13{,}500)}}{2(1)}$$

$$x = \frac{-105 \pm \sqrt{65{,}025}}{2} \quad \text{Simplify: } (105)^2 - 4(1)(-13{,}500) = 11{,}025 + 54{,}000 = 65{,}025.$$

$$x = \frac{-105 \pm 255}{2} \quad \text{Use a calculator: } \sqrt{65{,}025} = 255.$$

$$x = \frac{150}{2} \quad \text{or} \quad x = \frac{-360}{2}$$

$$x = 75 \quad \Big| \quad \cancel{x = -180}$$

Since the length of the hallway can't be negative, discard the solution $x = -180$.

State The length of the hallway from the classrooms to the office is 75 feet. The length of the hallway from the office to the cafeteria is $75 + 105 = 180$ feet. Instead

Self Check 5

RIGHT TRIANGLE The hypotenuse of a right triangle is 41 in. long. The longer leg is 31 inches longer than the shorter leg. Find the lengths of the sides of the triangle. 9 in., 40 in., 41 in.

Now Try **Problem 35**

Teaching Example 5 RIGHT TRIANGLE The hypotenuse of a right triangle is 10 in. long. The longer leg is 2 in. longer than the shorter leg. Find the lengths of the sides of the triangle.
Answer:
6 in., 8 in., 10 in.

of using the hallways, a distance of 75 + 180 = 255 feet, the students are taking the 195-foot shortcut to the cafeteria, a savings of (255 − 195), or 60 feet.

Check The length of the 180-foot hallway is 105 feet longer than the length of the 75-foot hallway. The sum of the squares of the lengths of the hallways is $75^2 + 180^2 = 38{,}025$. This equals the square of the length of the 195-foot shortcut. The answer checks.

Self Check 6

AIRPORT SHUTTLES A bus company shuttles 1,120 passengers daily between Rockford, Illinois, and O'Hare Airport. The current one-way fare is \$10. For each 25¢ increase in the fare, the company predicts that it will lose 48 passengers. What increase in fare will produce daily revenue of \$10,208? \$1

Now Try **Problem 74**

Teaching Example 6 TICKET PRICES A jazz group on tour has been drawing average crowds of 500 persons. It is projected that for every \$1 increase in the \$12 ticket price, the average attendance will decrease by 50. At what ticket price will nightly receipts be \$5,600?
Answer:
\$14

EXAMPLE 6 ***Mass Transit*** A bus company has 4,000 passengers daily, each currently paying a 75¢ fare. For each 15¢ fare increase, the company estimates that it will lose 50 passengers. If the company needs to bring in \$6,570 per day to stay in business, what fare must be charged to produce this amount of revenue?

Analyze To understand how a fare increase affects the number of passengers, let's consider what happens if there are two fare increases. We organize the data in a table. The fares are expressed in terms of dollars.

Number of increases	New fare	Number of passengers
One \$0.15 increase	\$0.75 + \$0.15(1) = \$0.90	4,000 − 50(1) = 3,950
Two \$0.15 increases	\$0.75 + \$0.15(2) = \$1.05	4,000 − 50(2) = 3,900

In general, the new fare will be the old fare (\$0.75) plus the number of fare increases times \$0.15. The number of passengers who will pay the new fare is 4,000 minus 50 times the number of \$0.15 fare increases.

Form If we let x = the number of \$0.15 fare increases necessary to bring in \$6,570 daily, then $\$(0.75 + 0.15x)$ is the fare that must be charged. The number of passengers who will pay this fare is $(4{,}000 - 50x)$. We can now form the equation.

The bus fare	times	the number of passengers who will pay that fare	equals	\$6,570.
$(0.75 + 0.15x)$	$\cdot$	$(4{,}000 - 50x)$	$=$	$6{,}570$

Solve

$$(0.75 + 0.15x)(4{,}000 - 50x) = 6{,}570$$

$$3{,}000 - 37.5x + 600x - 7.5x^2 = 6{,}570 \quad \text{Multiply the binomials.}$$

$$-7.5x^2 + 562.5x + 3{,}000 = 6{,}570 \quad \text{Combine like terms: } -37.5x + 600x = 562.5x.$$

$$-7.5x^2 + 562.5x - 3{,}570 = 0 \quad \text{Subtract 6,570 from both sides.}$$

$$7.5x^2 - 562.5x + 3{,}570 = 0 \quad \text{Multiply both sides by } -1 \text{ so that } a, 7.5, \text{ is positive.}$$

To solve this equation, we will use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-562.5) \pm \sqrt{(-562.5)^2 - 4(7.5)(3{,}570)}}{2(7.5)} \quad \text{Substitute 7.5 for } a, -562.5 \text{ for } b, \text{ and 3,570 for } c.$$

$$x = \frac{562.5 \pm \sqrt{209{,}306.25}}{15} \quad \text{Simplify: } (-562.5)^2 - 4(7.5)(3{,}570) = 316{,}406.25 - 107{,}100 = 209{,}306.25.$$

$$x = \frac{562.5 \pm 457.5}{15}$$

Use a calculator: $\sqrt{209{,}306.25} = 457.5$.

$$x = \frac{1{,}020}{15} \quad \text{or} \quad x = \frac{105}{15}$$

$$x = 68 \quad\quad x = 7$$

State If there are 7 fifteen-cent increases in the fare, the new fare will be \$0.75 + \$0.15(7) = \$1.80. If there are 68 fifteen-cent increases in the fare, the new fare will be \$0.75 + \$0.15(68) = \$10.95. Although this fare would bring in the necessary revenue, a \$10.95 bus fare is unreasonable, so we discard it.

Check A fare of \$1.80 will be paid by [4,000 − 50(7)] = 3,650 bus riders. The amount of revenue brought in would be \$1.80(3,650) = \$6,570. The answer checks.

EXAMPLE 7

Lawyers The number of lawyers N in the United States each year from 1980 to 2002 is approximated by $N = 222x^2 + 17{,}630x + 571{,}178$, where $x = 0$ corresponds to the year 1980, $x = 1$ corresponds to 1981, $x = 2$ corresponds to 1982, and so on. (Thus, $0 \le x \le 22$.) In what year does this model indicate that the United States had one million lawyers?

Strategy We will substitute 1,000,000 for N in the equation and solve for x.

WHY The value of x will give the number of years after 1980 that the United States had 1,000,000 lawyers.

Solution

$$N = 222x^2 + 17{,}630x + 571{,}178$$

$$1{,}000{,}000 = 222x^2 + 17{,}630x + 571{,}178$$ Replace N with 1,000,000.

$$0 = 222x^2 + 17{,}630x - 428{,}822$$ Subtract 1,000,000 from both sides so that the equation is in quadratic form.

We can simplify the computations by dividing both sides of the equation by 2, which is the greatest common factor of 222, 17,630, and 428,822.

$$111x^2 + 8{,}815x - 214{,}411 = 0$$ Divide both sides by 2.

We solve this equation using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-8{,}815 \pm \sqrt{(8{,}815)^2 - 4(111)(-214{,}411)}}{2(111)}$$ Substitute 111 for a, 8,815 for b, and −214,411 for c.

$$x = \frac{-8{,}815 \pm \sqrt{172{,}902{,}709}}{222}$$ Evaluate the expression within the radical.

$$x \approx \frac{4{,}334}{222} \quad \text{or} \quad x \approx \frac{-21{,}964}{222}$$ Use a calculator.

$$x \approx 19.5 \quad\quad x \approx -98.9$$ Since the model is defined only for $0 \le x \le 22$, we discard the second solution.

In 19.5 years after 1980, or midway through 1999, the United States had approximately 1,000,000 lawyers.

Self Check 7

STOPPING DISTANCE The number of feet that a car travels before stopping depends on the driver's reaction time and the braking distance. For one driver, the stopping distance d is given by $d = 0.04v^2 + 0.9v$, where v is the velocity of the car in miles per hour. Find the velocity if the stopping distance is 198 feet. 60 mph

Now Try Problem 79

Teaching Example 7 TOY ROCKETS A toy rocket is shot straight up with an initial velocity of 110 feet per second. Its height, in feet, t seconds after being launched is given by $h = -16t^2 + 110t$. Find the time(s) when the rocket is 175 feet in the air. Round to the nearest tenth of a second.
Answer:
2.5 sec, 4.4 sec

ANSWERS TO SELF CHECKS

1. $2, -\frac{1}{3}$ **2.** $\frac{1 \pm \sqrt{10}}{3}$; −0.72, 1.39 **3.** $-\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$ **4. a.** $6x^2 - 7x + 9 = 0$ **b.** $2x^2 - 4x - 5 = 0$ **c.** $4x^2 + 6x - 5 = 0$ **5.** 9 in., 40 in., 41 in. **6.** \$1 **7.** 60 mph

SECTION 8.2 STUDY SET

VOCABULARY

Fill in the blanks.

▶ **1.** An equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$ is a quadratic equation.

▶ **2.** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the quadratic formula.

CONCEPTS

3. Write each equation in quadratic form.

a. $x^2 + 2x = -5$ $x^2 + 2x + 5 = 0$

b. $3x^2 = -2x + 1$ $3x^2 + 2x - 1 = 0$

▶ **4.** For each quadratic equation, find a, b, and c.

a. $x^2 + 5x + 6 = 0$ 1, 5, 6

b. $8x^2 - x = 10$ 8, −1, −10

5. Determine whether each statement is true or false.

a. Any quadratic equation can be solved by using the quadratic formula. true

b. Any quadratic equation can be solved by completing the square. true

6. Find the error in the beginning of the solution shown below.

$\cancel{\text{Solve: } x^2 - 3x = 2}$

$\cancel{a = 1 \quad b = -3 \quad c = 2}$

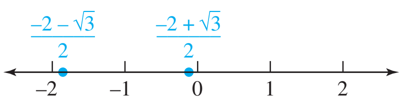
The equation wasn't written in quadratic form first: $c = -2$.

7. What form would the quadratic formula take if, in developing it, we began with the general quadratic equation expressed as $rc^2 + sc + t = 0$ where $r \neq 0$? $c = \dfrac{-s \pm \sqrt{s^2 - 4rt}}{2r}$

8. A student used the quadratic formula to solve a quadratic equation and obtained $x = \dfrac{-2 \pm \sqrt{3}}{2}$.

a. How many solutions does the equation have? What are they exactly? 2; $\dfrac{-2 + \sqrt{3}}{2}$, $\dfrac{-2 - \sqrt{3}}{2}$

b. Graph the solutions on a number line.



▶ **9.** Simplify each of the following solutions.

a. $x = \dfrac{3 \pm 6\sqrt{2}}{3}$ $x = 1 \pm 2\sqrt{2}$

b. $x = \dfrac{-12 \pm 4\sqrt{7}}{8}$ $x = \dfrac{-3 \pm \sqrt{7}}{2}$

10. For each of the following, write an equivalent equation so that the quadratic formula computations will be easier to perform.

a. $-5x^2 + 9x - 2 = 0$ $5x^2 - 9x + 2 = 0$

b. $\dfrac{1}{8}x^2 + \dfrac{1}{2}x - \dfrac{3}{4} = 0$ $x^2 + 4x - 6 = 0$

c. $45x^2 + 30x - 15 = 0$ $3x^2 + 2x - 1 = 0$

NOTATION

11. On a quiz, students were asked to write the quadratic formula. What is wrong with each answer?

a. $\cancel{x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}}$ The fraction bar wasn't drawn under both parts of the numerator.

b. $\cancel{x = \dfrac{-b\sqrt{b^2 - 4ac}}{2a}}$ A ± symbol wasn't written between b and the radical.

▶ **12.** In reading $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we say, "The opposite of b, plus or minus the square root of b squared minus 4 times a times c, all over $2a$."

GUIDED PRACTICE

Use the quadratic formula to solve each equation.
See Example 1.

13. $x^2 + 3x + 2 = 0$ −1, −2

14. $x^2 - 3x + 2 = 0$ 1, 2

15. $x^2 + 12x + 36 = 0$ −6, −6

▶ **16.** $y^2 - 18y + 81 = 0$ 9, 9

17. $2x^2 + 5x - 3 = 0$ $\frac{1}{2}, -3$

18. $6x^2 - x - 1 = 0$ $\frac{1}{2}, -\frac{1}{3}$

▶ **19.** $12t^2 - 5t - 2 = 0$ $\frac{2}{3}, -\frac{1}{4}$

20. $12z^2 + 5z - 3 = 0$ $\frac{1}{3}, -\frac{3}{4}$

Use the quadratic formula to solve each equation.
See Example 2.

21. $8u = -4u^2 - 3$ $-\frac{3}{2}, -\frac{1}{2}$

▶ **22.** $4t + 3 = 4t^2$ $\frac{3}{2}, -\frac{1}{2}$

23. $2x^2 - 1 = 3x$ $\frac{3 \pm \sqrt{17}}{4}$

24. $-9x = 2 - 3x^2$ $\frac{9 \pm \sqrt{105}}{6}$

Use the quadratic formula to solve each equation.
See Example 3.

25. $2x^2 + x + 1 = 0$ $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$

26. $2x^2 + 3x + 5 = 0$ $-\frac{3}{4} \pm \frac{\sqrt{31}}{4}i$

27. $3x^2 - 2x + 1 = 0$ $\frac{1}{3} \pm \frac{\sqrt{2}}{3}i$

▶ **28.** $3x^2 - 2x + 5 = 0$ $\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

▶ Selected exercises available online at **www.webassign.net/brookscole**

For each equation, write an equivalent equation so that the quadratic formula computation will be easier. See Example 4.

29. $-3x^2 - 6x - 1 = 0$ $3x^2 + 6x + 1 = 0$

30. $x^2 + \frac{4}{5}x - \frac{1}{5} = 0$ $5x^2 + 4x - 1 = 0$

31. $50x^2 - 100x + 300 = 0$ $x^2 - 2x + 6 = 0$

32. $x^2 = -\frac{5}{2}x + \frac{1}{2}$ $2x^2 + 5x - 1 = 0$

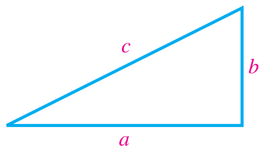
Use the Pythagorean theorem to find each value of x. See Example 5.

33. $a = x, b = 2x, c = 5$ $\sqrt{5}$

34. $a = x + 7, b = x, c = 13$ 5

35. $a = x, b = x + 2, c = 10$ 6

36. $a = x + 3, b = x, c = 15$ 9



TRY IT YOURSELF

37. $5x^2 + 5x + 1 = 0$ $\frac{-5 \pm \sqrt{5}}{10}$

38. $4w^2 + 6w + 1 = 0$ $\frac{-3 \pm \sqrt{5}}{4}$

39. $-16y^2 - 8y + 3 = 0$ $\frac{1}{4}, -\frac{3}{4}$

40. $-16x^2 - 16x - 3 = 0$ $-\frac{1}{4}, -\frac{3}{4}$

41. $x^2 - \frac{14}{15}x = \frac{8}{15}$ $\frac{4}{3}, -\frac{2}{5}$

42. $x^2 = -\frac{5}{4}x + \frac{3}{2}$ $\frac{3}{4}, -2$

43. $\frac{x^2}{2} + \frac{5}{2}x = -1$ $\frac{-5 \pm \sqrt{17}}{2}$

44. $\frac{x^2}{8} - \frac{x}{4} = \frac{1}{2}$ $1 \pm \sqrt{5}$

45. $-x^2 + 10x = 18$ $5 \pm \sqrt{7}$

46. $-3x = \frac{x^2}{2} + 2$ $-3 \pm \sqrt{5}$

47. $x^2 - 6x = 391$ $23, -17$

48. $-x^2 + 27x = -280$ $35, -8$

49. $x^2 - 2x + 2 = 0$ $1 \pm i$

50. $x^2 - 4x + 8 = 0$ $2 \pm 2i$

51. $x^2 - \frac{5}{3} = \frac{11}{6}x$ $-\frac{2}{3}, \frac{5}{2}$

52. $x^2 - \frac{1}{2} = \frac{2}{3}x$ $\frac{2 \pm \sqrt{22}}{6}$

53. $4a^2 + 4a + 5 = 0$ $-\frac{1}{2} \pm i$

54. $4b^2 + 4b + 17 = 0$ $-\frac{1}{2} \pm 2i$

55. $50x^2 + 30x - 10 = 0$ $\frac{-3 \pm \sqrt{29}}{10}$

56. $120b^2 + 120b - 40 = 0$ $\frac{-3 \pm \sqrt{21}}{6}$

57. $900x^2 - 8{,}100x = 1{,}800$ $\frac{9 \pm \sqrt{89}}{2}$

58. $-14x^2 + 21x = 49$ $\frac{3 \pm \sqrt{47}i}{4}$

59. $-0.6x^2 - 0.03 = -0.4x$ $\frac{10 \pm \sqrt{55}}{30}$

60. $2x^2 + 0.1x = 0.04$ $\frac{-1 \pm \sqrt{33}}{40}$

Use the quadratic formula and a calculator to solve each equation. Round answers to the nearest hundredth.

61. $x^2 + 8x + 5 = 0$ $-0.68, -7.32$

62. $2x^2 - x - 9 = 0$ $2.39, -1.89$

63. $3x^2 - 2x - 2 = 0$ $1.22, -0.55$

64. $81x^2 + 12x - 80 = 0$ $0.92, -1.07$

65. $0.7x^2 - 3.5x - 25 = 0$ $8.98, -3.98$

66. $-4.5x^2 + 0.2x + 3.75 = 0$ $0.94, -0.89$

APPLICATIONS

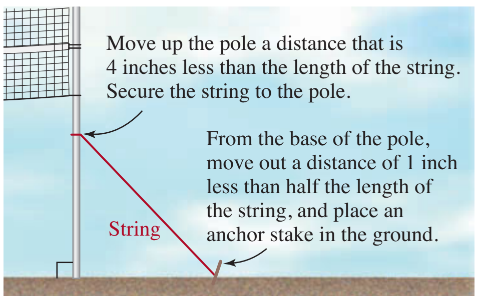
67. THEATER SCREENS The largest permanent movie screen is in the Panasonic IMAX Theater at Darling Harbor, Sydney, Australia. The rectangular screen has an area of 11,349 square feet. Find the dimensions of the screen if it is 20 feet longer than it is wide. 97 ft by 117 ft

68. ROCK CONCERTS During a 1997 tour, the rock group U2 used the world's largest LED (light emitting diode) electronic screen as part of the stage backdrop. The rectangular screen, with an area of 9,520 square feet, had a length that was 2 feet more than three times its width. Find the dimensions of the LED screen. 56 ft by 170 ft

69. CENTRAL PARK Central Park is one of New York's best-known landmarks. Rectangular in shape, its length is 5 times its width. When measured in miles, its perimeter numerically exceeds its area by 4.75. Find the dimensions of Central Park if we know that its width is less than 1 mile. 0.5 mi by 2.5 mi

70. HISTORY One of the most important cities of the ancient world was Babylon. Greek historians wrote that the city was square-shaped. Measured in miles, its area numerically exceeded its perimeter by about 124. Find its dimensions. (Round to the nearest tenth.) 13.3 mi by 13.3 mi

71. BADMINTON The person who wrote the instructions for setting up the badminton net shown below forgot to give the specific dimensions for securing the pole. How long is the support string? 34 in.



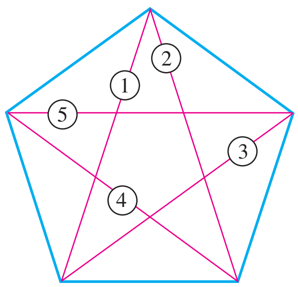
72. RIGHT TRIANGLES The hypotenuse of a right triangle is 2.5 units long. The longer leg is 1.7 units longer than the shorter leg. Find the lengths of the sides of the triangle. 0.7, 2.4, 2.5

73. SCHOOL DANCES Tickets to a school dance cost \$4, and the projected attendance is 300 persons. It is further projected that for every 10¢ increase in ticket price, the average attendance will decrease by 5. At what ticket price will the receipts from the dance be \$1,248? \$4.80 or \$5.20

74. TICKET SALES A carnival at a county fair normally sells three thousand 25¢ ride tickets on a Saturday. For each 5¢ increase in price, management estimates that 80 fewer tickets will be sold. What increase in ticket price will produce \$994 of revenue on Saturday? 10¢

75. MAGAZINE SALES The *Gazette*'s profit is \$20 per year for each of its 3,000 subscribers. Management estimates that the profit per subscriber will increase by 1¢ for each additional subscriber over the current 3,000. How many subscribers will bring a total profit of \$120,000? 4,000

76. POLYGONS The five-sided polygon called a *pentagon*, shown in the illustration, has 5 diagonals. The number of diagonals d of a polygon of n sides is given by the formula



$$d = \frac{n(n-3)}{2}$$

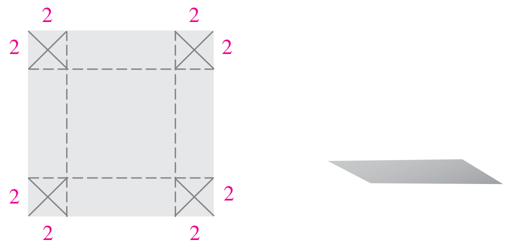
Find the number of sides of a polygon if it has 275 diagonals. 25 sides

77. INVESTMENT RATES A woman invests \$1,000 in a mutual fund for which interest is compounded annually at a rate r. After one year, she deposits an additional \$2,000. After two years, the balance in the account is

$$\$1{,}000(1 + r)^2 + \$2{,}000(1 + r)$$

If this amount is \$3,368.10, find r. 9%

78. METAL FABRICATION A box with no top is to be made by cutting a 2-inch square from each corner of the square sheet of metal shown in the next column. After bending up the sides, the volume of the box is to be 220 cubic inches. How large should the piece of metal be? Round to the nearest hundredth. 14.49 in. by 14.49 in.



79. RETIREMENT The labor force participation rate P (in percent) for men ages 55–64 from 1970 to 2000 is approximated by the quadratic equation $P = 0.03x^2 - 1.37x + 82.51$, where $x = 0$ corresponds to the year 1970, $x = 1$ corresponds to 1971, $x = 2$ corresponds to 1972, and so on. (Thus, $0 \le x \le 30$.) When does the model indicate that 75% of the men ages 55–64 were part of the workforce? early 1976

80. SPACE PROGRAM The yearly budget B (in billions of dollars) for the National Aeronautics and Space Administration (NASA) is approximated by the equation $B = 0.0596x^2 - 0.3811x + 14.2709$, where x is the number of years since 1995 and $0 \le x \le 9$. In what year does the model indicate that NASA's budget was about \$15 billion? 2003

WRITING

81. Explain why the quadratic formula, in most cases, is less tedious to use in solving a quadratic equation than is the method of completing the square.

82. On an exam, a student was asked to solve the equation $-4w^2 - 6w - 1 = 0$. Her first step was to multiply both sides of the equation by -1. She then used the quadratic formula to solve $4w^3 + 6w + 1 = 0$ instead. Is this a valid approach? Explain.

REVIEW

Change each radical to an exponential expression.

83. $\sqrt{n}$ $n^{1/2}$

84. $\sqrt[7]{\frac{3}{8}r^2s}$ $\left(\frac{3}{8}r^2s\right)^{1/7}$

85. $\sqrt[4]{3b}$ $(3b)^{1/4}$

86. $3\sqrt[3]{c^2 - d^2}$ $3(c^2 - d^2)^{1/3}$

Write each expression in radical form.

87. $t^{1/3}$ $\sqrt[3]{t}$

88. $\left(\frac{3}{4}m^2n^2\right)^{1/5}$ $\sqrt[5]{\frac{3}{4}m^2n^2}$

89. $(3t)^{1/4}$ $\sqrt[4]{3t}$

90. $(c^2 + d^2)^{1/2}$ $\sqrt{c^2 + d^2}$

SECTION 8.3

Quadratic Functions and Their Graphs

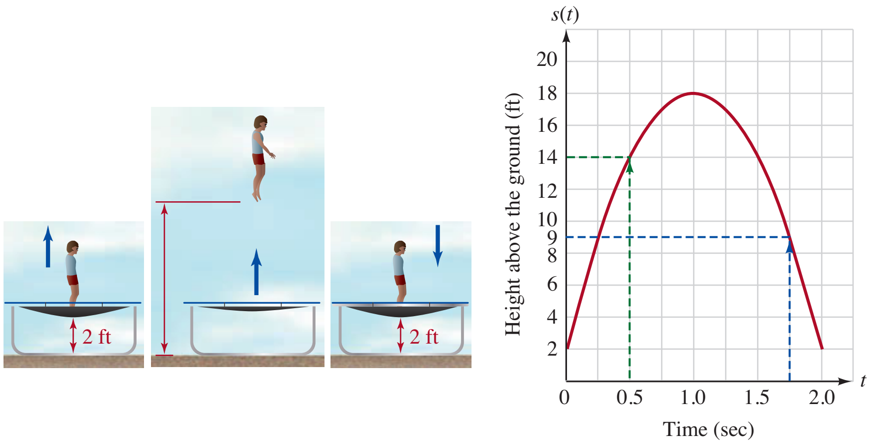
Objectives

1. Evaluate quadratic functions at a value.
2. Graph functions of the form $f(x) = ax^2$.
3. Graph functions of the form $f(x) = ax^2 + k$.
4. Graph functions of the form $f(x) = a(x - h)^2$.
5. Graph functions of the form $f(x) = a(x - h)^2 + k$.
6. Graph functions of the form $f(x) = ax^2 + bx + c$ by completing the square.
7. Find the vertex using $\frac{-b}{2a}$.
8. Determine minimum and maximum values.
9. Solve quadratic equations graphically.

The graph in the figure below shows a trampolinist's distance from the ground (in relation to time) as she bounds into the air and then falls back down to the trampoline.

From the graph, we can see that the trampolinist is 14 feet above the ground 0.5 second after bounding upward and that her height above the ground after 1.75 seconds is 9 feet.

The parabola shown is the graph of the *quadratic function* $s(t) = -16t^2 + 32t + 2$. In this section, we will discuss two forms in which quadratic functions are written and how to graph them.



1 Evaluate quadratic functions at a value.

Quadratic Functions

A quadratic function is a second-degree polynomial function of the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c represent real numbers and $a \neq 0$.

EXAMPLE 1 **Trampolines** The quadratic function $s(t) = -16t^2 + 32t + 2$ gives the distance (in feet) that the trampolinist shown in the figure above is from the ground, t seconds after bounding upward. How far is she from the ground after being in the air for $\frac{3}{4}$ second?

Strategy We will substitute 0.75 for the time t in $s(t)$ and simplify.

WHY To find her distance from the ground, we find the value of the function for $t = \frac{3}{4} = 0.75$.

Self Check 1

TRAMPOLINES Find the distance the trampolinist is from the ground after being in the air for $1\frac{1}{2}$ seconds. 14 ft

Now Try **Problem 17**

Teaching Example 1 TRAMPOLINES Find the distance the trampolinist is from the ground after being in the air for $1\frac{3}{4}$ seconds.
Use $s(t) = -16t^2 + 32t + 2$.
Answer:
9 ft

Solution

$$\begin{aligned} s(t) &= -16t^2 + 32t + 2 \\ s(0.75) &= -16(0.75)^2 + 32(0.75) + 2 \quad \text{Replace } t \text{ with } 0.75. \\ &= -9 + 24 + 2 \\ &= 17 \end{aligned}$$

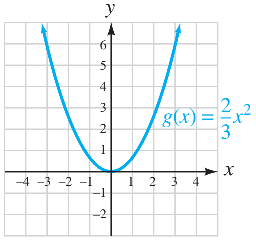
The trampolinist is 17 feet off the ground $\frac{3}{4}$ second after bounding upward.

We have seen that important information can be gained from the graph of a quadratic function. We now begin a discussion of how to graph such a function by considering the simplest case, quadratic functions of the form $f(x) = ax^2$.

2 Graph functions of the form $f(x) = ax^2$.

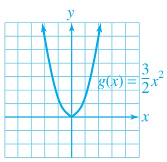
Self Check 2

Graph: $g(x) = \frac{2}{3}x^2$



Now Try **Problem 19**

Teaching Example 2 Graph:
$g(x) = \frac{3}{2}x^2$
Answer:



EXAMPLE 2

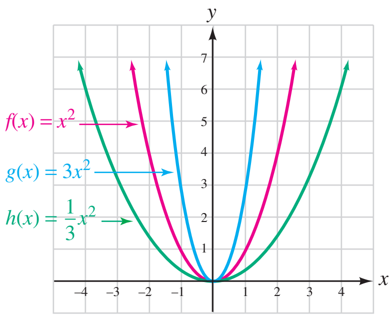
Graph: **a.** $f(x) = x^2$ **b.** $g(x) = 3x^2$ **c.** $h(x) = \frac{1}{3}x^2$

Strategy We can make a table of values for each function, plot each point, and connect them with a smooth curve.

WHY To graph an equation in two variables means to make a drawing that represents all of its solutions.

Solution

After graphing each curve, we note that the graph of $h(x) = \frac{1}{3}x^2$ is wider than the graph of $f(x) = x^2$, and that the graph of $g(x) = 3x^2$ is narrower than the graph of $f(x) = x^2$. In the function $f(x) = ax^2$, the smaller the value of $|a|$, the wider the graph.



$f(x) = x^2$		
x	$f(x)$	$(x, f(x))$
−2	4	(−2, 4)
−1	1	(−1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)

$g(x) = 3x^2$		
x	$g(x)$	$(x, g(x))$
−2	12	(−2, 12)
−1	3	(−1, 3)
0	0	(0, 0)
1	3	(1, 3)
2	12	(2, 12)

$h(x) = \frac{1}{3}x^2$		
x	$h(x)$	$(x, h(x))$
−2	$\frac{4}{3}$	$\left(-2, \frac{4}{3}\right)$
−1	$\frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	0	(0, 0)
1	$\frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\frac{4}{3}$	$\left(2, \frac{4}{3}\right)$

EXAMPLE 3

Graph: $f(x) = -3x^2$

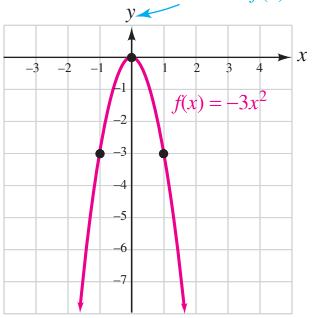
Strategy We can make a table of values for the function, plot each point, and connect them with a smooth curve.

WHY To graph an equation in two variables means to make a drawing that represents all of its solutions.

Solution

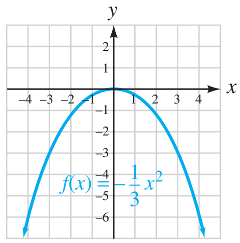
After graphing the curve, we see that the parabola opens downward and has the same shape as the graph of $g(x) = 3x^2$ in Example 2.

$f(x) = -3x^2$		
x	$f(x)$	$(x, f(x))$
-2	-12	$(-2, -12)$
-1	-3	$(-1, -3)$
0	0	$(0, 0)$
1	-3	$(1, -3)$
2	-12	$(2, -12)$



Self Check 3

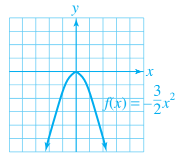
Graph: $f(x) = -\frac{1}{3}x^2$



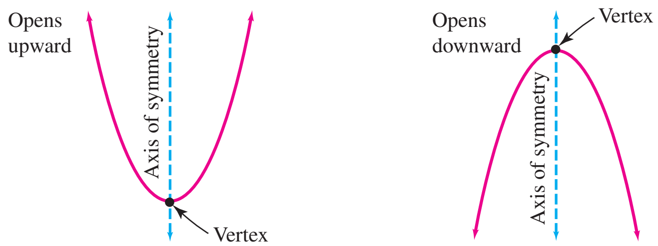
Now Try **Problem 22**

Teaching Example 3 Graph: $f(x) = -\frac{3}{2}x^2$

Answer:



The graphs of quadratic functions of the form $f(x) = ax^2$ are **parabolas.** The lowest point of a parabola that opens upward, or the highest point of a parabola that opens downward, is called the **vertex** of the parabola. The vertical line, called an **axis of symmetry,** that passes through the vertex divides the parabola into two congruent halves. If we fold the paper along the axis of symmetry, the two sides of the parabola will match.



The results of Examples 2 and 3 confirm the following facts.

The Graph of $f(x) = ax^2$

The graph of $f(x) = ax^2$ is a parabola opening upward when $a > 0$ and downward when $a < 0$, with vertex at the point $(0, 0)$ and axis of symmetry the line $x = 0$.

3 Graph functions of the form $f(x) = ax^2 + k$.

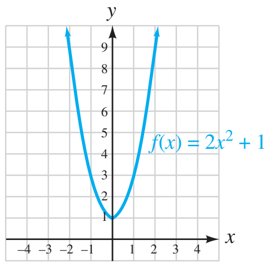
EXAMPLE 4 Graph: **a.** $f(x) = 2x^2$ **b.** $g(x) = 2x^2 + 3$ **c.** $h(x) = 2x^2 - 3$

Strategy We can make a table of values for each function, plot each point, and connect them with a smooth curve.

WHY To graph an equation in two variables means to make a drawing that represents all of its solutions.

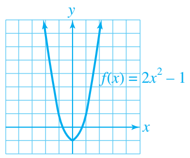
Self Check 4

Graph: $f(x) = 2x^2 + 1$



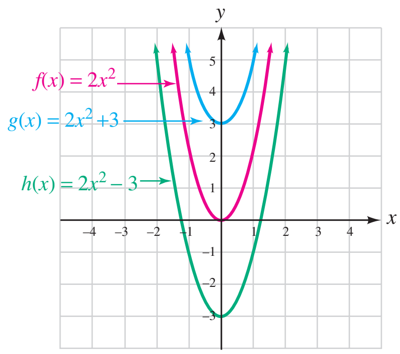
Now Try **Problem 24**

Teaching Example 4 Graph:
$f(x) = 2x^2 - 1$
Answer:



Solution

After graphing the curves, we note that the graph of $g(x) = 2x^2 + 3$ is identical to the graph of $f(x) = 2x^2$, except that it has been translated 3 units upward. The graph of $h(x) = 2x^2 - 3$ is identical to the graph of $f(x) = 2x^2$, except that it has been translated 3 units downward. In each case, the axis of symmetry is the line $x = 0$.



$f(x) = 2x^2$		
x	$f(x)$	$(x, f(x))$
-2	8	$(-2, 8)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	8	$(2, 8)$

$g(x) = 2x^2 + 3$		
x	$g(x)$	$(x, g(x))$
-2	11	$(-2, 11)$
-1	5	$(-1, 5)$
0	3	$(0, 3)$
1	5	$(1, 5)$
2	11	$(2, 11)$

$h(x) = 2x^2 - 3$		
x	$h(x)$	$(x, h(x))$
-2	5	$(-2, 5)$
-1	-1	$(-1, -1)$
0	-3	$(0, -3)$
1	-1	$(1, -1)$
2	5	$(2, 5)$

The results of Example 4 confirm the following facts.

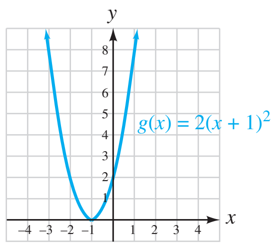
The Graph of $f(x) = ax^2 + k$

The graph of $f(x) = ax^2 + k$ is a parabola having the same shape as $f(x) = ax^2$ but translated upward k units if k is positive and downward $|k|$ units if k is negative. The vertex is at the point $(0, k)$, and the axis of symmetry is the line $x = 0$.

4 Graph functions of the form $f(x) = a(x - h)^2$.

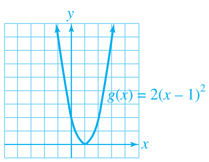
Self Check 5

Graph: $g(x) = 2(x + 1)^2$



Now Try **Problem 26**

Teaching Example 5 Graph:
$g(x) = 2(x - 1)^2$
Answer:



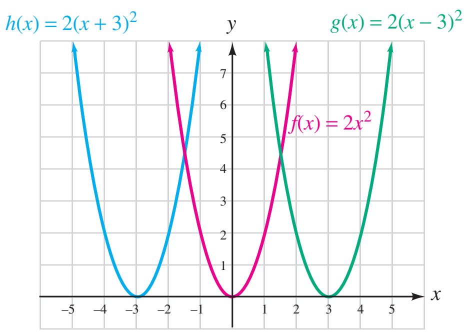
EXAMPLE 5 Graph: **a.** $f(x) = 2x^2$ **b.** $g(x) = 2(x - 3)^2$ **c.** $h(x) = 2(x + 3)^2$

Strategy We can make a table of values for each function, plot each point, and connect them with a smooth curve.

WHY To graph an equation in two variables means to make a drawing that represents all of its solutions.

Solution

After graphing the curves, we note that the graph of $g(x) = 2(x - 3)^2$ is identical to the graph of $f(x) = 2x^2$, except that it has been translated 3 units to the right. The graph of $h(x) = 2(x + 3)^2$ is identical to the graph of $f(x) = 2x^2$, except that it has been translated 3 units to the left.



$f(x) = 2x^2$		
x	$f(x)$	$(x, f(x))$
-2	8	$(-2, 8)$
-1	2	$(-1, 2)$
0	0	$(0, 0)$
1	2	$(1, 2)$
2	8	$(2, 8)$

$g(x) = 2(x - 3)^2$		
x	$g(x)$	$(x, g(x))$
1	8	$(1, 8)$
2	2	$(2, 2)$
3	0	$(3, 0)$
4	2	$(4, 2)$
5	8	$(5, 8)$

$h(x) = 2(x + 3)^2$		
x	$h(x)$	$(x, h(x))$
-5	8	$(-5, 8)$
-4	2	$(-4, 2)$
-3	0	$(-3, 0)$
-2	2	$(-2, 2)$
-1	8	$(-1, 8)$

The results of Example 5 confirm the following facts.

The Graph of $f(x) = a(x - h)^2$

The graph of $f(x) = a(x - h)^2$ is a parabola having the same shape as $f(x) = ax^2$ but translated h units to the right if h is positive and $|h|$ units to the left if h is negative. The vertex is at the point $(h, 0)$, and the axis of symmetry is the line $x = h$.

5 Graph functions of the form $f(x) = a(x - h)^2 + k$.

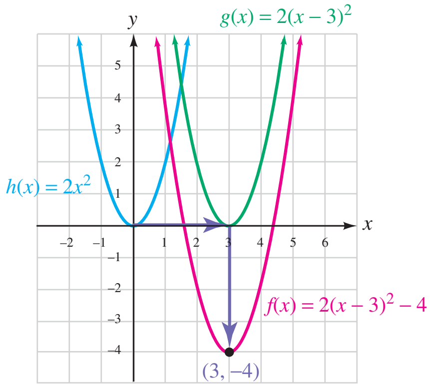
EXAMPLE 6 Graph: $f(x) = 2(x - 3)^2 - 4$

Strategy We will determine whether the graph opens upward or downward and find its vertex and axis of symmetry. Then we will plot some points and complete the graph.

WHY This method will be more efficient than plotting many points.

Solution

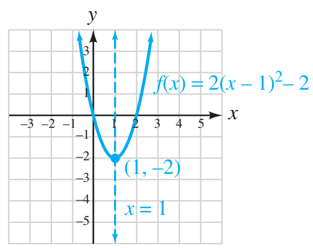
The graph of $f(x) = 2(x - 3)^2 - 4$ is identical to the graph of $g(x) = 2(x - 3)^2$, except that it has been translated 4 units downward. The graph of $g(x) = 2(x - 3)^2$ is identical to the graph of $h(x) = 2x^2$, except that it has been translated 3 units to the right. Thus, to graph $f(x) = 2(x - 3)^2 - 4$, we can graph $h(x) = 2x^2$ and shift it 3 units to the right and then 4 units downward, as shown.



The vertex of the graph is the point $(3, -4)$, and the axis of symmetry is the line $x = 3$.

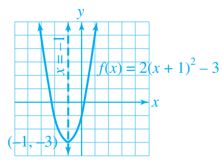
Self Check 6

Graph: $f(x) = 2(x - 1)^2 - 2$
Label the vertex and draw the axis of symmetry.



Now Try **Problem 28**

Teaching Example 6 Graph $f(x) = 2(x + 1)^2 - 3$. Label the vertex and draw the axis of symmetry.
Answer:



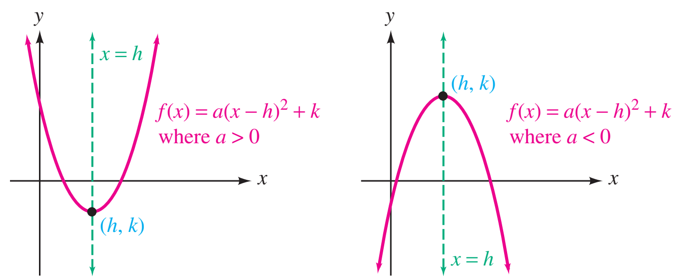
The results of Example 6 confirm the following facts.

Graphing a Quadratic Function in Standard Form

The graph of the quadratic function

$$f(x) = a(x - h)^2 + k \quad \text{where } a \neq 0$$

is a parabola with vertex at (h, k). The axis of symmetry is the line $x = h$. The parabola opens upward when $a > 0$ and downward when $a < 0$.



Self Check 7

Answer parts a, b, and c of Example 7 for the graph of: $f(x) = 6(x - 5)^2 + 1$

Now Try **Problem 31**

Self Check 7 Answers

a. upward
b. $(5, 1)$
c. $x = 5$

Teaching Example 7 Answer parts a, b, and c of Example 7 for the graph of: $f(x) = -2(x - 4)^2 + 5$
Answers:
a. downward
b. $(4, 5)$
c. $x = 4$

EXAMPLE 7 Consider the graph of: $f(x) = -3(x + 1)^2 - 4$

a. Does the graph open upward or downward?
b. What are the coordinates of the vertex?
c. What is the axis of symmetry?

Strategy We will compare the given equation with the standard form of a quadratic function to determine the values of a, h, and k.

WHY The coordinates of the vertex will be (h, k) and the equation of the axis of symmetry will be $x = h$. The graph will open upward if $a > 0$ or open downward if $a < 0$.

Solution

Rewriting the given function in $f(x) = a(x - h)^2 + k$ form, we have

Standard form requires a minus sign here.   Standard form requires a plus sign here.

$$f(x) = \underbrace{-3}_{a}[x - (\underbrace{-1}_{h})]^2 + (\underbrace{-4}_{k})$$

a. Since $a = -3 < 0$, the parabola opens downward.
b. The vertex is $(h, k) = (-1, -4)$.
c. The axis of symmetry is the line $x = h$. In this case, $x = -1$.

6 Graph functions of the form $f(x) = ax^2 + bx + c$ by completing the square.

To graph functions of the form $f(x) = ax^2 + bx + c$, we complete the square to write the function in the form $f(x) = a(x - h)^2 + k$.

EXAMPLE 8 Graph: $f(x) = 2x^2 - 4x - 1$

Strategy We will complete the square on x and write the equation of the function in standard form, $f(x) = a(x - h)^2 + k$.

WHY When the equation is in standard form we can identify a, h, and k from the equation. This information helps us sketch the graph.

Solution

Step 1 We complete the square on x to write the given function in the form $f(x) = a(x - h)^2 + k$.

$$f(x) = 2x^2 - 4x - 1$$

$$f(x) = 2(x^2 - 2x) - 1 \qquad \text{Factor 2 from } 2x^2 - 4x.$$

Now we complete the square on x by adding 1 within the parentheses. Since this adds 2 to the right-hand side, we also subtract 2 from the right-hand side.

By the distributive property, when 1 is added to the expression within the parentheses, $2 \cdot 1 = 2$ is added to the right-hand side.

Subtract 2 to counteract the addition of 2.

$$f(x) = 2(x^2 - 2x + 1) - 1 - 2$$

To complete the square within the parentheses, one-half of $-2 = -1$ and $(-1)^2 = 1$.

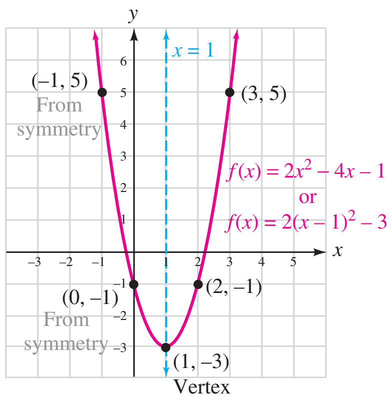
(1) $f(x) = 2(x - 1)^2 - 3$ Factor $x^2 - 2x + 1$ and combine like terms.

Step 2 From Equation 1, we can see that $h = 1$ and $k = -3$, so the vertex will be at the point $(1, -3)$, and the axis of symmetry is $x = 1$. We plot the vertex and axis of symmetry on the coordinate system shown below.

Step 3 Finally, we construct a table of values to determine several points on the parabola. Since the x-coordinate of the vertex is 1, we choose values for x close to 1 and on the same side of the axis of symmetry. After plotting $(2, -1)$ and $(3, 5)$, we use symmetry to locate two other points on the parabola: $(-1, 5)$ and $(0, -1)$. Then we draw the graph.

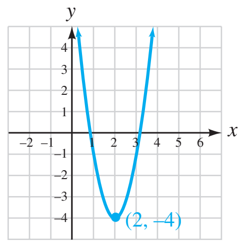
$f(x) = 2x^2 - 4x - 1$		
x	$f(x)$	$(x, f(x))$
2	-1	$(2, -1)$
3	5	$(3, 5)$

The x-coordinate of the vertex is 1. Choose values for x close to 1 and on the same side of the axis of symmetry.



Self Check 8

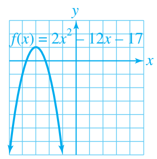
Graph: $f(x) = 3x^2 - 12x + 8$



$f(x) = 3x^2 - 12x + 8$

Now Try Problem 34

Teaching Example 8 Graph:
$f(x) = -2x^2 - 12x - 17$
Answer:



7 Find the vertex using $\frac{-b}{2a}$.

We can derive a formula for the vertex of the graph of $f(x) = ax^2 + bx + c$ by completing the square in the same manner as we did in Example 8. After using similar steps, the result is

$$f(x) = a\left[x - \left(\underset{h}{\underset{\uparrow}{-\frac{b}{2a}}}\right)\right]^2 + \underset{k}{\underset{\uparrow}{\frac{4ac - b^2}{4a}}}$$

The x-coordinate of the vertex is $-\frac{b}{2a}$. The y-coordinate of the vertex is $\frac{4ac - b^2}{4a}$. However, we can also find the y-coordinate of the vertex by substituting the x-coordinate, $-\frac{b}{2a}$, for x in the quadratic function.

Formula for the Vertex of a Parabola

The vertex of the graph of the quadratic function $f(x) = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and the axis of symmetry of the parabola is the line $x = -\frac{b}{2a}$.

In Example 8, for the function $f(x) = 2x^2 - 4x - 1$, $a = 2$ and $b = -4$. To find the vertex of its graph, we compute

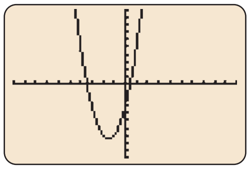
$$\begin{aligned} -\frac{b}{2a} &= -\frac{-4}{2(2)} \\ &= -\frac{-4}{4} \\ &= 1 \end{aligned} \qquad \begin{aligned} f\left(-\frac{b}{2a}\right) &= f(1) \\ &= 2(1)^2 - 4(1) - 1 \\ &= -3 \end{aligned}$$

The vertex is the point $(1, -3)$. This agrees with the result we obtained in Example 8 by completing the square.

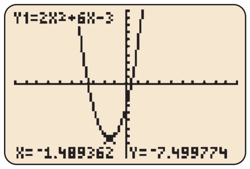
Using Your CALCULATOR Finding the Vertex

To graph the function $f(x) = 2x^2 + 6x - 3$ and find the coordinates of the vertex and the axis of symmetry of the parabola, we can use a graphing calculator with window settings of $[-10, 10]$ for x and $[-10, 10]$ for y. If we enter the function, we will obtain the graph shown in figure (a).

We then trace to move the cursor to the lowest point on the graph, as shown in figure (b). By zooming in, we can see that the vertex is the point $(-1.5, -7.5)$, or $\left(-\frac{3}{2}, -\frac{15}{2}\right)$, and that the line $x = -\frac{3}{2}$ is the axis of symmetry.



(a)



(b)

Some calculators have an fmin or fmax feature that can also be used to find the vertex. Consult your owner's manual for details.

Graphing a Quadratic Function $f(x) = ax^2 + bx + c$

We can determine much about the graph of $f(x) = ax^2 + bx + c$ from the coefficients a, b, and c. This information is summarized as follows:

- Determine whether the parabola opens upward or downward by finding the value of a.
- The x-coordinate of the vertex of the parabola is $x = -\frac{b}{2a}$.
- To find the y-coordinate of the vertex, substitute $-\frac{b}{2a}$ for x and find $f\left(-\frac{b}{2a}\right)$.
- The axis of symmetry is the vertical line passing through the vertex.
- The y-intercept is determined by the value of $f(x)$ when $x = 0$: the y-intercept is $(0, c)$.
- The x-intercepts (if any) are determined by the values of x that make $f(x) = 0$. To find them, solve the quadratic equation $ax^2 + bx + c = 0$.

8 Determine minimum and maximum values.

It is often useful to know the smallest or largest possible value a quantity can assume. For example, companies try to minimize their costs and maximize their profits. If the quantity is expressed by a quadratic function, the vertex of the graph of the function gives its minimum or maximum value.

EXAMPLE 9 ***Minimizing Costs*** A glassworks that makes lead crystal vases has daily production costs given by the function $C(x) = 0.2x^2 - 10x + 650$, where x is the number of vases made each day. How many vases should be produced to minimize the per-day costs? What will the costs be?

Strategy We will find the vertex of the graph of the quadratic function.

WHY The x-coordinate of the vertex indicates the number of vases to make to keep costs at a minimum, and the y-coordinate indicates the minimum cost.

Solution

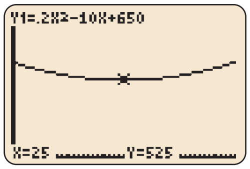
The graph of $C(x) = 0.2x^2 - 10x + 650$ is a parabola opening upward. The vertex is the lowest point on the graph. To find the vertex, we compute

$$-\frac{b}{2a} = -\frac{-10}{2(0.2)} \quad b = -10 \text{ and } a = 0.2.$$
$$= -\frac{-10}{0.4}$$
$$= 25$$

$$f\left(-\frac{b}{2a}\right) = f(25)$$
$$= 0.2(25)^2 - 10(25) + 650$$
$$= 525$$

The vertex is the point (25, 525), and it indicates that the costs are a minimum of \$525 when 25 vases are made daily.

To solve this problem with a graphing calculator with window settings of [0, 50] for x and [0, 1,000] for y, we graph the function $C(x) = 0.2x^2 - 10x + 650$. By using TRACE and ZOOM, we can locate the vertex of the graph. See the figure to the right. The coordinates of the vertex indicate that the minimum cost is \$525 when the number of vases produced is 25.



Self Check 9

MINIMIZING COSTS A manufacturing company has a daily production cost of $C(x) = 0.25x^2 - 10x + 800$, where x is the number of items produced and c is the cost. How many items should be produced to minimize the per day cost? What is the minimum cost?

Now Try **Problem 69**

Self Check 9 Answer
20, \$700

Teaching Example 9 PROFITS The profit function for a company is $P = -0.0025x^2 + 120x - 275{,}000$, where x is the number of units sold. What sales level will give a maximum profit?
Answer:
24,000 units

EXAMPLE 10 ***Maximizing Area*** A kennel owner wants to build the rectangular pen shown in the figure on the next page to house his dog. If he uses one side of his barn, find the maximum area that he can enclose with 80 feet of fencing.

Strategy We will find the vertex of the graph of the quadratic function.

WHY The w-coordinate of the vertex indicates the width of the pen, and the A-coordinate indicates maximum area.

Solution

We can let the width of the area be represented by w. Then the length is represented by $80 - 2w$. The function that gives the area the pen encloses is

$$A(w) = (80 - 2w)w \quad \text{To find the area of a rectangle, multiply its length and width.}$$

We can find the maximum value of A by determining the vertex of the graph of the function. This time, we will find the vertex by completing the square.

$$A(w) = 80w - 2w^2 \quad \text{Distribute the multiplication by } w.$$
$$= -2(w^2 - 40w) \quad \text{Factor out } -2.$$

Self Check 10

FENCES A farmer wants to fence in three sides of a rectangular field with 800 feet of fencing. The other side of the rectangle will be a river. If the enclosed area is to be maximum, find the dimensions of the field. 200 ft by 400 ft

Now Try **Problem 74**

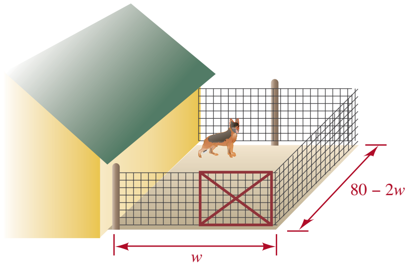
Teaching Example 10 FENCES A farmer wants to fence in three sides of a rectangular field with 1,200 feet of fence. The other side of the rectangle will be a river. If the enclosed area is to be maximum, find the dimensions of the field.
Answer:
300 ft by 600 ft

$= \mathbf{-2}(w^2 - 40w \mathbf{+ 400}) \mathbf{+ 800}$ Complete the square on w. Within the parentheses, add $\left(\frac{-40}{2}\right)^2 = 400$. Since this subtracts 800 from the right-hand side, add 800 to the right-hand side.

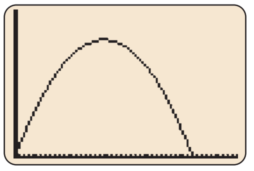
$= -2(w - 20)^2 + 800$ Factor $w^2 - 40w + 400$.

Thus, the coordinates of the vertex of the graph of the quadratic function are (20, 800), and the maximum area is 800 square feet.

To solve this problem with a graphing calculator with window settings of [0, 50] for x and [0, 1,000] for y, we graph the function $A(w) = -2w^2 + 80w$ to get the graph in figure (b). By using TRACE and ZOOM, we can determine the vertex of the graph, which shows that the maximum area is 800 square feet when the width is 20 feet.



(a)



(b)

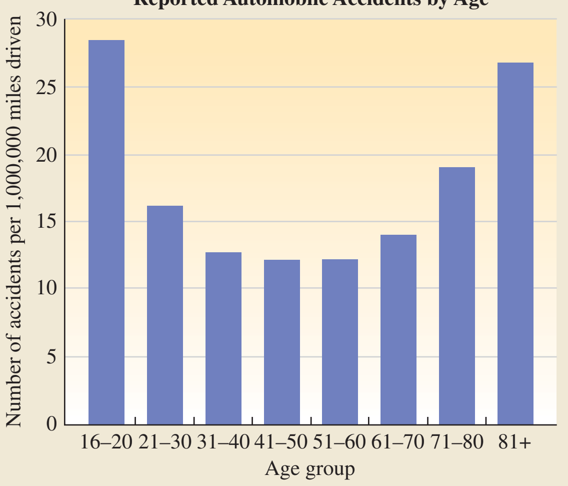
THINK IT THROUGH *Automobile Accidents*

"Motor vehicle accidents can have many causes, but they can usually be divided into negligence, intentional misconduct, or product liability."

Allen L. Rothenberg, InjuryLawyer.com

The graph below shows the results of a study that explored the relationship between the age of drivers and the number of traffic accidents in which they were involved. For example, we see that drivers 16–20 years old were involved in 28 reported accidents for every one million miles driven. On the graph, sketch a parabola that could be used to model the data and locate its vertex. What information about age of the driver and traffic accidents do the coordinates of the vertex give?

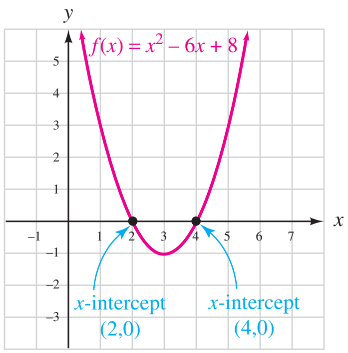
We estimate that 50-year-old drivers are involved in the least number of accidents, 12 per million miles driven.



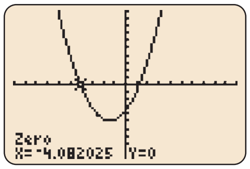
Source: Quality Planning Corporation

9 Solve quadratic equations graphically.

We can solve quadratic equations graphically. For example, the solutions of $x^2 - 6x + 8 = 0$ are the numbers x that will make y equal to 0 in the quadratic function $f(x) = x^2 - 6x + 8$. To find these numbers, we carefully inspect the graph of $f(x) = x^2 - 6x + 8$ and locate the points on the graph with a y-coordinate of 0. In the figure below, these points are (2, 0) and (4, 0), the x-intercepts of the graph. We can conclude that the x-coordinates of the x-intercepts, $x = 2$ and $x = 4$, are the solutions of $x^2 - 6x + 8 = 0$.

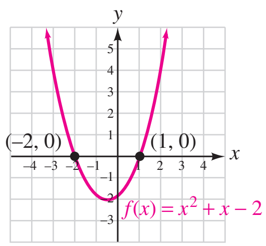


Using Your CALCULATOR Solving Quadratic Equations Graphically



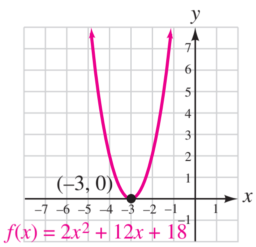
We can use a graphing calculator to find approximate solutions of quadratic equations. For example, the solutions of $0.7x^2 + 2x - 3.5 = 0$ are the numbers x that will make $y = 0$ in the quadratic function $f(x) = 0.7x^2 + 2x - 3.5$. To approximate these numbers, we graph the quadratic function and read the x-intercepts from the graph using the ZERO feature. In the figure, we see that the x-coordinate of the left-most x-intercept of the graph is given as -4.082025. This means that an approximate solution of the equation is -4.08. To find the positive x-intercept, we use similar steps.

When solving quadratic equations graphically, we must consider three possibilities. If the graph of the associated quadratic function has two x-intercepts, the quadratic equation has two real-number solutions. Figure (a) shows an example of this. If the graph has one x-intercept, as shown in figure (b), the equation has one real-number solution. Finally, if the graph does not have an x-intercept, as shown in figure (c), the equation does not have any real-number solutions.



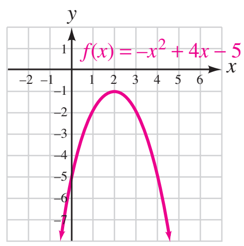
$x^2 + x - 2 = 0$ has two solutions, –2 and 1.

(a)



$2x^2 + 12x + 18 = 0$ has one solution, –3.

(b)

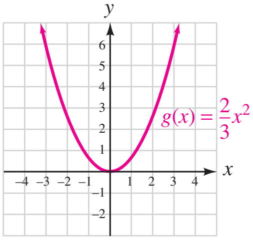


$-x^2 + 4x - 5 = 0$ has no real-number solutions.

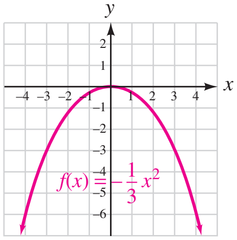
(c)

ANSWERS TO SELF CHECKS

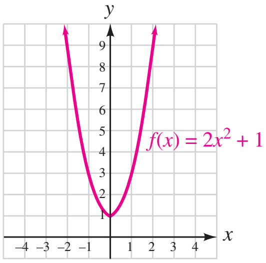
1. 14 ft **2.**



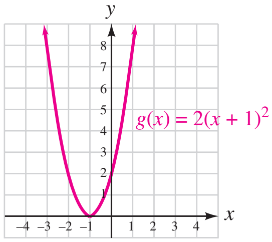
3.



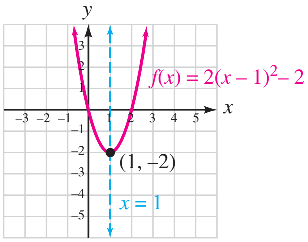
4.



5.

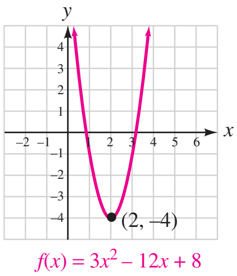


6.



7. a. upward **b.** (5, 1) **c.** $x = 5$

8.

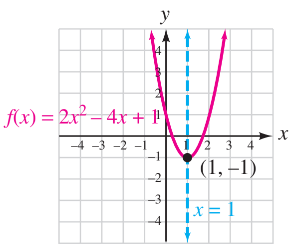


9. 20, \$700 **10.** 200 ft by 400 ft

SECTION 8.3 STUDY SET

VOCABULARY

Refer to the illustration and fill in the blanks.



1. The function graphed in the illustration is called a _quadratic_ function.

▶ **2.** The graph is called a _parabola_.

3. The lowest point on the graph, $(1, -1)$, is called the _vertex_ of the parabola.

4. The vertical line $x = 1$ divides the parabola into two halves. This line is called the _axis_ _of_ _symmetry_.

▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

Refer to the graph below.

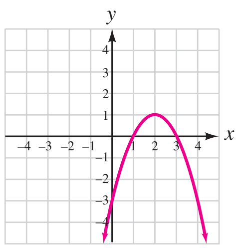
5. What do we call the curve? a parabola

6. Find the x-intercepts of the graph. (1, 0), (3, 0)

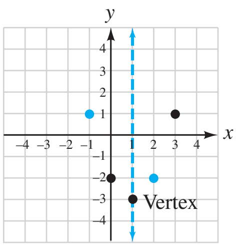
7. Find the y-intercept of the graph. $(0, -3)$

8. Find the vertex. (2, 1)

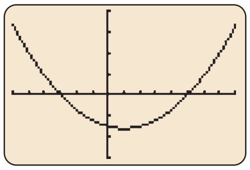
9. Find the axis of symmetry. $x = 2$



10. The vertex of a parabola is at $(1, -3)$, its y-intercept is at $(0, -2)$, and it passes through the point $(3, 1)$, as shown. Draw the axis of symmetry and use it to help determine two other points on the line.



11. Use the graph of $f(x) = \frac{x^2}{10} - \frac{x}{5} - \frac{3}{2}$, shown, to estimate the solutions of the quadratic equation $\frac{x^2}{10} - \frac{x}{5} - \frac{3}{2} = 0$. $-3, 5$

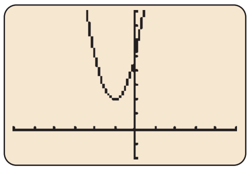


▶ **12.** Three quadratic equations are to be solved graphically. The graphs of their associated quadratic functions are shown below. Determine which graph indicates that the equation has

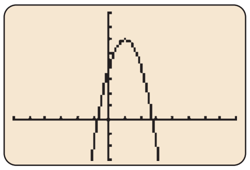
a. two real solutions. ii

b. one real solution. iii

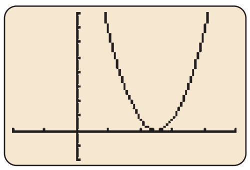
c. no real solutions. i



(i)



(ii)



(iii)

NOTATION

13. The function $f(x) = 2(x + 1)^2 + 6$ is written in the form $f(x) = a(x - h)^2 + k$. Is $h = -1$ or is $h = 1$? Explain.
$h = -1$; $f(x) = 2[x - (-1)]^2 + 6$

▶ **14.** The vertex of a quadratic function $f(x) = ax^2 + bx + c$ is given by the formula $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. Explain what is meant by the notation $f\left(-\frac{b}{2a}\right)$.
Substitute the value $-\frac{b}{2a}$ into the quadratic function for x.

GUIDED PRACTICE

Evaluate the function $s(t) = -16t^2 + 32t + 256$ for the following values of t. See Example 1.

15. $t = 0$ 256

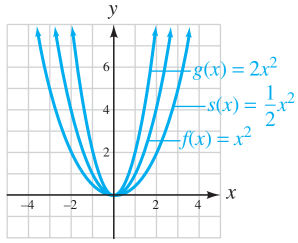
16. $t = 3$ 208

17. $t = 3.5$ 172

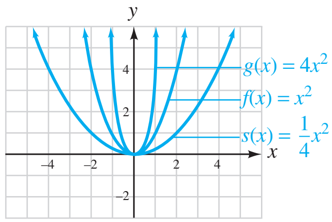
▶ **18.** $t = 4.5$ 76

Make a table of values for each function and graph them on the same coordinate system. See Example 2.

19. $f(x) = x^2, g(x) = 2x^2, s(x) = \frac{1}{2}x^2$

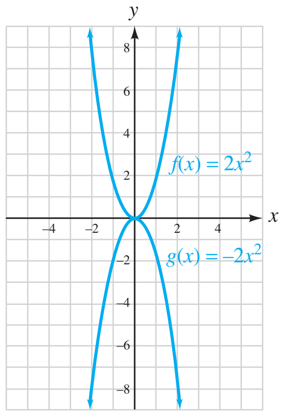


▶ **20.** $f(x) = x^2, g(x) = 4x^2, s(x) = \frac{1}{4}x^2$

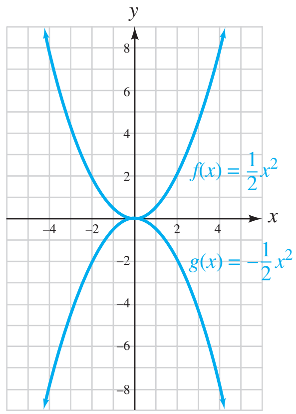


Graph each pair of functions on the same coordinate system. See Example 3.

21. $f(x) = 2x^2, g(x) = -2x^2$

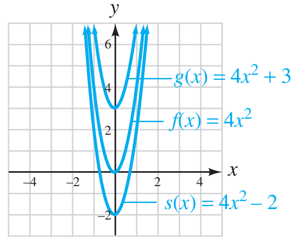


▶ **22.** $f(x) = \frac{1}{2}x^2, g(x) = -\frac{1}{2}x^2$

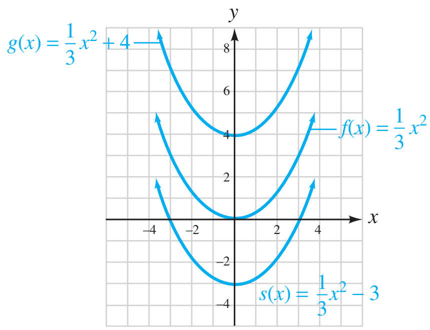


Graph each group of functions on the same coordinate system. See Example 4.

23. $f(x) = 4x^2, g(x) = 4x^2 + 3, s(x) = 4x^2 - 2$

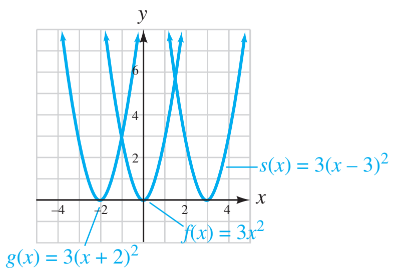


▶ **24.** $f(x) = \frac{1}{3}x^2, g(x) = \frac{1}{3}x^2 + 4, s(x) = \frac{1}{3}x^2 - 3$

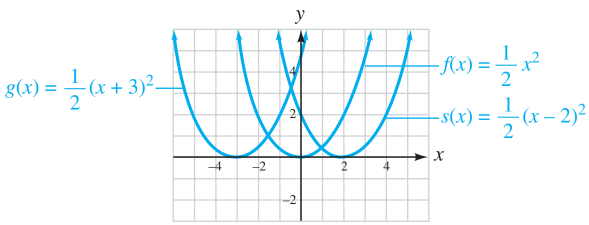


Graph each group of functions on the same coordinate system. See Example 5.

▶ **25.** $f(x) = 3x^2, g(x) = 3(x + 2)^2, s(x) = 3(x - 3)^2$

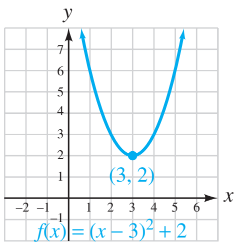


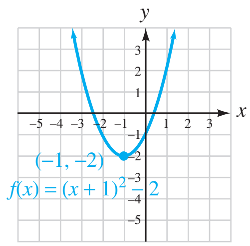
▶ **26.** $f(x) = \frac{1}{2}x^2, g(x) = \frac{1}{2}(x + 3)^2, s(x) = \frac{1}{2}(x - 2)^2$



Use translations to graph each function. See Example 6.

27. $f(x) = (x - 3)^2 + 2$ ▶ **28.** $f(x) = (x + 1)^2 - 2$





Consider the graph of each function. See Example 7.

a. ***Find the coordinates of the vertex.***

b. ***Find the axis of symmetry.***

c. ***Determine whether the graph will open upward or downward.***

29. $f(x) = (x - 1)^2 + 2$ (1, 2), $x = 1$, upward

▶ **30.** $f(x) = 2(x - 2)^2 - 1$ (2, −1), $x = 2$, upward

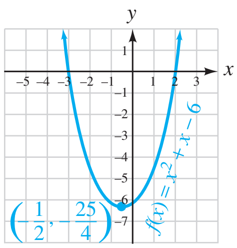
31. $f(x) = 2(x + 3)^2 - 4$ (−3, −4), $x = -3$, upward

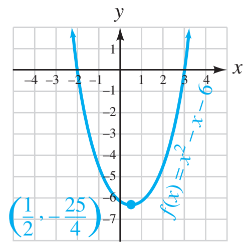
▶ **32.** $f(x) = -3(x + 1)^2 + 3$ (−1, 3), $x = -1$, downward

Graph each function and find the axis of symmetry. See Example 8.

33. $f(x) = x^2 + x - 6$ axis: $x = -\frac{1}{2}$

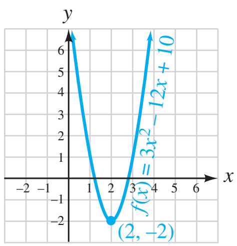
34. $f(x) = x^2 - x - 6$ axis: $x = \frac{1}{2}$

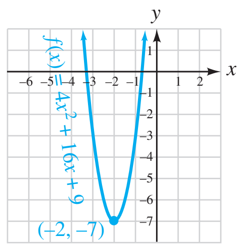




35. $f(x) = 3x^2 - 12x + 10$ axis: $x = 2$

▶ **36.** $f(x) = 4x^2 + 16x + 9$ axis: $x = -2$





TRY IT YOURSELF

Find the coordinates of the vertex and the axis of symmetry of the graph of each function. If necessary, complete the square on x to write the equation in the form $f(x) = a(x - h)^2 + k$. Do not graph the equation, but determine whether the graph will open upward or downward, and find the axis of symmetry and the coordinates of the vertex.

37. $f(x) = 2x^2 - 4x$ (1, −2), $x = 1$, upward

38. $f(x) = 3x^2 - 3$ (0, −3), $x = 0$, upward

39. $f(x) = -4x^2 + 16x + 5$ $(2, 21)$, $x = 2$, downward

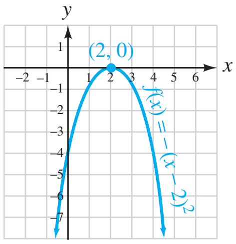
▶ **40.** $f(x) = 5x^2 + 20x + 25$ $(-2, 5)$, $x = -2$, upward

41. $f(x) = 3x^2 + 4x + 2$ $\left(-\frac{2}{3}, \frac{2}{3}\right)$, $x = -\frac{2}{3}$, upward

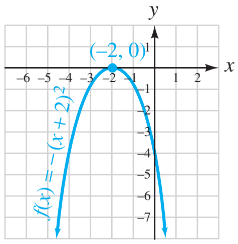
▶ **42.** $f(x) = -6x^2 + 5x - 7$ $\left(\frac{5}{12}, -\frac{143}{24}\right)$, $x = \frac{5}{12}$, downward

First determine the vertex and the axis of symmetry of the graph of the function. Then plot several points and complete the graph.

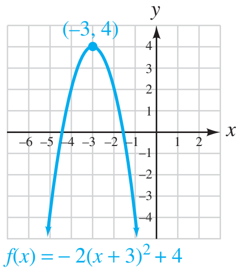
43. $f(x) = -(x - 2)^2$
$(2, 0)$ axis: $x = 2$



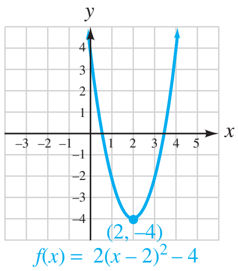
▶ **44.** $f(x) = -(x + 2)^2$
$(-2, 0)$ axis: $x = -2$



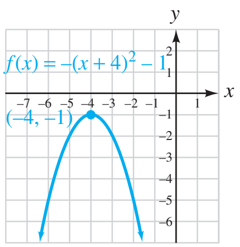
45. $f(x) = -2(x + 3)^2 + 4$
$(-3, 4)$ axis: $x = -3$



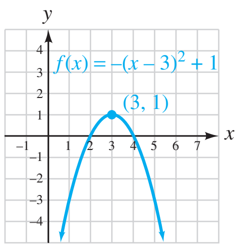
46. $f(x) = 2(x - 2)^2 - 4$
$(2, -4)$ axis: $x = 2$



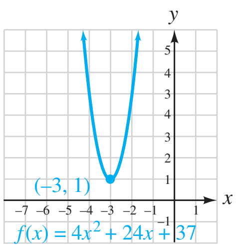
47. $f(x) = -(x + 4)^2 - 1$
$(-4, -1)$ axis: $x = -4$



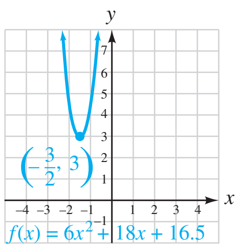
▶ **48.** $f(x) = -(x - 3)^2 + 1$
$(3, 1)$ axis: $x = 3$



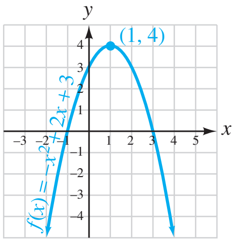
49. $f(x) = 4x^2 + 24x + 37$
$(-3, 1)$ axis: $x = -3$



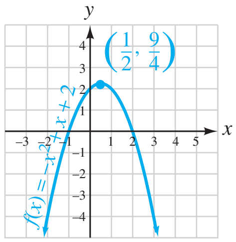
50. $f(x) = 6x^2 + 18x + 16.5$
$\left(-\frac{3}{2}, 3\right)$ axis: $x = -\frac{3}{2}$



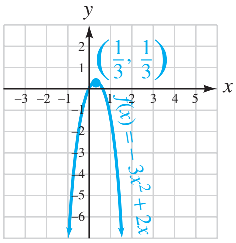
51. $f(x) = -x^2 + 2x + 3$
$(1, 4)$ axis: $x = 1$



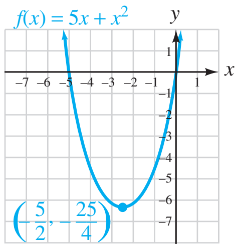
▶ **52.** $f(x) = -x^2 + x + 2$
$\left(\frac{1}{2}, \frac{9}{4}\right)$ axis: $x = \frac{1}{2}$



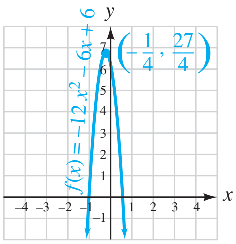
53. $f(x) = -3x^2 + 2x$
$\left(\frac{1}{3}, \frac{1}{3}\right)$ axis: $x = \frac{1}{3}$



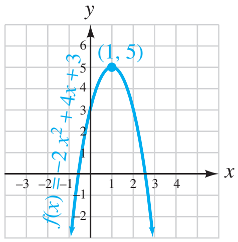
54. $f(x) = 5x + x^2$
$\left(-\frac{5}{2}, \frac{-25}{4}\right)$ axis: $x = -\frac{5}{2}$



55. $f(x) = -12x^2 - 6x + 6$
$\left(-\frac{1}{4}, \frac{27}{4}\right)$ axis: $x = -\frac{1}{4}$



▶ **56.** $f(x) = -2x^2 + 4x + 3$
$(1, 5)$ axis: $x = 1$



Use a graphing calculator to find the coordinates of the vertex of the graph of each quadratic function. Round to the nearest hundredth.

57. $f(x) = 2x^2 - x + 1$
$(0.25, 0.88)$

58. $f(x) = x^2 + 5x - 6$
$(-2.50, -12.25)$

59. $f(x) = 7 + x - x^2$
$(0.50, 7.25)$

60. $f(x) = 2x^2 - 3x + 2$
$(0.75, 0.88)$

Use a graphing calculator to solve each equation. If an answer is not exact, round to the nearest hundredth.

61. $x^2 + x - 6 = 0$
$2, -3$

62. $2x^2 - 5x - 3 = 0$
$3, -0.5$

63. $0.5x^2 - 0.7x - 3 = 0$
$-1.85, 3.25$

64. $2x^2 - 0.5x - 2 = 0$
$-0.88, 1.13$

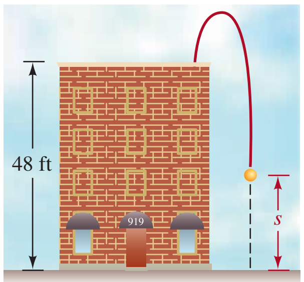
APPLICATIONS

65. FIREWORKS A fireworks shell is shot straight up with an initial velocity of 120 feet per second. Its height s after t seconds is given by the equation $s = 120t - 16t^2$. If the shell is designed to explode when it reaches its maximum height, how long after being fired, and at what height, will the fireworks appear in the sky? 3.75 sec, 225 ft

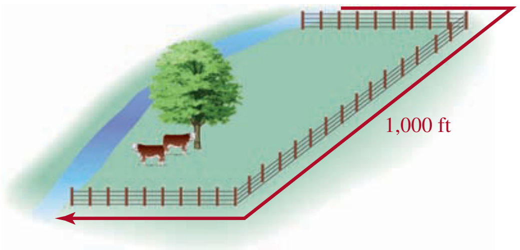
66. BALLISTICS From the top of the building, a ball is thrown straight up with an initial velocity of 32 feet per second. The equation

$$s = -16t^2 + 32t + 48$$

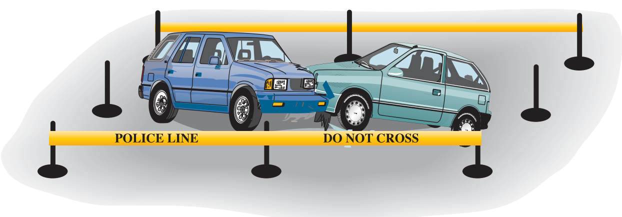
gives the height s of the ball t seconds after it is thrown. Find the maximum height reached by the ball and the time it takes for the ball to hit the ground. 64 ft, 3 sec



67. FENCING A FIELD A farmer wants to fence in three sides of a rectangular field shown below with 1,000 feet of fencing. The other side of the rectangle will be a river. If the enclosed area is to be maximum, find the dimensions of the field. 250 ft by 500 ft



68. POLICE INVESTIGATIONS A police officer seals off the scene of a car collision using a roll of yellow police tape that is 300 feet long, as shown below. What dimensions should be used to seal off the maximum rectangular area around the collision? What is the maximum area? 75 ft by 75 ft, 5,625 ft^2



69. OPERATING COSTS The cost C in dollars of operating a certain concrete-cutting machine is related to the number of minutes n the machine is run by the function

$$C(n) = 2.2n^2 - 66n + 655$$

For what number of minutes is the cost of running the machine a minimum? What is the minimum cost? 15 min, $160

70. WATER USAGE The height (in feet) of the water level in a reservoir over a 1-year period is modeled by the function

$$H(t) = 3.3(t - 9)^2 + 14$$

where $t = 1$ represents January, $t = 2$ represents February, and so on. How low did the water level get that year, and when did it reach the low mark? 14 ft, September

71. U.S. ARMY The function

$$N(x) = -0.0534x^2 + 0.337x + 0.969$$

gives the number of active-duty military personnel in the United States Army (in millions) for the years 1965–1972, where $x = 0$ corresponds to 1965, $x = 1$ corresponds to 1966, and so on. For this period, when was the army's personnel strength level at its highest, and what was it? Historically, can you explain why? 1968, 1.5 million; the U.S. involvement in the war in Vietnam was at its peak

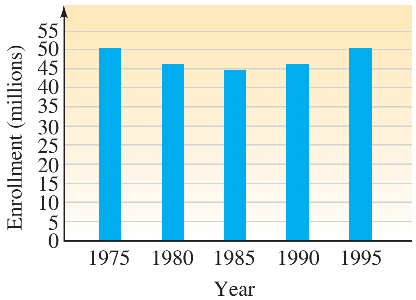
72. SCHOOL ENROLLMENTS The total annual enrollment (in millions) in U.S. elementary and secondary schools for the years 1975–1996 is given by the model

$$E(x) = 0.058x^2 - 1.162x + 50.604$$

where $x = 0$ corresponds to 1975, $x = 1$ corresponds to 1976, and so on.

a. For this period, when was enrollment the lowest? What was it? 1985, 44.8 million

b. Use the model to complete the bar graph below.



▶ **73.** MAXIMIZING REVENUE The revenue R received for selling x stereos is given by the formula

$$R = -\frac{x^2}{5} + 80x - 1{,}000$$

How many stereos must be sold to obtain the maximum revenue? Find the maximum revenue. 200, \$7,000

▶ **74.** MAXIMIZING REVENUE When priced at \$30 each, a toy has annual sales of 4,000 units. The manufacturer estimates that each \$1 increase in cost will decrease sales by 100 units. Find the unit price that will maximize total revenue. (*Hint:* Total revenue = price · the number of units sold.) \$35

WRITING

75. Use the example of a stream of water from a drinking fountain to explain the concepts of the vertex and the axis of symmetry of a parabola.

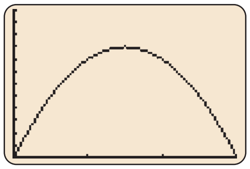
▶ **76.** What are some quantities that are good to maximize? What are some quantities that are good to minimize?

77. A table of values for $f(x) = 2x^2 - 4x + 3$ is shown on the right. Explain why it appears that the vertex of the graph of f is the point $(1, 1)$.

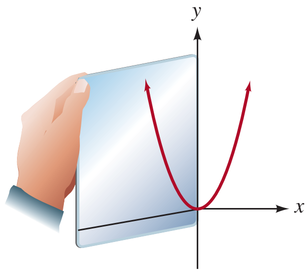
X	Y1
.25	2.125
.5	1.5
.75	1.125
1	1
1.25	1.125
1.5	1.5
1.75	2.125

X=.25

78. The illustration on the right shows the graph of the quadratic function $f(x) = -4x^2 + 12x$ with domain $[0, 3]$. Explain how the value of $f(x)$ changes as the value of x increases from 0 to 3.



79. A mirror is held against the y-axis of the graph of a quadratic function. What fact about parabolas does this illustrate?



80. Give a definition of the vertex of a parabola that opens upward.

REVIEW

Perform the operations. All variables represent positive numbers.

81. $\sqrt{8a}\sqrt{2a^3b}$ $4a^2\sqrt{b}$

82. $\left(\sqrt{23}\right)^2$ 23

83. $\dfrac{\sqrt{3}}{\sqrt{50}}$ $\dfrac{\sqrt{6}}{10}$

84. $\dfrac{3}{\sqrt[3]{9}}$ $\sqrt[3]{3}$

85. $3\left(\sqrt{5b} - \sqrt{3}\right)^2$ $15b - 6\sqrt{15b} + 9$

▶ **86.** $-2\sqrt{5b}\left(4\sqrt{2b} - 3\sqrt{3}\right)$ $-8b\sqrt{10} + 6\sqrt{15b}$

SECTION **8.4**

The Discriminant and Equations That Can Be Written in Quadratic Form

Objectives

1 Use the discriminant to determine number and type of solutions.

2 Solve equations that are in quadratic form.

3 Solve problems involving quadratic equations.

We have seen that solutions of the quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this section, we will examine the radicand within the quadratic formula to distinguish or "*discriminate*" among the three types of solutions—rational, irrational, or imaginary.

1 Use the discriminant to determine number and type of solutions.

The expression $b^2 - 4ac$ that appears under the radical symbol in the quadratic formula is called the **discriminant.** The discriminant can be used to predict what kind of solutions a quadratic equation has without solving it.

The Discriminant

For a quadratic equation of the form $ax^2 + bx + c = 0$ with real-number coefficients and $a \neq 0$, the expression $b^2 - 4ac$ is called the **discriminant** and can be used to determine the number and type of the solutions of the equation.

Discriminant: $b^2 - 4ac$	***Number and type of solutions***
Positive	Two different real numbers
0	One repeated solution, a rational number
Negative	Two different imaginary numbers that are complex conjugates

Discriminant: $b^2 - 4ac$	***Number and type of solutions***
A perfect square	Two different rational numbers
Positive and not a perfect square	Two different irrational numbers

Self Check 1

Determine the type of solutions for:

a. $x^2 + x - 1 = 0$

b. $4x^2 - 10x + 25 = 0$

Now Try **Problems 17 and 18**

Self Check 1 Answers

a. real numbers that are irrational and unequal

b. nonreal numbers that are complex conjugates

Teaching Example 1 Determine the type of solutions for:

a. $x^2 - x - 1 = 0$

b. $9x^2 - 6x + 1 = 0$

Answers:

a. real numbers that are irrational and unequal

b. one repeated solution, a rational number

EXAMPLE 1 Determine the type of solutions for each equation:

a. $x^2 + x + 1 = 0$ **b.** $3x^2 + 5x + 2 = 0$

Strategy We will identify the values of a, b, and c in each equation. Then we will use those values to compute $b^2 - 4ac$, the discriminant.

WHY Once we know whether the discriminant is positive, 0, or negative, and whether it is a perfect square, we can determine the number and type of the solutions of the equation.

Solution

a. For $x^2 + x + 1 = 0$, the discriminant is:

$$\begin{aligned} b^2 - 4ac &= 1^2 - 4(1)(1) && \text{Substitute: } a = 1, b = 1, \text{ and } c = 1. \\ &= -3 && \text{The result is a negative number.} \end{aligned}$$

Since $b^2 - 4ac < 0$, the solutions of $x^2 + x + 1 = 0$ are two nonreal complex numbers that are complex conjugates.

b. For $3x^2 + 5x + 2 = 0$, the discriminant is:

$$\begin{aligned} b^2 - 4ac &= 5^2 - 4(3)(2) && a = 3, b = 5, \text{ and } c = 2. \\ &= 25 - 24 \\ &= 1 && \text{The result is a positive number.} \end{aligned}$$

Since $b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square, the two solutions of $3x^2 + 5x + 2 = 0$ are two different rational numbers.

2 Solve equations that are in quadratic form.

We have discussed four methods that are used to solve quadratic equations. The table on the next page shows some advantages and disadvantages of each method.

Method	Advantages	Disadvantages	Examples
Factoring and the zero-factor property	It can be very fast. When each factor is set equal to 0, the resulting equations are usually easy to solve.	Some polynomials may be difficult to factor and others impossible.	$x^2 - 2x - 24 = 0$ $4a^2 - a = 0$
Square root property	It is the fastest way to solve equations of the form $ax^2 = n$ (n = a number) or $(ax + b)^2 = n$.	It only applies to equations that are in these forms.	$x^2 = 27$ $(2y + 3)^2 = 25$
Completing the square*	It can be used to solve any quadratic equation. It works well with equations of the form $x^2 + bx = n$, where b is even.	It involves more steps than the other methods. The algebra can be cumbersome if the leading coefficient is not 1.	$x^2 + 4x + 1 = 0$ $t^2 - 14t - 9 = 0$
Quadratic formula	It can be used to solve any quadratic equation.	It involves several computations where sign errors can be made. Often the result must be simplified.	$x^2 + 3x - 33 = 0$ $4s^2 - 10s + 5 = 0$

*The quadratic formula is just a condensed version of completing the square and is usually easier to use. However, you need to know how to complete the square because it is used in more advanced mathematics courses.

To determine the most efficient method for a given equation, we can use the following strategy.

Strategy for Solving Quadratic Equations

1. See whether the equation is in a form such that the **square root method** is easily applied.
2. See whether the equation is in a form such that the **completing the square method** is easily applied.
3. If neither step 1 nor step 2 is reasonable, write the equation in $ax^2 + bx + c = 0$ form.
4. See whether the equation can be solved using the **factoring method.**
5. If you can't factor, solve the equation by the **quadratic formula.**

Many nonquadratic equations can be written in quadratic form $(ax^2 + bx + c = 0)$ and solved using the techniques discussed in previous sections. For example, a careful inspection of the equation $x^4 - 5x^2 + 4 = 0$ leads to the following observations:

In the lead term, the exponent on the variable is the square of the exponent on the variable in the middle term.

$$x^4 - 5x^2 + 4 = 0$$

The last term is a constant.

Equations having the above characteristics are said to be *quadratic in form.* One way to solve them is to make a substitution. We will let $y = x^2$.

$$x^4 - 5x^2 + 4 = 0 \quad \text{This is the given equation.}$$
$$(x^2)^2 - 5(x^2) + 4 = 0 \quad \text{Write } x^4 \text{ as } (x^2)^2.$$
$$y^2 - 5y + 4 = 0 \quad \text{Replace each } x^2 \text{ with } y.$$

We can solve this quadratic equation by factoring.

$$(y - 4)(y - 1) = 0 \quad \text{Factor } y^2 - 5y + 4.$$
$$y - 4 = 0 \quad \text{or} \quad y - 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$y = 4 \quad \Big| \quad y = 1$$

These are not the solutions for x. To find x, we undo the earlier substitutions by replacing each y with x^2 and we solve for x.

$$x^2 = 4 \quad \text{or} \quad x^2 = 1$$
$$x = \pm\sqrt{4} \quad \Big| \quad x = \pm\sqrt{1} \quad \text{Use the square root property.}$$
$$x = \pm 2 \quad \Big| \quad x = \pm 1$$

This equation has four solutions: 1, −1, 2, and −2. Verify that each one satisfies the original equation.

Self Check 2

Solve: $x^4 - 3x^2 - 4 = 0$

Now Try **Problem 23**

Self Check 2 Answer
2, −2, i, $-i$

Teaching Example 2 Solve:
$x^4 + 15x^2 - 16 = 0$
Answer:
1, −1, $4i$, $-4i$

EXAMPLE 2

Solve: $x^4 - 8x^2 - 9 = 0$

Strategy Since the leading term, x^4, is the square of the expression x^2 in the middle term, we will substitute y for x^2.

WHY Our hope is that such a substitution will produce an equation that we can solve using one of the methods previously discussed.

Solution

If we write x^4 as $(x^2)^2$, the equation takes the form

$$(x^2)^2 - 8(x^2) - 9 = 0$$

which can be solved using substitution by letting $y = x^2$.

$$(x^2)^2 - 8(x^2) - 9 = 0$$
$$y^2 - 8y - 9 = 0 \quad \text{Substitute } y \text{ for } x^2.$$
$$(y - 9)(y + 1) = 0 \quad \text{Factor } y^2 - 8y - 9.$$
$$y - 9 = 0 \quad \text{or} \quad y + 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$y = 9 \quad \Big| \quad y = -1 \quad \text{Solve each equation.}$$

Since $y = x^2$, we have

$$x^2 = 9 \quad \text{or} \quad x^2 = -1$$
$$x = \pm 3 \quad \Big| \quad x = \pm i \quad \text{Use the square root property.}$$

The equation has four solutions: −3, 3, i, and $-i$. Check each one.

Self Check 3

Solve: $x + x^{1/2} - 6 = 0$ 4

Now Try **Problem 27**

Teaching Example 3 Solve:
$x - 12\sqrt{x} + 35 = 0$
Answer:
49, 25

EXAMPLE 3

Solve: $x - 7\sqrt{x} + 12 = 0$

Strategy Since the leading term, x, is the square of the expression $\sqrt{x}$ in the middle term, we will substitute y for $\sqrt{x}$.

WHY Our hope is that such a substitution will produce an equation that we can solve using one of the methods previously discussed.

Solution

This equation can be written in quadratic form, because the power of the variable of the lead term is the square of the variable factor of the middle term: $x = (\sqrt{x})^2$. If we let $y = \sqrt{x}$, then $y^2 = x$. With this substitution, the equation

$$x - 7\sqrt{x} + 12 = 0$$

becomes a quadratic equation that can be solved by factoring.

$$y^2 - 7y + 12 = 0 \quad \text{Substitute } y^2 \text{ for } x \text{ and } y \text{ for } \sqrt{x}.$$
$$(y - 3)(y - 4) = 0 \quad \text{Factor } y^2 - 7y + 12 = 0.$$
$$y - 3 = 0 \quad \text{or} \quad y - 4 = 0 \quad \text{Set each factor equal to 0.}$$
$$y = 3 \quad \bigg| \quad y = 4$$

Replace each y with $\sqrt{x}$ and solve the radical equations by squaring both sides.

$$\sqrt{x} = 3 \quad \text{or} \quad \sqrt{x} = 4$$
$$x = 9 \quad \bigg| \quad x = 16 \quad \text{Square both sides.}$$

Verify that 9 and 16 satisfy the original equation.

EXAMPLE 4

Solve: $2m^{2/3} - 2 = 3m^{1/3}$

Strategy We will write the equation in descending powers of m and look for a possible substitution to make.

WHY Our hope is that such a substitution will produce an equation that we can solve using one of the methods previously discussed.

Solution

After writing the equation in descending powers of m, we see that

$$2m^{2/3} - 3m^{1/3} - 2 = 0$$

can be written in quadratic form, because $m^{2/3} = (m^{1/3})^2$. We will use the substitution $y = m^{1/3}$.

$$2m^{2/3} - 3m^{1/3} - 2 = 0$$
$$2(m^{1/3})^2 - 3m^{1/3} - 2 = 0$$
$$2y^2 - 3y - 2 = 0 \quad \text{Replace } m^{1/3} \text{ with } y.$$
$$(2y + 1)(y - 2) = 0 \quad \text{Factor } 2y^2 - 3y - 2 = 0.$$
$$2y + 1 = 0 \quad \text{or} \quad y - 2 = 0 \quad \text{Set each factor equal to 0.}$$
$$y = -\frac{1}{2} \quad \bigg| \quad y = 2$$

Replace each y with $m^{1/3}$ and solve for m.

$$m^{1/3} = -\frac{1}{2} \quad \text{or} \quad m^{1/3} = 2$$
$$(m^{1/3})^3 = \left(-\frac{1}{2}\right)^3 \quad \bigg| \quad (m^{1/3})^3 = (2)^3 \quad \text{Recall that } m^{1/3} = \sqrt[3]{m}. \text{ To solve for } m \text{, cube both sides.}$$
$$m = -\frac{1}{8} \quad \bigg| \quad m = 8$$

Verify that $-\frac{1}{8}$ and 8 satisfy the original equation.

Self Check 4

Solve: $a^{2/3} = -3a^{1/3} + 10$

Now Try **Problem 31**

Self Check 4 Answer
$-125, 8$

Teaching Example 4 Solve: $n^{2/5} = n^{1/5} + 2$
Answer:
$32, -1$

Self Check 5

Solve:
$(n + 3)^2 - 6(n + 3) = -8$

Now Try **Problem 35**

Self Check 5 Answer
$-1, 1$

Teaching Example 5 Solve:
$(2a - 3)^2 + 4(2a - 3) = 12$
Answer:
$-\frac{3}{2}, \frac{5}{2}$

EXAMPLE 5 Solve: $(4t + 2)^2 - 30(4t + 2) + 224 = 0$

Strategy Since the leading term, $(4t + 2)^2$, is the square of the expression $4t + 2$ in the middle term, we will substitute y for $4t + 2$.

WHY Our hope is that such a substitution will produce an equation that we can solve using one of the methods previously discussed.

Solution

If we make the substitution $y = 4t + 2$, the given equation becomes

$$y^2 - 30y + 224 = 0$$

which can be solved by using the quadratic formula.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(224)}}{2(1)}$$ Substitute 1 for a, -30 for b, and 224 for c.

$$= \frac{30 \pm \sqrt{900 - 896}}{2}$$ Simplify within the radical.

$$= \frac{30 \pm 2}{2}$$ $\sqrt{900 - 896} = \sqrt{4} = 2$.

$$y = 16 \quad \text{or} \quad y = 14$$

To find t, we replace y with $4t + 2$ and solve for t.

$$\begin{aligned} 4t + 2 &= 16 \\ 4t &= 14 \\ t &= 3.5 \end{aligned} \quad \text{or} \quad \begin{aligned} 4t + 2 &= 14 \\ 4t &= 12 \\ t &= 3 \end{aligned}$$

Verify that 3.5 and 3 satisfy the original equation.

3 Solve problems involving quadratic equations.

Self Check 6

MOWING LAWNS Carly can mow a lawn in 1 hour less time than her friend Lindsey. Together they can finish the job in 5 hours. How long would it take Carly if she worked alone? ~9.5 hr

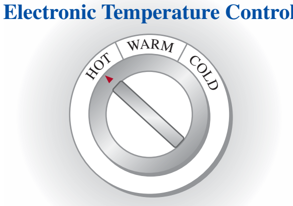
Now Try **Problem 38**

Teaching Example 6 WATER STORAGE TANKS Two pipes are used to fill a water storage tank. The first pipe can fill the tank in 4 hours, and the two pipes together can fill the tank in 2 hours less time than the second pipe alone. How long would it take for the two pipes together to fill the tank?
Answer:
2 hr

EXAMPLE 6 *Household Appliances*

A water temperature control on a washing machine is shown. When the warm setting is selected, both the hot and cold water inlets open to fill the tub in 2 minutes 15 seconds. When the cold temperature setting is chosen, the cold water inlet fills the tub 45 seconds faster than when the hot setting is used. How long does it take to fill the washing machine with hot water?

Electronic Temperature Control



Water Temp

Analyze The key to solving this problem is to determine how much of the tub is filled by each water temperature setting in 1 second. On the warm setting, when the hot and cold inlets are working together, the tub is filled in 2 minutes 15 seconds, or 135 seconds. So in 1 second, they fill $\frac{1}{135}$ of the tub.

Form Let $x =$ the number of seconds it takes to fill the tub when the hot temperature is chosen. In 1 second, the hot water inlet fills $\frac{1}{x}$ of the tub. Since the cold water inlet fills the tub in 45 seconds less time, the washing machine can be

filled with cold water in $(x - 45)$ seconds. In 1 second, $\frac{1}{x - 45}$ of the tub is filled by the cold water inlet. We can now form an equation.

What the hot water inlet pipe can do in 1 second	plus	what the cold water inlet pipe can do in 1 second	equals	what they can do together in 1 second.
$\frac{1}{x}$	$+$	$\frac{1}{x-45}$	$=$	$\frac{1}{135}$

Solve

$$\frac{1}{x} + \frac{1}{x - 45} = \frac{1}{135}$$

$$135x(x - 45)\left(\frac{1}{x} + \frac{1}{x - 45}\right) = 135x(x - 45)\left(\frac{1}{135}\right)$$ Multiply both sides by $135x(x - 45)$ to clear the equation of fractions.

$$135(x - 45) + 135x = x(x - 45)$$ Simplify.

$$135x - 6{,}075 + 135x = x^2 - 45x$$ Distribute the multiplication by 135 and by x.

$$270x - 6{,}075 = x^2 - 45x$$ Combine like terms.

$$0 = x^2 - 315x + 6{,}075$$ Subtract $270x$ from both sides. Add 6,075 to both sides.

To solve this equation, we can use the quadratic formula, with $a = 1$, $b = -315$, and $c = 6{,}075$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-315) \pm \sqrt{(-315)^2 - 4(1)(6{,}075)}}{2(1)}$$ Substitute 1 for a, -315 for b, and 6,075 for c.

$$= \frac{315 \pm \sqrt{99{,}225 - 24{,}300}}{2}$$ Simplify within the radical.

$$= \frac{315 \pm \sqrt{74{,}925}}{2}$$

$$x \approx \frac{589}{2} \quad \text{or} \quad x \approx \frac{41}{2}$$

$$x \approx 294 \qquad\quad x \approx 21$$

State We disregard the solution of 21 seconds, because this would imply that the cold water inlet fills the tub in a negative number of seconds $(21 - 45 = -24)$. Therefore, the hot water inlet fills the washing machine tub in about 294 seconds, which is 4 minutes 54 seconds.

Check Use estimation to check the result.

ANSWERS TO SELF CHECKS

1. a. real numbers that are irrational and unequal **b.** nonreal numbers that are complex conjugates **2.** $2, -2, i, -i$ **3.** 4 **4.** $-125, 8$ **5.** $-1, 1$ **6.** ~9.5 hr

SECTION 8.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. For the quadratic equation $ax^2 + bx + c = 0$, the discriminant is $b^2 - 4ac$.

2. When an equation is written in the form $ax^2 + bx + c = 0$, we say that it is written in quadratic form.

CONCEPTS

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c represent rational numbers, and fill in the blanks to make the statements true.

3. If $b^2 - 4ac < 0$, the solutions of the equation are nonreal complex conjugates.

4. If $b^2 - 4ac = 0$, the solutions of the equation are equal real numbers.

5. If $b^2 - 4ac$ is a perfect square, the solutions are rational numbers and unequal.

6. If $b^2 - 4ac$ is positive and not a perfect square, the solutions are irrational numbers and unequal.

Consider: $x^4 - 3x^2 + 2 = 0$

7. What is the relationship between the powers of x in the first two terms on the left-hand side? $x^4 = (x^2)^2$

8. Is this equation quadratic in form? yes

Consider: $x^{2/3} + 4x^{1/3} - 5 = 0$

9. What is the relationship between the powers of x in the first two terms on the left-hand side? $x^{2/3} = (x^{1/3})^2$

10. Is this equation quadratic in form? yes

NOTATION

Complete each solution.

11. To find the type of solutions for the equation $x^2 + 5x + 6 = 0$, we compute the discriminant.

$$\begin{aligned} b^2 - 4ac &= 5^2 - 4(1)(6) \\ &= 25 - 24 \\ &= 1 \end{aligned}$$

Since a, b, and c are rational numbers and the value of the discriminant is a perfect square, the solutions are rational numbers and unequal.

12. Change $\frac{3}{4} + x = \frac{3x - 50}{4(x - 6)}$ to quadratic form.

$$\begin{aligned} 4(x-6)\left(\frac{3}{4} + x\right) &= 4(x-6)\frac{3x - 50}{4(x-6)} \\ 3(x - 6) + 4x(x - 6) &= 3x - 50 \\ 3x - 18 + 4x^2 - 24x &= 3x - 50 \\ 4x^2 - 24x + 32 &= 0 \\ x^2 - 6x + 8 &= 0 \end{aligned}$$

GUIDED PRACTICE

Determine what type of solutions exist for each quadratic equation. Do not solve the equation. See Example 1.

13. $4x^2 - 4x + 1 = 0$ rational, equal

14. $6x^2 - 5x - 6 = 0$ rational, unequal

15. $5x^2 + x + 2 = 0$ complex conjugates

16. $3x^2 + 10x - 2 = 0$ irrational, unequal

17. $2x^2 = 4x - 1$ irrational, unequal

18. $9x^2 = 12x - 4$ rational, equal

19. $x(2x - 3) = 20$ rational, unequal

20. $x(x - 3) = -10$ complex conjugates

Solve each equation. See Example 2.

21. $x^4 - 17x^2 + 16 = 0$ $1, -1, 4, -4$

22. $x^4 - 10x^2 + 9 = 0$ $3, -3, 1, -1$

23. $x^4 + 5x^2 - 36 = 0$ $2, -2, 3i, -3i$

24. $x^4 - 15x^2 - 16 = 0$ $4, -4, i, -i$

Solve each equation. See Example 3.

25. $x - 13\sqrt{x} + 40 = 0$ 25, 64

26. $x - 9\sqrt{x} + 18 = 0$ 9, 36

27. $2x + \sqrt{x} - 3 = 0$ 1

28. $2x - \sqrt{x} - 1 = 0$ 1

Solve each equation. See Example 4.

29. $x^{2/3} + 5x^{1/3} + 6 = 0$ $-8, -27$

30. $x^{2/3} - 7x^{1/3} + 12 = 0$ 64, 27

31. $a^{2/3} - 2a^{1/3} - 3 = 0$ $-1, 27$

32. $r^{2/3} + 4r^{1/3} - 5 = 0$ $-125, 1$

Solve each equation. See Example 5.

33. $(c + 1)^2 - 4(c + 1) + 3 = 0$ 0, 2

34. $(b - 5)^2 - 4(b - 5) - 21 = 0$ 2, 12

35. $2(2x + 1)^2 - 7(2x + 1) + 6 = 0$ $\frac{1}{4}, \frac{1}{2}$

36. $3(2 - x)^2 + 10(2 - x) - 8 = 0$ $6, \frac{4}{3}$

Solve each equation. See Example 6.

37. $x + 5 + \dfrac{4}{x} = 0$

$-1, -4$

38. $x - 4 + \dfrac{3}{x} = 0$

$3, 1$

39. $\dfrac{1}{x + 2} + \dfrac{24}{x + 3} = 13$

$-1, -\frac{27}{13}$

40. $\dfrac{3}{x} + \dfrac{4}{x + 1} = 2$

$3, -\frac{1}{2}$

TRY IT YOURSELF

41. Use the discriminant to determine whether the solutions of $1{,}492x^2 + 1{,}776x - 2{,}000 = 0$ are real numbers. yes

42. Use the discriminant to determine whether the solutions of $1{,}776x^2 - 1{,}492x + 2{,}000 = 0$ are real numbers. no

Solve each equation, if possible.

43. $t^4 + 3t^2 = 28$

$2, -2, i\sqrt{7}, -i\sqrt{7}$

44. $3h^4 + h^2 - 2 = 0$

$\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}, i, -i$

45. $x^4 = 6x^2 - 5$

$1, -1, \sqrt{5}, -\sqrt{5}$

46. $2x^4 + 24 = 26x^2$

$1, -1, 2\sqrt{3}, -2\sqrt{3}$

47. $3x + 5\sqrt{x} + 2 = 0$

no solution

48. $3x - 4\sqrt{x} + 1 = 0$

$1, \frac{1}{9}$

49. $x - 6\sqrt{x} = -8$

$16, 4$

50. $x - 5x^{1/2} + 4 = 0$

$16, 1$

51. $x^{2/3} + 2x^{1/3} - 8 = 0$

$-64, 8$

52. $b + 8 = 6b^{1/2}$

$16, 4$

53. $(a + 1)^2 - 4(a + 1) - 8 = 0$

$1 - 2\sqrt{3}, 1 + 2\sqrt{3}$

54. $(k - 7)^2 + 6(k - 7) + 10 = 0$

$4 - i, 4 + i$

55. $1 - \dfrac{5}{x} = \dfrac{10}{x^2}$

$\frac{5 \pm \sqrt{65}}{2}$

56. $1 - \dfrac{3}{x} = \dfrac{5}{x^2}$

$\frac{3 \pm \sqrt{29}}{2}$

57. $\dfrac{2}{x - 1} + \dfrac{1}{x + 1} = 3$

$\frac{3 + \sqrt{57}}{6}, \frac{3 - \sqrt{57}}{6}$

58. $\dfrac{3}{x - 2} - \dfrac{1}{x + 2} = 5$

$\frac{1 + \sqrt{141}}{5}, \frac{1 - \sqrt{141}}{5}$

59. $x^{-4} - 2x^{-2} + 1 = 0$

$1, 1, -1, -1$

60. $4x^{-4} + 1 = 5x^{-2}$

$1, -1, 2, -2$

61. $x + \dfrac{2}{x - 2} = 0$

$1 \pm i$

62. $x + \dfrac{x + 5}{x - 3} = 0$

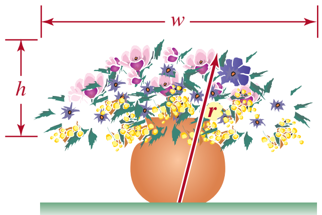
$1 \pm 2i$

APPLICATIONS

63. FLOWER ARRANGEMENTS A florist needs to determine the height h of the flowers shown in the next column. The radius r, the width w, and the height h of the circular-shaped arrangement are related by the formula

$$r = \frac{4h^2 + w^2}{8h}$$

If w is to be 34 inches and r is to be 18 inches, find h to the nearest tenth of an inch. 12.1 in.

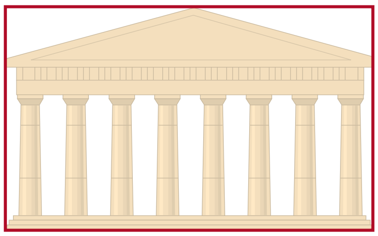


64. ARCHITECTURE A **golden rectangle** is one of the most visually appealing geometric forms. The Parthenon, built by the Greeks in the 5th century B.C. and shown in the illustration below, fits into a golden rectangle once its ruined triangular pediment is drawn in.

In a golden rectangle, the length l and width w must satisfy the equation

$$\frac{l}{w} = \frac{w}{l - w}$$

If a rectangular billboard is to have a width of 20 feet, what should its length be so that it is a golden rectangle? Round to the nearest tenth. 32.4 ft



65. SNOWMOBILING A woman drives a snowmobile 150 miles at a rate of r mph. She could have gone the same distance in 2 hours less time if she had increased her speed by 20 mph. Find r. 30 mph

66. BICYCLING Tina bicycles 160 miles at the rate of r mph. The same trip would have taken 2 hours longer if she had decreased her speed by 4 mph. Find r. 20 mph

67. CROWD CONTROL After a sold-out performance at a county fair, security guards have found that the grandstand area can be emptied in 6 minutes if both the east and west exits are opened. If just the east exit is used, it takes 4 minutes longer to clear the grandstand than it does if just the west exit is opened. How long does it take to clear the grandstand if everyone must file through the east exit? 14.3 min

68. PAPER ROUTES When a father, in a car, and his son, on a bicycle, work together to distribute the morning edition, it takes them 35 minutes to complete a paper route. Working alone, it takes the son 25 minutes longer than the father. To the nearest minute, how long does it take the son to cover the paper route on his bicycle? 85 min

WRITING

69. Describe how to predict what type of solutions the equation $3x^2 - 4x + 5 = 0$ will have.

70. Explain how the method of substitution is used in this section to solve equations.

REVIEW

Solve each equation.

71. $\frac{1}{4} + \frac{1}{t} = \frac{1}{2t}$ -2

72. $\frac{p-3}{3p} + \frac{1}{2p} = \frac{1}{4}$ 6

73. Find the slope of the line passing through $(-2, -4)$ and $(3, 5)$. $\frac{9}{5}$

74. Write the equation of the line passing through $(-2, -4)$ and $(3, 5)$ in general form. $9x - 5y = 2$

SECTION 8.5

Quadratic and Other Nonlinear Inequalities

Objectives

1. Solve quadratic inequalities.
2. Solve rational inequalities.
3. Graph nonlinear inequalities in two variables.

We have previously solved *linear* inequalities in one variable such as $2x + 3 > 8$ and $6x - 7 < 4x - 9$. To find their solution sets, we used properties of inequalities to isolate the variable on one side of the inequality.

In this section, we will solve *quadratic* inequalities in one variable such as $x^2 + x - 6 < 0$ and $x^2 + 4x \geq 5$. We will use an interval testing method on the number line to determine their solution sets.

1 Solve quadratic inequalities.

Recall that a quadratic equation can be written in the form $ax^2 + bx + c = 0$. If we replace the $=$ symbol with an inequality symbol, we have a quadratic inequality.

Quadratic Inequalities

A **quadratic inequality** can be written in one of the standard forms

$$ax^2 + bx + c < 0 \qquad ax^2 + bx + c > 0$$
$$ax^2 + bx + c \leq 0 \qquad ax^2 + bx + c \geq 0$$

where a, b, and c are real numbers and $a \neq 0$.

To solve a quadratic inequality in one variable, we will use the following steps to find the values of the variable that make the inequality true.

Solving Quadratic Inequalities

1. Write the inequality in standard form and solve its related quadratic equation.
2. Locate the solutions (called **critical numbers**) of the related quadratic equation on a number line.
3. Test each interval on the number line created in step 2 by choosing a test value from the interval and determining whether it satisfies the inequality. The solution set includes the interval(s) whose test value makes the inequality true.
4. Determine whether the endpoints of the intervals are included in the solution set.

EXAMPLE 1

Solve: $x^2 + x - 6 < 0$

Strategy We will solve the related quadratic equation $x^2 + x - 6 = 0$ by factoring to determine the critical numbers. These critical numbers will separate the number line into intervals.

WHY We can test each interval to see whether numbers in the interval are in the solution set of the inequality.

Solution

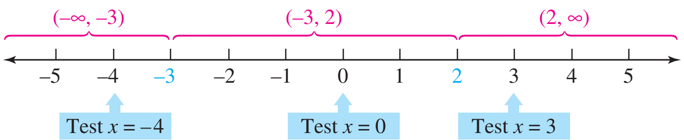
The expression $x^2 + x - 6$ can be positive, negative, or 0, depending on what value is substituted for x. Solutions of the inequality are x-values that make $x^2 + x - 6$ less than 0. To find them, we will follow the steps for solving quadratic inequalities.

Step 1 *Solve the related quadratic equation.* For the quadratic inequality $x^2 + x - 6 < 0$, the related quadratic equation is $x^2 + x - 6 = 0$.

$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0 \qquad \text{Factor the trinomial.}$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Set each factor equal to 0.}$$
$$x = -3 \quad \Big| \quad x = 2 \qquad \text{Solve each equation.}$$

The solutions of $x^2 + x - 6 = 0$ are -3 and 2. These solutions are the critical numbers.

Step 2 *Locate the critical numbers on a number line.* When we highlight -3 and 2 on a number line, they separate the number line into three intervals:



Step 3 *Test each interval.* To determine whether the numbers in $(-\infty, -3)$ are solutions of the inequality, we choose a number from that interval, substitute it for x, and see whether it satisfies $x^2 + x - 6 < 0$. *If one number in that interval satisfies the inequality, all numbers in that interval will satisfy the inequality.*

If we choose -4 from $(-\infty, -3)$, we have:

$$x^2 + x - 6 < 0 \qquad \text{This is the original inequality.}$$
$$(-4)^2 + (-4) - 6 \stackrel{?}{<} 0 \qquad \text{Substitute } -4 \text{ for } x.$$
$$16 + (-4) - 6 \stackrel{?}{<} 0$$
$$6 < 0 \qquad \text{False}$$

Since -4 does not satisfy the inequality, none of the numbers in $(-\infty, -3)$ are solutions.

To test the second interval, $(-3, 2)$, we choose $x = 0$.

$$x^2 + x - 6 < 0 \qquad \text{This is the original inequality.}$$
$$0^2 + 0 - 6 \stackrel{?}{<} 0 \qquad \text{Substitute 0 for } x.$$
$$-6 < 0 \qquad \text{True}$$

Since 0 satisfies the inequality, all of the numbers in $(-3, 2)$ are solutions.

To test the third interval, $(2, \infty)$, we choose $x = 3$.

$$x^2 + x - 6 < 0 \qquad \text{This is the original inequality.}$$
$$3^2 + 3 - 6 \stackrel{?}{<} 0 \qquad \text{Substitute 3 for } x.$$
$$9 + 3 - 6 \stackrel{?}{<} 0$$
$$6 < 0 \qquad \text{False}$$

Self Check 1

Solve: $x^2 + x - 12 < 0$

Now Try **Problem 15**

Self Check 1 Answer

$(-4, 3)$

Teaching Example 1 Solve: $x^2 - x - 12 < 0$

Answer:

$(-3, 4)$

Since 3 does not satisfy the inequality, none of the numbers in $(2, \infty)$ are solutions.

Step 4 *Are the endpoints included?* From the interval testing, we see that only numbers from $(-3, 2)$ satisfy $x^2 + x - 6 < 0$. The endpoints -3 and 2 are not included in the solution set because they do not satisfy the inequality. (Recall that -3 and 2 make $x^2 + x - 6$ equal to 0.) The solution set is the interval $(-3, 2)$ as graphed on the right.

Success Tip If a quadratic inequality contains $\le$ or $\ge$, the endpoints of the intervals are included in the solution set. If the inequality contains $<$ or $>$, they are not.

Self Check 2

Solve: $x^2 + 3x \ge 40$

Now Try **Problem 19**

Self Check 2 Answer

$(-\infty, -8] \cup [5, \infty)$

Teaching Example 2 Solve: $x^2 + 8x \ge -15$

Answer:

$(-8, -5] \cup [-3, \infty)$

EXAMPLE 2 Solve: $x^2 + 4x \ge 5$

Strategy This inequality is not in standard form because it does not have 0 on the right side. We will write it in standard form and solve its related quadratic equation to find any critical numbers. These critical numbers will separate the number line into intervals.

WHY We can then test each interval to see whether numbers in the interval are in the solution set of the inequality.

Solution

To get 0 on the right side, we subtract 5 from both sides.

$$x^2 + 4x \ge 5 \quad \text{This is the inequality to solve.}$$

$$x^2 + 4x - 5 \ge 0 \quad \text{Write the inequality in the equivalent form } ax^2 + bx + c \ge 0.$$

We can solve the related quadratic equation $x^2 + 4x - 5 = 0$ by factoring.

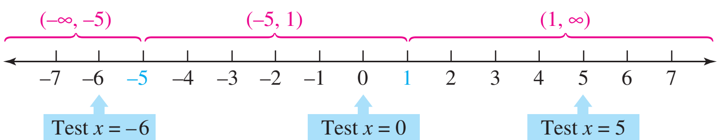
$$x^2 + 4x - 5 = 0$$

$$(x + 5)(x - 1) = 0 \quad \text{Factor the trinomial.}$$

$$x + 5 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = -5 \quad\Big|\quad x = 1$$

The critical numbers -5 and 1 separate the number line into three intervals. We pick a test value from each interval to see whether it satisfies $x^2 + 4x - 5 \ge 0$.

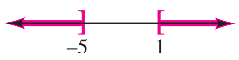


$$\begin{aligned} x^2 + 4x - 5 &\ge 0 \\ (-6)^2 + 4(-6) - 5 &\overset{?}{\ge} 0 \\ 7 &\ge 0 \quad \text{True} \end{aligned}$$

$$\begin{aligned} x^2 + 4x - 5 &\ge 0 \\ 0^2 + 4(0) - 5 &\overset{?}{\ge} 0 \\ -5 &\ge 0 \quad \text{False} \end{aligned}$$

$$\begin{aligned} x^2 + 4x - 5 &\ge 0 \\ 5^2 + 4(5) - 5 &\overset{?}{\ge} 0 \\ 40 &\ge 0 \quad \text{True} \end{aligned}$$

The numbers in the intervals $(-\infty, -5)$ and $(1, \infty)$ satisfy the inequality. Since the endpoints -5 and 1 also satisfy $x^2 + 4x - 5 \ge 0$, they are included in the solution set. (Recall that -5 and 1 make $x^2 + 4x - 5$ equal to 0.) Thus, the solution set is the union of two intervals: $(-\infty, -5] \cup [1, \infty)$. The graph of the solution set is shown on the right.



Success Tip When choosing a test value from an interval, pick a convenient number that makes the computations easy. When applicable, 0 is an obvious choice.

2 Solve rational inequalities.

Rational inequalities in one variable such as $\frac{9}{x} < 8$ and $\frac{x^2 + x - 2}{x - 4} \ge 0$ can also be solved using the interval testing method.

Solving Rational Inequalities

1. Write the inequality in standard form with a single quotient on the left side and 0 on the right side. Then solve its related rational equation.
2. Set the denominator equal to zero and solve that equation.
3. Locate the solutions (called *critical numbers*) found in steps 1 and 2 on a number line.
4. Test each interval on the number line created in step 3 by choosing a test value from the interval and determining whether it satisfies the inequality. The solution set includes the interval(s) whose test value makes the inequality true.
5. Determine whether the endpoints of the intervals are included in the solution set. Exclude any values that make the denominator 0.

EXAMPLE 3

Solve: $\dfrac{9}{x} < 8$

Strategy This rational inequality is not in standard form because it does not have 0 on the right side. We will write it in standard form and solve its related rational equation to find any critical numbers. These critical numbers will separate the number line into intervals.

WHY We can test each interval to see whether numbers in the interval are in the solution set of the inequality.

Solution

To get 0 on the right side, we subtract 8 from both sides. We then find a common denominator to write the left side as a single quotient.

$$\frac{9}{x} < 8 \quad \text{This is the inequality to solve.}$$

$$\frac{9}{x} - 8 < 0 \quad \text{Subtract 8 from both sides.}$$

$$\frac{9}{x} - 8 \cdot \frac{x}{x} < 0 \quad \text{To write the left side as a single quotient, build 8 to a fraction with denominator } x.$$

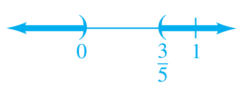
Self Check 3

Solve: $\dfrac{3}{x} < 5$

Now Try Problem 23

Self Check 3 Answer

$(-\infty, 0) \cup \left(\frac{3}{5}, \infty\right)$



Teaching Example 3 Solve: $\dfrac{12}{x} < 3$

Answer:
$(-\infty, 0) \cup (4, \infty)$

$$\frac{9}{x} - \frac{8x}{x} < 0$$

$$\frac{9 - 8x}{x} < 0 \quad \text{Subtract the numerators and keep the common denominator, } x.$$

Now we solve the related rational equation.

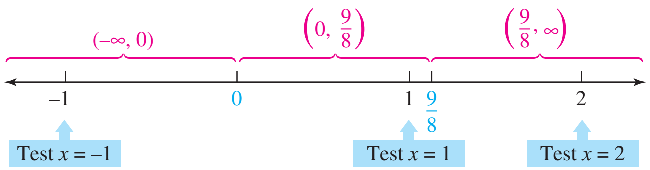
$$\frac{9 - 8x}{x} = 0$$

$$9 - 8x = 0 \quad \text{If } x \neq 0 \text{, we can clear the equation of the fraction by multiplying both sides by } x.$$

$$-8x = -9 \quad \text{Subtract 9 from both sides.}$$

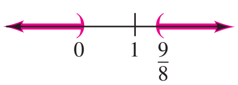
$$x = \frac{9}{8} \quad \text{This is a critical number.}$$

If we set the denominator of $\frac{9-8x}{x}$ equal to 0, we obtain a second critical number, $x = 0$. When graphed, the critical numbers 0 and $\frac{9}{8}$ separate the number line into three intervals. We pick a test value from each interval to see whether it satisfies $\frac{9-8x}{x} < 0$.



$\frac{9 - 8x}{x} < 0$	$\frac{9 - 8x}{x} < 0$	$\frac{9 - 8x}{x} < 0$
$\frac{9 - 8(-1)}{-1} \overset{?}{<} 0$	$\frac{9 - 8(1)}{1} \overset{?}{<} 0$	$\frac{9 - 8(2)}{2} \overset{?}{<} 0$
$-17 < 0$ True	$1 < 0$ False	$-\frac{7}{2} < 0$ True

The numbers in the intervals $(-\infty, 0)$ and $\left(\frac{9}{8}, \infty\right)$ satisfy the inequality. We do not include the endpoint 0 in the solution set, because it makes the denominator of the original inequality 0. Neither do we include $\frac{9}{8}$, because it does not satisfy $\frac{9-8x}{x} < 0$. (Recall that $\frac{9}{8}$ makes $\frac{9-8x}{x}$ equal to 0.) Thus, the solution set is the union of two intervals: $(-\infty, 0) \cup \left(\frac{9}{8}, \infty\right)$. Its graph is shown on the right.



Caution! When solving rational inequalities such as $\frac{9}{x} < 8$, a common error is to multiply both sides by x to clear it of the fraction. However, we don't know whether x is positive or negative, so we don't know whether or not to reverse the inequality symbol.

EXAMPLE 4

Solve: $\dfrac{x^2 + x - 2}{x - 4} \geq 0$

Strategy This inequality is in standard form. We will solve its related rational equation to find any critical numbers. These critical numbers will separate the number line into intervals.

WHY We can test each interval to see whether numbers in the interval are in the solution set of the inequality.

Solution

To solve the related rational equation, we proceed as follows:

$$\frac{x^2 + x - 2}{x - 4} = 0$$

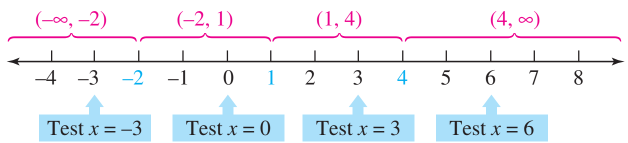
$x^2 + x - 2 = 0$ If $x \neq 4$, we can clear the equation of the fraction by multiplying both sides by $x - 4$.

$(x + 2)(x - 1) = 0$ Factor the trinomial.

$x + 2 = 0$ or $x - 1 = 0$ Set each factor equal to 0.

$x = -2$ | $x = 1$ These are critical numbers.

If we set the denominator of $\frac{x^2 + x - 2}{x - 4}$ equal to 0, we see that $x = 4$ is also a critical number. When graphed, the critical numbers, −2, 1, and 4, separate the number line into four intervals. We pick a test value from each interval to see whether it satisfies $\frac{x^2 + x - 2}{x - 4} \geq 0$.

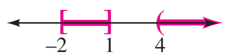


$\dfrac{(-3)^2 + (-3) - 2}{-3 - 4} \overset{?}{\geq} 0$ | $\dfrac{0^2 + 0 - 2}{0 - 4} \overset{?}{\geq} 0$ | $\dfrac{3^2 + 3 - 2}{3 - 4} \overset{?}{\geq} 0$ | $\dfrac{6^2 + 6 - 2}{6 - 4} \overset{?}{\geq} 0$

$-\dfrac{4}{7} \geq 0$ | $\dfrac{1}{2} \geq 0$ | $-10 \geq 0$ | $20 \geq 0$

False | True | False | True

The numbers in the intervals $(-2, 1)$ and $(4, \infty)$ satisfy the inequality. We include the endpoints −2 and 1 in the solution set because they satisfy the inequality. We do not include 4 because it makes the denominator of the inequality 0. Thus, the solution set is the union of two intervals $[-2, 1] \cup (4, \infty)$, as graphed on the right.



Self Check 4

Solve: $\dfrac{x + 2}{x^2 - 2x - 3} \geq 0$

Now Try Problem 27

Self Check 4 Answer

$[-2, -1) \cup (3, \infty)$

Teaching Example 4 Solve:
$\dfrac{x^2 - 2x - 8}{x + 3} \leq 0$

Answer:
$(-\infty, -3) \cup [-2, 4]$

EXAMPLE 5

Solve: $\dfrac{3}{x - 1} < \dfrac{2}{x}$

Strategy We will subtract $\frac{2}{x}$ from both sides to get 0 on the right side and solve the resulting related rational equation to find any critical numbers. These critical numbers will separate the number line into intervals.

WHY We can test each interval to see whether numbers in the interval are in the solution set of the inequality.

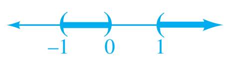
Self Check 5

Solve: $\dfrac{2}{x + 1} > \dfrac{1}{x}$

Now Try Problem 31

Self Check 5 Answer

$(-1, 0) \cup (1, \infty)$



Teaching Example 5 Solve: $\dfrac{4}{x-2} > \dfrac{5}{x}$

Answer: $(-\infty, 0) \cup (2, 10)$

Solution

$$\frac{3}{x-1} < \frac{2}{x} \qquad \text{This is the inequality to solve.}$$

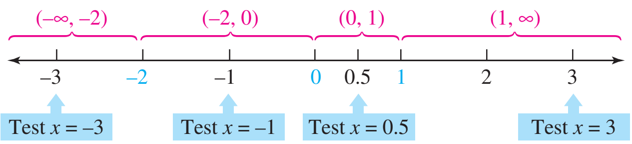
$$\frac{3}{x-1} - \frac{2}{x} < 0 \qquad \text{Subtract } \tfrac{2}{x} \text{ from both sides.}$$

$$\frac{3}{x-1} \cdot \frac{x}{x} - \frac{2}{x} \cdot \frac{x-1}{x-1} < 0 \qquad \text{To get a single quotient on the left side, build each rational expression to have the common denominator } x(x-1).$$

$$\frac{3x - 2x + 2}{x(x-1)} < 0 \qquad \text{Subtract the numerators and keep the common denominator.}$$

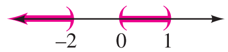
$$\frac{x+2}{x(x-1)} < 0 \qquad \text{Combine like terms.}$$

The only solution of the related rational equation $\frac{x+2}{x(x-1)} = 0$ is -2. Thus, -2 is a critical number. When we set the denominator equal to 0 and solve $x(x-1) = 0$, we find two more critical numbers, 0 and 1. These three critical numbers create four intervals to test.



$\frac{\mathbf{-3}+2}{\mathbf{-3}(\mathbf{-3}-1)} \overset{?}{<} 0$	$\frac{\mathbf{-1}+2}{\mathbf{-1}(\mathbf{-1}-1)} \overset{?}{<} 0$	$\frac{\mathbf{0.5}+2}{\mathbf{0.5}(\mathbf{0.5}-1)} \overset{?}{<} 0$	$\frac{\mathbf{3}+2}{\mathbf{3}(\mathbf{3}-1)} \overset{?}{<} 0$
$\frac{-1}{-3(-4)} \overset{?}{<} 0$	$\frac{1}{-1(-2)} \overset{?}{<} 0$	$\frac{2.5}{0.5(-0.5)} \overset{?}{<} 0$	$\frac{5}{3(2)} \overset{?}{<} 0$
$-\frac{1}{12} < 0$	$\frac{1}{2} < 0$	$-10 < 0$	$\frac{5}{6} < 0$
True	False	True	False

The numbers 0 and 1 are not included in the solution set because they make the denominator 0, and the number -2 is not included because it does not satisfy the inequality. The solution set is the union of two intervals $(-\infty, -2) \cup (0, 1)$, as graphed on the right.



Success Tip When the endpoints of an interval are consecutive integers, such as with the third interval (0, 1), we cannot choose an integer as a test value. For these cases, choose a fraction or decimal that lies within the interval.

Using Your CALCULATOR Solving Inequalities Graphically

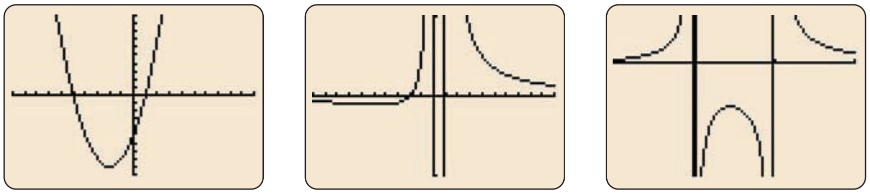
We can solve $x^2 + 4x \geq 5$ (Example 2) graphically by writing the inequality as $x^2 + 4x - 5 \geq 0$ and graphing the quadratic function $f(x) = x^2 + 4x - 5$, as shown in figure (a) on the next page. The solution set of the inequality will be those values of x for which the graph lies on or above the x-axis. We can trace to determine that this is the union of two intervals: $(-\infty, -5] \cup [1, \infty)$.

To solve $\frac{3}{x-1} < \frac{2}{x}$ (Example 5) graphically, we first write the inequality in the form $\frac{x+2}{x(x-1)} < 0$ and then graph the rational function $f(x) = \frac{x+2}{x(x-1)}$, as

shown in figure (b) below. The solution of the inequality will be those values of x for which the graph lies below the axis.

We can trace to see that the graph is below the x-axis when x is less than -2. Since we cannot see the graph in the interval $0 < x < 1$, we redraw the graph using window settings of $[-1, 2]$ for x and $[-25, 10]$ for y, as shown in figure (c).

Now we see that the graph is below the x-axis in the interval $(0, 1)$. Thus, the solution set of the inequality is the union of the two intervals: $(-\infty, -2) \cup (0, 1)$.



(a) (b) (c)

3 Graph nonlinear inequalities in two variables.

We have previously graphed linear inequalities in two variables such as $y > 3x + 2$ and $2x - 3y \leq 6$ using the following steps.

Graphing Inequalities in Two Variables

1. Graph the related equation to find the boundary line of the region. If the inequality allows equality (the symbol is either $\leq$ or $\geq$), draw the boundary as a solid line. If equality is not allowed ($<$ or $>$), draw the boundary as a dashed line.
2. Pick a test point that is on one side of the boundary line. (Use the origin if possible.) Replace x and y in the original inequality with the coordinates of that point. If the inequality is satisfied, shade the side that contains that point. If the inequality is not satisfied, shade the other side of the boundary.

We use the same procedure to graph *nonlinear* inequalities in two variables.

EXAMPLE 6 Graph: $y < -x^2 + 4$

Strategy We will graph the related equation $y = -x^2 + 4$ to establish a boundary parabola. Then we will determine which side of the boundary parabola represents the solution set of the inequality.

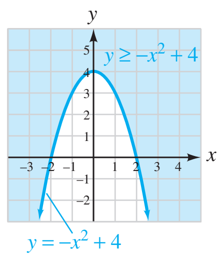
WHY To *graph a nonlinear inequality* in two variables means to draw a "picture" of the ordered pairs (x, y) that make the inequality true.

Solution

The graph of the boundary $y = -x^2 + 4$ is a parabola opening downward, with vertex at $(0, 4)$ and axis of symmetry $x = 0$ (the y-axis). Since the inequality contains an $<$ symbol and equality is not allowed, we draw the parabola using a dashed curve.

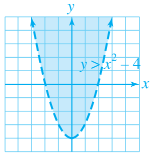
Self Check 6

Graph: $y \geq -x^2 + 4$



Now Try **Problem 35**

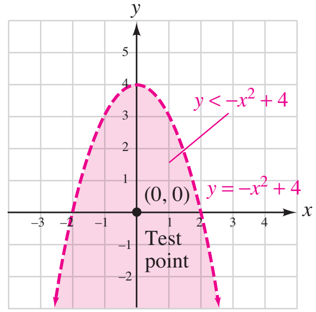
Teaching Example 6 Graph: $y > x^2 - 4$
Answer:



To determine which region to shade, we pick the test point (0, 0) and substitute its coordinates into the inequality. We shade the region containing (0, 0) because its coordinates satisfy $y < -x^2 + 4$.

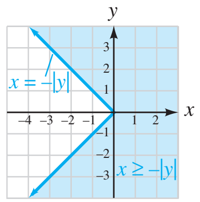
Graph the boundary	*Shading: Use the test point* **(0, 0)**
$y = -x^2 + 4$	$y < -x^2 + 4$
Compare to $y = a(x - h)^2 + k$	$0 \overset{?}{<} -0^2 + 4$
$a = -1$: Opens downward	$0 < 4$ True
$h = 0$ and $k = 4$: Vertex (0, 4)	Since $0 < 4$ is true, (0, 0) is a solution of $y < -x^2 + 4$.
Axis of symmetry $x = 0$	

x	y
1	3
2	0
−1	3
−2	0



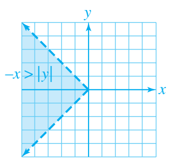
Self Check 7

Graph: $x \geq -|y|$



Now Try **Problem 39**

Teaching Example 7 Graph: $-x > |y|$
Answer:



EXAMPLE 7

Graph: $x \leq |y|$

Strategy We will graph the related equation $x = |y|$ to establish a boundary. Then we will determine which side of the boundary represents the solution set of the inequality.

WHY To *graph a nonlinear inequality* in two variables means to draw a "picture" of the ordered pairs (x, y) that make the inequality true.

Solution

To graph the boundary, $x = |y|$, we construct a table of solutions, as shown in figure (a). In figure (b), the boundary is graphed using a solid line because the inequality contains a $\leq$ symbol and equality is permitted. Since the origin is on the graph, we cannot use it as a test point. However, any other point, such as (1, 0), will do. We substitute 1 for x and 0 for y into the inequality to get

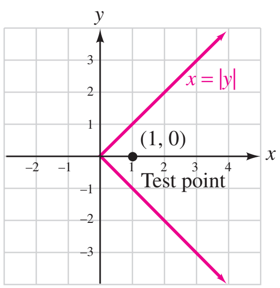
$$x \leq |y|$$
$$1 \overset{?}{\leq} |0|$$
$$1 \leq 0 \quad \text{False}$$

Since $1 \leq 0$ is a false statement, the point (1, 0) does not satisfy the inequality and is not part of the graph. Thus, the graph of $x \leq |y|$ is to the left of the boundary.

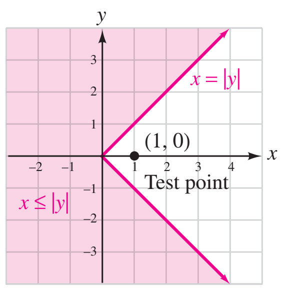
The complete graph is shown in figure (c).

$x = \lvert y \rvert$	
x	y
0	0
1	1
1	−1
2	2
2	−2

(a)



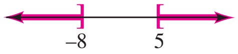
(b)

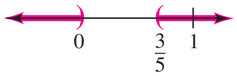


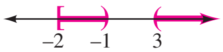
(c)

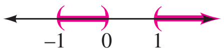
ANSWERS TO SELF CHECKS

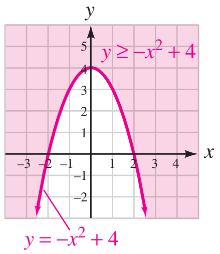
1. $(-4, 3)$ 

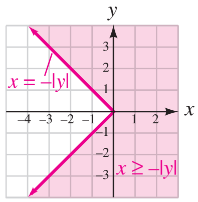
2. $(-\infty, -8] \cup [5, \infty)$ 

3. $(-\infty, 0) \cup \left(\frac{3}{5}, \infty\right)$ 

4. $[-2, -1) \cup (3, \infty)$ 

5. $(-1, 0) \cup (1, \infty)$ 

6. 

7. 

SECTION 8.5 STUDY SET

VOCABULARY

Fill in the blanks.

1. $x^2 + 3x - 18 < 0$ is an example of a quadratic inequality in one variable.

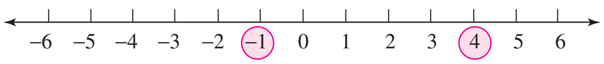
2. $\frac{x-1}{x^2 - x - 20} \le 0$ is an example of a rational inequality in one variable.

3. $y \le x^2 - 4x + 3$ is an example of a nonlinear inequality in two variables.

4. The set of real numbers greater than 3 can be represented using the interval notation $(3, \infty)$.

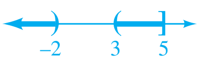
CONCEPTS

5. The critical numbers of a quadratic inequality are highlighted in red on the number line shown below. Use interval notation to represent each interval that must be tested to solve the inequality. $(-\infty, -1), (-1, 4), (4, \infty)$

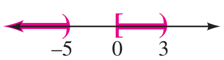


6. Graph each of the following solution sets.

a. $(-2, 4)$ 

b. $(-\infty, -2) \cup (3, 5]$ 

7. The graph of the solution set of a rational inequality in one variable is shown. Determine whether each of the following numbers is a solution of the inequality.



a. -10 yes **b.** -5 no

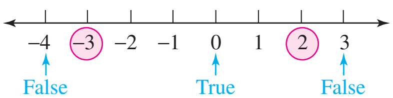
c. 0 yes **d.** 4 no

8. What are the critical numbers for each inequality?

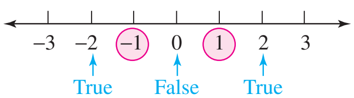
a. $x^2 - 2x - 48 \ge 0$ $-6, 8$

b. $\frac{x-3}{x(x+4)} > 0$ $-4, 0, 3$

9. a. The results after interval testing for a quadratic inequality containing a $>$ symbol are shown below. (The critical numbers are highlighted in red.) What is the solution set? $(-3, 2)$



b. The results after interval testing for a quadratic inequality containing a $\le$ symbol are shown below. (The critical numbers are highlighted in red.) What is the solution set? $(-\infty, -1] \cup [1, \infty)$



10. Fill in the blank to complete this important fact about the interval testing method discussed in this section: *If one number in an interval satisfies the inequality,* all *numbers in that interval will satisfy the inequality.*

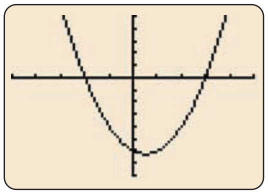
11. a. When graphing the solution of $y \le x^2 + 2x + 1$, should the boundary be solid or dashed? solid

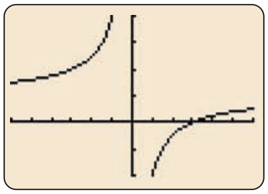
b. Does the test point (0, 0) satisfy the inequality? yes

Selected exercises available online at
www.webassign.net/brookscole

12. **a.** Estimate the solution of $x^2 - x - 6 > 0$ using the graph of $y = x^2 - x - 6$ shown in figure (a) below. $(-\infty, -2) \cup (3, \infty)$

b. Estimate the solution of $\frac{x-3}{x} \le 0$ using the graph of $y = \frac{x-3}{x}$ shown in figure (b) below. $(0, 3]$


(a)


(b)

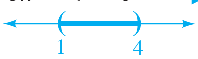
NOTATION

13. Write the quadratic inequality $x^2 - 6x \ge 7$ in standard form. $x^2 - 6x - 7 \ge 0$

▶ **14.** The solution set of a rational inequality consists of the intervals $(-1, 4]$ and $(7, \infty)$. When writing the solution set, what symbol is used between the two intervals? $\cup$

GUIDED PRACTICE

Solve each inequality. Write the solution set in interval notation and graph it. See Example 1.

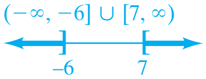
15. $x^2 - 5x + 4 < 0$
$(1, 4)$


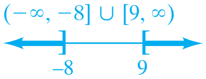
▶ **16.** $x^2 + 2x - 8 < 0$
$(-4, 2)$

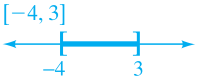

17. $x^2 - 8x + 15 > 0$
$(-\infty, 3) \cup (5, \infty)$

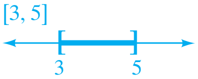

▶ **18.** $x^2 - 3x - 4 > 0$
$(-\infty, -1) \cup (4, \infty)$


Solve each inequality. Write the solution set in interval notation and graph it. See Example 2.

19. $x^2 - x \ge 42$
$(-\infty, -6] \cup [7, \infty)$


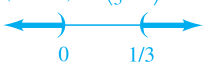
20. $x^2 - x \ge 72$
$(-\infty, -8] \cup [9, \infty)$


21. $x^2 + x \le 12$
$[-4, 3]$


▶ **22.** $x^2 - 8x \le -15$
$[3, 5]$


Solve each inequality. Write the solution set in interval notation and graph it. See Example 3.

23. $\frac{1}{x} < 2$
$(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$


▶ **24.** $\frac{1}{x} < 3$
$(-\infty, 0) \cup \left(\frac{1}{3}, \infty\right)$


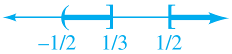
25. $\frac{5}{x} \ge -3$
$\left(-\infty, -\frac{5}{3}\right] \cup (0, \infty)$

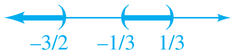

▶ **26.** $\frac{4}{x} \ge 8$
$\left(0, \frac{1}{2}\right]$


Solve each inequality. Write the solution set in interval notation and graph it. See Example 4.

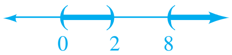
27. $\frac{x^2 - x - 12}{x - 1} < 0$
$(-\infty, -3) \cup (1, 4)$

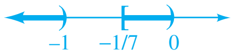

▶ **28.** $\frac{x^2 + x - 6}{x - 4} \ge 0$
$[-3, 2] \cup (4, \infty)$

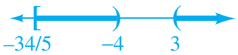

29. $\frac{6x^2 - 5x + 1}{2x + 1} \ge 0$
$\left(-\frac{1}{2}, \frac{1}{3}\right] \cup \left[\frac{1}{2}, \infty\right)$


▶ **30.** $\frac{6x^2 + 11x + 3}{3x - 1} < 0$
$\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right)$


Solve each inequality. Write the solution set in interval notation and graph it. See Example 5.

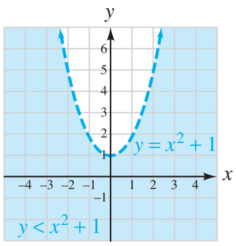
31. $\frac{3}{x - 2} < \frac{4}{x}$
$(0, 2) \cup (8, \infty)$


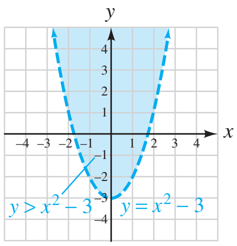
▶ **32.** $\frac{-6}{x + 1} \ge \frac{1}{x}$
$(-\infty, -1) \cup \left[-\frac{1}{7}, 0\right)$


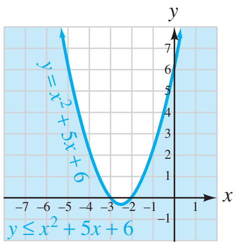
33. $\frac{7}{x - 3} \ge \frac{2}{x + 4}$
$\left[-\frac{34}{5}, -4\right) \cup (3, \infty)$


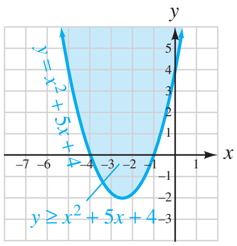
▶ **34.** $\frac{-5}{x - 4} < \frac{3}{x + 1}$
$\left(-1, \frac{7}{8}\right) \cup (4, \infty)$


Graph each inequality. See Example 6.

35. $y < x^2 + 1$


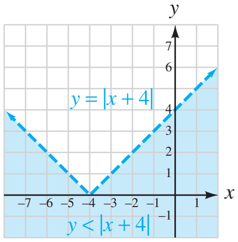
36. $y > x^2 - 3$


37. $y \le x^2 + 5x + 6$


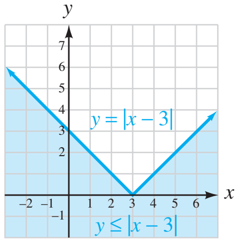
▶ **38.** $y \ge x^2 + 5x + 4$


Graph each inequality. See Example 7.

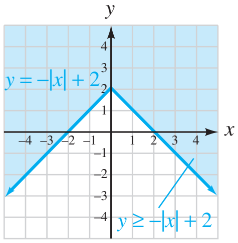
39. $y < |x + 4|$



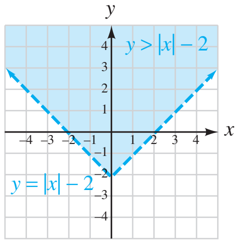
40. $y \le |x - 3|$



41. $y \ge -|x| + 2$



▶ **42.** $y > |x| - 2$



Use a graphing calculator to solve each inequality. Write the solution set in interval notation. See Using Your Calculator: Solving Inequalities Graphically.

43. $x^2 - 2x - 3 < 0$
$(-1, 3)$

44. $x^2 + x - 6 > 0$
$(-\infty, -3) \cup (2, \infty)$

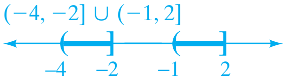
45. $\dfrac{x + 3}{x - 2} > 0$
$(-\infty, -3) \cup (2, \infty)$

46. $\dfrac{3}{x} < 2$
$(-\infty, 0) \cup \left(\frac{3}{2}, \infty\right)$

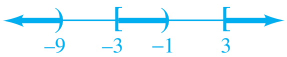
TRY IT YOURSELF

Solve each inequality. Write the solution set in interval notation and graph it.

47. $\dfrac{x}{x + 4} \le \dfrac{1}{x + 1}$
$(-4, -2] \cup (-1, 2]$



48. $\dfrac{x}{x + 9} \ge \dfrac{1}{x + 1}$
$(-\infty, -9) \cup [-3, -1) \cup [3, \infty)$



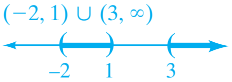
49. $x^2 \ge 9$
$(-\infty, -3] \cup [3, \infty)$

▶ **50.** $x^2 \ge 16$
$(-\infty, -4] \cup [4, \infty)$

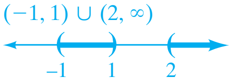
▶ **51.** $x^2 + 6x \ge -9$
$(-\infty, \infty)$

52. $x^2 + 8x < -16$
no solutions

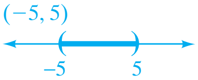
53. $\dfrac{x^2 + x - 2}{x - 3} > 0$
$(-2, 1) \cup (3, \infty)$



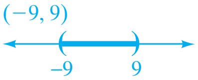
54. $\dfrac{x - 2}{x^2 - 1} > 0$
$(-1, 1) \cup (2, \infty)$



55. $2x^2 - 50 < 0$
$(-5, 5)$



▶ **56.** $3x^2 - 243 < 0$
$(-9, 9)$



57. $\dfrac{2x - 3}{3x + 1} < 0$
$\left(-\frac{1}{3}, \frac{3}{2}\right)$

58. $\dfrac{x - 5}{x + 1} < 0$
$(-1, 5)$

59. $x^2 - 6x + 9 < 0$
no solutions

60. $x^2 + 4x + 4 > 0$
$(-\infty, -2) \cup (-2, \infty)$

61. $\dfrac{5}{x + 1} > \dfrac{3}{x - 4}$
$(-1, -4) \cup \left(\frac{23}{2}, \infty\right)$

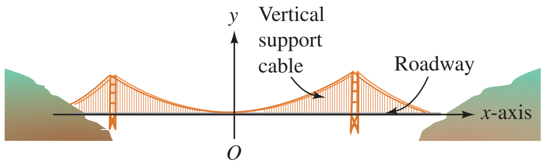
62. $\dfrac{3}{x - 2} \le -\dfrac{2}{x + 3}$
$(-\infty, -3) \cup [-1, 2)$

APPLICATIONS

▶ **63.** BRIDGES If an x-axis is superimposed over the roadway of the Golden Gate Bridge, with the origin at the center of the bridge, the length L in feet of a vertical support cable can be approximated by the formula

$$L = \frac{1}{9{,}000}x^2 + 5$$

For the Golden Gate Bridge, $-2{,}100 < x < 2{,}100$. For what intervals along the x-axis are the vertical cables more than 95 feet long? $(-2{,}100, -900) \cup (900, 2{,}100)$



▶ **64.** MALLS The number of people n in a mall is modeled by the formula

$$n = -100x^2 + 1{,}200x$$

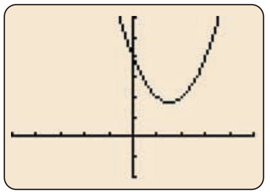
where x is the number of hours since the mall opened. If the mall opened at 9 A.M., when were there 2,000 or more people in it?
between 11 A.M. and 7 P.M., inclusive

WRITING

65. How are critical numbers used when solving a quadratic inequality in one variable?

▶ **66.** Explain how to graph $y \ge x^2$.

67. The graph of $f(x) = x^2 - 3x + 4$ is shown. Explain why the quadratic inequality $x^2 - 3x + 4 < 0$ has no solution.



68. Describe the following solution set of a rational inequality in words: $(-\infty, 4] \cup (6, 7)$.

REVIEW

Translate each statement into an equation.

69. x varies directly with y. $x = ky$

70. y varies inversely with t. $y = \frac{k}{t}$

71. t varies jointly with x and y. $t = kxy$

72. d varies directly with t and inversely with u^2. $d = \frac{kt}{u^2}$

STUDY SKILLS CHECKLIST

Preparing for the Chapter 8 Test

The material in Chapter 8 focused on quadratic equations, functions, and inequalities. Be sure to review the following checklist in addition to your other studying as you prepare for the exam over this material.

☐ When solving a quadratic equation using the factoring method, one side of the equation must be 0.

Solve:

$x^2 = 9x$	This is the equation to solve.
$x^2 - 9x = 0$	Subtract 9x from both sides to get 0 on the right side.
$x(x - 9) = 0$	Factor the right side.
$x = 0$ or $x - 9 = 0$	Set each factor equal to 0.
$x = 9$	Solve each linear equation.

☐ When using the square root property to solve a quadratic equation, always write the $\pm$ sign.

Solve:

$x^2 - 7 = 0$	This is the equation to solve.
$x^2 = 7$	Subtract 7 from both sides to isolate the x^2 term.
$\sqrt{x^2} = \pm\sqrt{7}$	Use the square root property, and remember the $\pm$.
$x = \pm\sqrt{7}$	Simplify the left side of the equation.

☐ When solving a quadratic equation using the completing the square method, make sure the coefficient of the x^2 term is 1 before you complete the square. If it is not 1, you must divide both sides of the equation by the coefficient of the x^2 to make it 1.

Solve by completing the square:

$2x^2 - 12x = 6$	This is the equation to solve.
$x^2 - 6x = 3$	Divide both sides by 2 to make the coefficient of the x^2 term 1.
$x^2 - 6x + 9 = 3 + 9$	Complete the square on the right side by $\frac{1}{2}(-6) = -3$, $(-3)^2 = 9$, add 9 to both sides.
$(x - 3)^2 = 12$	Factor the left side and simplify the right side of the equation.
$\sqrt{(x - 3)^2} = \pm\sqrt{12}$	Use the square root property, and remember the $\pm$.
$x - 3 = \pm 2\sqrt{3}$	Simplify the left and right sides of the equation.
$x = 3 \pm 2\sqrt{3}$	Isolate x by adding 3 to both sides of the equation.

(continued)

- ☐ We can find the vertex of the graph of a quadratic function by completing the square or by using the formula $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
- ☐ When solving a quadratic or other nonlinear inequality, write the inequality with all the terms on one side and 0 on the other side. Next, solve the related equation to get the critical numbers on the graph. Test each interval on the number line by choosing a test value in each interval and testing it in the original inequality. Lastly, determine whether the endpoints are included in the solution.

  Solve:

$$\frac{5}{x+6} \le \frac{2}{x} \quad \text{This is the inequality to solve.}$$

$$\frac{5}{x+6} - \frac{2}{x} \le 0 \quad \text{Subtract } \frac{2}{x} \text{ from both sides to get 0 on the right side.}$$

$$\frac{5}{x+6} \cdot \frac{\mathbf{x}}{\mathbf{x}} - \frac{2}{x} \cdot \frac{\mathbf{(x+6)}}{\mathbf{(x+6)}} \le 0 \quad \text{Build each expression so that it has a denominator of } x(x+6).$$

$$\frac{5x}{x(x+6)} - \frac{2x+12}{x(x+6)} \le 0 \quad \text{Multiply the numerators and the denominators.}$$

$$\frac{5x - 2x - 12}{x(x+6)} \le 0 \quad \text{Subtract the numerators, remember to use the distributive property } -(2x+12) = -1(2x+12) = -2x-12.$$

$$\frac{3x-12}{x(x+6)} \le 0 \quad \text{Simplify the left side of the inequality.}$$

  The only solution of the related rational equation is 4. The values that make the denominator 0 are 0 and -6. When graphed, these critical numbers separate the number line into four intervals. Pick a test value from each interval to see whether it satisfies the inequality. For this example, the solution is $(-\infty, -6) \cup (0, 4]$.

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

SECTION 8.1 The Square Root Property and Completing the Square

DEFINITIONS AND CONCEPTS	EXAMPLES
We can use the **square root property** to solve equations of the form $x^2 = c$, where $c > 0$. The two solutions are $x = \sqrt{c}$ or $x = -\sqrt{c}$ We can write $x = \sqrt{c}$ or $x = -\sqrt{c}$ in more compact form using **double-sign notation:** $x = \pm\sqrt{c}$	Solve: $x^2 = 24$ $x = \sqrt{24}$ or $x = -\sqrt{24}$ Use the square root property. $x = \pm\sqrt{24}$ Use double-sign notation. $x = \pm 2\sqrt{6}$ Simplify $\sqrt{24}$. The solutions are $2\sqrt{6}$ and $-2\sqrt{6}$. Solve: $(x-3)^2 = -81$ $x - 3 = \pm\sqrt{-81}$ Use the square root property and double-sign notation. $x = 3 \pm \sqrt{-81}$ To isolate x, add 3 to both sides. $x = 3 \pm 9i$ Simplify the radical expression. The solutions are $3 + 9i$ and $3 - 9i$.
To **complete the square** on $x^2 + bx$, add the square of one-half of the coefficient of x. $x^2 + bx + \left(\frac{1}{2}b\right)^2$	Complete the square on $x^2 + 8x$ and factor the resulting perfect-square trinomial. $x^2 + 8x + 16$ The coefficient of x is 8. To complete the square: $\frac{1}{2} \cdot 8 = 4$ and $4^2 = 16$. Add 16 to the binomial. Now we factor: $x^2 + 8x + 16 = (x+4)^2$
To **solve a quadratic equation in x by completing the square:** **1.** If necessary, divide both sides of the equation by the coefficient of x^2 to make its coefficient 1. **2.** Get all variable terms on one side of the equation and all constants on the other side. **3.** Complete the square. **4.** Factor the perfect-square trinomial. **5.** Solve the resulting equation by using the square root property. **6.** Check your answers in the original equation.	Solve: $3x^2 - 12x + 6 = 0$ $\frac{3x^2}{3} - \frac{12x}{3} + \frac{6}{3} = \frac{0}{3}$ To make the leading coefficient 1, divide both sides by 3. $x^2 - 4x + 2 = 0$ Do the divisions. $x^2 - 4x = -2$ Subtract 2 from both sides so that the constant term, -2, is on the right side. $x^2 - 4x + 4 = -2 + 4$ The coefficient of x is -4. To complete the square: $\frac{1}{2}(-4) = -2$ and $(-2)^2 = 4$. Add 4 to both sides. $(x-2)^2 = 2$ Factor the perfect-square trinomial on the left side. Add on the right side. $x - 2 = \pm\sqrt{2}$ Use the square root property. $x = 2 \pm \sqrt{2}$ To isolate x, add 2 to both sides. The solutions are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

REVIEW EXERCISES

Solve each equation by factoring.

1. $x^2 + 9x + 20 = 0$
$-5, -4$

2. $6x^2 + 17x + 5 = 0$
$-\frac{1}{3}, -\frac{5}{2}$

Solve each equation using the square root property.

3. $x^2 = 28$
$\pm 2\sqrt{7}$

4. $(t + 2)^2 = 36$
$4, -8$

5. $a^2 + 25 = 0$
$\pm 5i$

6. $5x^2 - 49 = 0$
$\pm \frac{7\sqrt{5}}{5}$

7. Solve $A = \pi r^2$ for r. Assume all variables represent positive numbers. Express the result in simplified radical form.
$r = \frac{\sqrt{\pi A}}{\pi}$

8. Complete the square on $x^2 - x$ and then factor the resulting perfect-square trinomial.
$x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

Solve each equation by completing the square.

9. $x^2 + 6x + 8 = 0$
$-4, -2$

10. $2x^2 - 6x + 3 = 0$
$\frac{3 \pm \sqrt{3}}{2}$

11. $6a^2 - 12a = -1$
$\frac{6 \pm \sqrt{30}}{6}$

12. $x^2 - 2x = -13$
$1 \pm 2i\sqrt{3}$

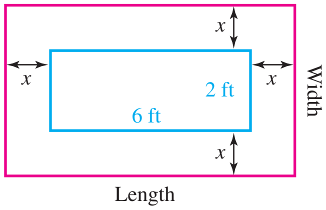
13. Explain why completing the square on $x^2 + 7x$ is more difficult than completing the square on $x^2 + 6x$.
Because 7 is an odd number and not divisible by 2, the computations involved in completing the square on $x^2 + 7x$ create fractions. The computations involved in completing the square on $x^2 + 6x$ do not.

14. Explain the error: $\frac{2 \pm \sqrt{7}}{2} = \frac{\overset{1}{\cancel{2}} \pm \sqrt{7}}{\underset{1}{\cancel{2}}}$

2 is not a factor of the numerator—it is a term. Only common factors of the numerator and denominator can be removed.

15. a. Write an expression that represents the width of the larger rectangle shown in red. $(2 + 2x)$ ft

b. Write an expression that represents the length of the larger rectangle shown in red. $(6 + 2x)$ ft



16. HAPPY NEW YEAR As part of a New Year's Eve celebration, a huge ball is to be dropped from the top of a 605-foot-tall building at the proper moment so that it strikes the ground at exactly 12:00 midnight. The distance d in feet traveled by a free-falling object in t seconds is given by the formula $d = 16t^2$. To the nearest second, when should the ball be dropped from the building?
6 seconds before midnight

SECTION 8.2 The Quadratic Formula

DEFINITIONS AND CONCEPTS	EXAMPLES
To **solve a quadratic equation in x using the quadratic formula:**	Solve: $3x^2 - 2x - 2 = 0$
1. Write the equation in general quadratic form: $ax^2 + bx + c = 0$.	Here, $a = 3$, $b = -2$, and $c = -2$.
2. Identify a, b, and c.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ This is the quadratic formula.
3. Substitute the values for a, b, and c in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and evaluate the right side to obtain the solutions.	$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)}$ Substitute 3 for a, -2 for b, and -2 for c.
	$x = \frac{2 \pm \sqrt{4 - (-24)}}{6}$ Evaluate the power and multiply within the radical. Multiply in the denominator.
	$x = \frac{2 \pm \sqrt{28}}{6}$ Add the opposite: $4 - (-24) = 4 + 24 = 28$.
	$x = \frac{2 \pm 2\sqrt{7}}{6}$ Simplify the radical: $\sqrt{28} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$.

$$x = \frac{\overset{1}{\cancel{2}}(1 \pm \sqrt{7})}{\underset{1}{\cancel{2}} \cdot 3}$$

Factor out the GCF, 2, from the two terms in the numerator. In the denominator, factor 6 as $2 \cdot 3$. Remove the common factor, 2.

$$x = \frac{1 \pm \sqrt{7}}{3}$$

The exact solutions are $\frac{1+\sqrt{7}}{3}$ and $\frac{1-\sqrt{7}}{3}$. We can use a calculator to approximate them. To two decimal places, they are 1.22 and -0.55.

When solving a quadratic equation using the quadratic formula, we can often **simplify the computations** by solving an equivalent equation that does not involve fractions or decimals, and whose leading coefficient is positive.

Before solving . . .	**do this . . .**	**to get this**
$-3x^2 + 5x - 1 = 0$	Multiply both sides by -1	$3x^2 - 5x + 1 = 0$
$x^2 + \frac{7}{8}x - \frac{1}{2} = 0$	Multiply both sides by 8	$8x^2 + 7x - 4 = 0$
$60x^2 - 40x + 90 = 0$	Divide both sides by 10	$6x^2 - 4x + 9 = 0$
$0.5x^2 + 1.6x + 7.1 = 0$	Multiply both sides by 10	$5x^2 + 16x + 71 = 0$

REVIEW EXERCISES

Solve each equation using the quadratic formula.

17. $2x^2 + 13x = 7$
$\frac{1}{2}, -7$

18. $-x^2 + 10x - 18 = 0$
$5 \pm \sqrt{7}$

19. $x^2 - 10x = 0$
0, 10

20. $3y^2 = 26y - 2$
$\frac{13 \pm \sqrt{163}}{3}$

21. $\frac{1}{3}p^2 + \frac{1}{2}p + \frac{1}{2} = 0$
$-\frac{3}{4} \pm \frac{\sqrt{15}}{4}i$

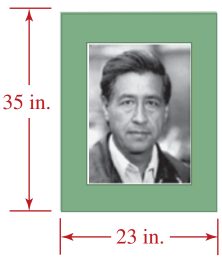
22. $3{,}000t^2 - 4{,}000t = -2{,}000$
$\frac{2}{3} \pm \frac{\sqrt{2}}{3}i$

23. $0.5x^2 + 0.3x - 0.1 = 0$
$\frac{-3 \pm \sqrt{29}}{10}$

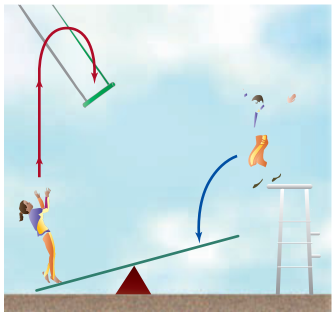
24. $x^2 - 3x - 27 = 0$
$\frac{3 \pm 3\sqrt{13}}{2}$

25. TUTORING A private tutoring company charges \$20 for a 1-hour session. Currently, 300 students are tutored each week. Since the company is losing money, the owner has decided to increase the price. For each 50¢ increase, she estimates that 5 fewer students will participate. If the company needs to bring in \$6,240 per week to stay in business, what price must be charged for a 1-hour tutoring session to produce this amount of revenue?
\$24 or \$26

26. POSTERS The specifications for a poster of Cesar Chavez call for a 615-square-inch photograph to be surrounded by a green border. The borders on the top and bottom of the poster are to be twice as wide as those on the sides. Find the width of each border. sides: 1.25 in. wide; top/bottom: 2.5 in. wide



27. ACROBATS To begin his routine on a trapeze, an acrobat is catapulted upward as shown in the illustration. His distance d (in feet) from the arena floor during this maneuver is given by the formula $d = -16t^2 + 40t + 5$, where t is the time in seconds since being launched. If the trapeze bar is 25 feet in the air, at what two times will he be able to grab it? Round to the nearest tenth. 0.7 sec, 1.8 sec



28. TRIANGLES The length of the longer leg of a right triangle exceeds the length of the shorter leg by 23 inches and the length of the hypotenuse is 65 inches. Find the length of each leg of the triangle. 33 in., 56 in.

SECTION 8.3 Quadratic Functions and Their Graphs

DEFINITIONS AND CONCEPTS	EXAMPLES

DEFINITIONS AND CONCEPTS

A **quadratic function** is a second-degree polynomial function of the form

$$f(x) = ax^2 + bx + c$$

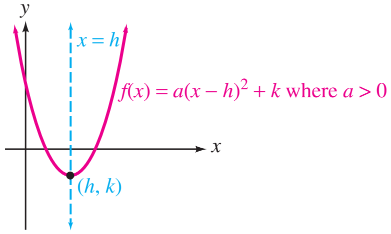
EXAMPLES

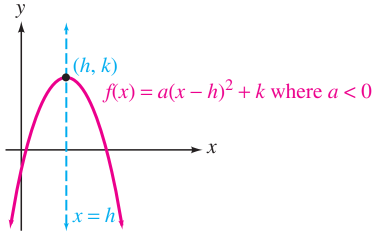
Quadratic functions:

$$f(x) = 2x^2 - 3x + 5,\ g(x) = -x^2 + 4x, \text{ and } s(x) = \frac{1}{4}x^2 - 10$$

DEFINITIONS AND CONCEPTS

The graph of the quadratic function $f(x) = a(x - h)^2 + k$ where $a \neq 0$ is a **parabola** with **vertex** at (h, k). The **axis of symmetry** is the line $x = h$. The parabola opens upward when $a > 0$ and downward when $a < 0$.





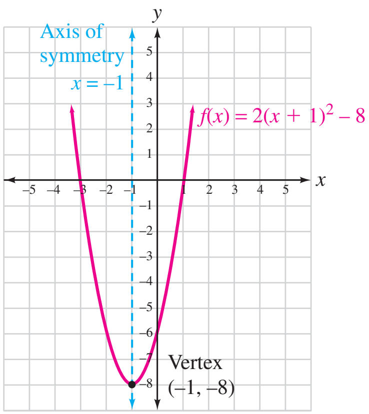
EXAMPLES

Graph: $f(x) = 2(x + 1)^2 - 8$

$$f(x) = 2[x - (-1)]^2 - 8$$

$$f(x) = a(x - \quad h)^2 + k$$

We see that $a = 2$, $h = -1$, and $k = -8$. The graph is a parabola with vertex $(h, k) = (-1, -8)$ and axis of symmetry $x = -1$. Since a is positive, the parabola opens upward.



DEFINITIONS AND CONCEPTS

The vertex of the graph of $f(x) = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and the axis of symmetry is the line

$$x = -\frac{b}{2a}$$

The y-coordinate of the vertex of the graph of a quadratic function gives the **minimum or maximum value** of the function.

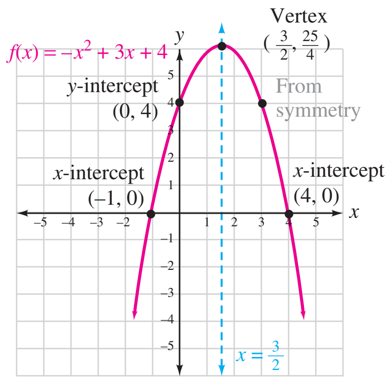
EXAMPLES

Graph: $f(x) = -x^2 + 3x + 4$

Here, $a = -1$, $b = 3$, and $c = 4$.

- Since $a < 0$, the graph opens downward.
- The x-coordinate of the vertex of the graph is

$$-\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$$



To find the y-coordinate of the vertex, we substitute $\frac{3}{2}$ for x in the function.

$$f(x) = -x^2 + 3x + 4$$

$$f\left(\frac{3}{2}\right) = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 4 = \frac{25}{4}$$

The vertex of the parabola is the point $\left(\frac{3}{2}, \frac{25}{4}\right)$.

DEFINITIONS AND CONCEPTS

The **y-intercept** is determined by the value of $f(x)$ when $x = 0$: the y-intercept is $(0, c)$.

EXAMPLES

- The y-intercept is the value of the function when $x = 0$. Thus, the y-intercept is $(0, 4)$.

To find the ***x*-intercepts,** let $f(x) = 0$ and solve $ax^2 + bx + c = 0$.

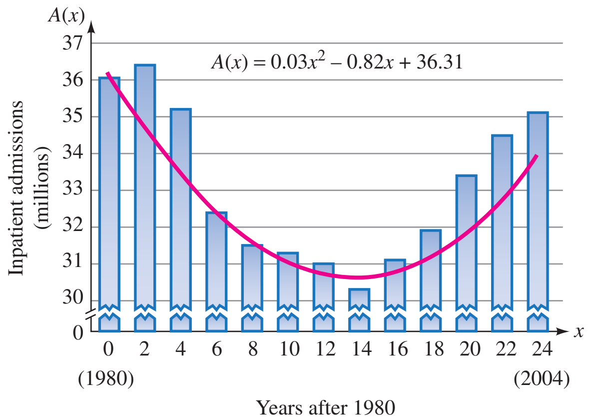
- To find the x-intercepts, we solve:

$$
\begin{aligned}
-x^2 + 3x + 4 &= 0 \\
x^2 - 3x - 4 &= 0 \quad \text{Multiply both sides by } -1. \\
(x + 1)(x - 4) &= 0 \quad \text{Factor.} \\
x = -1 \quad \text{or} \quad x &= 4
\end{aligned}
$$

The x-intercepts are $(-1, 0)$ and $(4, 0)$.

REVIEW EXERCISES

29. HOSPITALS The annual number of in-patient admissions to U.S. community hospitals for the years 1980–2004 can be modeled by the quadratic function $A(x) = 0.03x^2 - 0.82x + 36.31$, where $A(x)$ is the number of admissions in millions and x is the number of years after 1980. Use the function to estimate the number of in-patient admissions for the year 1992. Round to the nearest tenth of one million. 30.8 million

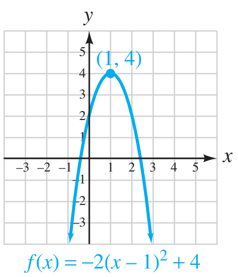


(Source: American Hospital Association)

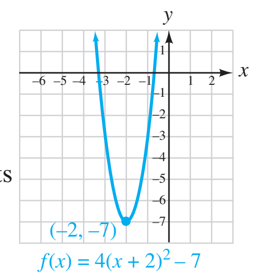
30. Fill in the blanks. The graph of the quadratic function $f(x) = a(x - h)^2 + k$ is a parabola with vertex at (h, k). The axis of symmetry is the line $x = h$. The parabola opens upward when $a > 0$ and downward when $a < 0$.

Graph each pair of functions on the same coordinate system.

31. Find the vertex and the axis of symmetry of the graph of $f(x) = -2(x - 1)^2 + 4$. Then plot several points and complete the graph. $(1, 4), x = 1$

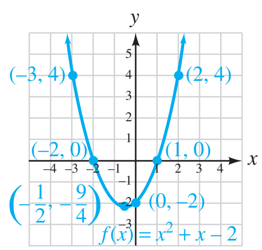


32. Complete the square to write $f(x) = 4x^2 + 16x + 9$ in the form $f(x) = a(x - h)^2 + k$. Determine the vertex and the axis of symmetry of the graph. Then plot several points and complete the graph. $f(x) = 4(x + 2)^2 - 7$; $(-2, -7), x = -2$



33. Find the vertex of the graph of $f(x) = -2x^2 + 4x - 8$ using the vertex formula. $(1, -6)$

34. First determine the coordinates of the vertex and the axis of symmetry of the graph of $f(x) = x^2 + x - 2$ using the vertex formula. Then determine the x- and y-intercepts of the graph. Finally, plot several points and complete the graph. $\left(-\frac{1}{2}, -\frac{9}{4}\right), x = -\frac{1}{2}$; $(-2, 0), (1, 0)$; $(0, -2)$

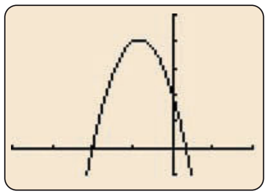


35. FARMING The number of farms in the United States for the years 1870–1970 is approximated by

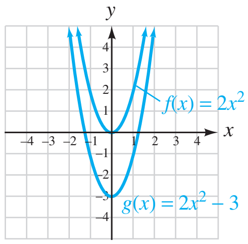
$$N(x) = -1{,}526x^2 + 155{,}652x + 2{,}500{,}200$$

where $x = 0$ represents 1870, $x = 1$ represents 1871, and so on. For this period, when was the number of U.S. farms a maximum? How many farms were there? 1921; 6,469,326

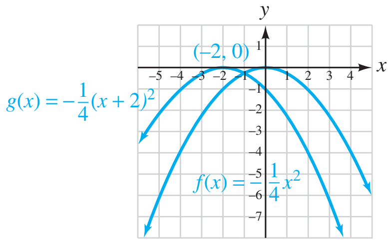
36. Estimate the solutions of $-3x^2 - 5x + 2 = 0$ from the graph of $f(x) = -3x^2 - 5x + 2$, shown here. $-2, \frac{1}{3}$



37. $f(x) = 2x^2, g(x) = 2x^2 - 3$



38. $f(x) = -\frac{1}{4}x^2, g(x) = -\frac{1}{4}(x + 2)^2$



SECTION 8.4 The Discriminant and Equations That Can Be Written in Quadratic Form

DEFINITIONS AND CONCEPTS	EXAMPLES
The **discriminant** predicts the type of solutions of $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$: **1.** If $b^2 - 4ac > 0$, there are two different real-number solutions. If $b^2 - 4ac$ is a perfect square, there are two different rational-number solutions. If $b^2 - 4ac$ is not a perfect square, there are two different irrational-number solutions. **2.** If $b^2 - 4ac = 0$, there is one repeated solution, a rational number. **3.** If $b^2 - 4ac < 0$, there are two different imaginary-number solutions that are complex conjugates.	In the quadratic equation $2x^2 - 5x - 3 = 0$, we have $a = 2$, $b = -5$, and $c = -3$. So the value of the discriminant is $b^2 - 4ac = (-5)^2 - 4(2)(-3) = 25 + 24 = 49$ Since the value of the discriminant is positive and a perfect square, the equation $2x^2 - 5x - 3 = 0$ has two different rational-number solutions.
Equations that contain an expression, the same expression squared, and a constant term are said to be **quadratic in form.** One method used to solve such equations is to make a **substitution.**	Solve: $x^{2/3} - 6x^{1/3} + 5 = 0$ The equation can be written in quadratic form: $(x^{1/3})^2 - 6x^{1/3} + 5 = 0$ We substitute y for $x^{1/3}$ and use factoring to solve the resulting quadratic equation. $y^2 - 6y + 5 = 0$ Let $y = x^{1/3}$. $(y - 1)(y - 5) = 0$ Factor. $y = 1$ or $y = 5$ Now we reverse the substitution $y = x^{1/3}$ and solve for x. $x^{1/3} = 1$ or $x^{1/3} = 5$ $(x^{1/3})^3 = (1)^3$ | $(x^{1/3})^3 = (5)^3$ $x = 1$ | $x = 125$ The solutions are 1 and 125. Check both in the original equation.

REVIEW EXERCISES

Use the discriminant to determine the number and type of solutions for each equation.

39. $3x^2 + 4x - 3 = 0$ two different irrational-number solutions

40. $4x^2 - 5x + 7 = 0$ two imaginary-number solutions that are complex conjugates

41. $3x^2 - 4x + \frac{4}{3} = 0$ one repeated solution, a rational number

42. $m(2m - 3) = 20$ two different rational-number solutions

Solve each equation.

43. $x - 13\sqrt{x} + 12 = 0$
1, 144

44. $a^{2/3} + a^{1/3} - 6 = 0$
8, -27

45. $3x^4 + x^2 - 2 = 0$
$i, -i, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}$

46. $\frac{6}{x + 2} + \frac{6}{x + 1} = 5$
$1, -\frac{8}{5}$

47. $(x - 7)^2 + 6(x - 7) + 10 = 0$
$4 \pm i$

48. $m^{-4} - 2m^{-2} + 1 = 0$
repeated solutions of -1 and 1

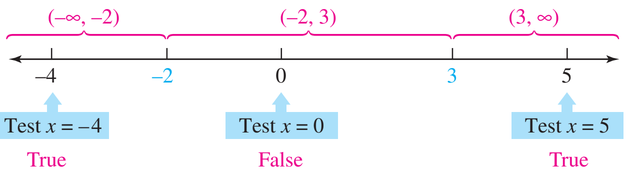
49. $4\left(\frac{x + 1}{x}\right)^2 + 12\left(\frac{x + 1}{x}\right) + 9 = 0$
a repeated solution of $-\frac{2}{5}$

50. $2m^{2/5} - 5m^{1/5} + 2 = 0$
$\frac{1}{32}$, 32

51. WEEKLY CHORES Working together, two sisters can do the yard work at their house in 45 minutes. When the older girl does it all herself, she can complete the job in 20 minutes less time than it takes the younger girl working alone. How long does it take the older girl to do the yard work?
about 81 min

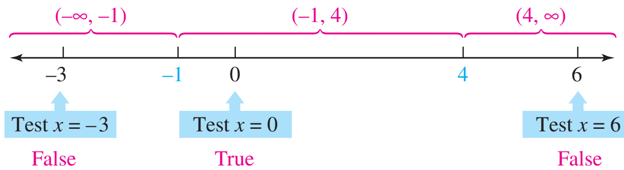
52. ROAD TRIPS A woman drives her automobile 150 miles at a rate of r mph. She could have gone the same distance in 2 hours less time if she had increased her speed by 20 mph. Find r.
30 mph

SECTION 8.5 Quadratic and Other Nonlinear Inequalities

DEFINITIONS AND CONCEPTS	EXAMPLES
To solve a quadratic inequality, get 0 on the right side and solve the related quadratic equation. Then locate the **critical numbers** on a number line, test each interval, and check the endpoints.	To solve $x^2 - x - 6 \geq 0$, we solve the related quadratic equation $x^2 - x - 6 = 0$. $x^2 - x - 6 = 0$ $(x - 3)(x + 2) = 0$ Factor. $x = 3$ or $x = -2$ These are the critical numbers that divide the number line into three intervals.  After testing each interval and noting that 3 and -2 satisfy the inequality, we see that the solution set is $(-\infty, -2] \cup [3, \infty)$.

To solve a rational inequality, get 0 on the right side and solve the related rational equation. Then locate the **critical numbers** (including any values that make the denominator 0) on a number line, test each interval, and check the endpoints.

To solve $\frac{x+1}{x-4} < 0$, we solve the related rational equation $\frac{x+1}{x-4} = 0$ to obtain the solution $x = -1$, which is a critical number. Another critical number is $x = 4$, the value that makes the denominator 0. These critical numbers divide the number line into three intervals.



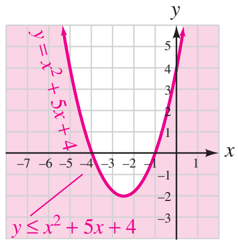
After testing each interval and noting that -1 and 4 do not satisfy the inequality, we see that the solution set is the interval $(-1, 4)$.

To graph a nonlinear inequality in two variables, first graph the boundary. Then use a test point to determine which side of the boundary to shade.

This is the graph of $y \leq x^2 + 5x + 4$.

Since the inequality contains the symbol $\leq$, and equality is allowed, we draw the parabola determined by $y = x^2 + 5x + 4$ using a solid line.

We shade the region containing the test point $(0, 0)$ because its coordinates satisfy $y \leq x^2 + 5x + 4$.



REVIEW EXERCISES

Solve each inequality. Write the solution set in interval notation and graph it.

53. $x^2 + 2x - 35 > 0$
$(-\infty, -7) \cup (5, \infty)$

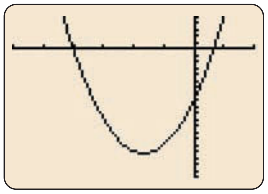
54. $x^2 \leq 81$
$[-9, 9]$

55. $\frac{3}{x} \leq 5$
$(-\infty, 0) \cup \left[\frac{3}{5}, \infty\right)$

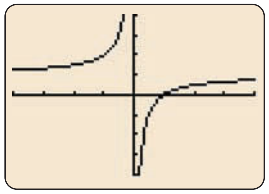
56. $\frac{2x^2 - x - 28}{x - 1} > 0$
$\left(-\frac{7}{2}, 1\right) \cup (4, \infty)$

57. Estimate the solution set of $3x^2 + 10x - 8 \leq 0$ from the graph of $f(x) = 3x^2 + 10x - 8$ shown in figure (a) in the next column. $\left[-4, \frac{2}{3}\right]$

58. Estimate the solution set of $\frac{x-1}{x} > 0$ from the graph of $f(x) = \frac{x-1}{x}$ shown in figure (b).
$(-\infty, 0) \cup (1, \infty)$



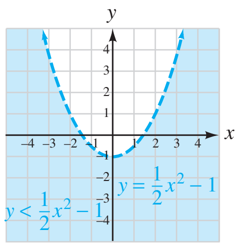
(a)



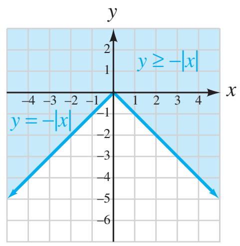
(b)

Graph each inequality.

59. $y < \frac{1}{2}x^2 - 1$



60. $y \geq -|x|$



CHAPTER 8 TEST

1. Fill in the blanks.
   a. An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, is called a quadratic equation.
   b. When we add 81 to $x^2 + 18x$, we say that we have completed the square on $x^2 + 18x$.
   c. The lowest point on a parabola that opens upward, or the highest point on a parabola that opens downward, is called the vertex of the parabola.
   d. $\frac{x-5}{x^2-x-56} > 0$ is an example of a rational inequality in one variable.
   e. $y \leq x^2 - 4x + 3$ is an example of a nonlinear inequality in two variables.

2. Solve $x^2 - 63 = 0$ using the square root property. Approximate the solutions to the nearest hundredth. $\pm 3\sqrt{7} \approx \pm 7.94$

Solve each equation using the square root property.

3. $(a + 7)^2 = 50$
   $-7 \pm 5\sqrt{2}$

4. $m^2 + 4 = 0$
   $\pm 2i$

5. Add a number to make $x^2 + 11x$ a perfect-square trinomial. Then factor the result.
   $x^2 + 11x + \frac{121}{4} = \left(x + \frac{11}{2}\right)^2$

6. Solve $4x^2 - 16x + 15 = 0$ by completing the square.
   $\frac{3}{2}, \frac{5}{2}$

Use the quadratic formula to solve each equation.

7. $4x^2 + 4x - 1 = 0$ $\frac{-1 \pm \sqrt{2}}{2}$

8. $\frac{1}{8}t^2 - \frac{1}{4}t = \frac{1}{2}$ $1 \pm \sqrt{5}$

9. $-t^2 + 4t - 13 = 0$
   $2 \pm 3i$

10. $0.01x^2 = -0.08x - 0.15$
    $-5, -3$

Solve each equation by any method.

11. $2y - 3\sqrt{y} + 1 = 0$
    $1, \frac{1}{4}$

12. $3 = m^{-2} - 2m^{-1}$
    $-1, \frac{1}{3}$

13. $x^4 - x^2 - 12 = 0$
    $2, -2, i\sqrt{3}, -i\sqrt{3}$

14. $4\left(\frac{x+2}{3x}\right)^2 - 4\left(\frac{x+2}{3x}\right) - 3 = 0$
    $-\frac{4}{5}, \frac{4}{7}$

15. $\frac{1}{n+2} = \frac{1}{3} - \frac{1}{n}$
    $2 \pm \sqrt{10}$

16. $5a^{2/3} + 11a^{1/3} = -2$
    $-8, -\frac{1}{125}$

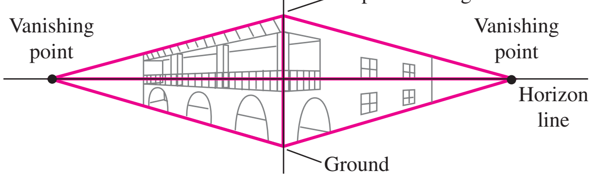
17. Solve $E = mc^2$ for c. Assume that all variables represent positive numbers. Express any radical in simplified form.
    $c = \frac{\sqrt{Em}}{m}$

18. Use the discriminant to determine the number and type of solutions for each equation.
    a. $3x^2 + 5x + 17 = 0$
    two different imaginary-number solutions that are complex conjugates
    b. $9m^2 - 12m = -4$
    one repeated solution, a rational number

19. TABLECLOTHS In 1990, Sportex of Highland, Illinois, made what was at the time the world's longest tablecloth. Find the dimensions of the rectangular tablecloth if it covered an area of 6,759 square feet and its length was 8 feet more than 332 times its width.
    4.5 ft by 1,502 ft

20. COOKING Working together, a chef and his assistant can make a pastry dessert in 25 minutes. When the chef makes it himself, it takes him 8 minutes less time than it takes his assistant working alone. How long does it take the chef to make the dessert?
    about 46 min

21. DRAWING An artist uses four equal-sized right triangles to block out a perspective drawing of an old hotel. See the illustration below. For each triangle, the leg on the horizontal line is 14 inches longer than the leg on the center line. The length of each hypotenuse is 26 inches. On the centerline of the drawing, what is the length of the segment extending from the ground to the top of the building?
    20 in.



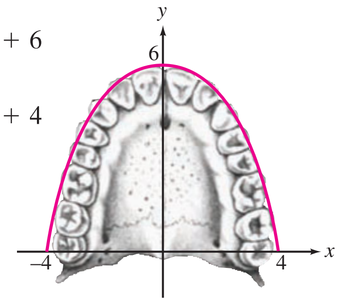
22. ANTHROPOLOGY Anthropologists refer to the shape of the human jaw as a *parabolic dental arcade.* Which function is the best mathematical model of the parabola shown in the illustration? iii

i. $f(x) = -\frac{3}{8}(x - 4)^2 + 6$

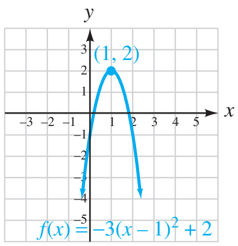
ii. $f(x) = -\frac{3}{8}(x - 6)^2 + 4$

iii. $f(x) = -\frac{3}{8}x^2 + 6$

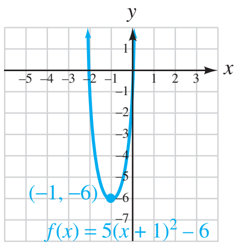
iv. $f(x) = \frac{3}{8}x^2 + 6$



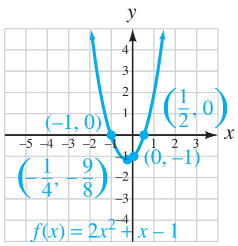
23. Find the vertex and the axis of symmetry of the graph of $f(x) = -3(x - 1)^2 + 2$. Then plot several points and complete the graph.
$(1, -2), x = 1$



24. Complete the square to write the function $f(x) = 5x^2 + 10x - 1$ in the form $f(x) = a(x - h)^2 + k$. Determine the vertex and the axis of symmetry of the graph. Then plot several points and complete the graph.
$f(x) = 5(x + 1)^2 - 6; (-1, -6), x = -1$



25. First determine the coordinates of the vertex and the axis of symmetry of the graph of $f(x) = 2x^2 + x - 1$ using the vertex formula. Then determine the x- and y-intercepts of the graph. Finally, plot several points and complete the graph.
$\left(-\frac{1}{4}, -\frac{9}{8}\right), x = -\frac{1}{4}, (-1, 0), \left(\frac{1}{2}, 0\right); (0, -1)$



26. DISTRESS SIGNALS A flare is fired directly upward into the air from a boat that is experiencing engine problems. The height of the flare (in feet) above the water, t seconds after being fired, is given by the formula $h = -16t^2 + 112t + 15$. If the flare is designed to explode when it reaches its highest point, at what height will this occur? 211 ft

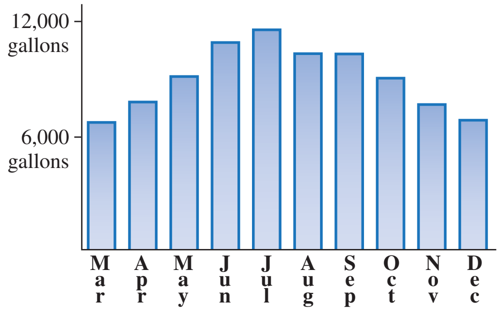
Solve each inequality. Write the solution set in interval notation and then graph it.

27. $x^2 - 2x > 8$ $(-\infty, -2) \cup (4, \infty)$

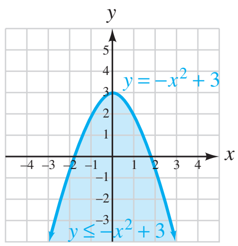
28. $\frac{x - 2}{x + 3} \le 0$ $(-3, 2]$

29. WATER USAGE The average amount of water used per month by a single-family residential customer in Tucson, Arizona, for the year 2004 is modeled by the function $W(m) = -235m^2 + 2{,}095m + 6{,}540$, where $W(m)$ is the number of gallons and m is the number of months *after March.* Use the function to approximate the average number of gallons of water used in July, which is typically Tucson's warmest month. (Based on data from the City of Tucson Water Department) 11,160 gal

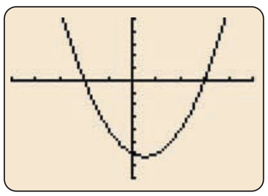
Tucson: Average Water Usage by Month
Single-Family Residental Customer, 2004



30. Graph: $y \le -x^2 + 3$



31. The graph of a quadratic function of the form $f(x) = ax^2 + bx + c$ is shown. Estimate the solutions of the corresponding quadratic equation $ax^2 + bx + c = 0$. $-2, 3$



32. See Exercise 31. Estimate the solution of the quadratic inequality $ax^2 + bx + c \le 0$. $[-2, 3]$

CHAPTERS 1–8 CUMULATIVE REVIEW

1. Solve: $3(x + 2) - 2 = -(5 + x) + x$ [Section 1.5]
-3

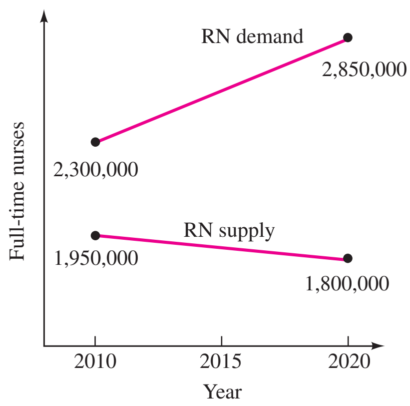
2. PHARMACISTS How many liters of a 1% glucose solution should a pharmacist mix with 2 liters of a 5% glucose solution to obtain a 2% glucose solution? [Section 1.8]
6 L

Find an equation of the line with the given properties. Write the equation in slope–intercept form.

3. Slope 3, passing through $(-2, -4)$ [Section 2.4]
$y = 3x + 2$

4. Parallel to the graph of $2x + 3y = 6$ and passing through $(0, -2)$ [Section 2.4]
$y = -\frac{2}{3}x - 2$

5. SHORTAGE OF NURSES Use the data in the graph to find the projected rates of change in the supply and demand for registered nurses (RNs) in the United States for the years 2010–2020. [Section 2.3]
supply: a decrease of 15,000 nurses per year; demand: an increase of 55,000 nurses per year



(Source: American Hospital Association)

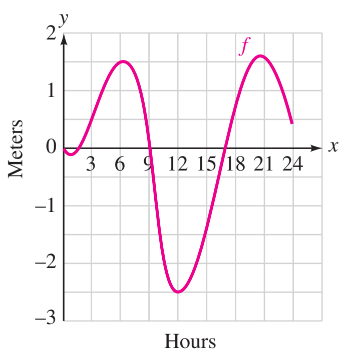
6. TIDES The illustration shows the graph of a function f, which gives the height of the tide for a 24-hour period in Seattle, Washington. (Note that military time is used on the x-axis: 3 A.M. = 3, noon = 12, 9 P.M. = 21, and so on.) [Section 2.6]

a. Find the domain of the function.
domain: $[0, 24]$

b. Find $f(6)$.
1.5

c. What information does $f(12)$ give?
At noon, the low tide mark was -2.5 m.

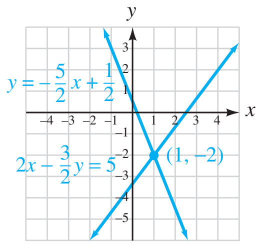
d. Estimate the values of x for which $f(x) = 0$.
0, 2, 9, 17



7. Solve the system by graphing:

$$\begin{cases} y = -\dfrac{5}{2}x + \dfrac{1}{2} \\ 2x - \dfrac{3}{2}y = 5 \end{cases}$$

[Section 3.1] $(1, -2)$



8. Use substitution to solve the system:

$$\begin{cases} x - y = -5 \\ 3x - 2y = -7 \end{cases}$$ [Section 3.2] $(3, 8)$

9. Solve the system:

$$\begin{cases} x - y + z = 4 \\ x + 2y - z = -1 \\ x + y - 3z = -2 \end{cases}$$ [Section 3.3] $(2, -1, 1)$

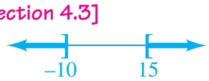
10. Evaluate the determinant: $\begin{vmatrix} -6 & -2 \\ 15 & 4 \end{vmatrix}$ [Section 3.5] 6

Solve each inequality. Write the solution set in interval notation and then graph it.

11. $5(-2x + 2) > 20 - x$ [Section 4.1]
$\left(-\infty, -\frac{10}{9}\right)$

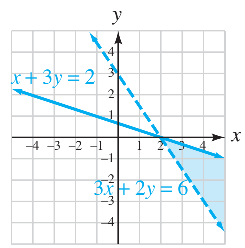
12. $5x - 3 \geq 2$ and $6 \geq 4x - 3$ [Section 4.2]
$\left[1, \frac{9}{4}\right]$

13. $|2x - 5| \geq 25$ [Section 4.3]
$(-\infty, -10] \cup [15, \infty)$



14. Graph the solution set of the system:

$$\begin{cases} 3x + 2y > 6 \\ x + 3y \leq 2 \end{cases}$$ [Section 4.5]



15. Simplify: $\left(\dfrac{2a^2b^3c^{-4}}{5a^{-2}b^{-1}c^3}\right)^{-3}$ [Section 5.1] $\dfrac{125c^{21}}{8a^{12}b^{12}}$

16. Write each number in scientific notation and perform the operations. Give the answer in scientific notation and in standard notation: $\dfrac{(1{,}280{,}000{,}000)(2{,}700{,}000)}{240{,}000}$
[Section 5.2] 1.44×10^{10}; 14,400,000,000

Perform the indicated operations.

17. $(-8.9t^3 - 2.4t) - (2.1t^3 + 0.8t^2 - t)$ [Section 5.3]
$-11t^3 - 0.8t^2 - 1.4t$

18. $(2a - b)(4a^2 + 2ab + b^2)$ [Section 5.4]
$8a^3 - b^3$

Factor each expression.

19. $x^2 + 4y - xy - 4x$ [Section 5.5] $(x - y)(x - 4)$

20. $x^4 - 16y^4$ [Section 5.6] $(x^2 + 4y^2)(x + 2y)(x - 2y)$

21. $8x^6 + 125y^3$ [Section 5.6] $(2x^2 + 5y)(4x^4 - 10x^2y + 25y^2)$

22. $30a^4 - 4a^3 - 16a^2$ [Section 5.7] $2a^2(3a + 2)(5a - 4)$

23. $49s^6 - 84s^3n^2 + 36n^4$ [Section 5.7] $(7s^3 - 6n^2)^2$

24. $x^2 + 10x + 25 - y^8$ [Section 5.8] $(x + 5 + y^4)(x + 5 - y^4)$

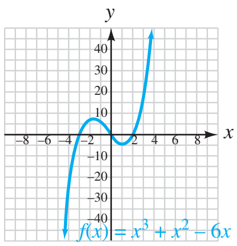
Solve each equation.

25. $(m + 4)(2m + 3) - 22 = 10m$ [Section 5.9] $2, -\frac{5}{2}$

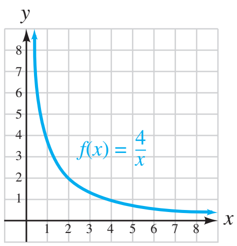
26. $6a^3 - 2a = a^2$ [Section 5.9] $0, \frac{2}{3}, -\frac{1}{2}$

Graph each function and give its domain and range.

27. $f(x) = x^3 + x^2 - 6x$
[Section 5.3]
D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



28. $f(x) = \dfrac{4}{x}$ for $x > 0$
[Section 6.1]
D: $(0, \infty)$, R: $(0, \infty)$



29. Simplify: $\dfrac{6x^2 - 7x - 5}{2x^2 + 5x + 2}$ [Section 6.1] $\frac{3x - 5}{x + 2}$

30. Divide: $\dfrac{x^3 + y^3}{x^3 - y^3} \div \dfrac{x^2 - xy + y^2}{x^2 + xy + y^2}$ [Section 6.2] $\frac{x + y}{x - y}$

31. Perform the operations: $\dfrac{1}{x + y} - \dfrac{1}{x - y} + \dfrac{2y}{x^2 - y^2}$
[Section 6.3] 0

32. Simplify: $\dfrac{\dfrac{1}{r^2 + 4r + 4}}{\dfrac{r}{r + 2} + \dfrac{r}{r + 2}}$ [Section 6.4] $\frac{1}{2r(r + 2)}$

33. Divide: $\dfrac{24x^6y^7 - 12x^5y^{12} + 36xy}{48x^2y^3}$ [Section 6.5]
$\frac{x^4y^4}{2} - \frac{x^3y^9}{4} + \frac{3}{4xy^2}$

34. Divide: $3a - 4\overline{)15a^3 - 29a^2 + 16}$ [Section 6.5]
$5a^2 - 3a - 4$

35. Solve: $\dfrac{x - 4}{x - 3} + \dfrac{x - 2}{x - 3} = x - 3$ [Section 6.7]
5; 3 is extraneous

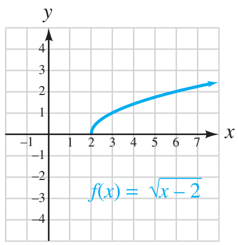
36. Solve for R: $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$ [Section 6.7]
$R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

37. SINKS A sink has two faucets, one for cold water and one for hot water. It can be filled by the cold-water faucet in 30 seconds and by the hot-water faucet in 45 seconds. How long will it take to fill the sink if both faucets are opened? [Section 6.7] 18 sec

38. SNOW REMOVAL A state highway department uses a 7-to-2 sand-to-salt mix in the winter months for spreading across roadways covered with snow and ice. If they have 6 tons of salt in storage, how many tons of sand should be added to obtain the proper mix? [Section 6.8] 21 tons

39. DELIVERIES The costs of a delivery company vary jointly with the number of trucks in service and the number of hours they are used. When 8 trucks are used for 12 hours each, the costs are \$3,600. Find the costs of using 20 trucks, each for 12 hours. [Section 6.8] \$9,000

40. Graph the function $f(x) = \sqrt{x - 2}$ and give its domain and range.
[Section 7.1] D: $[2, \infty)$, R: $[0, \infty)$



Simplify each expression.

41. $\sqrt[3]{-27x^3}$ [Section 7.1]
$-3x$

42. $\sqrt{48t^3}$ [Section 7.1]
$4t\sqrt{3t}$

43. $-3\sqrt[4]{32} - 2\sqrt[4]{162} + 5\sqrt[4]{48}$ [Section 7.2]
$-12\sqrt[4]{2} + 10\sqrt[4]{3}$

44. $3\sqrt{2}\left(2\sqrt{3} - 4\sqrt{12}\right)$ [Section 7.3]
$-18\sqrt{6}$

45. $\dfrac{\sqrt{x} + 2}{\sqrt{x} - 1}$ [Section 7.3]
$\frac{x + 3\sqrt{x} + 2}{x - 1}$

46. $\dfrac{5}{\sqrt[3]{x}}$ [Section 7.3]
$\frac{5\sqrt[3]{x^2}}{x}$

47. $64^{-2/3}$ [Section 7.4]
$\frac{1}{16}$

48. $\dfrac{x^{5/3}x^{1/2}}{x^{3/4}}$ [Section 7.4]
$x^{17/12}$

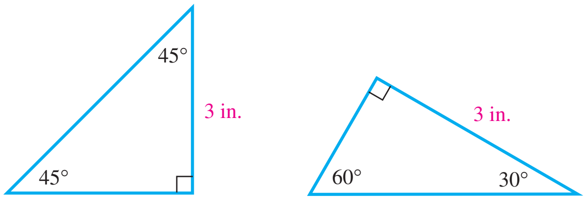
Solve each equation.

49. $5\sqrt{x + 2} = x + 8$ [Section 7.5] 2, 7

50. $\sqrt{x} + \sqrt{x + 2} = 2$ [Section 7.5] $\frac{1}{4}$

51. Find the length of the hypotenuse of the right triangle in figure (a). [Section 7.6] $3\sqrt{2}$ in.

52. Find the length of the hypotenuse of the right triangle in figure (b). [Section 7.6] $2\sqrt{3}$ in.



(a) (b)

53. Find the distance between $(-2, 6)$ and $(4, 14)$. [Section 7.6] 10

54. Simplify: i^{43} [Section 7.7] $-i$

Perform the indicated operations. Write each result in a + bi form.

55. $\left(-7 + \sqrt{-81}\right) - \left(-2 - \sqrt{-64}\right)$ [Section 7.7] $-5 + 17i$

56. $\dfrac{5}{3 - i}$ [Section 7.7] $\frac{3}{2} + \frac{1}{2}i$

57. $(2 + i)^2$ [Section 7.7] $3 + 4i$

58. $\dfrac{-4}{6i^7}$ [Section 7.7] $0 - \frac{2}{3}i$

Solve each equation.

59. $x^2 = 28$ [Section 8.1] $\pm 2\sqrt{7}$

60. $(x - 19)^2 = -5$ [Section 8.1] $19 \pm i\sqrt{5}$

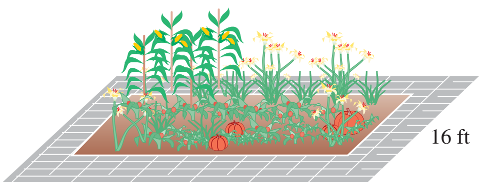
61. Use the method of completing the square to solve $2x^2 - 6x + 3 = 0$. [Section 8.1] $\frac{3 \pm \sqrt{3}}{2}$

62. Use the quadratic formula to solve $a^2 - \frac{2}{5}a = -\frac{1}{5}$. [Section 8.2] $\frac{1}{5} \pm \frac{2}{5}i$

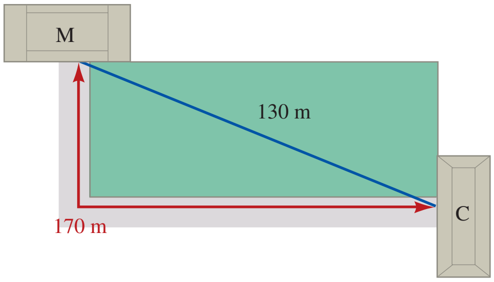
63. COMMUNITY GARDENS Residents of a community can work their own 16-ft × 24-ft plot of city-owned land if they agree to the following conditions:

- The area of the garden cannot exceed 180 square feet.
- A path of uniform width must be maintained around the garden.

Find the dimensions of the largest possible garden. [Section 8.2] 10 ft by 18 ft



64. SIDEWALKS A 170-meter-long sidewalk from the mathematics building M to the student center C is shown in red in the illustration. However, students prefer to walk directly from M to C, across a lawn. How long are the two segments of the existing sidewalk? [Section 8.2] 50 m and 120 m

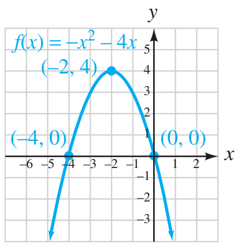


Solve each equation.

65. $t^{2/3} - t^{1/3} = 6$ [Section 8.3] $-8, 27$

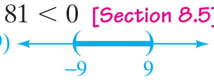
66. $x^{-4} - 2x^{-2} + 1 = 0$ [Section 8.3] repeated solutions of -1 and 1

67. First determine the vertex and the axis of symmetry of the graph of $f(x) = -x^2 - 4x$ using the vertex formula. Then determine the x- and y-intercepts of the graph. Finally, plot several points and complete the graph. [Section 8.4] $(-2, 4), x = -2; (-4, 0), (0, 0); (0, 0)$



Solve each inequality. Write the solution set in interval notation and then graph it.

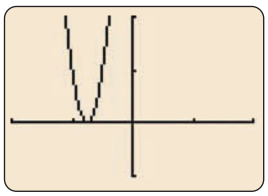
68. $x^2 - 81 < 0$ [Section 8.5] $(-9, 9)$



69. $\dfrac{1}{x + 1} \geq \dfrac{x}{x + 4}$ [Section 8.5] $(-4, -2] \cup (-1, 2]$

70. **a.** The graph of $f(x) = 16x^2 + 24x + 9$ is shown below. Estimate the solution(s) of $16x^2 + 24x + 9 = 0$. [Section 8.5] $-\frac{3}{4}$

b. Use the graph to determine the solution of $16x^2 + 24x + 9 < 0$. no solution



Exponential and Logarithmic Functions

9

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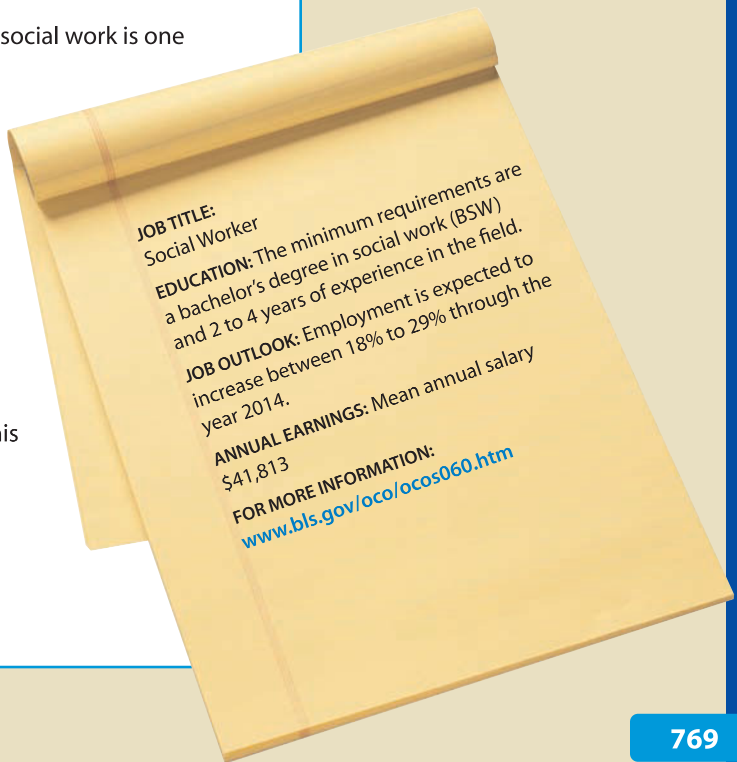
from Campus to Careers

Social Worker

For those with a desire to help improve other people's lives, social work is one career option to consider. Social workers offer guidance and counseling to people in crisis. They must be critical thinkers—able to use their logic and reasoning to brainstorm alternative solutions to problems faced by their clients. Social workers use their mathematical skills to construct family budgets, plan personnel schedules, gather and interpret data, and comprehend the statistical methods used in research studies.

Social workers often use occupational test results when counseling their clients about employment options. In **Problem 57** of **Study Set 9.4,** you will use concepts from this chapter to interpret the "learning curve" of a factory trainee.



Objectives

1. Add, subtract, multiply, divide functions.
2. Find the composition of functions.
3. Define the identity function.
4. Use composite functions to solve problems.

SECTION 9.1
Algebra and Composition of Functions

Just as it is possible to perform operations on real numbers, it is possible to perform operations on functions. We will begin by showing how to add, subtract, multiply, and divide functions. Then we will consider another method of combining functions, called *composition of functions.*

1 Add, subtract, multiply, divide functions.

We now consider how functions can be added, subtracted, multiplied, and divided.

Operations on Functions

If the domains and ranges of functions f and g are subsets of the real numbers, then

The **sum** of f and g, denoted as $\boldsymbol{f + g}$, is defined by

$$(f + g)(x) = f(x) + g(x)$$

The **difference** of f and g, denoted as $\boldsymbol{f - g}$, is defined by

$$(f - g)(x) = f(x) - g(x)$$

The **product** of f and g, denoted as $\boldsymbol{f \cdot g}$, is defined by

$$(f \cdot g)(x) = f(x)g(x)$$

The **quotient** of f and g, denoted as $\boldsymbol{f / g}$, is defined by

$$(f/g)(x) = \frac{f(x)}{g(x)} \quad \text{where } g(x) \neq 0$$

The domain of each of these functions is the set of real numbers x that are in the domain of both f and g. In the case of the quotient, there is the further restriction that $g(x) \neq 0$.

Self Check 1

Let $f(x) = 3x - 2$ and $g(x) = 2x^2 + 3x$. Find:
a. $(f + g)(x)$
b. $(f - g)(x)$

Now Try Problems 17 and 19

Self Check 1 Answers
a. $2x^2 + 6x - 2$
b. $-2x^2 - 2$

Teaching Example 1 Let $f(x) = 5x - 3$ and $g(x) = 4x^2 - 2x + 1$. Find:
a. $(f + g)(x)$
b. $(f - g)(x)$
Answers:
a. $4x^2 + 3x - 2$
b. $-4x^2 + 7x - 4$

EXAMPLE 1 Let $f(x) = 2x^2 + 1$ and $g(x) = 5x - 3$. Find each function and its domain: **a.** $(f + g)(x)$ **b.** $(f - g)(x)$

Strategy We will add and subtract the functions as if they were polynomials.

WHY We add because of the plus symbol in $f + g$, and we subtract because of the minus symbol in $f - g$.

Solution

a.
$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= (2x^2 + 1) + (5x - 3) && \text{Replace } f(x) \text{ with } 2x^2 + 1 \text{ and } g(x) \text{ with } 5x - 3. \\ &= 2x^2 + 1 + 5x - 3 && \text{Drop the parentheses.} \\ &= 2x^2 + 5x - 2 && \text{Combine like terms.}\end{aligned}$$

The domain of $f + g$ is the set of real numbers that are in the domain of both f and g. Since the domain of both f and g is the interval $(-\infty, \infty)$, the domain of $f + g$ is also the interval $(-\infty, \infty)$.

b. $(f - g)(x) = f(x) - g(x)$
$= (2x^2 + 1) - (5x - 3)$
$= 2x^2 + 1 - 5x + 3$
$= 2x^2 - 5x + 4$ Combine like terms.

Since the domain of both f and g is $(-\infty, \infty)$, the domain of $f - g$ is also the interval $(-\infty, \infty)$.

EXAMPLE 2 Let $f(x) = 2x^2 + 1$ and $g(x) = 5x - 3$. Find each function and its domain: **a.** $(f \cdot g)(x)$ **b.** $(f/g)(x)$

Strategy We will multiply and divide the functions as if they were polynomials.

WHY We multiply because of the raised dot $f \cdot g$, and we divide because of the fraction bar in f / g.

Solution

a. $(f \cdot g)(x) = f(x)g(x)$
$= (2x^2 + 1)(5x - 3)$
$= 10x^3 - 6x^2 + 5x - 3$ Multiply the binomials.

The domain of $f \cdot g$ is the set of real numbers that are in the domain of both f and g. Since the domain of both f and g is the interval $(-\infty, \infty)$, the domain of $f \cdot g$ is also the interval $(-\infty, \infty)$.

b. $(f/g)(x) = \dfrac{f(x)}{g(x)}$
$= \dfrac{2x^2 + 1}{5x - 3}$

Since the denominator of the fraction cannot be 0, $x \neq \frac{3}{5}$. Thus, the domain of f/g is the interval $\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)$.

Self Check 2

Let $f(x) = 2x^2 - 3$ and $g(x) = x^2 + 1$. Find each function and its domain:
a. $(f \cdot g)(x)$
b. $(f/g)(x)$

Now Try **Problems 21 and 22**

Self Check 2 Answers
a. $2x^4 - x^2 - 3$, D: $(-\infty, \infty)$
b. $\dfrac{2x^2 - 3}{x^2 + 1}$, D: $(-\infty, \infty)$

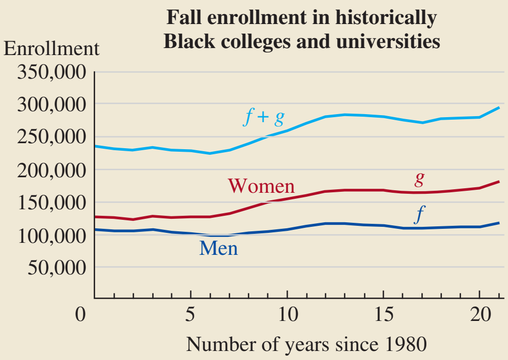
Teaching Example 2 Let $f(x) = 3x^2 - 1$ and $g(x) = 2x + 5$. Find each function and its domain:
a. $(f \cdot g)(x)$
b. $(f/g)(x)$
Answers:
a. $6x^3 + 15x^2 - 2x - 5$, D: $(-\infty, \infty)$
b. $\dfrac{3x^2 - 1}{2x + 5}$, D: $\left(-\infty, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$

THINK IT THROUGH *Black Colleges and Universities*

"Historically Black Colleges and Universities (HBCUs) provide an academic atmosphere which recognizes, responds, and appreciates the diverse background and qualifications of thousands of students that annually enroll at these institutions."

Wilvena T. McDowell, Assistant Direction of Admissions, North Carolina A&T State University

In the illustration, the graph of function f gives the number of men and the graph of function g gives the number of women enrolled in historically Black colleges and universities. The variable x represents the number of years since 1980. Sketch the graph of the function $f + g$ on the same coordinate system and explain its significance.

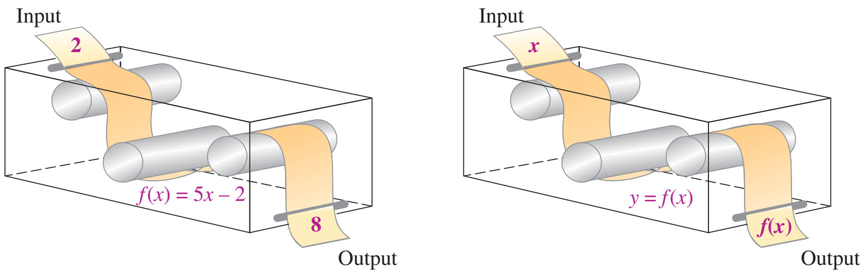


Source: National Center for Educational Statistics, 2004

The graph of $f + g$ gives the total number of students enrolled in historically Black colleges and universities.

2 Find the composition of functions.

We have seen that a function can be represented by a machine: We input a number from the domain, and a number from the range comes out. For example, if we put the number 2 into the machine shown in figure (a), the number $f(2) = 8$ comes out. In general, if we put x into the machine shown in figure (b), the value $f(x)$ comes out.

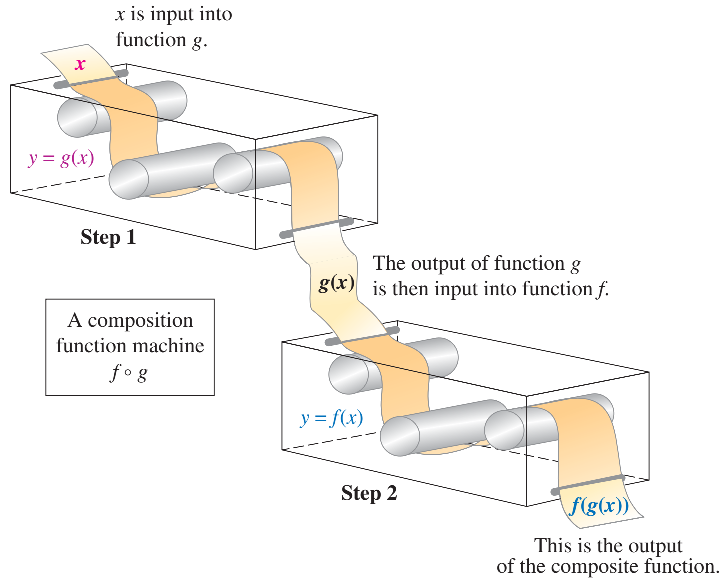


(a) (b)

Often one quantity is a function of a second quantity that depends, in turn, on a third quantity. For example, the cost of a car trip is a function of the gasoline consumed. The amount of gasoline consumed, in turn, is a function of the number of miles driven. Such chains of dependence can be analyzed mathematically as **compositions of functions.**

Suppose that $y = f(x)$ and $y = g(x)$ define two functions. Any number x in the domain of g will produce the corresponding value $g(x)$ in the range of g. If $g(x)$ is in the domain of function f, then $g(x)$ can be substituted into f, and a corresponding value $f(g(x))$ will be determined. This two-step process defines a new function, called a **composite function,** denoted by $f \circ g$. (This is read as "f composed with g.")

The function machines shown below illustrate the composition $f \circ g$. When we put a number x into the function g, $g(x)$ comes out. The value $g(x)$ goes into function f, which transforms $g(x)$ into $f(g(x))$. (This is read as "f of g of x.") If the function machines for g and f were connected to make a single machine, that machine would be named $f \circ g$.



To be in the domain of the composite function $f \circ g$, a number x has to be in the domain of g. Also, the output of g must be in the domain of f. Thus, the domain of $f \circ g$ consists of those numbers x that are in the domain of g, and for which $g(x)$ is in the domain of f.

Composite Functions

The **composite function $f \circ g$** is defined by

$$(f \circ g)(x) = f(g(x))$$

For example, if $f(x) = 4x$ and $g(x) = 3x + 2$, then

$$\begin{aligned}(f \circ g)(x) &= f(\mathbf{g(x)}) \\ &= f(\mathbf{3x + 2}) \\ &= 4(\mathbf{3x + 2}) \\ &= 12x + 8\end{aligned} \qquad \begin{aligned}(g \circ f)(x) &= g(\mathbf{f(x)}) \\ &= g(\mathbf{4x}) \\ &= 3(\mathbf{4x}) + 2 \\ &= 12x + 2\end{aligned}$$

Different results

Caution! Note that in the previous example, $(f \circ g)(x) \neq (g \circ f)(x)$. This shows that the composition of functions is not commutative. Also note that $(f \circ g)(x)$ is not equal to $(f \cdot g)(x)$: $(f \circ g)(x) = 12x + 8$, but $(f \cdot g)(x) = 12x^2 + 8x$.

EXAMPLE 3

Let $f(x) = 2x + 1$ and $g(x) = x - 4$. Find:

a. $(f \circ g)(9)$ **b.** $(f \circ g)(x)$ **c.** $(g \circ f)(-2)$

Strategy In part a, we will find $f(g(9))$. In part b, we will find $f(g(x))$. In part c, we will find $g(f(-2))$.

WHY To evaluate a composition function written with the circle $\circ$ notation, we rewrite it using nested parentheses notation.

Solution

a. $(f \circ g)(9)$ means $f(g(9))$. In figure (a) on the next page, function g receives the number 9, subtracts 4, and releases the number $g(9) = 5$. Then 5 goes into the f function, which doubles 5 and adds 1. The final result, 11, is the output of the composite function $f \circ g$:

Read as "f of g of 9."

$$(f \circ g)(9) = f(\mathbf{g(9)}) = f(\mathbf{5}) = 2(5) + 1 = 11$$

Thus, $(f \circ g)(9) = 11$.

b. $(f \circ g)(x)$ means $f(g(x))$. In figure (a) on the next page, function g receives the number x, subtracts 4, and releases the number $x - 4$. Then $x - 4$ goes into the f function, which doubles $x - 4$ and adds 1. The final result, $2x - 7$, is the output of the composite function $f \circ g$.

Read as "f of g of x."

$$(f \circ g)(x) = f(\mathbf{g(x)}) = f(\mathbf{x - 4}) = 2(x - 4) + 1 = 2x - 7$$

Thus, $(f \circ g)(x) = 2x - 7$.

c. $(g \circ f)(-2)$ means $g(f(-2))$. In figure (b), function f receives the number -2, doubles it and adds 1, and releases -3 into the g function. Function g subtracts 4 from -3 and outputs a final result of -7. Thus,

Self Check 3

Let $f(x) = 3x - 1$ and $g(x) = x + 5$. Find:

a. $(f \circ g)(4)$

b. $(f \circ g)(x)$

c. $(g \circ f)(-2)$

Now Try Problems 34 and 40

Self Check 3 Answers

a. 26 **b.** $3x + 14$ **c.** -2

Teaching Example 3 Let $f(x) = 5x + 2$ and $g(x) = 2x - 1$. Find:

a. $(f \circ g)(3)$

b. $(f \circ g)(x)$

c. $(g \circ f)(-2)$

Answers:

a. 27

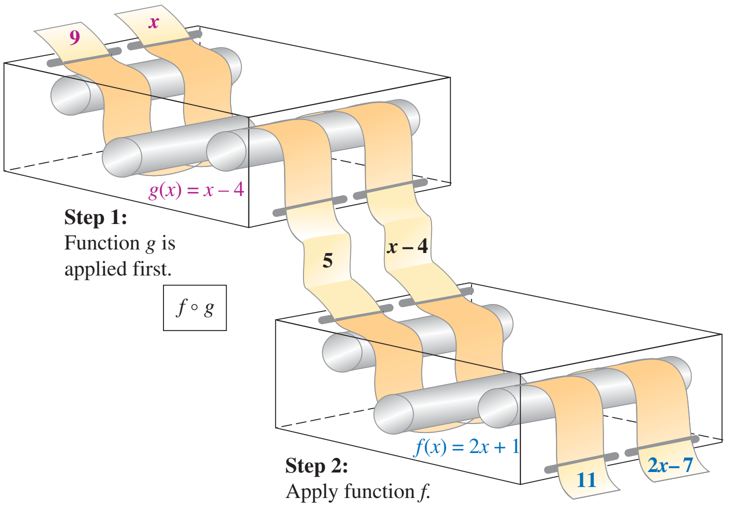
b. $10x - 3$

c. -17

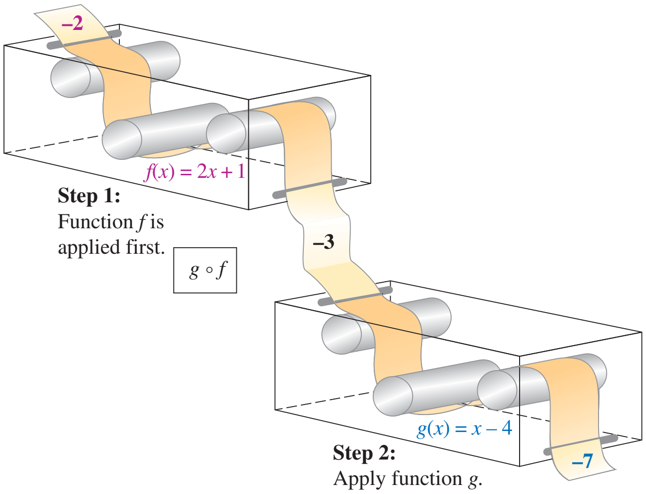
Read as "g of f of -2."

$$(g \circ f)(-2) = g(\mathbf{f(-2)}) = g(\mathbf{-3}) = -3 - 4 = -7$$

Thus, $(g \circ f)(-2) = -7$.



(a)



(b)

3 Define the identity function.

The **identity function** is defined by the equation $I(x) = x$. Under this function, the value that is assigned to any real number x is x itself. For example $I(2) = 2$, $I(-3) = -3$, and $I(7.5) = 7.5$. If f is any function, the composition of f with the identity function is just the function f:

$$(f \circ I)(x) = (I \circ f)(x) = f(x)$$

EXAMPLE 4 Let f be any function and let I be the identity function, $I(x) = x$. Show that: **a.** $(f \circ I)(x) = f(x)$ **b.** $(I \circ f)(x) = f(x)$

Strategy In part a, we will find $f(I(x))$. In part b, we will find $I(f(x))$.

WHY To find a composition function written with the circle $\circ$ notation, we rewrite it using nested parentheses notation.

Self Check 4

Let $f(x) = x^2 + 1$ and let I be the identity function, $I(x) = x$. Find:

a. $(f \circ I)(x)$ $x^2 + 1$

b. $(I \circ f)(x)$ $x^2 + 1$

Now Try Problems 41 and 43

Solution

a. $(f \circ I)(x)$ means $f(I(x))$. Because $I(x) = (x)$, we have

$$(f \circ I)(x) = f(\boldsymbol{I(x)}) = f(\boldsymbol{x})$$

b. $(I \circ f)(x)$ means $I(f(x))$. Because I passes any number through unchanged, we have $I(f(x)) = f(x)$ and

$$(I \circ f)(x) = I(\boldsymbol{f(x)}) = \boldsymbol{f(x)}$$

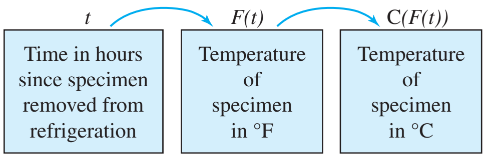
Teaching Example 4 Let $f(x) = 5x^2 + 3$ and let I be the identity function, $I(x) = x$. Find:
a. $(f \circ I)(x)$
b. $(I \circ f)(x)$
Answers:
a. $5x^2 + 3$
b. $5x^2 + 3$

4 Use composite functions to solve problems.

EXAMPLE 5 ***Biological Research*** A laboratory specimen is stored in a refrigeration unit at a temperature of 15° Fahrenheit. Biologists remove the specimen and warm it at the rate of 3°F per hour. Express the sample's Celsius temperature as a function of the time t since it was removed from refrigeration.

Strategy We will express the Fahrenheit temperature of the specimen as a function of the time t since it was removed from the refrigerator. Then we will express the Celsius temperature of the specimen as a function of its Fahrenheit temperature and find the composition of the two functions.

WHY The Celsius temperature of the specimen is a function of its Fahrenheit temperature. Its Fahrenheit temperature is a function of the time since it was removed from refrigeration. This chain of dependence suggests we write a composition of functions.



Solution

The temperature of the specimen is 15°F when the time $t = 0$. Because it warms at a rate of 3°F per hour, its initial temperature of 15° increases by $3t$°F in t hours. The Fahrenheit temperature of the specimen is given by the function

$$F(t) = 3t + 15$$

The Celsius temperature C is a function of this Fahrenheit temperature F, given by the function

$$C(F) = \frac{5}{9}(F - 32)$$

To express the specimen's Celsius temperature as a function of *time,* we find the composite function

$$\begin{aligned}
(C \circ F)(t) &= C(\boldsymbol{F(t)}) \\
&= C(\boldsymbol{3t + 15}) && \text{Substitute } 3t + 15 \text{ for } F(t). \\
&= \frac{5}{9}[(\boldsymbol{3t + 15}) - 32] && \text{Substitute } 3t + 15 \text{ for } F \text{ in } \tfrac{5}{9}(F - 32). \\
&= \frac{5}{9}(3t - 17) && \text{Simplify.}
\end{aligned}$$

The composite function, $C(t) = \frac{5}{9}(3t - 17)$, finds the temperature of the specimen in degrees Celsius t hours after it is removed from refrigeration.

Self Check 5

WEATHER FORECASTING A low-pressure area is bringing in colder weather for the next 12 hours. The temperature is now 86° Fahrenheit and is expected to fall 3° every 2 hours. Write a composition function that expresses the Celsius temperature as a function of the number of hours from now.

Now Try **Problem 63**

Self Check 5 Answer

$C(t) = 30 - \frac{5}{6}t$

Teaching Example 5 WEATHER FORECASTING A changing weather pattern is bringing in warmer weather for the next 24 hours. The temperature is now 50° Fahrenheit and is expected to increase 6° every 5 hours. Write a composition function that expresses the Fahrenheit temperature as a function of the number of hours from now.
Answer:
$C(t) = 10 + \frac{2}{3}t$

ANSWERS TO SELF CHECKS

1. a. $2x^2 + 6x - 2$ **b.** $-2x^2 - 2$ **2. a.** $2x^4 - x^2 - 3$, D: $(-\infty, \infty)$ **b.** $\frac{2x^2 - 3}{x^2 + 1}$, D: $(-\infty, \infty)$ **3. a.** 26 **b.** $3x + 14$ **c.** -2 **4.** $x^2 + 1$ **5.** $C(t) = 30 - \frac{5}{6}t$

SECTION 9.1 STUDY SET

VOCABULARY

Fill in the blanks.

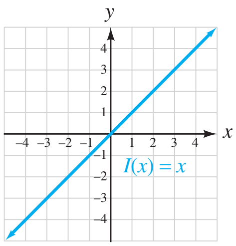
1. The __sum__ of f and g, denoted as $f + g$, is defined by $(f + g)(x) =$ $f(x) + g(x)$.
2. The __difference__ of f and g, denoted as $f - g$, is defined by $(f - g)(x) =$ $f(x) - g(x)$.
3. The __product__ of f and g, denoted as $f \cdot g$, is defined by $(f \cdot g)(x) =$ $f(x)g(x)$.
4. The __quotient__ of f and g, denoted as f/g, is defined by $(f/g)(x) = \frac{f(x)}{g(x)}$ $(g \neq 0)$.
5. In Exercises 1–3, the __domain__ of each function is the set of real numbers x that are in the domain of both f and g.
6. The __composite__ function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$.
7. Under the __identity__ function, the value that is assigned to any real number x is x itself.
8. When reading the notation $f(g(x))$, we say "f __of__ g __of__ x."

CONCEPTS

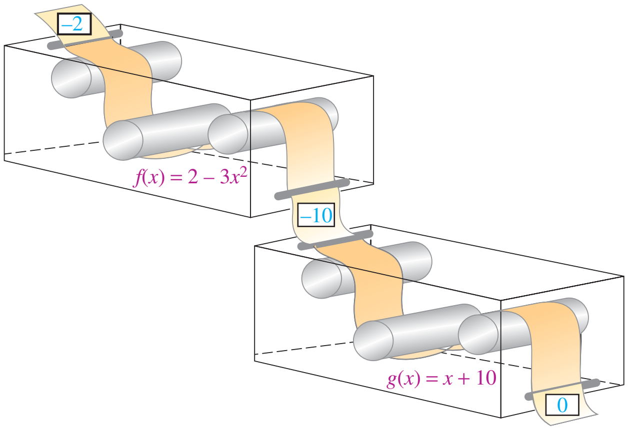
Fill in the blanks.

9. $(f \circ g)(x) = f\big(g(x)\big)$
10. To find $f(g(x))$, we first find $g(x)$ and then substitute that value for x in $f(x)$.
11. If $f(x) = 3x + 1$ and $g(x) = 1 - 2x$, find $f(g(3))$ and $g(f(3))$. $-14, -19$
12. Is the composition of functions commutative? Explain. no
13. Complete the table of values for the identity function, $I(x) = x$. Then graph it.

x	$I(x)$
-3	-3
-2	-2
-1	-1
0	0
1	1
2	2
3	3



14. Fill in the blanks in the drawing of the function machines below that show how to compute $g(f(-2))$.



NOTATION

Complete each solution.

15. Let $f(x) = 3x - 1$ and $g(x) = 2x + 3$. Find $f \cdot g$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= (3x - 1)(2x + 3)\\ &= 6x^2 + 9x - 2x - 3\\ &= 6x^2 + 7x - 3\end{aligned}$$

16. Let $f(x) = 3x - 1$ and $g(x) = 2x + 3$. Find $f \circ g$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f\big(2x + 3\big)\\ &= 3\big(2x + 3\big) - 1\\ &= 6x + 9 - 1\\ &= 6x + 8\end{aligned}$$

GUIDED PRACTICE

Let $f(x) = 3x$ and $g(x) = 4x$. Find each function and its domain. See Examples 1–2.

17. $(f + g)(x)$ $7x, (-\infty, \infty)$
18. $(f - g)(x)$ $-x, (-\infty, \infty)$
19. $(g - f)(x)$ $x, (-\infty, \infty)$
20. $(g + f)(x)$ $7x, (-\infty, \infty)$
21. $(f \cdot g)(x)$ $12x^2, (-\infty, \infty)$
22. $(f/g)(x)$ $\frac{3}{4}, (-\infty, 0) \cup (0, \infty)$

Selected exercises available online at **www.webassign.net/brookscole**

23. $(g/f)(x)$
$\frac{4}{3}, (-\infty, 0) \cup (0, \infty)$

▶ **24.** $(g \cdot f)(x)$
$12x^2, (-\infty, \infty)$

Let $f(x) = 2x + 1$ and $g(x) = x - 3$. Find each function and its domain. **See Examples 1–2.**

25. $(f + g)(x)$
$3x - 2, (-\infty, \infty)$

26. $(f - g)(x)$
$x + 4, (-\infty, \infty)$

27. $(f \cdot g)(x)$
$2x^2 - 5x - 3, (-\infty, \infty)$

▶ **28.** $(f/g)(x)$
$\frac{2x+1}{x-3}, (-\infty, 3) \cup (3, \infty)$

29. $(g - f)(x)$
$-x - 4, (-\infty, \infty)$

30. $(g + f)(x)$
$3x - 2, (-\infty, \infty)$

31. $(g/f)(x)$
$\frac{x-3}{2x+1}, \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

32. $(g \cdot f)(x)$
$2x^2 - 5x - 3, (-\infty, \infty)$

Let $f(x) = 2x + 1$ and $g(x) = x^2 - 1$. Find each value. **See Example 3.**

33. $(f \circ g)(2)$ 7

▶ **34.** $(f \circ g)(-3)$ 17

35. $(f \circ g)(x)$ $2x^2 - 1$

▶ **36.** $(f \circ g)(2x)$ $8x^2 - 1$

37. $(g \circ f)(0)$ 0

▶ **38.** $(g \circ f)\left(\frac{1}{3}\right)$ $\frac{16}{9}$

39. $(g \circ f)(x)$ $4x^2 + 4x$

40. $(g \circ f)(2x)$ $16x^2 + 8x$

Let $f(x) = 7x + 5$ and $I(x) = x$. Find each composition. **See Example 4.**

41. $(f \circ I)(x)$ $7x + 5$

▶ **42.** $(I \circ f)(x)$ $7x + 5$

Let $f(x) = -3x + 6$ and $I(x) = x$. Find each composition. **See Example 4.**

43. $(f \circ I)(x)$ $-3x + 6$

▶ **44.** $(I \circ f)(x)$ $-3x + 6$

TRY IT YOURSELF

Let $f(x) = 3x - 2$ and $g(x) = 2x^2 + 1$. Find each function and its domain.

45. $(f - g)(x)$
$-2x^2 + 3x - 3, (-\infty, \infty)$

▶ **46.** $(f + g)(x)$
$2x^2 + 3x - 1, (-\infty, \infty)$

47. $(f/g)(x)$
$\frac{3x-2}{2x^2+1}, (-\infty, \infty)$

48. $(f \cdot g)(x)$
$6x^3 - 4x^2 + 3x - 2, (-\infty, \infty)$

Let $f(x) = x^2 - 1$ and $g(x) = x^2 - 4$. Find each function and its domain.

49. $(f - g)(x)$
$3, (-\infty, \infty)$

50. $(f + g)(x)$
$2x^2 - 5, (-\infty, \infty)$

51. $(g/f)(x)$
$\frac{x^2-4}{x^2-1}, (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

▶ **52.** $(g \cdot f)(x)$
$x^4 - 5x^2 + 4, (-\infty, \infty)$

Let $f(x) = 3x - 2$ and $g(x) = x^2 + x$. Find each value.

53. $(f \circ g)(4)$ 58

▶ **54.** $(g \circ f)(4)$ 110

55. $(g \circ f)(-3)$ 110

56. $(f \circ g)(-3)$ 16

57. $(g \circ f)(0)$ 2

▶ **58.** $(f \circ g)(0)$ -2

▶ **59.** $(g \circ f)(x)$ $9x^2 - 9x + 2$

60. $(f \circ g)(x)$ $3x^2 + 3x - 2$

61. If $f(x) = x + 1$ and $g(x) = 2x - 5$, show that $(f \circ g)(x) \neq (g \circ f)(x)$.

62. If $f(x) = x^2 + 1$ and $g(x) = 3x^2 - 2$, show that $(f \circ g)(x) \neq (g \circ f)(x)$.

APPLICATIONS

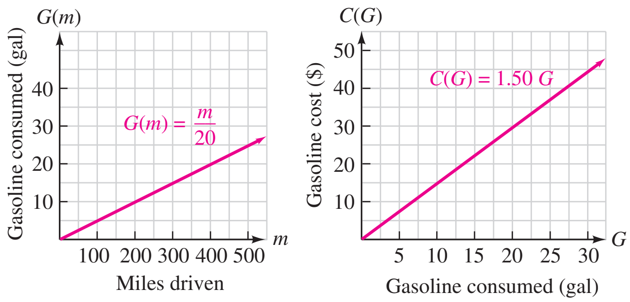
▶ **63.** METALLURGY A molten alloy must be cooled slowly to control crystallization. When removed from the furnace, its temperature is 2,700°F, and it will be cooled at 200° per hour. Express the Celsius temperature as a function of the number of hours t since cooling began. $C(t) = \frac{5}{9}(2{,}668 - 200t)$

▶ **64.** WEATHER FORECASTING A high-pressure area promises increasingly warmer weather for the next 48 hours. The temperature is now 34° Celsius and is expected to rise 1° every 6 hours. Express the Fahrenheit temperature as a function of the number of hours from now. (*Hint:* $F = \frac{9}{5}C + 32$.) $F(t) = \frac{3}{10}t + \frac{466}{5}$

▶ **65.** MILEAGE COSTS

a. Use the graphs below to determine the cost of the gasoline consumed if a family drove 500 miles on a summer vacation. about \$37.50

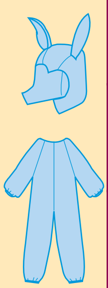
b. Write a composition function that expresses the cost of the gasoline consumed on the vacation as a function of the miles driven. $C(m) = \frac{1.50m}{20}$



▶ **66.** COSTUMES The tables on the back of the pattern package shown below can be used to determine the number of yards of material needed to make a rabbit costume for a child.

a. How many yards of material are needed if the child's chest measures 29 inches? $3\frac{3}{4}$ yd

b. In this exercise, one quantity is a function of a second quantity that depends, in turn, on a third quantity. Explain this dependence.
The amount of material needed depends on the size of the pattern used. The size of the pattern to use depends on the chest measurement of the child.



PATTERN 9810 **Simplicity**

Costumes have front zipper, long raglan sleeves, elastic sleeve and leg casings, hood. Fabrics: Fleece or suede

BODY MEASUREMENTS

Chest (in.)	21	22	23	25	26	27	$28\frac{1}{2}$	29	30
Pattern Size	2	3	4	6	7	8	10	11	12

YARDAGE NEEDED

Pattern Size	2-4	6-8	10-12
Yards	$2\frac{5}{8}$	$3\frac{3}{8}$	$3\frac{3}{4}$

WRITING

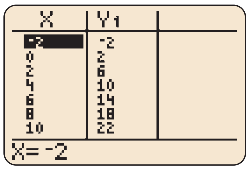
67. Problem 65 illustrates a chain of dependence between the cost of the gasoline, the gasoline consumed, and the miles driven. Describe another chain of dependence that could be represented by a composition function.

68. Explain how to add, subtract, multiply, and divide two functions.

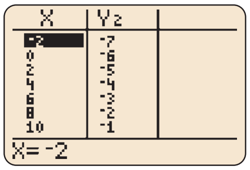
69. Write out in words how to say each of the following:

$(f \circ g)(2) \qquad g(f(-8))$

70. If $Y_1 = f(x)$ and $Y_2 = g(x)$, explain how to use the tables shown below to find $g(f(2))$.



(a)



(b)

REVIEW

Simplify each expression.

71. $\dfrac{3x^2 + x - 14}{4 - x^2}$ $\quad -\dfrac{3x+7}{x+2}$

72. $\dfrac{2x^3 + 14x^2}{3 + 2x - x^2} \cdot \dfrac{x^2 - 3x}{x}$ $\quad -\dfrac{2x^2(x+7)}{x+1}$

73. $\dfrac{x^2 - 2x - 8}{3x^2 - x - 12} \div \dfrac{3x^2 + 5x - 2}{3x - 1}$ $\quad \dfrac{x-4}{3x^2 - x - 12}$

74. $\dfrac{x - 1}{1 + \dfrac{x}{x - 2}}$ $\quad \dfrac{x-2}{2}$

Objectives

1. Determine whether a function is a one-to-one function.
2. Use the horizontal line test to determine whether a function is a one-to-one function.
3. Find the equation of the inverse of a function.
4. Find the composition of a function and its inverse.
5. Graph a function and its inverse.

SECTION 9.2

Inverse Functions

The function defined by $C(F) = \frac{5}{9}(F - 32)$ can be used to convert degrees Fahrenheit to degrees Celsius. If we input a Fahrenheit reading, the output is a Celsius reading. For example, if we substitute 41° for F, we will obtain a Celsius reading of 5°:

$$C(F) = \frac{5}{9}(F - 32)$$

$$C(41) = \frac{5}{9}(41 - 32) \qquad \text{Substitute 41 for } F.$$

$$= \frac{5}{9}(9)$$

$$= 5$$

If we want to find a Fahrenheit reading from a Celsius reading, we need a function into which we can input a Celsius reading and have a Fahrenheit reading come out. Such a function is $F(C) = \frac{9}{5}C + 32$, which takes the Celsius reading of 5° and turns it back into a Fahrenheit reading of 41°.

$$F(C) = \frac{9}{5}C + 32$$

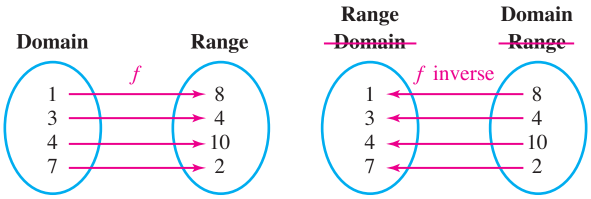
$$F(5) = \frac{9}{5}(5) + 32 \qquad \text{Substitute 5 for } C.$$

$$= 41$$

These two functions do opposite things. The first turns 41°F into 5° C, and the second turns 5° C back into 41°F. For this reason, we say that the functions are *inverses* of each other. In this section, we will show how to find inverses of one-to-one functions.

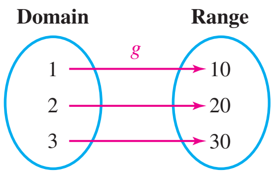
1 Determine whether a function is a one-to-one function.

In figure (a) below, the arrow diagram defines a function f. If we reverse the arrows as shown in figure (b), we obtain a new correspondence where the range of f becomes the domain of the new correspondence, and the domain of f becomes the range. The new correspondence is a function because to each member of the domain, there corresponds exactly one member of the range. We call this new correspondence the **inverse** of f, or f inverse.



(a) (b)

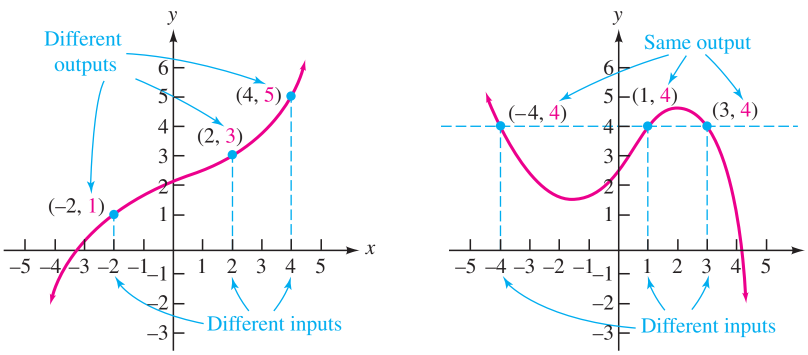
This reversing process does not always produce a function. For example, if we reverse the arrows in function g defined by the diagram in figure (a) below, the resulting correspondence is not a function. This is because to the number 2 in the domain, there corresponds two members of the range: 8 and 4.



(a) (b)

The question that arises is, "What must be true of an original function to guarantee that the reversing process produces a function?" The answer is: *the original function must be one-to-one.*

We have seen that a function assigns to each input exactly one output. For some functions, different inputs are assigned different outputs, as shown in figure (a). For other functions, different inputs are assigned the *same* output, as in figure (b). When each output corresponds to exactly one input, as in figure (a), we say the function is *one-to-one.*



A one-to-one function
(a)

Not a one-to-one function
(b)

One-to-One Functions

A function is called **one-to-one** if different inputs determine different outputs.

Self Check 1

Determine whether $f(x) = 2x + 3$ is one-to-one. Explain why or why not.

Now Try **Problems 29 and 30**

Self Check 1 Answer
Yes, because different input values determine different output values.

Teaching Example 1 Determine whether $f(x) = |x|$ is one-to-one.
Answer:
No, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2$

EXAMPLE 1 Determine whether each function is one-to-one:

a. $f(x) = x^2$ **b.** $f(x) = x^3$

Strategy We will determine whether different inputs have different outputs.

WHY If different inputs have different outputs, the function is one-to-one. If different inputs have the same output, the function is not one-to-one.

Solution

a. The function $f(x) = x^2$ is not one-to-one, because different input values x can determine the same output value y. For example, inputs of 3 and -3 produce the same output value of 9.

$$f(3) = 3^2 = 9 \qquad \text{and} \qquad f(-3) = (-3)^2 = 9$$

b. The function $f(x) = x^3$ is one-to-one, because different input values x determine different output values y. This is because different numbers have different cubes.

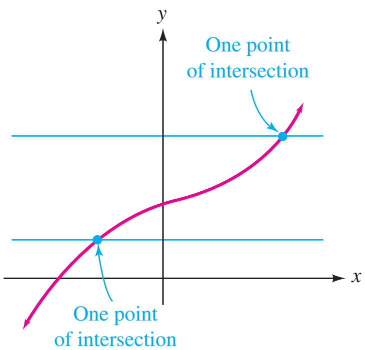
2 Use the horizontal line test to determine whether a function is a one-to-one function.

To determine whether a function is one-to-one, it is often easier to view its graph rather than its defining equation. If two (or more) points on the graph of a function have the same y-coordinate, the function is not one-to-one. This observation suggests the following horizontal line test.

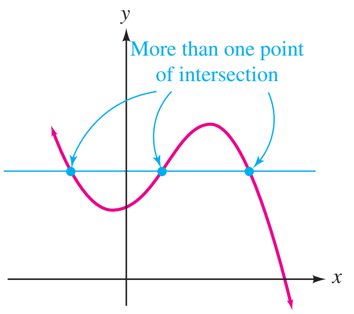
The Horizontal Line Test

A function is one-to-one if every horizontal line that intersects its graph does so only once. Otherwise, the function is not one-to-one.

In figure (a), since the graph of the function (shown in red) passes the horizontal line test, the function is one-to-one. In figure (b), the graph of the function does not pass the horizontal line test. Therefore, the function is not one-to-one.



A one-to-one function
(a)



Not a one-to-one function
(b)

EXAMPLE 2 Use the horizontal line test to determine whether the graphs in the figure represent one-to-one functions.

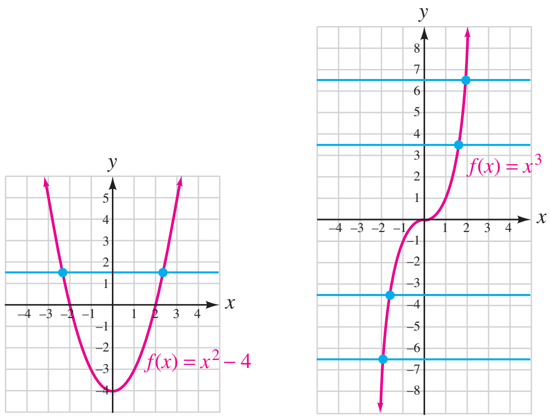
Strategy We will draw horizontal lines through the graph of the function and see how many times each line intersects the graph.

WHY If each horizontal line intersects the graph of the function exactly once, the graph represents a one-to-one function. If any horizontal line intersects the graph more than once, the graph does not represent a one-to-one function.

Solution

a. Refer to the graph in figure (a). Because at least one horizontal line intersects the graph twice, the graph does not represent a one-to-one function.

b. Refer to figure (b). Because every horizontal line that can be drawn intersects the graph exactly once, the graph represents a one-to-one function.

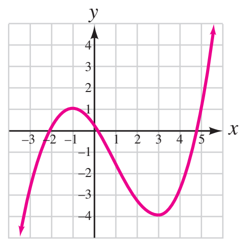


(a) (b)

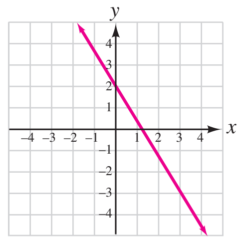
Self Check 2

Use the horizontal line test to determine whether the following graphs represent one-to-one functions.

a.



b.



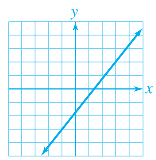
Now Try Problems 33 and 37

Self Check 2 Answers

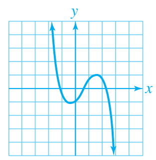
a. no **b.** yes

Teaching Example 2 Use the horizontal line test to determine whether the following graphs represent one-to-one functions.

a.



b.

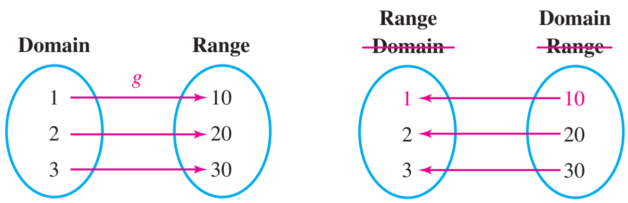


Answers:

a. yes **b.** no

3 Find the equation of the inverse of a function.

If f is the function determined by the ordered pairs in the table shown in figure (a) it turns the number 1 into 10, 2 into 20, and 3 into 30. Since the inverse of f must turn 10 back into 1, 20 back into 2, and 30 back into 3, it consists of the ordered pairs shown in figure (b).



Function f		
x	y	(x, y)
1	10	(1, 10)
2	20	(2, 20)
3	30	(3, 30)

(a)

Inverse of f		
x	y	(x, y)
10	1	(10, 1)
20	2	(20, 2)
30	3	(30, 3)

(b)

The x- and y-coordinates are interchanged.

We note that the domain of f and the range of its inverse is $\{1, 2, 3\}$. The range of f and the domain of its inverse is $\{10, 20, 30\}$.

This example illustrates that to form the inverse of a function f, we simply interchange the coordinates of each ordered pair that determines f. When the inverse of a function f is also a function, we call it ***f* inverse** and denote it with the symbol f^{-1}.

Caution! The symbol $f^{-1}(x)$ is read as "the inverse of $f(x)$" or just "f inverse." The -1 in the notation $f^{-1}(x)$ is not an exponent. Remember that $f^{-1}(x) \neq \frac{1}{f(x)}$.

The Inverse of a Function

If f is a one-to-one function consisting of ordered pairs of the form (x, y), the **inverse of *f*,** denoted f^{-1}, is a one-to-one function consisting of ordered pairs of the form (y, x).

When a one-to-one function is defined by an equation, we use the following method to find the equation of its inverse.

Finding the Inverse of a One-to-One Function

If a function is one-to-one, we find its inverse as follows:

1. If the function is written using function notation, replace $f(x)$ with y.
2. Interchange the variables x and y.
3. Solve the resulting equation for y.
4. We can then substitute $f^{-1}(x)$ for y to denote the inverse function.

Self Check 3

If $f(x) = -5x - 3$, find the inverse of f and tell whether it is a function.

Now Try **Problem 40**

Self Check 3 Answer

$f^{-1}(x) = \frac{-x - 3}{5}$, yes

Teaching Example 3 If $f(x) = \frac{2}{3}x + 1$, find the inverse of f and tell whether it is a function.

Answer:

$f(x) = \frac{3x - 3}{2}$, yes

EXAMPLE 3 If $f(x) = 4x + 2$, find the inverse of f and tell whether it is a function.

Strategy We will find the equation of the inverse by replacing $f(x)$ with y, interchanging x and y, and solving for y.

WHY If a function takes an input of x and gives an output y, its inverse function has the reverse effect.

Solution

We proceed as follows:

$$f(x) = 4x + 2$$

$$y = 4x + 2 \quad \text{Step 1: Replace } f(x) \text{ with } y.$$

$$x = 4y + 2 \quad \text{Step 2: Interchange the variables } x \text{ and } y.$$

To decide whether the inverse $x = 4y + 2$ is a function, we solve for y (step 3).

$$x = 4y + 2$$

$$x - 2 = 4y \quad \text{Subtract 2 from both sides.}$$

(1) $$y = \frac{x - 2}{4} \quad \text{Divide both sides by 4 and write } y \text{ on the left-hand side.}$$

Because each input x that is substituted into Equation 1 gives one output y, the inverse of f is a function, so we can express it in the form

$$f^{-1}(x) = \frac{x - 2}{4} \qquad \text{Step 4: Substitute } f^{-1}(x) \text{ for } y.$$

4 Find the composition of a function and its inverse.

To emphasize an important relationship between a function and its inverse, we substitute some number x, such as $x = 3$, into the function $f(x) = 4x + 2$ of Example 3. The corresponding value of y produced is

$$f(3) = 4(3) + 2 = 14$$

If we substitute 14 into the inverse function, f^{-1}, the corresponding value of y that is produced is

$$f^{-1}(14) = \frac{14 - 2}{4} = 3$$

Thus, the function f turns 3 into 14, and the inverse function f^{-1} turns 14 back into 3. In general, the composition of a function and its inverse is the identity function.

To prove that $f(x) = 4x + 2$ and $f^{-1}(x) = \frac{x-2}{4}$ are inverse functions, we must show that their composition (in both directions) is the identity function:

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{x-2}{4}\right) \\ &= 4\left(\frac{x-2}{4}\right) + 2 \\ &= x - 2 + 2 \\ &= x\end{aligned} \qquad \begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(4x + 2) \\ &= \frac{4x + 2 - 2}{4} \\ &= \frac{4x}{4} \\ &= x\end{aligned}$$

Thus, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, which is the identity function $I(x)$.

5 Graph a function and its inverse.

EXAMPLE 4 Find the inverse function of the one-to-one function $f(x) = -\frac{3}{2}x + 3$. Graph the function and its inverse on one coordinate system. Also graph $y = x$.

Strategy We will find the equation of the inverse by replacing $f(x)$ with y, interchanging x and y, and solving for y. Then we will graph $f(x)$, $f^{-1}(x)$, and $y = x$ on one coordinate system.

WHY If a function takes an input of x and gives an output y, its inverse function has the reverse effect.

Solution

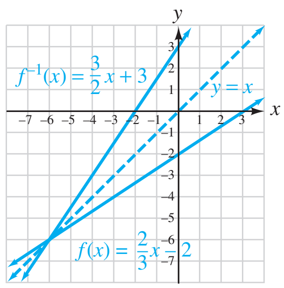
To find the inverse function of $f(x) = -\frac{3}{2}x + 3$, we begin by replacing $f(x)$ with y.

$$y = -\frac{3}{2}x + 3$$

Then we interchange the variables x and y and solve the equation for y.

Self Check 4

Find the inverse of the one-to-one function defined by $f(x) = \frac{2}{3}x - 2$. Graph the function and its inverse on one coordinate system. Also graph $y = x$.

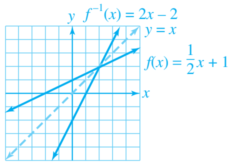


Now Try **Problem 44**

Self Check 4 Answer

$f^{-1}(x) = \frac{3}{2}x + 3$

Teaching Example 4 Find the inverse function of the one-to-one function $f(x) = \frac{1}{2}x + 1$. Graph the function, its inverse, and $y = x$ on one coordinate system.



Answer:
$f^{-1}(x) = 2x - 2$

$$x = -\frac{3}{2}y + 3$$

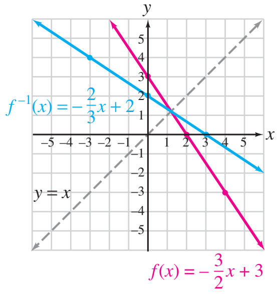
$$x - 3 = -\frac{3}{2}y \qquad \text{Subtract 3 from both sides.}$$

$$2x - 6 = -3y \qquad \text{Multiply both sides by 2.}$$

$$-\frac{2}{3}x + 2 = y \qquad \text{Divide both sides by } -3.$$

$$-\frac{2}{3}x + 2 = f^{-1}(x) \qquad \text{Substitute } f^{-1}(x) \text{ for } y.$$

The inverse of $f(x) = -\frac{3}{2}x + 3$ is $f^{-1}(x) = -\frac{2}{3}x + 2$. The graphs of the function, its inverse, and the line $y = x$ appear in the figure to the right.



$f(x) = -\frac{3}{2}x + 3$		
x	$f(x)$	
0	3	→ (0, 3)
2	0	→ (2, 0)
4	−3	→ (4, −3)

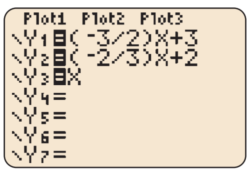
$f^{-1}(x) = -\frac{2}{3}x + 2$		
x	$f(x)$	
3	0	→ (3, 0)
0	2	→ (0, 2)
−3	4	→ (−3, 4)

In the figure above, we see that the graphs of $f(x) = -\frac{3}{2}x + 3$ and $f^{-1}(x) = -\frac{2}{3}x + 2$ are symmetric about the line $y = x$. That is, the graphs are mirror images (reflections) of each other with reference to the line $y = x$. This is always the case with a function and its inverse, because when the coordinates (a, b) satisfy an equation, the coordinates (b, a) will satisfy its inverse.

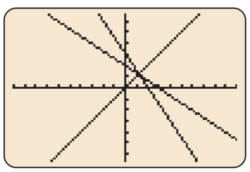
Using Your CALCULATOR Graphing the Inverse of a Function

We can use a graphing calculator for an informal check of the result found in Example 4. First, we enter $f(x) = -\frac{3}{2}x + 3$ as $Y_1 = -\frac{3}{2}x + 3$. Then we enter what we believe to be the inverse function, as $Y_2 = -\frac{2}{3}x + 2$, as well as the equation $y = x$ as $Y_3 = x$. See figure (a). Before graphing, we adjust the display so that the graphing grid will be composed of squares. (The standard window display is rectangular, having a grid with a unit width different from the unit height.) Check your owner's manual for directions on how to program the calculator to graph in the square mode.

In figure (b), it appears that the two graphs are symmetric about the line $y = x$. Although it is not definitive, this visual check does help to validate that $f^{-1}(x) = -\frac{2}{3}x + 2$ is the inverse of the function $f(x) = -\frac{3}{2}x + 3$.



(a)



(b)

In each example so far, the inverse of a function has been another function. This is not always true, as the following example will show.

EXAMPLE 5 Find the inverse of the function determined by $f(x) = x^2$.

Strategy We will find the equation of the inverse by replacing $f(x)$ with y, interchanging x and y, and solving for y.

WHY If a function takes an input of x and gives an output y, its inverse has the reverse effect.

Solution

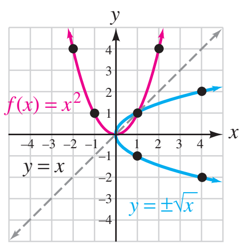
$y = x^2$ Replace $f(x)$ with y.

$x = y^2$ Interchange x and y.

$y = \pm\sqrt{x}$ Use the square root property and write y on the left-hand side.

When the inverse $y = \pm\sqrt{x}$ is graphed, we see that the graph (in blue) does not pass the vertical line test. Thus, it is not a function.

The graph of $f(x) = x^2$ is also shown in the figure. Note that it is not one-to-one. The graphs of $f(x) = x^2$ and $y = \pm\sqrt{x}$ are symmetric about the line $y = x$.



Self Check 5 Find the inverse of the function determined by $f(x) = 4x^2$.

Now Try **Problem 48**

Self Check 5 Answer

$y = \pm\frac{\sqrt{x}}{2}$

Teaching Example 5 Find the inverse of the function determined by $f(x) = 9x^2$.
Answer:
$y = \pm\frac{\sqrt{x}}{3}$

One-to-One Functions and Inverses

If a function is one-to-one, its inverse is a function.

EXAMPLE 6 Find the inverse of $f(x) = x^3$.

Strategy We will find the equation of the inverse by replacing $f(x)$ with y, interchanging x and y, and solving for y.

WHY If a function takes an input of x and gives an output y, its inverse function has the reverse effect.

Solution

Since $f(x) = x^3$ is a one-to-one function (see Example 2(b)), its inverse is a function. To find the inverse, we proceed as follows:

$y = x^3$ Replace $f(x)$ with y.

$x = y^3$ Interchange the variables x and y.

$\sqrt[3]{x} = y$ Take the cube root of both sides.

We note that to each number x there corresponds one real cube root. Thus, $y = \sqrt[3]{x}$ represents a function. Using $f^{-1}(x)$ notation, the inverse of $f(x) = x^3$ is

$$f^{-1}(x) = \sqrt[3]{x}$$

Self Check 6 Find the inverse of $f(x) = x^5$.

Now Try **Problem 52**

Self Check 6 Answer

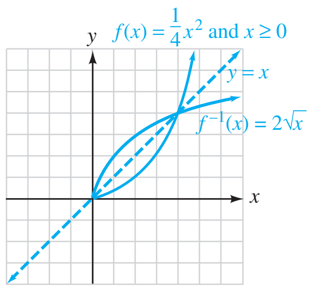
$f^{-1}(x) = \sqrt[5]{x}$

Teaching Example 6 Find the inverse of $f(x) = x^7$.
Answer:
$f^{-1}(x) = \sqrt[7]{x}$

If a function is not one-to-one, it is often possible to make it one-to-one by restricting its domain. For example, we have seen that $f(x) = x^2$ is not a one-to-one function. However, if we restrict the domain of $f(x) = x^2$ to $x \geq 0$ (only 0 and positive input values for x), the result is a one-to-one function whose inverse is also a function.

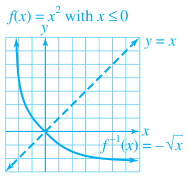
Self Check 7

Find the inverse of the function defined by $f(x) = \frac{1}{4}x^2$ with $x \geq 0$. Graph the function and its inverse on one set of coordinate axes.



Now Try **Problem 56**

Teaching Example 7 Find the inverse of the function defined by $f(x) = x^2$ with $x \leq 0$. Graph the function and its inverse on one set of coordinate axes.
Answer:



EXAMPLE 7 Find the inverse of the function defined by $f(x) = x^2$ with $x \geq 0$. Graph the function and its inverse on one set of coordinate axes.

Strategy We will find the equation of the inverse by replacing $f(x)$ with y, interchanging x and y, and solving for y.

WHY If a function takes an input of x and gives an output y, its inverse function has the reverse effect.

Solution

The inverse of the function $f(x) = x^2$ with $x \geq 0$ is

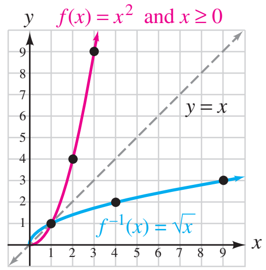
$$x = y^2 \quad \text{with } y \geq 0 \qquad \text{Interchange the variables } x \text{ and } y.$$

Then we solve for y.

$$y = \pm\sqrt{x} \quad \text{with } y \geq 0$$

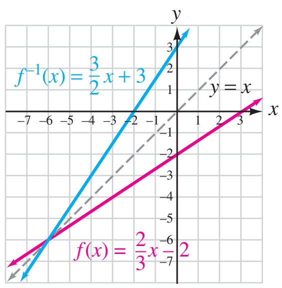
Since $y \geq 0$, we can discard the possibility that $y = -\sqrt{x}$. Thus, the inverse of $f(x) = x^2$ with $x \geq 0$ is $f^{-1}(x) = \sqrt{x}$.

The graphs of the function and its inverse appear in the figure to the right. The line $y = x$ is included so that we can see that the graphs are symmetric about the line $y = x$.

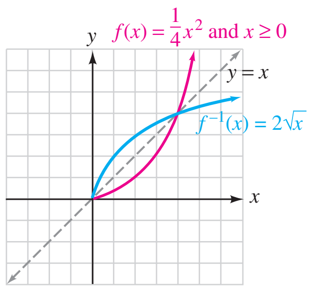


ANSWERS TO SELF CHECKS

1. Yes, because different input values give different output values. **2. a.** no **b.** yes **3.** $f^{-1}(x) = \frac{-x-3}{5}$, yes **4.**



5. $y = \pm\frac{\sqrt{x}}{2}$ **6.** $f^{-1}(x) = \sqrt[5]{x}$ **7.**



SECTION 9.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. A function is called _one-to-one_ if each input determines a different output.

▶ **2.** The _horizontal_ line test can be used to determine whether the graph of a function represents a one-to-one function.

3. The functions f and f^{-1} are _inverses_.

▶ Selected exercises available online at **www.webassign.net/brookscole**

4. An input value is an element of a function's domain. An output value is an element of a function's range.

CONCEPTS

Fill in the blanks.

5. If every horizontal line that intersects the graph of a function does so only once, the function is one-to-one.

6. If any horizontal line that intersects the graph of a function does so more than once, the function is not one-to-one.

7. If a function turns an input of 2 into an output of 5, the inverse function will turn an input of 5 into an output of 2.

8. The graphs of a function and its inverse are symmetrical about the line $y = x$.

9. $(f \circ f^{-1})(x) = x$.

10. To find the inverse of the function $f(x) = 2x - 3$, we begin by replacing $f(x)$ with y, and then we interchange x and y.

11. Use the table of values of the one-to-one function f to complete a table of values for f^{-1}.

x	$f(x)$
−6	−3
−4	−2
0	0
2	1
8	4

x	$f^{-1}(x)$
−3	−6
−2	−4
0	0
1	2
4	8

12. How can we tell that function f is not one-to-one from the table of values? $f(-2) = f(2) = 4$

x	$f(x)$
−2	4
−1	1
0	0
2	4
3	9

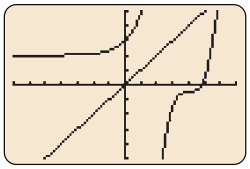
13. Is the inverse of a function always a function? no

14. Name four points that the line $y = x$ passes through. (1, 1), (2, 2), (3, 3), (4, 4) (answers may vary)

15. If f is a one-to-one function, and if $f(2) = 6$, what is $f^{-1}(6)$? 2

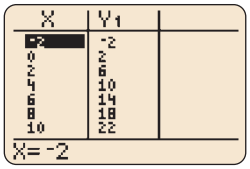
16. If the point $(2, -4)$ is on the graph of the one-to-one function f, then what point is on the graph of f^{-1}? $(-4, 2)$

17. Two functions are shown on a calculator display along with the line $y = x$. Explain why we know that the functions are not inverses of each other.

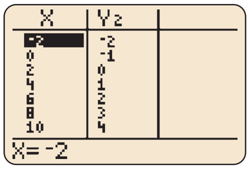


The graphs are not symmetric about the line $y = x$.

18. A table of values for a function f is shown in illustration (a). A table of values for f^{-1} is shown in illustration (b). Explain how to use the tables to find $f^{-1}(f(4))$ and $f(f^{-1}(2))$. $f(4) = 10$ and $f^{-1}(10) = 4$; $f^{-1}(2) = 0$ and $f(0) = 2$



(a)



(b)

NOTATION

Complete each solution.

19. Find the inverse of $f(x) = 2x - 3$.

$y = 2x - 3$ Replace $f(x)$ with y.

$x = 2y - 3$ Interchange the variables x and y.

$x + 3 = 2y$ Add 3 to both sides.

$\frac{x + 3}{2} = y$ Divide both sides by 2.

The inverse of $f(x) = 2x - 3$ is $f^{-1}(x) = \frac{x + 3}{2}$

20. Find the inverse of $f(x) = \sqrt[3]{x} + 2$.

$y = \sqrt[3]{x} + 2$ Replace $f(x)$ with y.

$x = \sqrt[3]{y} + 2$ Interchange the variables x and y.

$x - 2 = \sqrt[3]{y}$ Subtract 2 from both sides.

$(x - 2)^3 = y$ Cube both sides.

The inverse of $f(x) = \sqrt[3]{x} + 2$ is $f^{-1}(x) = (x - 2)^3$.

21. The symbol $f^{-1}(x)$ is read as "the inverse of f" or "f inverse."

22. Explain the difference in the meaning of the -1 in the notation $f^{-1}(x)$ as compared to x^{-1}.
In $f^{-1}(x)$, the -1 denotes the inverse function. In x^{-1}, the -1 is a negative exponent.

GUIDED PRACTICE

Determine whether the set of ordered pairs (x, y) is a function. Then find the inverse of the set of ordered pairs (x, y) and determine whether the inverse is a function. See Objective 1.

23. $\{(3, 2), (2, 1), (1, 0)\}$ yes, $\{(2, 3), (1, 2), (0, 1)\}$, yes

24. $\{(4, 1), (5, 1), (6, 1), (7, 1)\}$
yes, $\{(1, 4), (1, 5), (1, 6), (1, 7)\}$, no

25. $\{(1, 1), (2, 1), (3, 1), (4, 1)\}$
yes, $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$, no

▶ **26.** $\{(1, 2), (2, 3), (1, 3), (1, 5)\}$
no, $\{(2, 1), (3, 2), (3, 1), (5, 1)\}$, no

Determine whether each function is one-to-one. See Example 1.

27. $f(x) = 2x$ yes

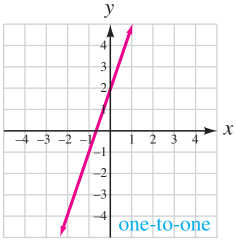
▶ **28.** $f(x) = |x|$ no

29. $f(x) = x^4$ no

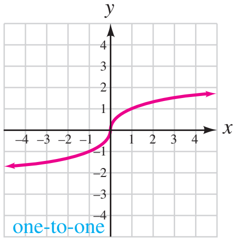
▶ **30.** $f(x) = x^3 + 1$ yes

Each graph represents a function. Use the horizontal line test to determine whether each function is one-to-one. See Example 2.

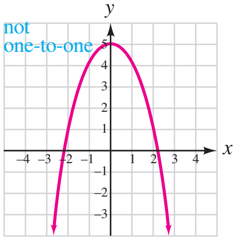
31.



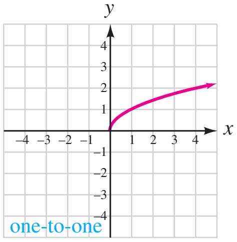
32.



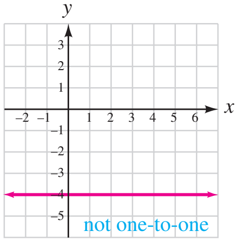
33.



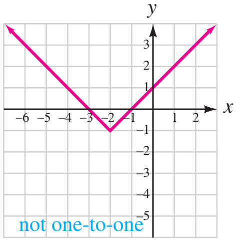
▶ **34.**



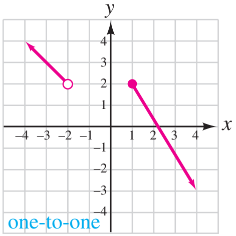
35.



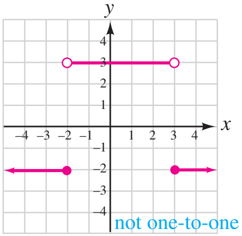
▶ **36.**



37.



38.



Find the inverse of the function and express it using $f^{-1}(x)$ notation. See Example 3.

39. $f(x) = \frac{4}{5}x - 4$
$f^{-1}(x) = \frac{5}{4}x + 5$

▶ **40.** $f(x) = 5x - 1$
$f^{-1}(x) = \frac{1}{5}x + \frac{1}{5}$

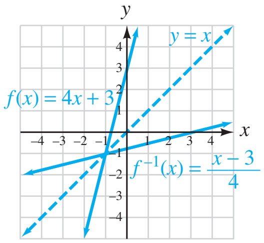
41. $f(x) = \frac{x}{5} + \frac{4}{5}$
$f^{-1}(x) = 5x - 4$

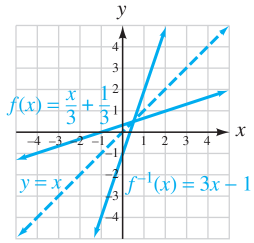
▶ **42.** $f(x) = \frac{x}{3} - \frac{1}{3}$
$f^{-1}(x) = 3x + 1$

Find the inverse of each one-to-one function. Then graph the function and its inverse on one coordinate system. Show the line of symmetry on the graph. See Example 4.

43. $f(x) = 4x + 3$

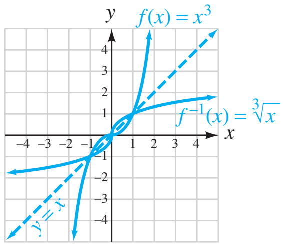
▶ **44.** $f(x) = \frac{x}{3} + \frac{1}{3}$

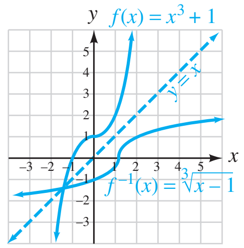




45. $f(x) = x^3$

▶ **46.** $f(x) = x^3 + 1$





Find the inverse of the function and determine whether it is a function. If it is a function, express it using $f^{-1}(x)$ notation. See Examples 5–6.

47. $f(x) = x^2 + 4$
$y = \pm\sqrt{x - 4}$, no

▶ **48.** $f(x) = x^2 + 5$
$y = \pm\sqrt{x - 5}$, no

49. $f(x) = x^3$
$f^{-1}(x) = \sqrt[3]{x}$, yes

▶ **50.** $xy = 4$
$f^{-1}(x) = \frac{4}{x}$, yes

51. $f(x) = |x|$
$x = |y|$, no

52. $f(x) = \sqrt[3]{x}$
$f^{-1}(x) = x^3$, yes

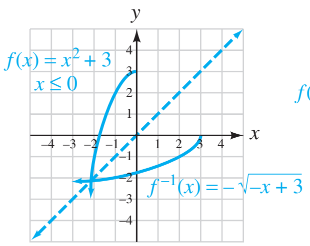
53. $f(x) = 2x^3 - 3$
$f^{-1}(x) = \sqrt[3]{\frac{x + 3}{2}}$, yes

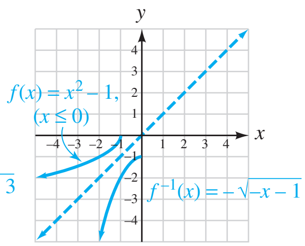
▶ **54.** $f(x) = \frac{3}{x^3} - 1$
$f^{-1}(x) = \sqrt[3]{\frac{3}{x + 1}}$, yes

Find the inverse of each function. Graph the function and its inverse on one set of coordinate axes. See Example 7.

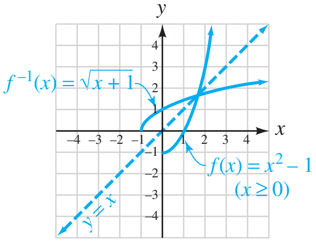
55. $f(x) = -x^2 + 3, (x \le 0)$

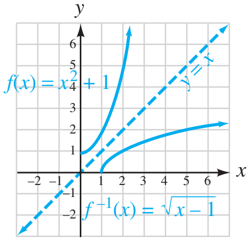
56. $f(x) = -x^2 - 1, (x \le 0)$





57. $f(x) = x^2 - 1, (x \geq 0)$ ▶ **58.** $f(x) = x^2 + 1, (x \geq 0)$





TRY IT YOURSELF

Find the inverse of each one-to-one function and express it using $f^{-1}(x)$ notation.

59. $f(x) = 5x - 1$
$f^{-1}(x) = \frac{x+1}{5}$

60. $f(x) = 2x + 9$
$f^{-1}(x) = \frac{x-9}{2}$

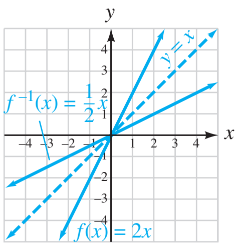
61. $f(x) = \dfrac{x-4}{5}$
$f^{-1}(x) = 5x + 4$

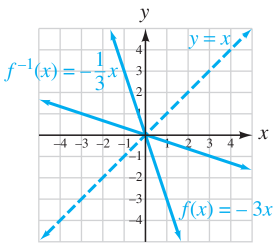
▶ **62.** $f(x) = \dfrac{2x+6}{3}$
$f^{-1}(x) = \frac{3}{2}x - 3$

Find the inverse of each function. Then graph the function and its inverse on one coordinate system. Show the line of symmetry on the graph.

63. $f(x) = 2x$
$f^{-1}(x) = \frac{1}{2}x$

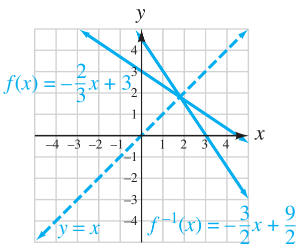
▶ **64.** $f(x) = -3x$
$f^{-1}(x) = -\frac{1}{3}x$

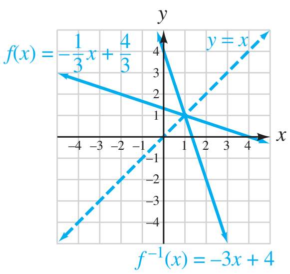




65. $f(x) = -\dfrac{2}{3}x + 3$
$f^{-1}(x) = -\frac{3}{2}x + \frac{9}{2}$

66. $f(x) = -\dfrac{1}{3}x + \dfrac{4}{3}$
$f^{-1}(x) = -3x + 4$



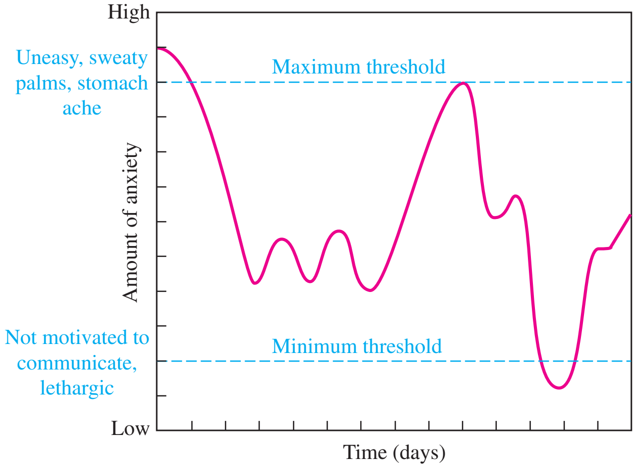


APPLICATIONS

▶ **67.** INTERPERSONAL RELATIONSHIPS Feelings of anxiety in a relationship can increase or decrease, depending on what is going on in the relationship. The graph in the next column shows how a person's anxiety might vary as a relationship develops over time.

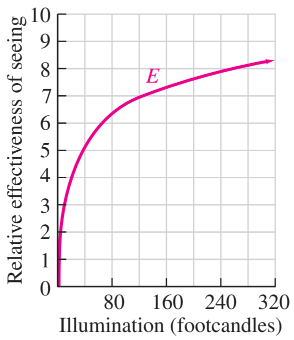
a. Is this the graph of a function? Is its inverse a function?
yes, no

b. Does each anxiety level correspond to exactly one point in time? Use the dashed lined labeled *Maximum threshold* to explain.
No. Twice during this period, the person's anxiety level was at the maximum threshold value.



Source: Gudykunst, *Building Bridges: Interpersonal Skills for a Changing World* (Houghton Mifflin, 1994)

▶ **68.** LIGHTING LEVELS The ability of the eye to see detail increases as the level of illumination increases. This relationship can be modeled by a function E, whose graph is shown.



Based on information from *World Book Encyclopedia*

a. From the graph, determine $E(240)$. 7.9

b. Is function E one-to-one? Does E have an inverse? yes, yes

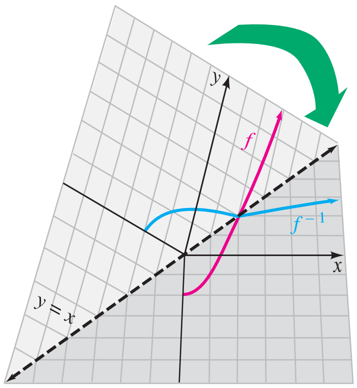
c. If the effectiveness of seeing in an office is 7, what is the illumination in the office? How can this question be asked using inverse function notation? 120 footcandles; What is $E^{-1}(7)$?

WRITING

69. What does it mean when we say that one graph is symmetric to the other about the line $y = x$?

70. Explain the purpose of the horizontal line test.

71. In the illustration below, a function f and its inverse f^{-1} have been graphed on the same coordinate system. Explain what concept can be demonstrated by folding the graph paper on the dotted line.



72. Write in words how to read the notation:

$$f^{-1}(x) = \frac{1}{2}x - 3$$

REVIEW

Simplify each expression and write the result in the form $a + bi$.

73. $3 - \sqrt{-64}$
$3 - 8i$

74. $(2 - 3i) + (4 + 5i)$
$6 + 2i$

75. $(3 + 4i)(2 - 3i)$
$18 - i$

76. $\dfrac{6 + 7i}{3 - 4i}$
$-\frac{2}{5} + \frac{9}{5}i$

77. $(6 - 8i)^2$
$-28 - 96i$

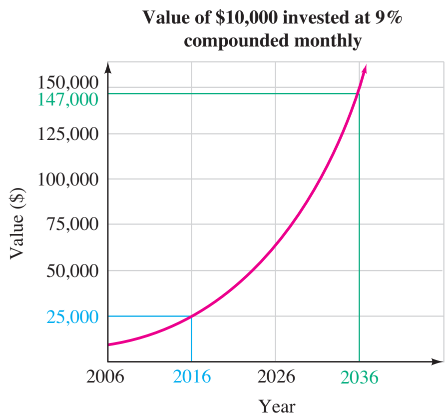
78. i^{100}
$1 + 0i$

Objectives

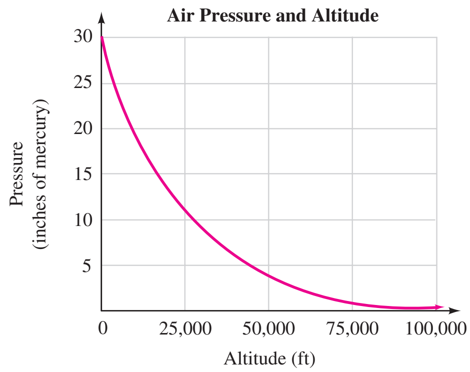
1. Simplify expressions containing irrational exponents.
2. Define exponential functions.
3. Graph exponential functions.
4. Graph exponential functions with vertical or horizontal translations.
5. Use exponential functions in applications involving growth or decay.

SECTION 9.3
Exponential Functions

The graph in figure (a) below shows the balance in a bank account in which \$10,000 was invested in 2006 at 9%, compounded monthly. The graph shows that in the year 2016, the value of the account will be approximately \$25,000, and in the year 2036, the value will be approximately \$147,000. The rapidly rising red curve is the graph of a function called an *exponential function.*



(a)



(b)

If you have ever climbed a high mountain or gone up in an airplane that does not have a pressurized cabin, you have probably felt the effects of low air pressure. The graph in figure (b) shows how atmospheric pressure decreases with increasing altitude. The rapidly falling red curve is also the graph of an *exponential function.*

Exponential functions are used to model many other situations, such as population growth, the spread of an epidemic, the temperature of a heated object as it cools, and radioactive decay. Before we can discuss exponential functions in more detail, we must define irrational exponents.

1 Simplify expressions containing irrational exponents.

We have discussed expressions of the form b^x, where x is a rational number.

$8^{1/2}$ means "the square root of 8."

$5^{1/3}$ means "the cube root of 5."

$3^{-2/5} = \frac{1}{3^{2/5}}$ means "the reciprocal of the fifth root of 3^2."

To give meaning to b^x when x is an irrational number, we consider the expression

$5^{\sqrt{2}}$ where $\sqrt{2}$ is the irrational number 1.414213562 . . .

We can successively approximate $5^{\sqrt{2}}$ by the following rational powers:

$5^{1.4}$, $5^{1.41}$, $5^{1.414}$, $5^{1.4142}$, $5^{1.41421}$, . . .

Using concepts from advanced mathematics, it can be shown that there is exactly one number that these powers approach. We define $5^{\sqrt{2}}$ to be that number. This process can be used to approximate $5^{\sqrt{2}}$ as many decimal places as desired.

Any other positive irrational exponent can be defined in the same manner, and negative irrational exponents can be defined using reciprocals. Thus, if b is positive, b^x has meaning for any real number x.

We can use a calculator to obtain a very good approximation of an exponential expression with an irrational exponent.

Using Your CALCULATOR **Evaluating Exponential Expressions**

To find the value of $5^{\sqrt{2}}$ with a scientific calculator, we enter these numbers and press these keys:

5 $\boxed{y^x}$ 2 $\boxed{\sqrt{}}$ = 9.738517742

With a graphing calculator, we enter these numbers and press these keys:

5 $\boxed{\wedge}$ $\boxed{\text{2nd}}$ $\boxed{\sqrt{}}$ 2 $\boxed{)}$ $\boxed{\text{ENTER}}$

5^√(2)
9.738517742

It can be shown that all of the familiar rules of exponents are also true for irrational exponents.

EXAMPLE 1 Simplify: **a.** $\left(5^{\sqrt{2}}\right)^{\sqrt{2}}$ **b.** $b^{\sqrt{3}} \cdot b^{\sqrt{12}}$

Strategy We will use the power rule $(x^m)^n = x^{mn}$ to simplify part a and the product rule $x^m x^n = x^{m+n}$ to simplify part b.

WHY These rules for exponents hold true for irrational exponents.

Solution

a. $(5^{\sqrt{2}})^{\sqrt{2}} = 5^{\sqrt{2}\sqrt{2}}$ Keep the base and multiply the exponents.

$= 5^2$ $\sqrt{2}\sqrt{2} = \sqrt{4} = 2$.

$= 25$

b. $b^{\sqrt{3}} \cdot b^{\sqrt{12}} = b^{\sqrt{3}+\sqrt{12}}$ Keep the base and add the exponents.

$= b^{\sqrt{3}+2\sqrt{3}}$ Simplify: $\sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$.

$= b^{3\sqrt{3}}$ Combine like radicals: $\sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$.

Self Check 1

Simplify:

a. $(3^{\sqrt{2}})^{\sqrt{8}}$ 81

b. $b^{\sqrt{2}} \cdot b^{\sqrt{18}}$ $b^{4\sqrt{2}}$

Now Try **Problems 21 and 27**

Teaching Example 1 Simplify:
a. $(2^{\sqrt{3}})^{\sqrt{3}}$ **b.** $x^{\sqrt{3}} \cdot x^{\sqrt{75}}$
Answers:
a. 8 **b.** $x^{6\sqrt{3}}$

2 Define exponential functions.

If $b > 0$ and $b \neq 1$, the function $f(x) = b^x$ is an **exponential function.** Since x can be any real number, its domain is the set of real numbers. This is the interval $(-\infty, \infty)$. Because b is positive, the value of $f(x)$ is positive, and the range is the set of positive numbers. This is the interval $(0, \infty)$.

Since $b \neq 1$, an exponential function cannot be the constant function $f(x) = 1^x$, in which $f(x) = 1$ for every real number x.

Exponential Functions

An **exponential function with base b** is defined by the equation

$$f(x) = b^x \quad \text{or} \quad y = b^x$$

where $b > 0$, $b \neq 1$, and x is a real number.

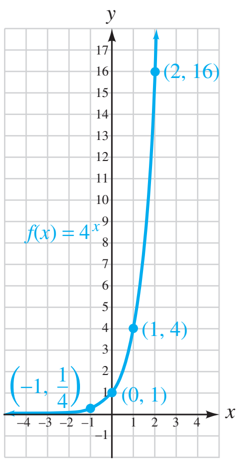
The **domain** of $f(x) = b^x$ is the interval $(-\infty, \infty)$, and the **range** is the interval $(0, \infty)$.

3 Graph exponential functions.

Since the domain and range of $f(x) = b^x$ are sets of real numbers, we can graph exponential functions on a rectangular coordinate system.

Self Check 2

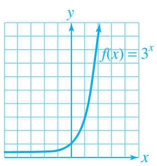
Graph: $f(x) = 4^x$



Now Try **Problem 33**

Teaching Example 2 Graph: $f(x) = 3^x$

Answer:



EXAMPLE 2 Graph: $f(x) = 2^x$

Strategy We will graph $f(x) = 2^x$ by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

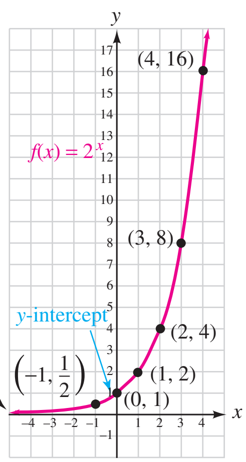
To graph $f(x) = 2^x$, we make a table of values for the function, plot the points, and join them with a smooth curve, as shown in the figure below. For example, if $x = -1$, we have

$$f(x) = 2^x$$
$$f(-1) = 2^{-1} \quad \text{Substitute } -1 \text{ for } x.$$
$$= \frac{1}{2}$$

The point $\left(-1, \frac{1}{2}\right)$ is on the graph of $f(x) = 2^x$.

$f(x) = 2^x$		
x	$f(x)$	$(x, f(x))$
-1	$\frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)
4	16	(4, 16)

The graph steadily approaches the x-axis, but never touches or crosses it.



From the graph, we can verify that the domain of $f(x) = 2^x$ is the interval $(-\infty, \infty)$ and the range is the interval $(0, \infty)$. Since the graph passes the horizontal line test, the function is one-to-one.

Note that as x decreases, the values of $f(x)$ decrease and approach 0. Thus, the x-axis is a horizontal asymptote of the graph. The graph does not have an x-intercept, the y-intercept is $(0, 1)$, and the graph passes through the point $(1, 2)$.

Success Tip We have previously graphed the linear function $f(x) = 2x$ and the squaring function $f(x) = x^2$. For the exponential function $f(x) = 2^x$, note that the variable is in the exponent.

EXAMPLE 3

Graph: $f(x) = \left(\frac{1}{2}\right)^x$

Strategy We will graph $f(x) = \left(\frac{1}{2}\right)^x$ by creating a table of function values and plotting the corresponding ordered pairs.

WHY After drawing a smooth curve through the plotted points, we will have the graph.

Solution

We make a table of values for the function. For example, if $x = -4$, we have

$$f(x) = \left(\frac{1}{2}\right)^x$$

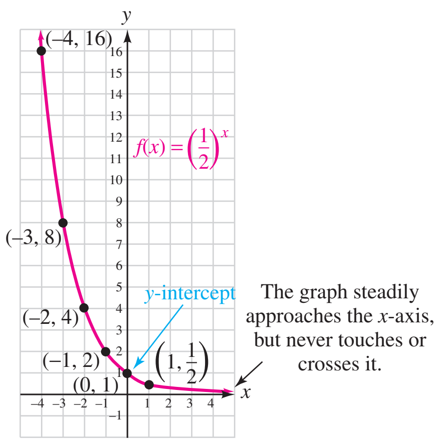
$$f(-4) = \left(\frac{1}{2}\right)^{-4} \quad \text{Substitute } -4 \text{ for } x.$$

$$= \left(\frac{2}{1}\right)^4$$

$$= 16$$

The point $(-4, 16)$ is on the graph of $f(x) = \left(\frac{1}{2}\right)^x$. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ appears in the figure below.

$f(x) = \left(\frac{1}{2}\right)^x$		
x	$f(x)$	$(x, f(x))$
-4	16	$(-4, 16)$
-3	8	$(-3, 8)$
-2	4	$(-2, 4)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	$\frac{1}{2}$	$\left(1, \frac{1}{2}\right)$

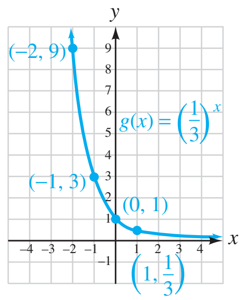


From the graph, we can verify that the domain of $f(x) = \left(\frac{1}{2}\right)^x$ is the interval $(-\infty, \infty)$ and the range is the interval $(0, \infty)$. Since the graph passes the horizontal line test, the function is one-to-one.

Note that as x increases, the values of $f(x)$ decrease and approach 0. Thus, the x-axis is a horizontal asymptote of the graph. The graph does not have an x-intercept, the y-intercept is $(0, 1)$, and the graph passes through the point $\left(1, \frac{1}{2}\right)$.

Self Check 3

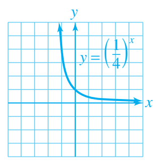
Graph: $g(x) = \left(\frac{1}{3}\right)^x$



Now Try **Problem 35**

Teaching Example 3 Graph: $f(x) = \left(\frac{1}{4}\right)^x$

Answer:



Examples 2 and 3 illustrate the following properties of exponential functions.

Properties of Exponential Functions

The domain of the exponential function $f(x) = b^x$ is the interval $(-\infty, \infty)$.

The range is the interval $(0, \infty)$.

The graph has a y-intercept of $(0, 1)$.

The x-axis (the line $y = 0$) is an asymptote of the graph.

The graph of $f(x) = b^x$ passes through the point $(1, b)$.

In Example 2 (where $b = 2$), the values of y increase as the values of x increase. Since the graph rises as we move to the right, we call the function an *increasing function*. When $b > 1$, the larger the value of b, the steeper the curve.

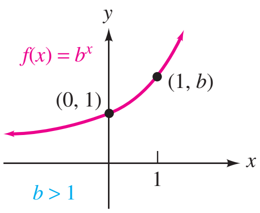
In Example 3 $\left(\text{where } b = \frac{1}{2}\right)$, the values of y decrease as the values of x increase. Since the graph drops as we move to the right, we call the function a *decreasing function*. When $0 < b < 1$, the smaller the value of b, the steeper the curve.

In general, the following is true.

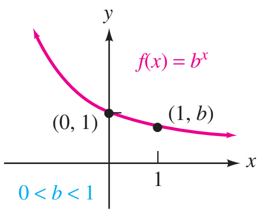
Increasing and Decreasing Functions

If $b > 1$, then $f(x) = b^x$ is an **increasing function.**

If $0 < b < 1$, then $f(x) = b^x$ is a **decreasing function.**



Increasing function

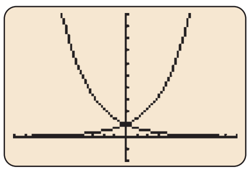


Decreasing function

An exponential function with base b is either increasing (for $b > 1$) or decreasing ($0 < b < 1$). Since different real numbers x determine different values of b^x, exponential functions are one-to-one.

Using Your CALCULATOR Graphing Exponential Functions

To use a graphing calculator to graph $f(x) = \left(\frac{2}{3}\right)^x$ and $g(x) = \left(\frac{3}{2}\right)^x$, we enter the right-hand sides of the equations after the symbols $Y_1 =$ and $Y_2 =$. The screen will show the following equations.



$Y_1 = (2/3)$ ^X

$Y_2 = (3/2)$ ^X

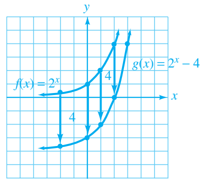
If we use window settings of $[-10, 10]$ for x and $[-2, 10]$ for y and press the $\boxed{\text{GRAPH}}$ key, we will obtain the display shown.

We note that the graph of $f(x) = \left(\frac{2}{3}\right)^x$ passes through the points (0, 1) and $\left(1, \frac{2}{3}\right)$. Since $\frac{2}{3} < 1$, the function is decreasing.

The graph of $g(x) = \left(\frac{3}{2}\right)^x$ passes through the points (0, 1) and $\left(1, \frac{3}{2}\right)$. Since $\frac{3}{2} > 1$, the function is increasing.

Since both graphs pass the horizontal line test, each function is one-to-one.

Teaching Example 4 Graph $f(x) = 2^x$ and $g(x) = 2^x - 4$ on one set of axes and describe the translation.
Answer:
The graph of $f(x) = 2^x$ is translated 4 units downward.



The graphs of many functions are translations of basic exponential function graphs.

4 Graph exponential functions with vertical and horizontal translations.

EXAMPLE 4 On one set of axes, graph $f(x) = 2^x$ and $g(x) = 2^x + 3$, and describe the translation.

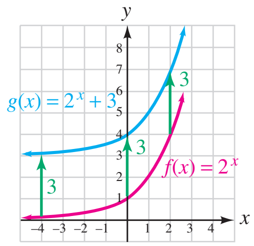
Strategy We will graph $f(x) = 2^x$. Then we will graph $g(x) = 2^x + 3$ by translating the graph of $f(x) = 2^x$ up 3 units.

WHY The addition of 3 in $g(x) = 2^x + 3$ causes a vertical shift of the graph of the base-2 exponential function 3 units upward.

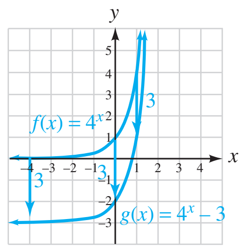
Solution
The graph of $g(x) = 2^x + 3$ is identical to the graph of $f(x) = 2^x$, except that it is translated 3 units upward. We note that the asymptote of the graph of $g(x) = 2^x + 3$ is the line $y = 3$.

$f(x) = 2^x$		
x	$f(x)$	$(x, f(x))$
-4	$\frac{1}{16}$	$\left(-4, \frac{1}{16}\right)$
0	1	(0, 1)
2	4	(2, 4)

$g(x) = 2^x + 3$		
x	$g(x)$	$(x, g(x))$
-4	$3\frac{1}{16}$	$\left(-4, 3\frac{1}{16}\right)$
0	4	(0, 4)
2	7	(2, 7)



Self Check 4
Graph $f(x) = 4^x$ and $g(x) = 4^x - 3$, and describe the translation.



Now Try Problem 44

Self Check 4 Answer
The graph of $f(x) = 4^x$ is translated 3 units downward.

EXAMPLE 5 On one set of axes, graph $f(x) = 2^x$ and $g(x) = 2^{x+3}$, and describe the translation.

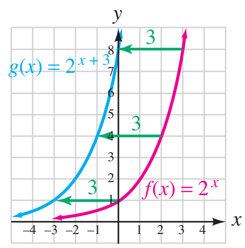
Strategy We will graph $f(x) = 2^x$. Then we will graph $g(x) = 2^{x+3}$ by translating the graph of $f(x) = 2^x$ left 3 units.

WHY The addition of 3 in $g(x) = 2^{x+3}$ causes a horizontal shift of the graph of the base-2 exponential function 3 units left.

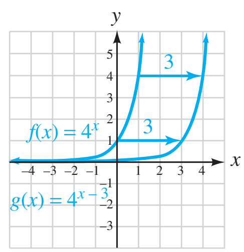
Solution
The graph of $g(x) = 2^{x+3}$ is identical to the graph of $f(x) = 2^x$, except that it is translated 3 units to the left.

$f(x) = 2^x$		
x	$f(x)$	$(x, f(x))$
-1	$\frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	1	(0, 1)
1	2	(1, 2)

$g(x) = 2^{x+3}$		
x	$g(x)$	$(x, g(x))$
-1	4	$(-1, 4)$
0	8	(0, 8)
1	16	(1, 16)



Self Check 5
On one set of axes, graph $f(x) = 4^x$ and $g(x) = 4^{x-3}$, and describe the translation.

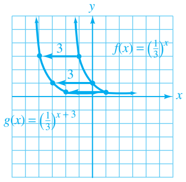


Now Try Problem 46

Self Check 5 Answer
The graph of $f(x) = 4^x$ is translated 3 units to the right.

Teaching Example 5 On one set of axes graph $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^{x+3}$ and describe the translation.

Answer:
The graph of $f(x) = \left(\frac{1}{3}\right)^x$ is translated 3 units left.



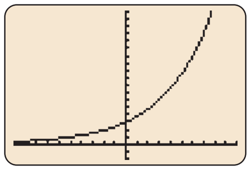
The graphs of $f(x) = kb^x$ and $f(x) = b^{kx}$ are vertical and horizontal stretchings of the graph of $f(x) = b^x$. To graph these functions, we can plot several points and join them with a smooth curve, or we can use a graphing calculator.

Using Your CALCULATOR **Graphing Exponential Functions**

To use a graphing calculator to graph the exponential function $f(x) = 3(2^{x/3})$, we enter the right-hand side of the equation after the symbol $Y_1 =$. The display will show the equation

$$Y_1 = 3(2 \text{ ^ } (X/3))$$

If we use window settings of $[-10, 10]$ for x and $[-2, 18]$ for y and press the graph key, we will obtain the graph shown.



5 Use exponential functions in applications involving growth or decay.

If we deposit $\$P$ in an account paying an annual interest rate r, we can find the amount A in the account at the end of t years by using the formula $A = P + Prt$, or $A = P(1 + rt)$.

Suppose that we deposit \$500 in such an account that pays interest every six months. Then $P = 500$, and after six months $\left(\frac{1}{2} \text{ year}\right)$, the amount in the account will be

$$\begin{aligned} A &= 500(1 + rt) \\ &= 500\left(1 + r \cdot \frac{1}{2}\right) \quad \text{Substitute } \tfrac{1}{2} \text{ for } t. \\ &= 500\left(1 + \frac{r}{2}\right) \end{aligned}$$

The account will begin the second six-month period with a value of $\$500\left(1 + \frac{r}{2}\right)$. After the second six-month period, the amount in the account will be

$$\begin{aligned} A &= P(1 + rt) \\ A &= \left[500\left(1 + \frac{r}{2}\right)\right]\left(1 + r \cdot \frac{1}{2}\right) \quad \text{Substitute } 500\left(1 + \tfrac{r}{2}\right) \text{ for } P \text{ and } \tfrac{1}{2} \text{ for } t. \\ &= 500\left(1 + \frac{r}{2}\right)\left(1 + \frac{r}{2}\right) \\ &= 500\left(1 + \frac{r}{2}\right)^2 \end{aligned}$$

At the end of a third six-month period, the amount in the account will be

$$A = 500\left(1 + \frac{r}{2}\right)^3$$

In this discussion, the earned interest is deposited back in the account and also earns interest. When this is the case, we say that the account is earning **compound interest.** This example illustrates the following formula for compound interest.

Formula for Compound Interest

If $\$P$ is deposited in an account and interest is paid k times a year at an annual rate r, the amount A in the account after t years is given by

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

EXAMPLE 6 ***Saving for College*** To save for college, parents invest \$12,000 for their newborn child in a mutual fund that should average a 10% annual return. If the quarterly dividends are reinvested, how much will be available in 18 years?

Strategy We will substitute 12,000 for P, 0.10 for r, 4 for k, and 18 for t in the formula for compound interest and find A.

WHY The words *compounded quarterly* indicate that we should use the compound interest formula.

Solution

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

$$A = 12{,}000\left(1 + \frac{0.10}{4}\right)^{4(18)} \quad \text{Express } r = 10\% \text{ as a decimal.}$$

$$= 12{,}000(1 + 0.025)^{72} \quad \text{Divide: } \frac{0.10}{4} = 0.025\text{, multiply: } 4(18) = 72.$$

$$= 12{,}000(1.025)^{72} \quad \text{Add: } 1 + 0.025 = 1.025.$$

$$= 71{,}006.74 \quad \text{Use a scientific calculator and press these keys: } 1.025\ y^x\ 72 = \times\ 12{,}000 =.$$

In 18 years, the account will be worth \$71,006.74.

Self Check 6

SAVING FOR COLLEGE In Example 6, how much would be available if the parents invested \$20,000?

\$118,344.56

Now Try Problem 51

Teaching Example 6 SAVING FOR COLLEGE In Example 6, how much would be available if the parents invested \$10,000?

Answer:

\$59,172.28

Using Your CALCULATOR Solving Investment Problems

Suppose \$1 is deposited in an account earning 6% annual interest, compounded monthly. To use a graphing calculator to estimate how much will be in the account in 100 years, we can substitute 1 for P, 0.06 for r, and 12 for k in the formula

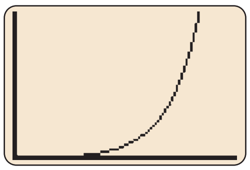
$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

$$A = 1\left(1 + \frac{0.06}{12}\right)^{12t}$$

and simplify to get

$$A = (1.005)^{12t}$$

We now graph $A = (1.005)^{12t}$ using window settings of [0, 120] for t and [0, 400] for A to obtain the graph shown. We can then trace and zoom to estimate that \$1 grows to be approximately \$397 in 100 years. From the graph, we can see that the money grows slowly in the early years and rapidly in the later years.



In business applications, the initial amount of money deposited is often called the **present value** (PV). The amount to which the money will grow is called the **future value** (FV). The interest rate used for each compounding period is the **periodic interest rate** (i), and the number of times interest is compounded is the number of **compounding periods** (n). Using these definitions, we have an alternate formula for compound interest.

Formula for Compound Interest

$$FV = PV(1 + i)^n$$

This alternate formula appears on business calculators. To use this formula to solve Example 6, we proceed as follows:

$$FV = PV(1 + i)^n$$

$$FV = 12{,}000(1 + 0.025)^{72} \quad i = \frac{0.10}{4} = 0.025 \text{ and } n = 4(18) = 72.$$

$$= 71{,}006.74 \quad \text{Use a calculator to evaluate the expression.}$$

Example 6 is an application illustrating exponential growth. In the next example, we see an application of exponential decay.

Self Check 7

INTERNET ACCESS If the trend described in Example 7 continues, what ratio of students to instructional computers with Internet access does the function predict for the year 2006? 3.29

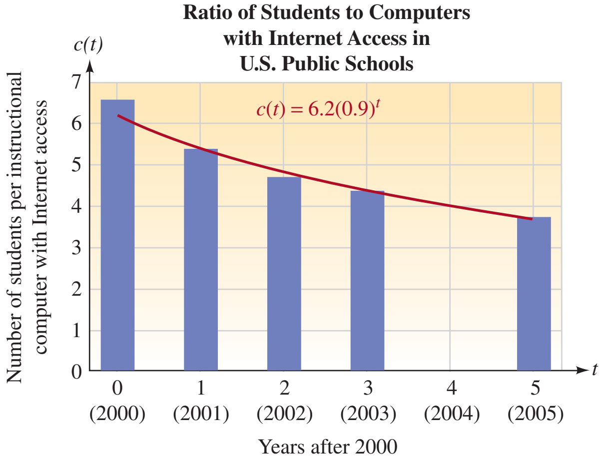
Now Try **Problem 66**

Teaching Example 7 INTERNET ACCESS If the trend described in Example 7 continues, what ratio of students to instructional computers with Internet access does the function predict for the year 2010?
Answer:
2.16

EXAMPLE 7 ***Internet Access*** The exponential function $c(t) = 6.2(0.9)^t$ approximates the ratio of students to instructional computers with Internet access in U.S. public schools, where t is the number of years after 2000 and $0 \le t \le 5$. Use the function to answer the following questions.

a. The National Center for Educational Statistics did not survey the public schools in 2004. What ratio does the exponential function give for 2004?

b. If the current trend continues, what ratio of students to instructional computers with Internet access does the function predict for the year 2014?



Source: National Center for Educational Stastistics

Strategy We will substitute 4 and 14 into the function $c(t) = 6.2(0.9)^t$ for t.

WHY The year 2004 is 4 years after 2000 and the year 2014 is 14 years after 2000.

Solution

a. To determine the ratio for the year 2004 as given by the exponential function, we evaluate it for $t = 4$.

$$c(\mathbf{4}) = 6.2(0.9)^{\mathbf{4}}$$
$$\approx 4.1 \qquad \text{Use a calculator.}$$

In the year 2004, the exponential function predicts the ratio was approximately 4.1 students to each computer with Internet access.

b. To predict the ratio for the year 2014, we evaluate the function for $t = 14$.

$$c(\mathbf{14}) = 6.2(0.9)^{\mathbf{14}}$$
$$\approx 1.4 \qquad \text{Use a calculator.}$$

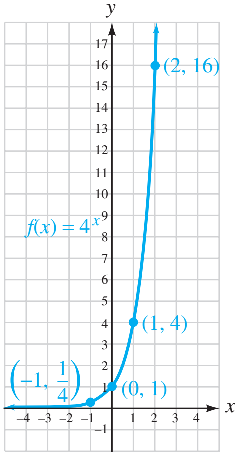
If the trend continues, in 2014, the ratio will be approximately 1.4 students to each computer with Internet access.

Success Tip $c(t) = 6.2(0.9)^t$ is a decreasing function because the base, 0.9, is such that $0 < 0.9 < 1$.

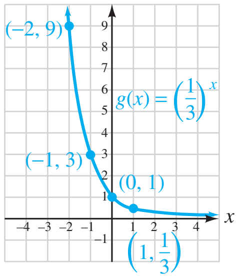
ANSWERS TO SELF CHECKS

1. a. 81 **b.** $b^{4\sqrt{2}}$

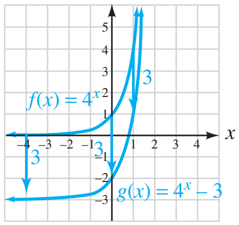
2.



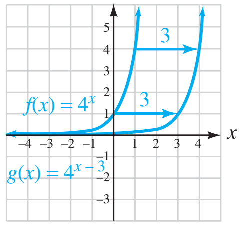
3.



4.



5.



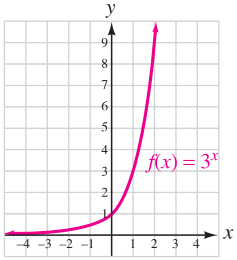
6. \$118,344.56

7. 3.29

SECTION 9.3 STUDY SET

VOCABULARY

Refer to the graph of $f(x) = 3^x$ shown below.

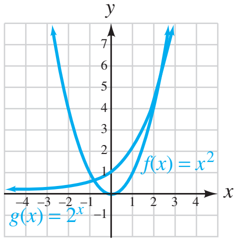


1. What type of function is $f(x) = 3^x$? exponential
2. Find the domain of the function. $(-\infty, \infty)$
3. Find the range of the function. $(0, \infty)$
4. Find the y-intercept of the graph. $(0, 1)$
5. Find the x-intercept of the graph. none
6. Find an asymptote of the graph. the x-axis ($y = 0$)
7. Is $f(x)$ an increasing or a decreasing function? increasing
8. The graph passes through the point $(1, y)$. What is y? 3

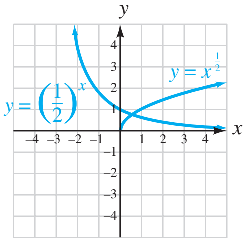
▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

9. Graph the functions $f(x) = x^2$ and $g(x) = 2^x$ on the same set of coordinate axes.



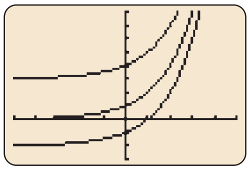
10. Graph $y = x^{1/2}$ and $y = \left(\frac{1}{2}\right)^x$ on the same set of coordinate axes.



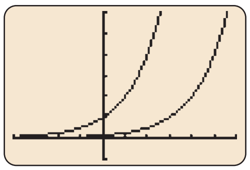
11. What are the two formulas that are used to determine the amount of money in a savings account that is earning compound interest?
$A = P\left(1 + \frac{r}{k}\right)^{kt}$, $FV = PV(1 + i)^n$

▶ **12.** Explain the order in which the expression $20{,}000(1.036)^{72}$ should be evaluated.
First find the power, then multiply.

13. The illustration shows the graph of $f(x) = 2^x$ as well as two vertical translations of that graph. Using the notation $g(x)$ for one translation and $h(x)$ for the other, write the defining equation for each function.
$g(x) = 2^x + 3$, $h(x) = 2^x - 2$



▶ **14.** The illustration shows the graph of $f(x) = 2^x$ as well as a horizontal translation of that graph. Using the notation $g(x)$ for the translation, write its defining equation.
$g(x) = 2^{x-3}$



NOTATION

15. In $A = P\left(1 + \frac{r}{k}\right)^{kt}$, what is the base, and what is the exponent? $\left(1 + \frac{r}{k}\right)$, kt

▶ **16.** For an exponential function of the form $f(x) = b^x$, what are the restrictions on b? $b > 0$, $b \neq 1$

GUIDED PRACTICE

Use a calculator to find each value to four decimal places. See Objective 1.

17. $2^{\sqrt{2}}$ 2.6651

▶ **18.** $7^{\sqrt{2}}$ 15.6729

19. $5^{\sqrt{5}}$ 36.5548

▶ **20.** $6^{\sqrt{3}}$ 22.2740

Simplify each expression. See Example 1.

▶ **21.** $(2^{\sqrt{3}})^{\sqrt{3}}$ 8

▶ **22.** $(3^{\sqrt{5}})^{\sqrt{5}}$ 243

23. $(b^{\sqrt{5}})^{\sqrt{5}}$ b^5

24. $(b^{\sqrt{7}})^{\sqrt{7}}$ b^7

▶ **25.** $7^{\sqrt{3}}7^{\sqrt{12}}$ $7^{3\sqrt{3}}$

▶ **26.** $3^{\sqrt{2}}3^{\sqrt{18}}$ $3^{4\sqrt{2}}$

27. $(b^{\sqrt{5}})^{\sqrt{50}}$ $b^{5\sqrt{10}}$

28. $(b^{\sqrt{2}})^{\sqrt{24}}$ $b^{4\sqrt{3}}$

29. $\dfrac{3^{2\sqrt{7}}}{3^{\sqrt{7}}}$ $3^{\sqrt{7}}$

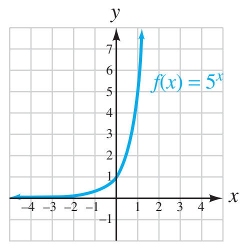
30. $\dfrac{5^{6\sqrt{2}}}{5^{4\sqrt{2}}}$ $5^{2\sqrt{2}}$

31. $5^{-\sqrt{5}}$ $\frac{1}{5^{\sqrt{5}}}$

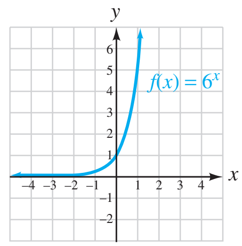
32. $4^{-\sqrt{2}}$ $\frac{1}{4^{\sqrt{2}}}$

Graph each exponential function. See Examples 2–3.

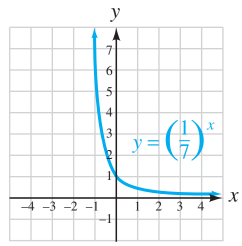
▶ **33.** $f(x) = 5^x$



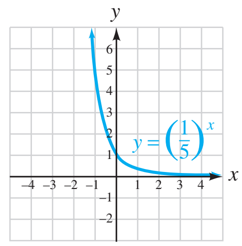
34. $f(x) = 6^x$



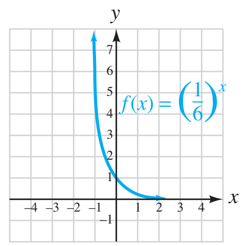
35. $y = \left(\dfrac{1}{7}\right)^x$



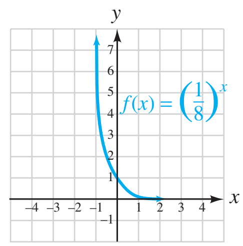
36. $y = \left(\dfrac{1}{5}\right)^x$



37. $f(x) = \left(\dfrac{1}{6}\right)^x$

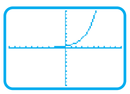


▶ **38.** $f(x) = \left(\dfrac{1}{8}\right)^x$

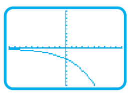


Use a graphing calculator to graph each function. Determine whether the function is an increasing or a decreasing function. See Objective 3.

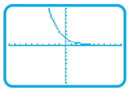
39. $f(x) = \dfrac{1}{2}(3^{x/2})$
increasing



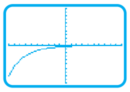
▶ **40.** $f(x) = -3(2^{x/3})$
decreasing



41. $f(x) = 2(3^{-x/2})$
decreasing

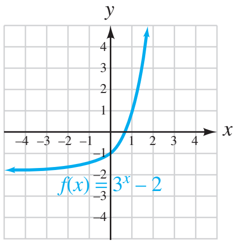


▶ **42.** $f(x) = -\frac{1}{4}(2^{-x/2})$
increasing

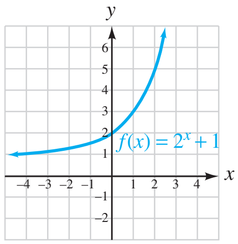


Graph each exponential function by plotting points or using a translation. See Examples 4–5.

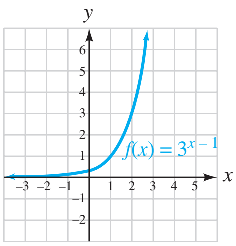
43. $f(x) = 3^x - 2$



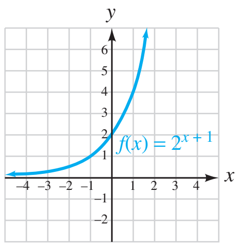
44. $f(x) = 2^x + 1$



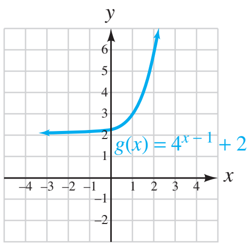
45. $f(x) = 3^{x-1}$



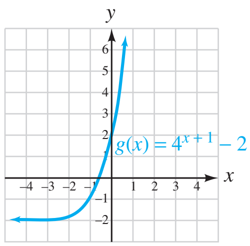
46. $f(x) = 2^{x+1}$



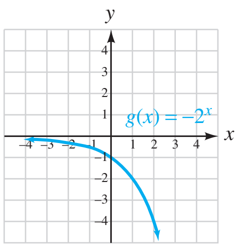
▶ **47.** $g(x) = 4^{x-1} + 2$



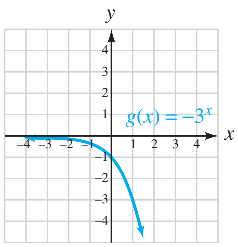
48. $g(x) = 4^{x+1} - 2$



49. $g(x) = -2^x$



50. $g(x) = -3^x$



APPLICATIONS

Assume that there are no deposits or withdrawals.

51. COMPOUND INTEREST An initial deposit of \$10,000 earns 8% interest, compounded quarterly. How much will be in the account after 10 years?
\$22,080.40

▶ **52.** COMPOUND INTEREST An initial deposit of \$10,000 earns 8% interest, compounded monthly. How much will be in the account after 10 years?
\$22,196.40

53. COMPARING INTEREST RATES How much more interest could \$1,000 earn in 5 years, compounded quarterly, if the annual interest rate were $5\frac{1}{2}\%$ instead of 5%?
\$32.03

▶ **54.** COMPARING SAVINGS PLANS Which institution in the illustration provides the better investment?
Fidelity

Fidelity Savings & Loan
Earn 5.25%
compounded monthly

Union Trust
Money Market Account
paying 5.35%
compounded annually

▶ **55.** COMPOUND INTEREST If \$1 had been invested on July 4, 1776, at 5% interest, compounded annually, what would it be worth on July 4, 2076?
\$2,273,996.13

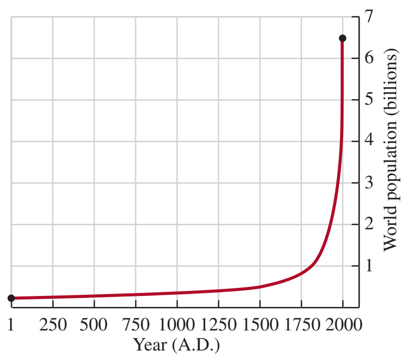
▶ **56.** FREQUENCY OF COMPOUNDING \$10,000 is invested in each of two accounts, both paying 6% annual interest. In the first account, interest compounds quarterly, and in the second account, interest compounds daily. Find the difference between the accounts after 20 years.
\$291.27

▶ **57.** WORLD POPULATION See the illustration below.

a. Estimate when the world's population reached $\frac{1}{2}$ billion and when it reached 1 billion.
about 1500, about 1825

b. Estimate the world's population in the year 2000.
6.5 billion

c. What type of function does it appear could be used to model the population growth?
exponential



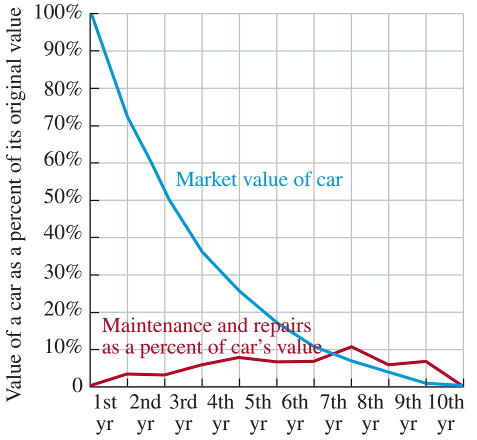
Source: *The Blue Planet* (Wiley, 1995)

58. VALUE OF A CAR The following graph shows how the value of the average car depreciates as a percent of its original value over a 10-year period. It also shows the yearly maintenance costs as a percent of the car's value.

a. When is the car worth half of its purchase price?
at the end of the 2nd year

b. When is the car worth a quarter of its purchase price?
at the end of the 4th year

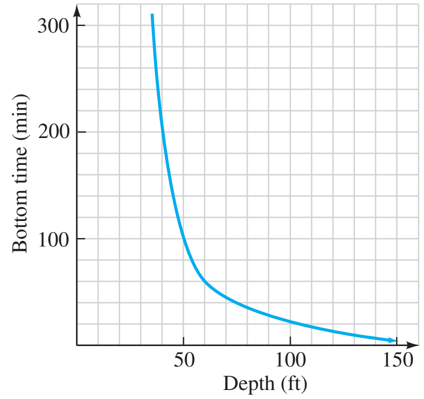
c. When do the average yearly maintenance costs surpass the value of the car?
during the 7th year



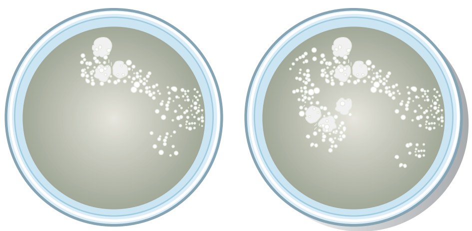
Source: U.S. Department of Transportation

59. DIVING *Bottom time* is the time a scuba diver spends descending plus the actual time spent at a certain depth. Complete the graph in the next column of the bottom time limits given in the table below.

Bottom Time Limits			
Depth (ft)	**Bottom time (min)**	**Depth (ft)**	**Bottom time (min)**
30	no limit	80	40
35	310	90	30
40	200	100	25
50	100	110	20
60	60	120	15
70	50	130	10



60. BACTERIAL CULTURES A colony of 6 million bacteria was determined to be growing in the culture medium shown in illustration (a). If the population P of bacteria after t hours is given by the function $P(t) = 6{,}000{,}000(2.3)^t$, find the population in the culture later in the day using the information given in illustration (b). 167,904,600



Date: 8-16-08
Time: 6:00 AM
(a)

Date: 8-16-08
Time: 10:00 AM
(b)

61. RADIOACTIVE DECAY A radioactive material decays according to the formula $A = A_0\left(\frac{2}{3}\right)^t$, where A_0 is the initial amount present and t is measured in years. Find the amount present in 5 years. $\frac{32}{243}A_0$

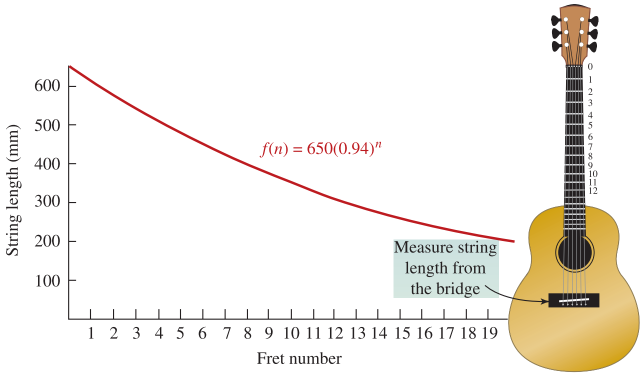
62. DISCHARGING A BATTERY The charge remaining in a battery decreases as the battery discharges. The charge C (in coulombs) after t days is given by the formula $C = (3 \times 10^{-4})(0.7)^t$. Find the charge after 5 days. 5.0421×10^{-5} coulombs

63. POPULATION GROWTH The population of North Rivers is decreasing exponentially according to the formula $P = 3{,}745(0.93)^t$, where t is measured in years from the present date. Find the population in 6 years, 9 months. 2,295

64. SALVAGE VALUE A small business purchased a computer for \$4,700. It is expected that its value each year will be 75% of its value in the preceding year. If the business disposes of the computer after 5 years, find its salvage value (the value after 5 years). \$1,115.33

▶ **65.** LOUISIANA PURCHASE In 1803, the United States negotiated the Louisiana Purchase with France. The country doubled its territory by adding 827,000 square miles of land for \$15 million. If the land appreciated at the rate of 6% each year, what would one square mile of land be worth in 2005? about \$2,346,230

66. GUITARS The frets on the neck of a guitar are placed so that pressing a string against them determines the string's vibrating length. The exponential function $f(n) = 650(0.94)^n$ gives the vibrating length (in millimeters) of a string on a certain guitar for the fret number n. Find the length of the vibrating string when a guitarist holds down a string on the 7th fret. about 422 mm



WRITING

67. If world population is increasing exponentially, why is there cause for concern?

▶ **68.** How do the graphs of $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$ differ? How are they similar?

REVIEW

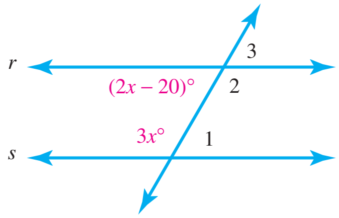
Refer to the illustration below, in which lines r and s are parallel.

69. Find x. 40

▶ **70.** Find the measure of $\angle 1$. 60°

71. Find the measure of $\angle 2$. 120°

▶ **72.** Find the measure of $\angle 3$. 60°



SECTION 9.4
Base-*e* Exponential Functions

Objectives

1. Define *e* and identify the formula for exponential growth/decay.
2. Define the natural exponential function.
3. Graph the natural exponential function.
4. Use base-*e* exponential functions in applications involving growth or decay.

Any positive number can be used as a base of an exponential function. However, some bases are used more often than others. An exponential function that has many applications is one whose base is an irrational number represented by the letter e. In this section, we will show how to evaluate e, graph the base-e exponential function, and discuss one of its applications in analyzing population growth.

1 Define *e* and identify the formula for exponential growth/decay.

If a bank pays interest twice a year, we say that interest is compounded semiannually. If it pays interest four times a year, we say that interest is compounded quarterly. If it pays interest continuously (infinitely many times in a year), we say that interest is compounded continuously.

To develop the formula for continuous compound interest, we start with the formula

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$ *The formula for compound interest: r is the annual rate and k is the number of times per year interest is paid.*

and let $rn = k$. Since r and k are positive numbers, so is n.

$$A = P\left(1 + \frac{r}{rn}\right)^{rnt}$$

We can then simplify the fraction $\frac{r}{rn}$ and use the commutative property of multiplication to change the order of the factors in the exponent.

$$A = P\left(1 + \frac{1}{n}\right)^{nrt}$$

Finally, we can use a property of exponents to write the formula as

$$\textbf{(1)}\quad A = P\left[\left(1 + \frac{1}{n}\right)^{n}\right]^{rt} \qquad \text{Use the property } a^{mn} = (a^m)^n.$$

To find the value of $\left(1 + \frac{1}{n}\right)^n$, we evaluate it for several values of n, as shown.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
4	2.44140625...
12	2.61303529...
365	2.71456748...
1,000	2.71692393...
100,000	2.71826830...
1,000,000	2.71828137...

The results illustrate that as n gets larger, the value of $\left(1 + \frac{1}{n}\right)^n$ approaches the number 2.71828.... This number is called e, which has the following value.

$$e = 2.718281828459\ldots$$

In continuous compound interest, k (the number of compoundings) is infinitely large. Since k, r, and n are all positive, r is a fixed rate, and $k = rn$, as k gets very large (approaches infinity), so does n. Therefore, we can replace $\left(1 + \frac{1}{n}\right)^n$ in Equation 1 with e to get

$$A = Pe^{rt}$$

Formula for Exponential Growth/Decay

If a quantity P increases or decreases at an annual rate r, compounded continuously, the amount A after t years is given by

$$A = Pe^{rt}$$

If time is measured in years, then r is called the **annual growth rate.** If r is negative, the growth represents a decrease.

2 Define the natural exponential function.

Of all possible bases for an exponential function, e is the most convenient for problems involving growth or decay. Since these situations often occur in natural settings, we call $f(x) = e^x$ the *natural exponential function.*

The Natural Exponential Function

The function defined by $f(x) = e^x$ or $y = e^x$ is the **natural exponential function** (or the **base-*e* exponential function**), where $e = 2.71828\ldots$. The domain of $f(x) = e^x$ is the interval $(-\infty, \infty)$. The range is the interval $(0, \infty)$.

The $\boxed{e^x}$ key on a calculator is used to enter the natural exponential function.

Using Your CALCULATOR **The Natural Exponential Function Key**

To compute the amount to which \$12,000 will grow if invested for 18 years at 10% annual interest, compounded continuously, we substitute 12,000 for P, 0.10 for r, and 18 for t in the formula for exponential growth:

$$A = Pe^{rt}$$
$$A = 12{,}000e^{0.10(18)}$$
$$= 12{,}000e^{1.8}$$

To evaluate this expression, we enter these numbers and press these keys on a scientific calculator:

1.8 $\boxed{e^x}$ $\boxed{\times}$ 12000 $\boxed{=}$ 72595.76957

Using a graphing calculator, we enter these numbers and press these keys:

12000 $\boxed{\times}$ $\boxed{\text{2nd}}$ $\boxed{e^x}$ 1.8 $\boxed{)}$ $\boxed{\text{ENTER}}$

12000 × e^(1.8)
72595.76957

After 18 years, the account will contain \$72,595.77. This is \$1,589.03 more than the result in Example 6 in the previous section, where interest was compounded quarterly.

EXAMPLE 1 ***Investing*** If \$25,000 accumulates interest at an annual rate of 8%, compounded continuously, find the balance in the account in 50 years.

Strategy We will substitute 25,000 for P, 0.08 for r, and 50 for t in the formula $A = Pe^{rt}$ and calculate the value for A.

WHY The words *compounded continuously* indicate that we should use the exponential growth/decay formula.

Solution

$A = Pe^{rt}$ This is the exponential growth formula.

$A = 25{,}000e^{0.08(50)}$ 8% = 0.08.

$= 25{,}000e^4$

$\approx 1{,}364{,}953.75$ Use a calculator.

In 50 years, the balance will be \$1,364,953.75—more than a million dollars.

Self Check 1

INVESTING In Example 1, find the balance in 60 years. \$3,037,760.44

Now Try **Problem 23**

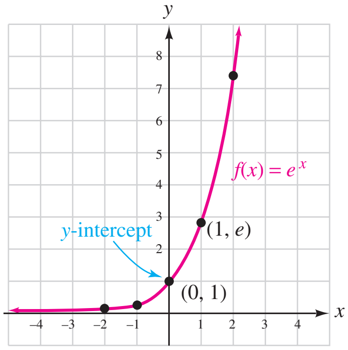
Teaching Example 1 INVESTING In Example 1, find the balance in 55 years.
Answer:
\$2,036,271.72

3 Graph the natural exponential function.

To graph $f(x) = e^x$, we plot several points and join them with a smooth curve, as shown in the figure on the next page. By looking at the graph, we can verify that it is an increasing function. The domain is the interval $(-\infty, \infty)$, and the range is the interval $(0, \infty)$. Note that as x decreases, the values of $f(x)$ decrease and approach 0. Thus, the x-axis is a horizontal asymptote of the graph.

$f(x) = e^x$		
x	$f(x)$	$(x, f(x))$
-2	0.1	$(-2, 0.1)$
-1	0.4	$(-1, 0.4)$
0	1	$(0, 1)$
1	2.7	$(1, 2.7)$
2	7.4	$(2, 7.4)$

The outputs can be found using the e^x key on a calculator. They are rounded to the nearest tenth to make point plotting easier.



We can illustrate the effects of vertical and horizontal translations of the natural exponential function by using a graphing calculator.

Using Your CALCULATOR Translations of the Natural Exponential Function

Figure (a) shows the calculator graphs of $f(x) = e^x$, $g(x) = e^x + 5$, and $h(x) = e^x - 3$. To graph these functions with window settings of $[-3, 6]$ for x and $[-5, 15]$ for y, we enter the right-hand sides of the equations after the symbols $Y_1 =$, $Y_2 =$, and $Y_3 =$. The display will show

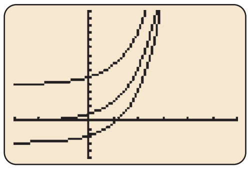
$$Y_1 = e^\wedge(X)$$
$$Y_2 = e^\wedge(X) + 5$$
$$Y_3 = e^\wedge(X) - 3$$

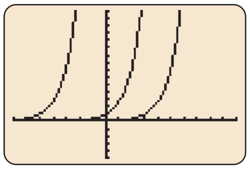
After graphing these functions, we can see that the graph of $g(x) = e^x + 5$ is 5 units above the graph of $f(x) = e^x$, and that the graph of $h(x) = e^x - 3$ is 3 units below the graph of $f(x) = e^x$.

Figure (b) shows the calculator graphs of $f(x) = e^x$, $g(x) = e^{x+5}$, and $h(x) = e^{x-3}$. To graph these functions with window settings of $[-7, 10]$ for x and $[-5, 15]$ for y, we enter the right-hand sides of the equations after the symbols $Y_1 =$, $Y_2 =$, and $Y_3 =$. The display will show

$$Y_1 = e^\wedge(X)$$
$$Y_2 = e^\wedge(X + 5)$$
$$Y_3 = e^\wedge(X - 3)$$

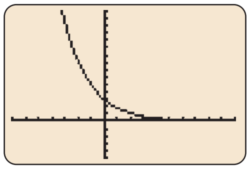
After graphing these functions, we can see that the graph of $g(x) = e^{x+5}$ is 5 units to the left of the graph of $f(x) = e^x$, and that the graph of $h(x) = e^{x-3}$ is 3 units to the right of the graph of $f(x) = e^x$.


(a)


(b)

Using Your CALCULATOR **Graphing Exponential Functions**

The figure shows the calculator graph of $f(x) = 3e^{-x/2}$. To graph this function with window settings of $[-7, 10]$ for x and $[-5, 15]$ for y, we enter the right-hand side of the equation after the symbol $Y_1 =$. The display will show the equation



$$Y_1 = 3(e\wedge(-X/2))$$

Explain why the graph has a y-intercept of (0, 3).

4 Use base-*e* exponential functions in applications involving growth or decay.

An equation based on the natural exponential function provides a model for **population growth.** In the **Malthusian model for population growth,** the future population of a colony is related to the present population by the formula $A = Pe^{rt}$.

EXAMPLE 2 ***City Planning*** The population of a city is currently 15,000, but changing economic conditions are causing the population to decrease 3% each year. If this trend continues, find the population in 30 years.

Strategy We will substitute 15,000 for P, -0.03 for r, and 30 for t in the formula $A = Pe^{rt}$ and calculate the value for A.

WHY Since the population is decreasing 3% each year, the annual growth rate is -3%, or -0.03 in decimal form.

Solution

$$\begin{aligned} A &= Pe^{rt} \\ A &= 15{,}000e^{-0.03(30)} \\ &= 15{,}000e^{-0.9} \\ &\approx 6.099 \end{aligned}$$

In 30 years, the expected population will be 6,099.

Self Check 2

CITY PLANNING In Example 2, find the population in 50 years. 3,347

Now Try **Problem 34**

Teaching Example 2 CITY PLANNING In Example 2, find the population in 40 years.
Answer:
4,518

The English economist Thomas Robert Malthus (1766–1834) pioneered in population study. He believed that poverty and starvation were unavoidable, because the human population tends to grow exponentially, but the food supply tends to grow linearly.

EXAMPLE 3 ***Food Shortages*** Suppose that a country with a population of 1,000 people is growing exponentially according to the formula

$$P = 1{,}000e^{0.02t} \quad \text{The annual growth rate is } 2\% = 0.02.$$

where t is in years. Furthermore, assume that the food supply F, measured in adequate food per day per person, is growing linearly according to the formula

$$F = 30.625t + 2{,}000 \quad (t \text{ is time in years})$$

In how many years will the population outstrip the food supply?

Strategy We will use a graphing calculator to graph these two functions and find the point where the graphs intersect.

WHY This will be the point where the food supply is exactly adequate to feed the population.

Self Check 3

FOOD SHORTAGES In Example 3, suppose that the population grows at a 3% rate. Use a graphing calculator to determine for how many years the food supply will be adequate.

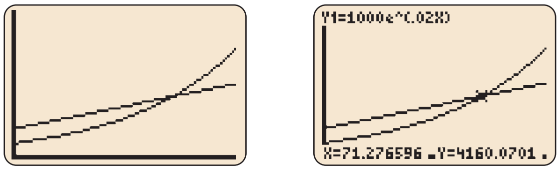
Now Try **Problem 38**

Self Check 3 Answer
about 39 years

Teaching Example 3 FOOD SHORTAGES In Example 3, suppose that the population grows at a 2.5% rate. Use a graphing calculator to determine for how many years the food supply will be adequate.
Answer:
about 51 years

Solution

We can use a graphing calculator with window settings of [0, 100] for x and [0, 10,000] for y. After graphing the functions, we obtain figure (a) on the next page. If we trace, as in figure (b), we can find the point where the two graphs intersect. From the graph, we can see that the food supply will be adequate for about 71 years. At that time, the population of approximately 4,200 people will begin to have problems.



(a) (b)

Self Check 4

BAKING In Example 4, find the temperature of the cake 10 minutes after it is removed from the oven. 104.4°

Now Try **Problem 55**

Teaching Example 4 BAKING In Example 4, find the temperature of the cake 15 minutes after it is removed from the oven.
Answer:
82.8°

EXAMPLE 4 *Baking*

A mother takes a cake out of the oven and sets it on a rack to cool. The function $T(t) = 68 + 220e^{-0.18t}$ gives the cake's temperature in degrees Fahrenheit after it has cooled for t minutes. If her children will be home from school in 20 minutes, will the cake have cooled enough for the children to eat it?

Strategy We will substitute 20 for t in the formula $T(t) = 68 + 220e^{-0.18t}$ and calculate the value for T.

WHY The formula gives the temperature T of the cake in degrees Fahrenheit after it has been cooled for t minutes.

Solution

When the children arrive home, the cake will have cooled for 20 minutes. To find the temperature of the cake at that time, we need to find $T(20)$.

$$\begin{aligned} T(t) &= 68 + 220e^{-0.18t} \\ T(20) &= 68 + 220e^{-0.18(20)} && \text{Substitute 20 for } t. \\ &= 68 + 220e^{-3.6} \\ &\approx 74.0 && \text{Use a calculator.} \end{aligned}$$

When the children return home, the temperature of the cake will be about 74°, and it can be eaten.

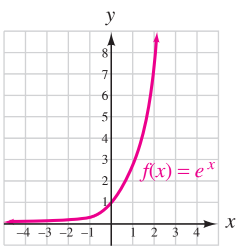
ANSWERS TO SELF CHECKS

1. \$3,037,760.44 **2.** 3,347 **3.** About 39 years **4.** 104.4°

SECTION 9.4 STUDY SET

VOCABULARY

Refer to the graph of $f(x) = e^x$ in the illustration below.



1. What is the name of the function $f(x) = e^x$?
the natural exponential function

▶ **2.** Find the domain of the function.
$(-\infty, \infty)$

3. Find the range of the function.
$(0, \infty)$

▶ **4.** Find the y-intercept of the graph.
(0, 1)

5. Find the x-intercept of the graph.
none

▶ Selected exercises available online at **www.webassign.net/brookscole**

6. Find the asymptote of the graph. the x-axis ($y = 0$)

7. Is f an increasing or a decreasing function? increasing

8. The graph passes through the point $(1, y)$. Find y. e

CONCEPTS

Fill in the blanks.

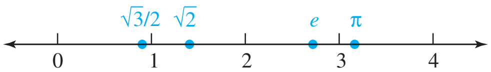
9. In continuous compound interest, the number of compoundings is infinitely large.

10. The formula for continuous compound interest is $A = Pe^{rt}$.

11. To two decimal places, the value of e is 2.72.

12. If n gets larger and larger, the value of $\left(1 + \frac{1}{n}\right)^n$ approaches the value of e.

13. Graph each irrational number on the number line: $\left\{\pi, e, \sqrt{2}, \frac{\sqrt{3}}{2}\right\}$.

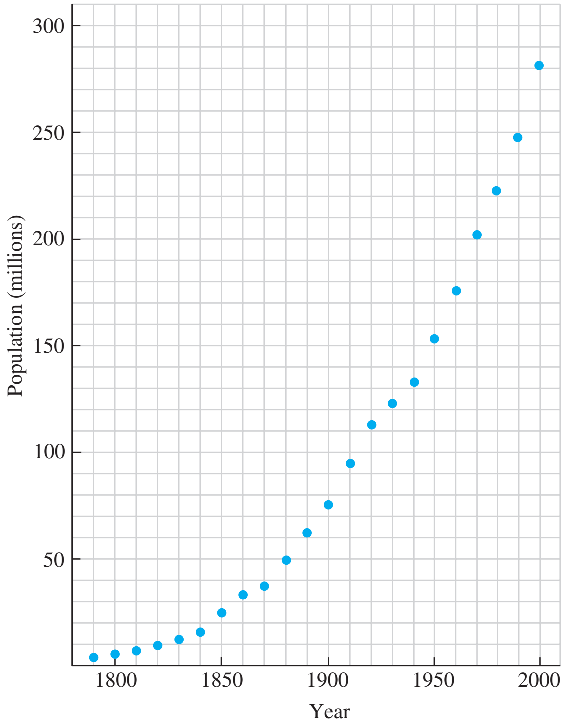


14. Complete the table of values. Round to the nearest hundredth.

x	−2	−1	0	1	2
e^x	0.14	0.37	1	2.72	7.39

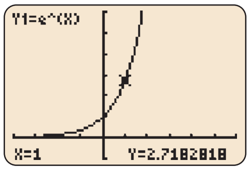
15. POPULATION OF THE UNITED STATES Complete the graph in the next column of the U.S. census population figures shown in the table below (in millions). What type of function does it appear could be used to model the population?
an exponential function

Year	Population	Year	Population
1790	3.9	1900	76.0
1800	5.3	1910	92.2
1810	7.2	1920	106.0
1820	9.6	1930	123.2
1830	12.9	1940	132.1
1840	17.0	1950	151.3
1850	23.1	1960	179.3
1860	31.4	1970	203.3
1870	38.5	1980	226.5
1880	50.1	1990	248.7
1890	62.9	2000	281.4

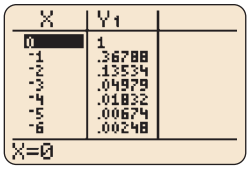


16. What is the Malthusian population growth formula? $A = Pe^{rt}$

17. The function $f(x) = e^x$ is graphed, and the TRACE feature is used. What is the y-coordinate of the point on the graph having an x-coordinate of 1? What is the name given this number? 2.7182818. . .; e



18. The calculator display shows a table of values for $f(x) = e^x$. As x decreases, what happens to the values of $f(x)$ listed in the Y_1 column? Will the value of $f(x)$ ever be 0 or negative? they decrease; no



NOTATION

Evaluate A in the formula $A = Pe^{rt}$ for the following values of r and t.

19. $P = 1{,}000$, $r = 0.09$, and $t = 10$

$$A = 1{,}000\, e^{(0.09)(10)}$$
$$= 1{,}000e^{0.9}$$
$$\approx 1{,}000(2.459603111)$$
$$\approx 2{,}459.603111$$

▶ **20.** $P = 1{,}000$, $r = 0.12$, and $t = 50$

$$A = 1{,}000e^{(0.12)(50)}$$
$$= 1{,}000\,e^{6}$$
$$\approx 1{,}000(403.4287935)$$
$$\approx 403{,}428.7935$$

GUIDED PRACTICE

Find A using the formula $A = Pe^{rt}$ given the following values of P, r, and t. Round to the nearest hundredth. See Example 1.

21. $P = 5{,}000$, $r = 8\%$, $t = 20$ years 24,765.16

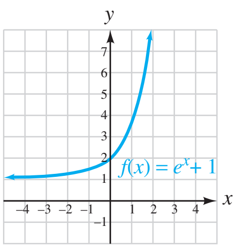
22. $P = 15{,}000$, $r = 6\%$, $t = 40$ years 165,347.65

23. $P = 20{,}000$, $r = 10.5\%$, $t = 50$ years 3,811,325.37

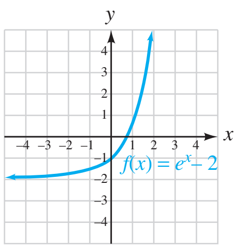
▶ **24.** $P = 25{,}000$, $r = 6.5\%$, $t = 100$ years 16,628,540.83

Graph each function. Check your work with a graphing calculator. Compare each graph to the graph of $f(x) = e^x$. See Objective 3.

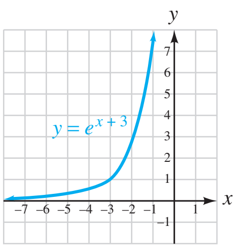
25. $f(x) = e^x + 1$



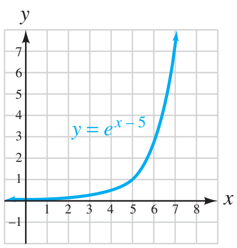
▶ **26.** $f(x) = e^x - 2$



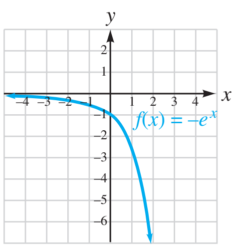
27. $y = e^{x+3}$



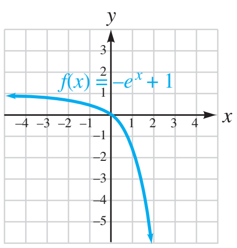
28. $y = e^{x-5}$



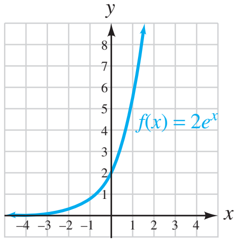
29. $f(x) = -e^x$



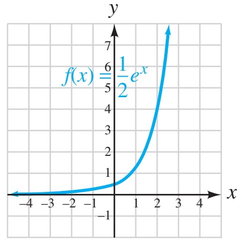
30. $f(x) = -e^x + 1$



31. $f(x) = 2e^x$



▶ **32.** $f(x) = \frac{1}{2}e^x$



Find A using the formula $A = Pe^{rt}$ given the following values of P, r, and t. Round to the nearest hundredth. See Example 2.

33. $P = 15{,}895$, $r = -2\%$, $t = 16$ years 11,542.14

34. $P = 33{,}999$, $r = -4$, $t = 21$ years 14,677.73

▶ **35.** $P = 565$, $r = -0.5\%$, $t = 8$ years 542.85

▶ **36.** $P = 110$, $r = -0.25\%$, $t = 9$ years 107.55

Suppose that a population of 500 is growing exponentially according to the formula $P = 500e^{rt}$, where t is the time in years and r is the rate of growth. Further assume that the food supply F is growing linearly according to the formula $F = 10t + 1{,}000$, where t is the time in years. For the following growth rates, how long will it take for the population to exceed the food supply? Use a calculator window of $X = [0, 300]$ and $Y = [0, 5{,}000]$. See Example 3.

37. $r = 0.01$
about 168 years

▶ **38.** $r = 0.015$
about 88 years

39. $r = 0.02$
about 57 years

40. $r = 0.03$
about 32 years

APPLICATIONS

In Exercises 41–46, assume that there are no deposits or withdrawals.

▶ **41.** CONTINUOUS COMPOUND INTEREST An initial investment of \$5,000 earns 8.2% interest, compounded continuously. What will the investment be worth in 12 years?
\$13,375.68

▶ **42.** CONTINUOUS COMPOUND INTEREST An initial investment of \$2,000 earns 8% interest, compounded continuously. What will the investment be worth in 15 years?
\$6,640.23

▶ **43.** COMPARING COMPOUNDING METHODS An initial deposit of \$5,000 grows at an annual rate of 8.5% for 5 years. Compare the final balances resulting from annual compounding and continuous compounding.
\$7,518.28 from annual compounding, \$7,647.95 from continuous compounding

44. COMPARING COMPOUNDING METHODS An initial deposit of \$30,000 grows at an annual rate of 8% for 20 years. Compare the final balances resulting from annual compounding and continuous compounding.
\$139,828.71 from annual compounding, \$148,590.97 from continuous compounding

45. DETERMINING THE INITIAL DEPOSIT An account now contains \$11,180 and has been accumulating interest at 7% annual interest, compounded continuously, for 7 years. Find the initial deposit.
\$6,849.16

46. DETERMINING THE PREVIOUS BALANCE An account now contains \$3,610 and has been accumulating interest at 8% annual interest, compounded continuously. How much was in the account 4 years ago?
\$2,621.40

47. WORLD POPULATION GROWTH The population of Earth is approximately 6.1 billion people and is growing at an annual rate of 1.4%. Use the exponential growth model to find the world population in 30 years.
about 9.3 billion

48. HIGHS AND LOWS Somalia, in eastern Africa, has one of the greatest population growth rates in the world. Bulgaria, in southeastern Europe, has one of the smallest. Use the exponential growth model to complete the table.

Country	Population 2003	Annual growth rate	Estimated population 2015
Somalia	8,025,190	3.43%	12,111,863
Bulgaria	7,537,929	−1.09%	6,613,728

Source: nationmaster.com

49. POPULATION GROWTH The growth of a population is modeled by

$$P = 173e^{0.03t}$$

How large will the population be when $t = 20$? 315

50. POPULATION DECLINE The decline of a population is modeled by

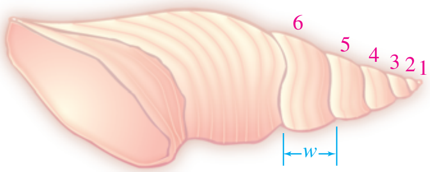
$$P = (1.2 \times 10^6)e^{-0.008t}$$

How large will the population be when $t = 30$? 9.44×10^3

51. OCEANOGRAPHY The width w (in millimeters) of successive growth spirals of the sea shell *Catapulus voluto,* shown in the illustration below, is given by the exponential function

$$w = 1.54e^{0.503n}$$

where n is the spiral number. Find the width, to the nearest tenth of a millimeter, of the sixth spiral. 31.5 mm

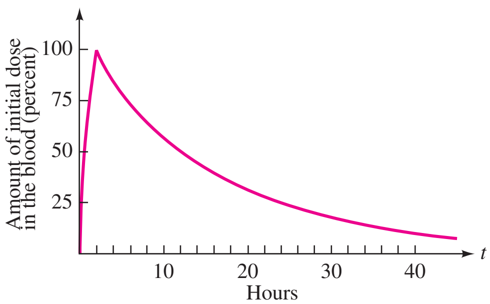


52. EPIDEMICS The spread of hoof and mouth disease through a herd of cattle can be modeled by the formula

$$P = P_0e^{0.27t} \quad (t \text{ is in days})$$

If a rancher does not act quickly to treat two cases, how many cattle will have the disease in 12 days? 51

53. HALF-LIFE OF A DRUG The quantity of a prescription drug in the bloodstream of a patient t hours after it is administered can be modeled by an exponential function. (See the graph below.) Use the graph to determine the time it takes to eliminate half of the initial dose from the body. 12 hr



54. MEDICINE The concentration x of a certain prescription drug in an organ after t minutes is given by

$$x = 0.08(1 - e^{-0.1t})$$

Find the concentration of the drug at 30 minutes. 0.076

55. SKYDIVING Before the parachute opens, a skydiver's velocity v in meters per second is given by

$$v = 50(1 - e^{-0.2t})$$

Find the velocity after 20 seconds of free fall. 49 mps

56. FREE FALL After t seconds a certain falling object has a velocity v given by

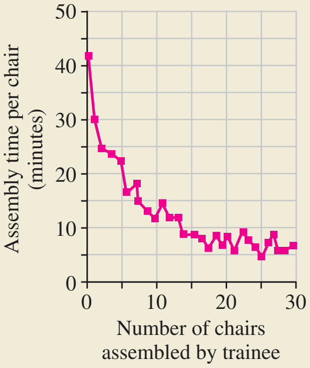
$$v = 50(1 - e^{-0.3t})$$

Which is falling faster after 2 seconds—the object or the skydiver in Exercise 55? the object

57. LEARNING CURVE Social workers often work with learning curves. For example, the learning curve shown below illustrates that as a factory worker trainee assembles more chairs, the assembly time per chair decreases in an exponential way. If factory standards require an average assembly time of 10 minutes or less, how many chairs must a trainee assemble to meet factory standards? 14

from Campus to Careers
Social Worker

© iStockphoto.com/Bo 1982



58. LEARNING CURVES If the factory in Exercise 57 changes its standard and requires an average assembly time of 20 minutes or less, how many chairs must a trainee assemble to meet factory standards? 14

Use a graphing calculator to solve each problem.

59. FOOD SUPPLY In Example 3, suppose that better farming methods changed the formula for food growth to $y = 31x + 2{,}000$. How long would the food supply be adequate? about 72 yr

60. FOOD SUPPLY In Example 3, suppose that a birth-control program changed the formula for population growth to $P = 1{,}000e^{0.01t}$. How long would the food supply be adequate? about 215 yr

WRITING

61. Explain why the graph of $y = e^x - 5$ is five units below the graph of $y = e^x$.

62. A feature article in a newspaper stated that the sport of snowboarding was growing *exponentially.* Explain what the author of the article meant by that.

REVIEW

Simplify each expression. Assume that all variables represent positive numbers.

63. $\sqrt{240x^5}$ $4x^2\sqrt{15x}$

64. $\sqrt[3]{-125x^5y^4}$ $-5xy\sqrt[3]{x^2y}$

65. $4\sqrt{48y^3} - 3y\sqrt{12y}$ $10y\sqrt{3y}$

66. $\sqrt[4]{48z^5} + \sqrt[4]{768z^5}$ $6z\sqrt[4]{3z}$

Objectives

1 Define logarithms.

2 Write logarithmic equations in exponential form.

3 Write exponential equations in logarithmic form.

4 Evaluate logarithmic expressions.

5 Graph logarithmic functions.

6 Use logarithmic functions in applications.

SECTION 9.5 Logarithmic Functions

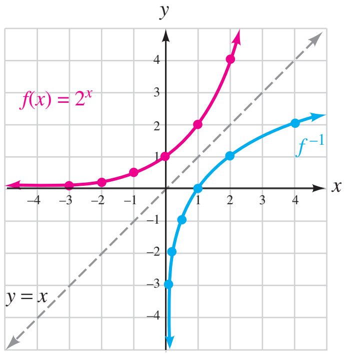
A function that is closely related to the exponential function is the *logarithmic function.* It can be used to solve many application problems from fields such as electronics, seismology (the study of earthquakes), business, and population growth.

1 Define logarithms.

The graph of the exponential function $f(x) = 2^x$ is shown in red on the next page. Since it passes the horizontal line test, it is a one-to-one function and has an inverse. To graph f^{-1}, we interchange the coordinates of the ordered pairs in the table, plot those points, and draw a smooth curve through them, as shown in blue. As expected, the graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

To graph f^{-1}, interchange each pair of coordinates.

$f(x) = 2^x$			
x	$f(x)$		
-3	$\frac{1}{8}$	$\rightarrow \left(-3, \frac{1}{8}\right)$	$\left(\frac{1}{8}, -3\right)$
-2	$\frac{1}{4}$	$\rightarrow \left(-2, \frac{1}{4}\right)$	$\left(\frac{1}{4}, -2\right)$
-1	$\frac{1}{2}$	$\rightarrow \left(-1, \frac{1}{2}\right)$	$\left(\frac{1}{2}, -1\right)$
0	1	$\rightarrow (0, 1)$	$(1, 0)$
1	2	$\rightarrow (1, 2)$	$(2, 1)$
2	4	$\rightarrow (2, 4)$	$(4, 2)$
3	8	$\rightarrow (3, 8)$	$(8, 3)$



To write an equation for the inverse of $f(x) = 2^x$, we proceed as follows:

$f(x) = 2^x$

$y = 2^x$ Replace $f(x)$ with y.

$x = 2^y$ Interchange the variables x and y.

We cannot solve the equation for y because we have not discussed methods for solving equations with a variable in an exponent. However, we can translate the relationship $x = 2^y$ into words:

$y =$ the power to which we raise 2 to get x

If we substitute $f^{-1}(x)$ for y, we see that

$f^{-1}(x) =$ the power to which we raise 2 to get x

If we define the symbol $\log_2 x$ to mean *the power to which we raise 2 to get x*, we can write the equation for the inverse as

$f^{-1}(x) = \log_2 x$ Read $\log_2 x$ as "the logarithm, base 2, of x" or "log, base 2, of x."

The Language of Algebra The abbreviation log is used for the word *logarithm*. A *logarithm* is an exponent.

We have found that the inverse of the exponential function $f(x) = 2^x$ is $f^{-1}(x) = \log_2 x$. To find the inverse of exponential functions with other bases, such as $f(x) = 3^x$ and $f(x) = 10^x$, we define logarithm in the following way.

Definition of Logarithm

For all positive numbers b, where $b \neq 1$, and all positive numbers x,

$y = \log_b x$ is equivalent to $x = b^y$

The **domain of the logarithmic function** is the interval $(0, \infty)$. The **range** is the interval $(-\infty, \infty)$.

Success Tip Here are examples of inverses of other exponential functions:

$f(x) = 3^x \qquad f^{-1}(x) = \log_3 x$

$g(x) = 10^x \qquad g^{-1}(x) = \log_{10} x$

This definition guarantees that any pair (x, y) that satisfies the logarithmic equation $y = \log_b x$ also satisfies the exponential equation $x = b^y$. Because of this relationship, a statement written in logarithmic form can be written in an equivalent exponential form, and vice versa. The following diagram will help you remember the respective positions of the exponent and base in each form.

Exponent

$$y = \log_b x \qquad x = b^y$$

Base

2 Write logarithmic equations in exponential form.

The following shows several pairs of equivalent equations.

Logarithmic equation	*Exponential equation*
$\log_2 8 = 3$	$2^3 = 8$
$\log_3 81 = 4$	$3^4 = 81$
$\log_4 4 = 1$	$4^1 = 4$
$\log_5 \frac{1}{125} = -3$	$5^{-3} = \frac{1}{125}$

Self Check 1

Write $\log_2 128 = 7$ as an exponential equation. $2^7 = 128$

Now Try **Problem 21**

Teaching Example 1 Write each logarithmic equation as an exponential equation.

a. $\log_5 25 = 2$ **b.** $\log_{2/3} \frac{8}{27} = 3$

c. $\log_2 \frac{1}{8} = -3$

Answers:

a. $5^2 = 25$ **b.** $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

c. $2^{-3} = \frac{1}{8}$

EXAMPLE 1 Write each logarithmic equation as an exponential equation:

a. $\log_4 64 = 3$ **b.** $\log_7 \sqrt{7} = \frac{1}{2}$ **c.** $\log_6 \frac{1}{36} = -2$

Strategy To write an equivalent exponential equation, we will determine which number will serve as the base and which will serve as the exponent.

WHY We can then use the definition of logarithm to move from one form to the other: $\log_b x = y$ is equivalent to $x = b^y$.

Solution

Exponent

a. $\log_4 64 = 3$ is equivalent to $4^3 = 64$.

Base

Exponent

b. $\log_7 \sqrt{7} = \frac{1}{2}$ is equivalent to $7^{1/2} = \sqrt{7}$.

Base

Exponent

c. $\log_6 \frac{1}{36} = -2$ is equivalent to $6^{-2} = \frac{1}{36}$.

Base

3 Write exponential equations in logarithmic Form.

Self Check 2

Write $9^{-1} = \frac{1}{9}$ as a logarithmic equation. $\log_9 \frac{1}{9} = -1$

Now Try **Problem 29**

EXAMPLE 2 Write each exponential equation as a logarithmic equation:

a. $8^0 = 1$ **b.** $6^{1/3} = \sqrt[3]{6}$ **c.** $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Strategy To write an equivalent logarithmic equation, we will determine which number will serve as the base and where we will place the exponent.

WHY We can then use the definition of logarithm to move from one form to the other: $x = b^y$ is equivalent to $\log_b x = y$.

Solution

a. $\overset{\text{Exponent}}{8^0 = 1}$ is equivalent to $\log_8 1 = 0$ (Base: 8)

b. $6^{1/3} = \sqrt[3]{6}$ is equivalent to $\log_6 \sqrt[3]{6} = \dfrac{1}{3}$ (Exponent: $\frac{1}{3}$; Base: 6)

c. $\left(\dfrac{1}{4}\right)^2 = \dfrac{1}{16}$ is equivalent to $\log_{1/4} \dfrac{1}{16} = 2$ (Exponent: 2; Base: $\frac{1}{4}$)

Teaching Example 2 Write each exponential equation as a logarithmic equation.

a. $3^{1/2} = \sqrt{3}$ **b.** $\left(\dfrac{1}{5}\right)^{-1} = 5$

c. $2^0 = 1$

Answers:

a. $\log_3 \sqrt{3} = \frac{1}{2}$ **b.** $\log_{1/5} 5 = -1$

c. $\log_2 1 = 0$

Certain logarithmic equations can be solved by writing them as exponential equations.

EXAMPLE 3

Solve each equation for x:

a. $\log_x 25 = 2$ **b.** $\log_3 x = -3$ **c.** $\log_{1/2} \dfrac{1}{16} = x$

Strategy To solve each logarithmic equation, we will instead write and solve an equivalent exponential equation.

WHY The resulting exponential equation is easier to solve because the variable term is often isolated on one side.

Solution

a. Since $\log_x 25 = 2$ is equivalent to $x^2 = 25$, we can solve $x^2 = 25$ to find x.

$$x^2 = 25$$
$$x = \pm\sqrt{25} \quad \text{Use the square root property.}$$
$$x = \pm 5$$

In the expression $\log_x 25$, the base of the logarithm is x. Because the base must be positive, we discard -5 and we have

$$x = 5$$

To check the solution of 5, we verify that $\log_5 25 = 2$.

b. Since $\log_3 x = -3$ is equivalent to $3^{-3} = x$, we can solve $3^{-3} = x$ to find x.

$$3^{-3} = x$$
$$\frac{1}{3^3} = x$$
$$x = \frac{1}{27}$$

To check the solution of $\frac{1}{27}$, we verify that $\log_3 \frac{1}{27} = -3$.

Self Check 3

Solve each equation for x:

a. $\log_x 49 = 2$ 7

b. $\log_{1/3} x = 2$ $\frac{1}{9}$

c. $\log_6 216 = x$ 3

Now Try **Problems 37, 39, and 41**

Teaching Example 3 Solve for x:

a. $\log_x 9 = 2$

b. $\log_2 x = 4$

c. $\log_{1/7} \dfrac{1}{49} = x$

Answers:

a. 3 **b.** 16 **c.** 2

c. Since $\log_{1/2} \frac{1}{16} = x$ is equivalent to $\left(\frac{1}{2}\right)^x = \frac{1}{16}$, we can solve $\left(\frac{1}{2}\right)^x = \frac{1}{16}$ to find x.

$$\left(\frac{1}{2}\right)^x = \frac{1}{16}$$

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^4 \quad \text{Write } \tfrac{1}{16} \text{ as a power of } \tfrac{1}{2} \text{ to match the bases: } \tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{16}.$$

$$x = 4 \quad \text{Since the bases are the same, and since exponential functions are one-to-one, the exponents must be equal.}$$

To check the solution of 4, we verify that $\log_{1/2} \frac{1}{16} = 4$.

4 Evaluate logarithmic expressions.

In the previous examples, we have seen that the logarithm of a number is an exponent. In fact,

$\log_b x$ is the exponent to which b is raised to get x.

Translating this statement into symbols, we have

$$b^{\log_b x} = x$$

Self Check 4

Evaluate each logarithmic expression:

a. $\log_9 81$ 2

b. $\log_4 \frac{1}{16}$ -2

c. $\log_9 3$ $\frac{1}{2}$

Now Try **Problems 61 and 63**

Teaching Example 4 Evaluate each logarithmic expression:

a. $\log_9 81$ **b.** $\log_{11} \frac{1}{121}$

c. $\log_8 2$

Answers:

a. 2 **b.** -2 **c.** $\frac{1}{3}$

EXAMPLE 4 Evaluate each logarithmic expression:

a. $\log_8 64$ **b.** $\log_3 \frac{1}{3}$ **c.** $\log_4 2$

Strategy After identifying the base, we will ask "To what power must the base be raised to get the other number?"

WHY That power is the value of the logarithmic expression.

Solution

a. $\log_8 64 = 2$ Ask: "To what power must we raise 8 to get 64?" Since $8^2 = 64$, the answer is the 2nd power.

b. $\log_3 \frac{1}{3} = -1$ Ask: "To what power must we raise 3 to get $\frac{1}{3}$?" Since $3^{-1} = \frac{1}{3}$, the answer is the -1 power.

c. $\log_4 2 = \frac{1}{2}$ Ask: "To what power must we raise 4 to get 2?" Since $\sqrt{4} = 4^{1/2} = 2$, the answer is the $\frac{1}{2}$ power.

For computational purposes and in many applications, we will use base-10 logarithms (also called **common logarithms**). When the base b is not indicated in the notation $\log x$, we assume that $b = 10$:

$$\log x \quad \text{means} \quad \log_{10} x$$

The table below shows several pairs of equivalent statements involving base-10 logarithms.

Logarithmic form	*Exponential form*	
$\log 100 = 2$	$10^2 = 100$	Read log 100 as "log of 100."
$\log \frac{1}{10} = -1$	$10^{-1} = \frac{1}{10}$	
$\log 1 = 0$	$10^0 = 1$	

In general, we have

$$\log_{10} 10^x = x$$

EXAMPLE 5

Evaluate each logarithmic expression, if possible:

a. $\log 1{,}000$ **b.** $\log \frac{1}{100}$ **c.** $\log 10$ **d.** $\log(-10)$

Strategy After identifying the base, we will ask "To what power must 10 be raised to get the other number?"

WHY That power is the value of the logarithmic expression.

Solution

a. $\log 1{,}000 = 3$ Ask: "To what power must we raise 10 to get 1,000?" Since $10^3 = 1{,}000$, the answer is the 3rd power.

b. $\log \frac{1}{100} = -2$ Ask: "To what power must we raise 10 to get $\frac{1}{100}$?" Since $10^{-2} = \frac{1}{100}$, the answer is the -2 power.

c. $\log 10 = 1$ Ask: "To what power must we raise 10 to get 10?" Since $10^1 = 10$, the answer is the 1st power.

d. To find $\log(-10)$, we must find a power of 10 such that $10^? = -10$. There is no such number. Thus, $\log(-10)$ is undefined.

Self Check 5

Evaluate each expression:

a. $\log 10{,}000$ 4

b. $\log \frac{1}{1{,}000}$ -3

c. $\log 0$ undefined

Now Try **Problems 65 and 67**

Teaching Example 5 Evaluate each logarithmic expression, if possible:

a. $\log 100$ **b.** $\log \frac{1}{10{,}000}$

c. $\log(-100)$

Answers:

a. 2 **b.** -4 **c.** undefined

Many logarithmic expressions cannot be evaluated by inspection. For example, to find $\log 2.34$, we ask, "To what power must we raise 10 to get 2.34?" This answer isn't obvious. In such cases, we use a calculator.

Using Your CALCULATOR Evaluating Logarithms

To find log 2.34 with a scientific calculator we enter

2.34 [LOG] .369215857

On some calculators, the [10^x] key also serves as the [LOG] key when [2nd] or [SHIFT] is pressed. This is because $f(x) = 10^x$ and $f(x) = \log x$ are inverses.

To use a graphing calculator, we enter

[LOG] 2.34 [)] [ENTER]

```
log(2.34)
       .369215857
```

To four decimal places, $\log 2.34 = 0.3692$. This means, $10^{0.3692} \approx 2.34$.

If we attempt to evaluate logarithmic expressions such as log 0, or the logarithm of a negative number, such as $\log(-5)$, an error message like the following will be displayed.

```
Error
```

```
ERR:DOMAIN
1:QUIT
2:Go to
```

```
ERR:NONREAL ANS
1:QUIT
2:Go to
```

EXAMPLE 6

Solve $\log x = 0.3568$ and round to four decimal places.

Strategy To solve this logarithmic equation, we will instead write and solve an equivalent exponential equation.

WHY The resulting exponential equation is easier to solve because the variable term is isolated on one side.

Self Check 6

Solve $\log x = 1.87737$ and round to four decimal places. 75.3998

Now Try **Problem 77**

Teaching Example 6 Solve $\log x = 0.7429$ and round to four decimal places.
Answer:
5.5322

Solution

The equation $\log x = 0.3568$ is equivalent to $10^{0.3568} = x$. Since we cannot determine $10^{0.3568}$ by inspection, we will use a calculator to find an approximate solution. We enter

$10\ y^x\ .3568\ =$

The display reads 2.274049951 . To four decimal places,

$x = 2.2740$

If your calculator has a 10^x key, enter .3568 and press it to get the same result. The solution is 2.2740. To check, use your calculator to verify that $\log 2.2740 \approx 0.3568$.

5 Graph logarithmic functions.

Because an exponential function defined by $f(x) = b^x$ is one-to-one, it has an inverse function that is defined by $x = b^y$. When we write $x = b^y$ in the equivalent form $y = \log_b x$, the result is called a *logarithmic function.*

Logarithmic Functions

If $b > 0$ and $b \neq 1$, the **logarithmic function with base *b*** is defined by the equations

$$f(x) = \log_b x \quad \text{or} \quad y = \log_b x$$

The domain of $f(x) = \log_b x$ is the interval $(0, \infty)$ and the range is the interval $(-\infty, \infty)$.

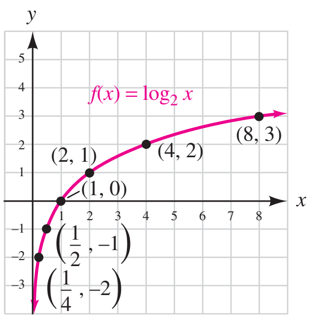
Since every logarithmic function is the inverse of a one-to-one exponential function, logarithmic functions are one-to-one.

We can plot points to graph logarithmic functions. For example, to graph $f(x) = \log_2 x$, we construct a table of function values, plot the resulting ordered pairs, and draw a smooth curve through the points to get the graph, as shown in figure (a). To graph $f(x) = \log_{1/2} x$, we use the same method, as shown in figure (b).

$f(x) = \log_2 x$

x	$f(x)$	
$\frac{1}{4}$	-2	$\rightarrow \left(\frac{1}{4}, -2\right)$
$\frac{1}{2}$	-1	$\rightarrow \left(\frac{1}{2}, -1\right)$
1	0	$\rightarrow (1, 0)$
2	1	$\rightarrow (2, 1)$
4	2	$\rightarrow (4, 2)$
8	3	$\rightarrow (8, 3)$

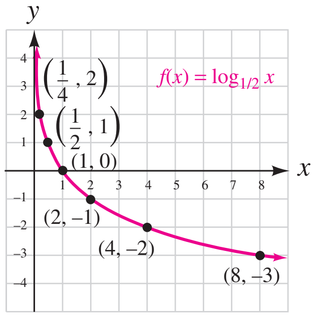
Because the base of the function is 2, choose values for x that are integer powers of 2.



(a)

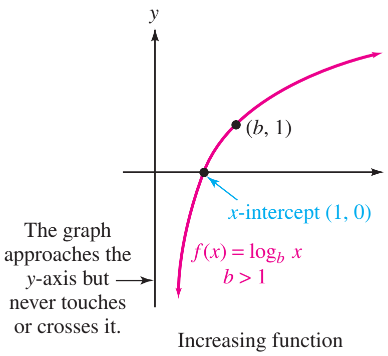
$f(x) = \log_{1/2} x$		
x	$f(x)$	
$\frac{1}{4}$	2	$\rightarrow \left(\frac{1}{4}, 2\right)$
$\frac{1}{2}$	1	$\rightarrow \left(\frac{1}{2}, 1\right)$
1	0	$\rightarrow (1, 0)$
2	-1	$\rightarrow (2, -1)$
4	-2	$\rightarrow (4, -2)$
8	-3	$\rightarrow (8, -3)$

Because the base of the function is $\frac{1}{2}$, choose values for x that are integers powers of $\frac{1}{2}$.

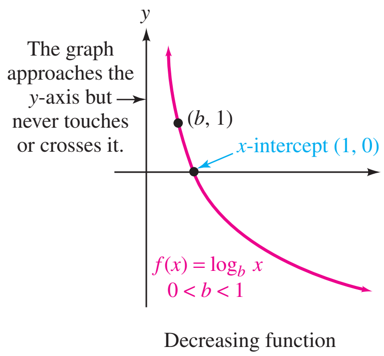


(b)

The graphs of all logarithmic functions are similar to those shown below. If $b > 1$, the logarithmic function is increasing, as in figure (a). If $0 < b < 1$, the logarithmic function is decreasing, as in figure (b).



(a)



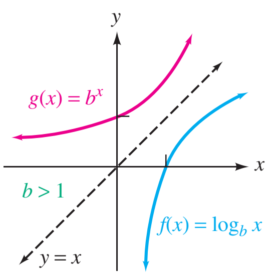
(b)

Properties of Logarithmic Functions

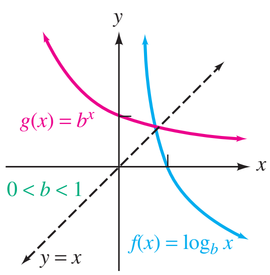
The graph of $f(x) = \log_b x$ (or $y = \log_b x$) has the following properties.

1. It passes through the point $(1, 0)$.
2. It passes through the point $(b, 1)$.
3. The y-axis (the line $x = 0$) is an asymptote.
4. The domain is the interval $(0, \infty)$ and the range is the interval $(-\infty, \infty)$.

The exponential and logarithmic functions are inverses of each other, so their graphs have symmetry about the line $y = x$. The graphs of $f(x) = \log_b x$ and $g(x) = b^x$ are shown in figure (a) when $b > 1$ and in figure (b) when $0 < b < 1$.



(a)



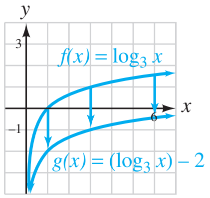
(b)

The graphs of many functions involving logarithms are translations of the basic logarithmic graphs.

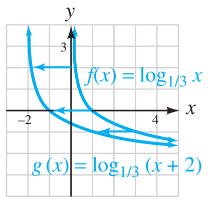
Self Check 7

Graph each function by using a translation.

a. $g(x) = (\log_3 x) - 2$

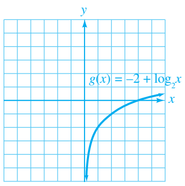


b. $g(x) = \log_{1/3}(x + 2)$

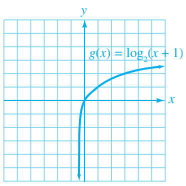


Now Try **Problems 89 and 91**

Teaching Example 7 Graph each function by using a translation:
a. $g(x) = -2 + \log_2 x$
b. $g(x) = \log_2 (x + 1)$
Answers:
a.



b.



EXAMPLE 7 Graph each function by using a translation:

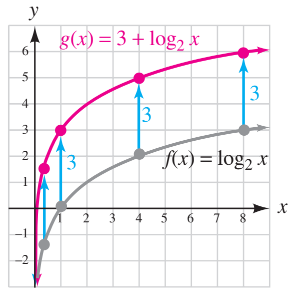
a. $g(x) = 3 + \log_2 x$ **b.** $g(x) = \log_{1/2}(x - 1)$

Strategy We will graph $g(x) = 3 + \log_2 x$ by translating the graph of $f(x) = \log_2 x$ upward 3 units. We will graph $g(x) = \log_{1/2}(x - 1)$ by translating the graph of $f(x) = \log_{1/2} x$ to the right 1 unit.

WHY The addition of 3 in $g(x) = \mathbf{3 +} \log_2 x$ causes a vertical shift of the graph of the base-2 logarithmic function 3 units upward. The subtraction of 1 from x in $g(x) = \log_{1/2}(x \mathbf{- 1})$ causes a horizontal shift of the graph of the base-$\frac{1}{2}$ logarithmic function 1 unit to the right.

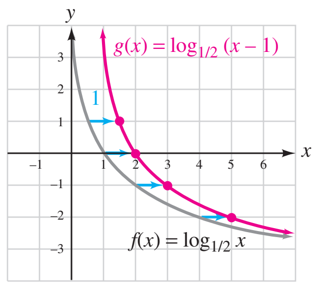
Solution

a. The graph of $g(x) = 3 + \log_2 x$ will be the same shape as the graph of $f(x) = \log_2 x$, except that it is shifted 3 units upward.



To graph $g(x) = 3 + \log_2 x$, translate each point on the graph of $f(x) = \log_2 x$ up 3 units.

b. The graph of $g(x) = \log_{1/2}(x - 1)$ will be the same shape as the graph of $f(x) = \log_{1/2} x$, except it is shifted 1 unit to the right.



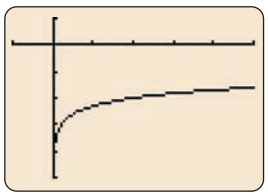
To graph $g(x) = \log_{1/2}(x - 1)$, translate each point on the graph of $f(x) = \log_{1/2} x$ to the right 1 unit.

To graph more complicated logarithmic functions, a graphing calculator is a useful tool.

Using Your CALCULATOR **Graphing Logarithmic Functions**

To use a calculator to graph the logarithmic function $f(x) = -2 + \log_{10} \frac{x}{2}$, we enter the right side of the equation after the symbol $Y_1 =$. The display will show the equation

$$Y_1 = -2 + \log(X/2)$$



If we use window settings of $[-1, 5]$ for x and $[-4, 1]$ for y and press the $\boxed{\text{GRAPH}}$ key, we will obtain the graph shown.

6 Use logarithmic functions in applications.

Logarithmic functions, like exponential functions, can be used to model certain types of growth and decay. Common logarithms are used in electrical engineering to express the voltage gain (or loss) of an electronic device such as an amplifier. The unit of gain (or loss), called the **decibel,** is defined by a logarithmic relation.

Decibel Voltage Gain

If E_O is the output voltage of a device and E_I is the input voltage, the decibel voltage gain of the device (db gain) is given by

$$\text{db gain} = 20 \log \frac{E_O}{E_I}$$

EXAMPLE 8 ***db Gain*** If the input to an amplifier is 0.5 volt and the output is 40 volts, find the decibel voltage gain of the amplifier.

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Strategy We will substitute into the formula for db gain and evaluate the right side using a calculator.

WHY We can use this formula to find the db gain because we are given the input voltage E_I and the output voltage E_O.

Solution

We can find the decibel voltage gain by substituting 0.5 for E_I and 40 for E_O into the formula for db gain:

$$\text{db gain} = 20 \log \frac{E_O}{E_I}$$

$$\text{db gain} = 20 \log \frac{40}{0.5}$$

$$= 20 \log 80 \quad \text{Divide: } \frac{40}{0.5} = 80.$$

$$\approx 38 \quad \text{Use a calculator: 20 log means } 20 \cdot \log 80.$$

The amplifier provides a 38-decibel voltage gain.

Self Check 8

dB GAIN If the input to an amplifier is 0.6 volt and the output is 40 volts, find the decibel voltage gain of the amplifier. about 36 db

Now Try **Problem 97**

Teaching Example 8 dB GAIN If the input to an amplifier is 0.7 volt and the output is 40 volts, find the decibel voltage gain of the amplifier.
Answer:
about 35 db

In seismology, common logarithms are used to measure the intensity of earthquakes on the **Richter scale.** The intensity of an earthquake is given by the following logarithmic function.

Richter Scale

If R is the intensity of an earthquake, A is the amplitude (measured in micrometers) of the ground motion, and P is the period (the time of one oscillation of the Earth's surface measured in seconds), then

$$R = \log \frac{A}{P}$$

EXAMPLE 9 ***Earthquakes*** Find the measure on the Richter scale of an earthquake with an amplitude of 5,000 micrometers (0.5 centimeter) and a period of 0.1 second.

Strategy We will substitute into the formula for intensity of an earthquake and evaluate the right side using a calculator.

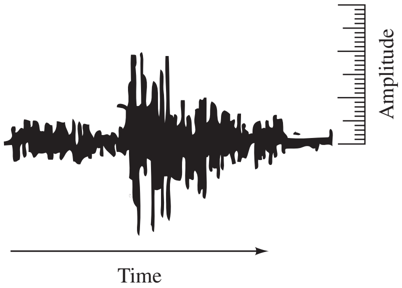
Self Check 9

EARTHQUAKES Find the measure on the Richter scale of an earthquake with an amplitude of 4,000 micrometers (0.4 centimeter) and a period of 0.2 second. 4.3

Now Try **Problem 101**

Teaching Example 9
EARTHQUAKES Find the measure on the Richter scale of an earthquake with an amplitude of 6,000 micrometers (0.6 centimeter) and a period of 0.2 second.
Answer:
about 4.5

WHY We can use this formula to find the intensity of the earthquake because we are given the amplitude A and the period P.



Solution

We substitute 5,000 for A and 0.1 for P in the Richter scale formula and proceed as follows:

$$R = \log \frac{A}{P}$$

$$R = \log \frac{5{,}000}{0.1}$$

$$= \log 50{,}000 \qquad \text{Divide: } \frac{5{,}000}{0.1} = 50{,}000.$$

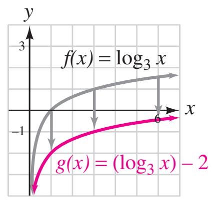
$$\approx 4.698970004 \qquad \text{Use a calculator.}$$

The earthquake measures about 4.7 on the Richter scale.

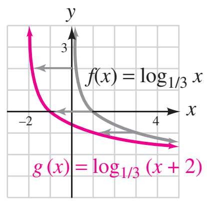
The Language of Algebra The Richter scale was developed in 1935 by Charles F. Richter of the California Institute of Technology.

ANSWERS TO SELF CHECKS

1. $2^7 = 128$ **2.** $\log_9 \frac{1}{9} = -1$ **3. a.** 7 **b.** $\frac{1}{9}$ **c.** 3 **4. a.** 2 **b.** -2 **c.** $\frac{1}{2}$
5. a. 4 **b.** -3 **c.** undefined **6.** 75.3998
7. a.



b.

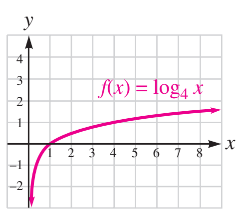


8. about 36 db **9.** 4.3

SECTION 9.5 STUDY SET

VOCABULARY

Refer to the graph of $f(x) = \log_4 x$.



1. What type of function is $f(x) = \log_4 x$? logarithmic
2. What is the domain of the function? $(0, \infty)$
3. What is the range of the function? $(-\infty, \infty)$
4. **a.** What is the y-intercept of the graph? none
   b. What is the x-intercept of the graph? $(1, 0)$
5. Is f a one-to-one function? yes
6. ▶ What is an asymptote of the graph? the y-axis $(x = 0)$
7. Is f an increasing or a decreasing function? increasing
8. The graph passes through the point $(4, y)$. What is y? 1

CONCEPTS

Fill in the blanks.

9. The equation $y = \log_b x$ is equivalent to the exponential equation x = b^y.
10. ▶ $\log_b x$ is the exponent to which b is raised to get x.
11. The functions $f(x) = \log_{10} x$ and $f(x) = 10^x$ are inverse functions.
12. The inverse of an exponential function is called a logarithmic function.

▶ Selected exercises available online at **www.webassign.net/brookscole**

Complete the table of values, where possible.

13. $f(x) = \log x$

x	$f(x)$
100	2
$\frac{1}{100}$	-2

▶ **14.** $f(x) = \log_5 x$

x	$f(x)$
25	2
$\frac{1}{25}$	-2

15. $f(x) = \log_6 x$

Input	Output
6	1
-6	undefined
0	undefined

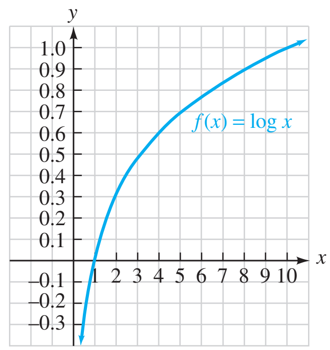
16. $f(x) = \log_8 x$

Input	Output
8	1
-8	undefined
0	undefined

17. **a.** Use a calculator to complete the table of values for $f(x) = \log x$. Round to the nearest hundredth.

x	$f(x)$
0.5	-0.30
1	0
2	0.30
4	0.60
6	0.78
8	0.90
10	1

b. Graph $f(x) = \log x$. Note that the units on the x- and y-axes are different.



18. For each function, determine its inverse, $f^{-1}(x)$.

a. $f(x) = 10^x$
$f^{-1}(x) = \log x$

b. $f(x) = 3^x$
$f^{-1}(x) = \log_3 x$

c. $f(x) = \log x$
$f^{-1}(x) = 10^x$

d. $f(x) = \log_2 x$
$f^{-1}(x) = 2^x$

NOTATION

Fill in the blanks.

19. **a.** $\log x = \log_{10} x$ ▶ **b.** $\log_{10} 10^x = x$

▶ **20.** **a.** We read $\log_5 25$ as "log, base 5, of 25."

b. We read $\log x$ as "log of x."

GUIDED PRACTICE

Write each logarithmic equation as an exponential equation. See Example 1.

21. $\log_3 81 = 4$
$3^4 = 81$

22. $\log_7 7 = 1$
$7^1 = 7$

23. $\log_{10} 10 = 1$
$10^1 = 10$

▶ **24.** $\log_{10} 100 = 2$
$10^2 = 100$

25. $\log_4 \frac{1}{64} = -3$
$4^{-3} = \frac{1}{64}$

26. $\log_6 \frac{1}{36} = -2$
$6^{-2} = \frac{1}{36}$

▶ **27.** $\log_5 \sqrt{5} = \frac{1}{2}$
$5^{1/2} = \sqrt{5}$

28. $\log_7 \sqrt[3]{7} = \frac{1}{3}$
$7^{1/3} = \sqrt[3]{7}$

Write each exponential equation as a logarithmic equation. See Example 2.

29. $8^2 = 64$
$\log_8 64 = 2$

30. $10^3 = 1{,}000$
$\log_{10} 1{,}000 = 3$

▶ **31.** $4^{-2} = \frac{1}{16}$
$\log_4 \frac{1}{16} = -2$

32. $3^{-4} = \frac{1}{81}$
$\log_3 \frac{1}{81} = -4$

33. $\left(\frac{1}{2}\right)^{-5} = 32$
$\log_{1/2} 32 = -5$

▶ **34.** $\left(\frac{1}{3}\right)^{-3} = 27$
$\log_{1/3} 27 = -3$

35. $x^y = z$
$\log_x z = y$

36. $m^n = p$
$\log_m p = n$

Solve for x. See Example 3.

37. $\log_x 81 = 2$ 9

38. $\log_x 9 = 2$ 3

39. $\log_8 x = 2$ 64

40. $\log_7 x = 0$ 1

41. $\log_5 125 = x$ 3

42. $\log_4 16 = x$ 2

43. $\log_5 x = -2$ $\frac{1}{25}$

44. $\log_3 x = -4$ $\frac{1}{81}$

▶ **45.** $\log_{36} x = -\frac{1}{2}$ $\frac{1}{6}$

▶ **46.** $\log_{27} x = -\frac{1}{3}$ $\frac{1}{3}$

47. $\log_x 0.01 = -2$ 10

48. $\log_x 0.001 = -3$ 10

49. $\log_{27} 9 = x$ $\frac{2}{3}$

50. $\log_{12} x = 0$ 1

51. $\log_x 5^3 = 3$ 5

52. $\log_x 5 = 1$ 5

53. $\log_{100} x = \frac{3}{2}$ 1,000

54. $\log_x \frac{1}{1{,}000} = -\frac{3}{2}$ 100

55. $\log_x \frac{1}{64} = -3$ 4

▶ **56.** $\log_x \frac{1}{100} = -2$ 10

57. $\log_8 x = 0$ 1

58. $\log_4 8 = x$ $\frac{3}{2}$

59. $\log_x \frac{\sqrt{3}}{3} = \frac{1}{2}$ $\frac{1}{3}$

▶ **60.** $\log_x \frac{9}{4} = 2$ $\frac{3}{2}$

Evaluate each logarithmic expression. See Examples 4 and 5.

61. $\log_2 8$ 3

62. $\log_3 9$ 2

63. $\log_4 16$ 2

64. $\log_6 216$ 3

65. $\log 1{,}000{,}000$ 6

66. $\log 100{,}000$ 5

▶ **67.** $\log \dfrac{1}{10}$ -1

68. $\log \dfrac{1}{10{,}000}$ -4

69. $\log_{1/2} \dfrac{1}{32}$ 5

70. $\log_{1/3} \dfrac{1}{81}$ 4

71. $\log_9 3$ $\frac{1}{2}$

▶ **72.** $\log_{125} 5$ $\frac{1}{3}$

Use a calculator to find each value. Give answers to four decimal places. **See Using Your Calculator: Evaluating Logarithms.**

73. $\log 3.25$ 0.5119

74. $\log 0.57$ -0.2441

75. $\log 0.00467$ -2.3307

▶ **76.** $\log 375.876$ 2.5750

Use a calculator to solve each equation. Round answers to four decimal places. **See Example 6.**

77. $\log x = 3.7813$ 6,043.6597

78. $\log x = 2.8945$ 784.3321

79. $\log x = -0.7630$ 0.1726

80. $\log x = -1.3587$ 0.0438

▶ **81.** $\log x = -0.5$ 0.3162

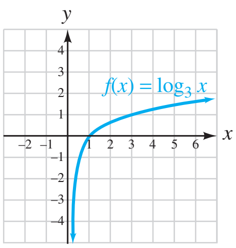
82. $\log x = -0.926$ 0.1186

83. $\log x = -1.71$ 0.0195

84. $\log x = 1.4023$ 25.2522

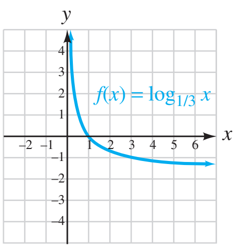
Graph each function. Determine whether each function is an increasing or a decreasing function. **See Objective 5.**

85. $f(x) = \log_3 x$



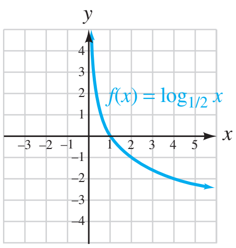
increasing

▶ **86.** $f(x) = \log_{1/3} x$



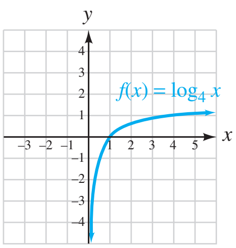
decreasing

87. $y = \log_{1/2} x$



decreasing

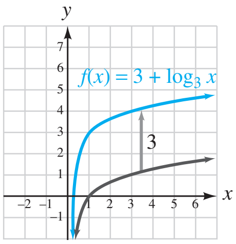
88. $y = \log_4 x$



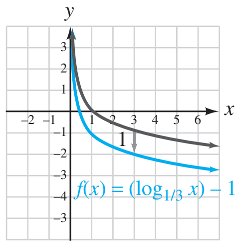
increasing

Graph each function by plotting points or by using a translation. (The basic logarithmic functions graphed in Exercises 85–88 will be helpful.) **See Example 7.**

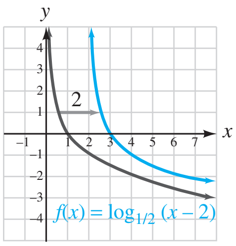
89. $f(x) = 3 + \log_3 x$



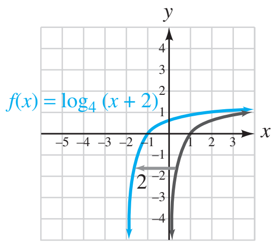
90. $f(x) = (\log_{1/3} x) - 1$



91. $y = \log_{1/2} (x - 2)$

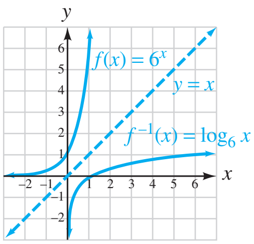


▶ **92.** $y = \log_4 (x + 2)$

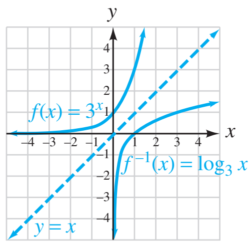


Graph each pair of inverse functions on the same coordinate system. Draw the axis of symmetry. **See Objective 1.**

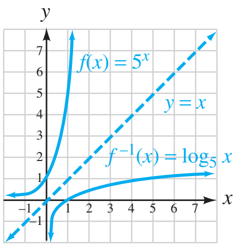
93. $f(x) = 6^x$
$f^{-1}(x) = \log_6 x$



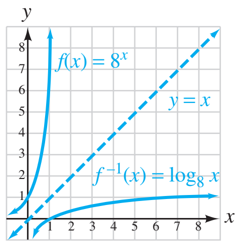
94. $f(x) = 3^x$
$f^{-1}(x) = \log_3 x$



▶ **95.** $f(x) = 5^x$
$f^{-1}(x) = \log_5 x$



96. $f(x) = 8^x$
$f^{-1}(x) = \log_8 x$

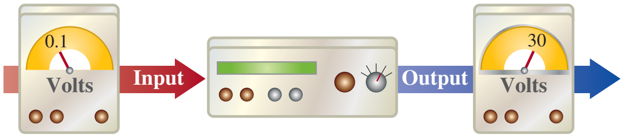


APPLICATIONS

97. INPUT VOLTAGE Find the db gain of an amplifier if the input voltage is 0.71 volt when the output voltage is 20 volts. 29 db

98. OUTPUT VOLTAGE Find the db gain of an amplifier if the output voltage is 2.8 volts when the input voltage is 0.05 volt. 35 db

99. db GAIN Find the db gain of the amplifier shown below. 49.5 db



100. db GAIN An amplifier produces an output of 80 volts when driven by an input of 0.12 volt. Find the amplifier's db gain. 56.5 db

101. THE RICHTER SCALE An earthquake has amplitude of 5,000 micrometers and a period of 0.2 second. Find its measure on the Richter scale. 4.4

102. EARTHQUAKES Find the period of an earthquake with amplitude of 80,000 micrometers that measures 6 on the Richter scale. 0.08 sec

103. EARTHQUAKES An earthquake with a period of $\frac{1}{4}$ second measures 4 on the Richter scale. Find its amplitude. 2,500 micrometers

104. EARTHQUAKES In 1985, Mexico City experienced an earthquake of magnitude 8.1 on the Richter scale. In 1989, the San Francisco Bay area was rocked by an earthquake measuring 7.1. By what factor must the amplitude of an earthquake change to increase its severity by 1 point on the Richter scale? (Assume that the period remains constant.) a factor of 10

105. CHILDREN'S HEIGHT The function $h(A) = 29 + 48.8\log(A + 1)$ gives the percent of the adult height a male child A years old has attained. If a boy is 9 years old, what percent of his adult height will he have reached? 77.8%

106. DEPRECIATION In business, equipment is often depreciated using the double declining-balance method. In this method, a piece of equipment with a life expectancy of N years, costing $\$C$, will depreciate to a value of $\$V$ in n years, where n is given by the formula

$$n = \frac{\log V - \log C}{\log\left(1 - \frac{2}{N}\right)}$$

A computer that cost \$37,000 has a life expectancy of 5 years. If it has depreciated to a value of \$8,000, how old is it? 3 yr old

107. INVESTING If $\$P$ is invested at the end of each year in an annuity earning annual interest at a rate r, the amount in the account will be $\$A$ after n years, where

$$n = \frac{\log\left[\frac{Ar}{P} + 1\right]}{\log(1 + r)}$$

If \$1,000 is invested each year in an annuity earning 12% annual interest, how long will it take for the account to be worth \$20,000? 10.8 yr

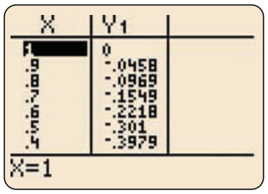
108. GROWTH OF MONEY If \$5,000 is invested each year in an annuity earning 8% annual interest, how long will it take for the account to be worth \$50,000? (See Exercise 107.) 7.6 yr

WRITING

109. Explain the mathematical relationship between $f(x) = \log x$ and $g(x) = 10^x$.

110. Explain why it is impossible to find the logarithm of a negative number.

111. A table of solutions for $f(x) = \log x$ is shown. As x decreases and gets close to 0, what happens to the values of $f(x)$?



112. What question should be asked when evaluating the expression $\log_4 16$?

REVIEW

Solve each equation.

113. $\sqrt[3]{6x + 4} = 4$ 10

114. $\sqrt{3x + 4} = \sqrt{7x + 2}$ $\frac{1}{2}$

115. $\sqrt{a + 1} - 1 = 3a$ 0; $-\frac{5}{9}$ does not check

116. $3 - \sqrt{t - 3} = \sqrt{t}$ 4

Objectives

1. Define base-e logarithms.
2. Evaluate natural logarithmic expressions.
3. Graph the natural logarithmic function.
4. Use natural logarithmic functions in applications.

SECTION 9.6
Base-e Logarithmic Functions

We have seen the importance of e in modeling the growth and decay of natural events. Just as $f(x) = e^x$ is called the natural exponential function, its inverse, the base-e logarithmic function, is called the *natural logarithmic function.* Natural logarithmic functions have many applications. They play a very important role in advanced mathematics courses, such as calculus.

1 Define base-e logarithms.

Of all possible bases for a logarithmic function, e is the most convenient for problems involving growth or decay. Since these situations occur often in natural settings, base-e logarithms are called **natural logarithms** or **Napierian logarithms** after John Napier (1550–1617). They are usually written as $\ln x$ rather than $\log_e x$:

$\ln x$ means $\log_e x$ Read ln x letter-by-letter as "ℓ. . .n. . . of x."

In general, the logarithm of a number is an exponent. For natural logarithms,

ln x is the exponent to which e is raised to get x.

Translating this statement into symbols, we have

$$e^{\ln x} = x$$

> ***Caution!*** Because of the font used to print the natural log of x, some students initially misread the notation as $\ln x$. In handwriting, $\ln x$ should look like *lnx*.

2 Evaluate natural logarithmic expressions.

Self Check 1

Evaluate each expression:

a. $\ln e^3$ 3

b. $\ln \frac{1}{e}$ -1

c. $\ln \sqrt[3]{e}$ $\frac{1}{3}$

Now Try **Problems 19, 23, and 25**

Teaching Example 1 Evaluate each natural logarithmic expression:

a. $\ln e^5$ **b.** $\ln \sqrt[5]{e}$

c. $\ln \frac{1}{e^5}$

Answers:

a. 5 **b.** $\frac{1}{5}$ **c.** -5

EXAMPLE 1 Evaluate each natural logarithmic expression:

a. $\ln e$ **b.** $\ln \frac{1}{e^2}$ **c.** $\ln 1$ **d.** $\ln \sqrt{e}$

Strategy Since the base is e in each case, we will ask "To what power must e be raised to get the given number?"

WHY That power is the value of the logarithmic expression.

Solution

a. $\ln e = 1$ Ask: "To what power must we raise e to get e?" Since $e^1 = e$, the answer is: the 1st power.

b. $\ln \frac{1}{e^2} = -2$ Ask: "To what power must we raise e to get $\frac{1}{e^2}$?" Since $e^{-2} = \frac{1}{e^2}$, the answer is: the -2 power.

c. $\ln 1 = 0$ Ask: "To what power must we raise e to get 1?" Since $e^0 = 1$, the answer is: the 0 power.

d. $\ln \sqrt{e} = \frac{1}{2}$ Ask: "To what power must we raise e to get $\sqrt{e}$?" Since $e^{1/2} = \sqrt{e}$, the answer is: the $\frac{1}{2}$ power.

Many natural logarithmic expressions are not as easy to evaluate as those in the previous example. For example, to find ln 2.34, we ask, "To what power must we raise e to get 2.34?" The answer isn't obvious. In such cases, we use a calculator.

Using Your CALCULATOR **Evaluating Base-e (Natural) Logarithms**

To find ln 2.34 with a scientific calculator, we enter

2.34 [LN]   .850150929

On some calculators, the [e^x] key also serves as the [LN] key when [2nd] or [SHIFT] is pressed. This is because $f(x) = e^x$ and $g(x) = \ln x$ are inverses.

To use a direct-entry or graphing calculator, we enter

[LN] 2.34 [)] [ENTER]   ln(2.34)
.8501509294

To four decimal places, [ln] $2.34 = 0.8502$. This means that $e^{0.8502} \approx 2.34$.

If we attempt to evaluate logarithmic expressions such as ln 0, or the logarithm of a negative number, such as ln (-5), then one of the following error statements will be displayed.

```
Error
```

```
ERR:DOMAIN
1:QUIT
2:Go to
```

```
ERR:NONREAL ANS
1:QUIT
2:Go to
```

EXAMPLE 2 Find each value to four decimal places:

a. ln 17.32 **b.** ln (-0.05)

Strategy We will use the LN key on the calculator to find an approximation for each value.

WHY The LN key on the calculator means $\ln(x)$.

Solution

a. Enter these numbers and press these keys:

Scientific calculator	Direct-entry graphing calculator
17.32 [LN]	[LN] 17.32 [)] [ENTER]

Either way, the result is 2.851861903.

b. Enter these numbers and press these keys:

Scientific calculator	Direct-entry graphing calculator
0.05 [+/−] [LN]	[LN] [(−)] 0.05 [)] [ENTER]

Either way, we obtain an error, because we cannot take the logarithm of a negative number.

Self Check 2

Find each value to four decimal places:

a. ln π 1.1447

b. ln 0 no value

Now Try **Problems 31 and 32**

Teaching Example 2 Find each value to four decimal places:

a. ln 2 **b.** ln 3 **c.** ln 1

Answers:

a. 0.6931 **b.** 1.0986 **c.** 0

EXAMPLE 3 Solve each equation: **a.** $\ln x = 1.335$ **b.** $\ln x = -5.5$

Give each result to four decimal places.

Strategy We will write each equation in equivalent exponential form.

WHY The resulting exponential equation in each case is easier to solve because the variable term is isolated on one side.

Self Check 3

Solve:

a. $\ln x = 1.9344$ 6.9199

b. $-3 = \ln x$ 0.0498

Give each result to four decimal places.

Now Try **Problems 42 and 44**

Teaching Example 3 Solve each equation:
a. $\ln x = 3.2178$ **b.** $\ln x = -4.3$
Give each result to four decimal places.
Answers:
a. 24.9731 **b.** 0.0136

Solution

a. Since the base of the natural logarithmic function is e, the equation $\ln x = 1.335$ is equivalent to $e^{1.335} = x$. To use a reverse-entry scientific calculator to find x, press these keys:

1.335 $\boxed{e^x}$

The display will read 3.799995946. To four decimal places,

$$x = 3.8000$$

b. The equation $\ln x = -5.5$ is equivalent to $e^{-5.5} = x$. To use a reverse-entry scientific calculator to find x, press these keys:

5.5 $\boxed{+/-}$ $\boxed{e^x}$

The display will read 0.004086771. To four decimal places,

$$x = 0.0041$$

3 Graph the natural logarithmic function.

The equation $y = \ln x$ is equivalent to the equation $x = e^y$. When we write $x = e^y$ in equivalent form $y = \ln x$, the result is called the *natural logarithmic function.*

The Natural Logarithmic Function

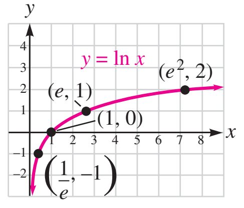
The **natural logarithmic function** with base e is defined by the equations

$$f(x) = \ln x \qquad \text{or} \qquad y = \ln x, \quad \text{where } \ln x = \log_e x.$$

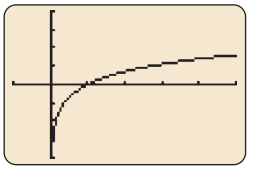
The domain of $f(x) = \ln x$ is the interval $(0, \infty)$, and the range is the interval $(-\infty, \infty)$.

To get the graph of $\ln x$, we can plot points that satisfy the equation $x = e^y$ and join them with a smooth curve, as shown in figure (a). Figure (b) shows the calculator graph of $y = \ln x$.

$y = \ln x$		
x	y	(x, y)
$\frac{1}{e}$	-1	$\left(\frac{1}{e}, -1\right)$
1	0	$(1, 0)$
e	1	$(e, 1)$
e^2	2	$(e^2, 2)$

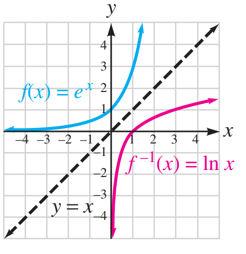


(a)



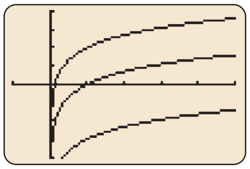
(b)

The exponential function and the natural logarithmic function are inverse functions. The figure below shows that their graphs are symmetric to the line $y = x$.



Using Your CALCULATOR **Graphing Base-e Logarithmic Functions**

Many graphs of logarithmic functions involve translations of the graph of $f(x) = \ln x$. For example, the figure shows calculator graphs of the functions $f(x) = \ln x$, $g(x) = \ln x + 2$, and $h(x) = \ln x - 3$.

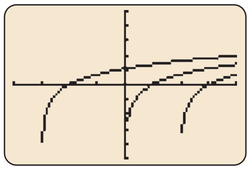


The graph of $g(x) = \ln x + 2$ is 2 units above the graph of $f(x) = \ln x$.

The graph of $h(x) = \ln x - 3$ is 3 units below the graph of $f(x) = \ln x$.

The figure below shows the calculator graph of the functions $f(x) = \ln x$, $g(x) = \ln (x - 2)$, and $h(x) = \ln (x + 3)$.

The graph of $h(x) = \ln (x + 3)$ is 3 units to the left of the graph of $f(x) = \ln x$.



The graph of $g(x) = \ln (x - 2)$ is 2 units to the right of the graph of $f(x) = \ln x$.

4 Use natural logarithmic functions in applications.

Base-e logarithms have many applications. If a population grows exponentially at a certain annual rate, the time required for the population to double is called the **doubling time.** It is given by the following formula.

Formula for Doubling Time

If r represents the annual rate, compounded continuously, and t represents time required for a population to double, then

$$t = \frac{\ln 2}{r}$$

EXAMPLE 4 ***Doubling Time*** The population of Earth is growing at the approximate rate of 1.17% per year. If this rate continues, how long will it take for the population to double?

Strategy We will substitute 0.0117 for r in the formula for doubling time and evaluate the right side using a calculator.

WHY We can use this formula because we are given the annual rate of continuous compounding.

Solution

Because the population is growing at the rate of 1.17% per year, we substitute 0.0117 for r in the formula for doubling time and simplify.

$$t = \frac{\ln 2}{r}$$

$$t = \frac{\ln 2}{0.0117}$$

$$\approx 59.24334877 \quad \text{Use a calculator. Find ln 2 first. Then divide the result by 0.0117.}$$

The population will double in about 59 years.

Self Check 4

DOUBLING TIME See Example 4. If the population's annual growth rate could be reduced to 1.1% per year, what would be the doubling time? about 63 years

Now Try **Problem 51**

Teaching Example 4 DOUBLING TIME See Example 4. If the population's annual growth rate could be reduced to 1% per year, what would be the doubling time?
Answer:
about 69 years

Self Check 5

DOUBLING TIME In Example 5, how long will it take to double at 9%, compounded continuously?

Now Try **Problem 52**

Self Check 5 Answer
about 7.7 years

Teaching Example 5 DOUBLING TIME In Example 5, how long will it take to double at 7% compounded continuously?
Answer:
about 9.9 years

EXAMPLE 5 ***Doubling Time*** How long will it take \$1,000 to double at an annual rate of 8%, compounded continuously?

Strategy We will substitute 0.08 for r in the formula for doubling time and evaluate the right side using a calculator. In this case, the information that the original amount is \$1,000 is unnecessary.

WHY We can use this formula because we are given the annual rate of continuous compounding.

Solution
We substitute 0.08 for r and simplify:

$$t = \frac{\ln 2}{r}$$

$$t = \frac{\ln 2}{\mathbf{0.08}}$$

$$\approx 8.664339757$$ Use a calculator. Find ln 2 first. Then divide the result by 0.08.

It will take about $8\frac{2}{3}$ years for the money to double.

ANSWERS TO SELF CHECKS
1. a. 3 **b.** −1 **c.** $\frac{1}{3}$ **2. a.** 1.1447 **b.** no value **3. a.** 6.9199 **b.** 0.0498
4. about 63 years **5.** about 7.7 years

SECTION 9.6 STUDY SET

VOCABULARY

Fill in the blanks.

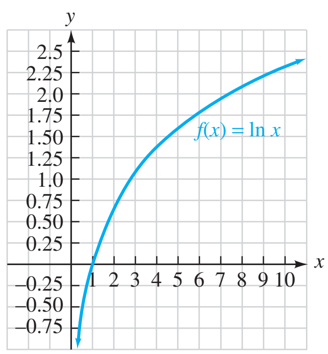
1. Base-e logarithms are often called _natural_ logarithms.
2. $f(x) = \ln x$ and $f(x) = e^x$ are _inverse_ functions.

CONCEPTS

3. Use a calculator to complete the table of values for $f(x) = \ln x$. Round to the nearest hundredth.

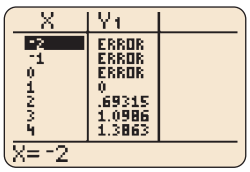
x	0.5	1	2	3	4	5	6	7	8	9	10
$f(x)$	−0.69	0	0.69	1.10	1.39	1.61	1.79	1.95	2.08	2.20	2.30

4. Graph $f(x) = \ln x$. (See Exercise 3.) Note that the units on the x- and y-axes are different.



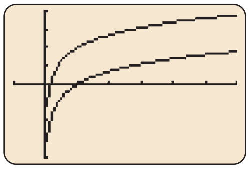
Fill in the blanks.

5. The graph of $f(x) = \ln x$ has the _y-axis_ as an asymptote.
6. The domain of the function $f(x) = \ln x$ is the interval $(0, \infty)$.
7. The range of the function $f(x) = \ln x$ is the interval $(-\infty, \infty)$.
8. The graph of $f(x) = \ln x$ passes through the point $(1, 0)$.
9. The statement $y = \ln x$ is equivalent to the exponential statement $e^y = x$.
10. The logarithm of a negative number is _undefined_.
11. A table of values for $f(x) = \ln x$ is shown. Explain why ERROR appears in the Y_1 column for the first three entries.
The logarithm of a negative number or 0 is not defined.

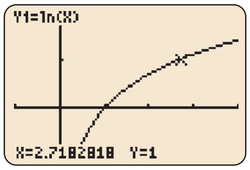


Selected exercises available online at
www.webassign.net/brookscole

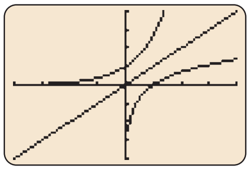
12. The illustration shows the graph of $f(x) = \ln x$, as well as a vertical translation of that graph. Using the notation $g(x)$ for the translation, write the defining equation for the function. $g(x) = 2 + \ln x$



13. In the illustration, $f(x) = \ln x$ was graphed, and the TRACE feature was used. Find the x-coordinate of the point on the graph having a y-coordinate of 1. What is the name given this number? 2.7182818…, e



14. The graphs of $f(x) = \ln x$, $f(x) = e^x$, and $y = x$ are shown. What phrase is used to describe the relationship between the graphs?
symmetric about the line $y = x$



NOTATION

Fill in the blanks.

15. ln 2 means $\log_{e}$ 2.

▶ **16.** log 2 means $\log_{10}$ 2.

17. If a population grows exponentially at a rate r, the time it will take the population to double is given by the formula $t = \frac{\ln 2}{r}$.

▶ **18.** To evaluate a base-10 logarithm with a calculator, use the LOG key. To evaluate a base-e logarithm, use the LN key.

GUIDED PRACTICE

***Evaluate each natural logarithmic expression without using a calculator.* See Example 1.**

19. $\ln e^5$ 5

20. $\ln e^2$ 2

▶ **21.** $\ln e^6$ 6

22. $\ln e^4$ 4

23. $\ln \frac{1}{e^6}$ -6

24. $\ln \frac{1}{e^3}$ -3

25. $\ln \sqrt[4]{e}$ $\frac{1}{4}$

26. $\ln \sqrt[6]{e}$ $\frac{1}{6}$

▶ **27.** $\ln \sqrt[3]{e^2}$ $\frac{2}{3}$

28. $\ln \sqrt[4]{e^3}$ $\frac{3}{4}$

29. $\ln e^{-7}$ -7

30. $\ln e^{-10}$ -10

***Use a calculator to evaluate each expression, if possible. Express all answers to four decimal places.* See Example 2.**

31. ln 35.15 3.5596

▶ **32.** ln 0.675 -0.3930

33. ln 0.00465 -5.3709

▶ **34.** ln 378.96 5.9374

35. ln 1.72 0.5423

▶ **36.** ln 2.7 0.9933

37. $\ln(-0.1)$ undefined

▶ **38.** $\ln(-10)$ undefined

***Solve each equation for x. Express all answers to four decimal places.* See Example 3.**

39. $\ln x = 1.4023$ 4.0645

▶ **40.** $\ln x = 2.6490$ 14.1399

41. $\ln x = 4.24$ 69.4079

▶ **42.** $\ln x = 0.926$ 2.5244

43. $\ln x = -3.71$ 0.0245

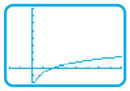
▶ **44.** $\ln x = -0.28$ 0.7558

45. $1.001 = \ln x$ 2.7210

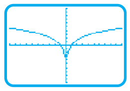
▶ **46.** $\ln x = -0.001$ 0.9990

***Use a graphing calculator to graph each function.* See Objective 3.**

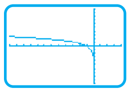
47. $y = \ln\left(\frac{1}{2}x\right)$



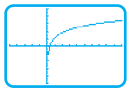
48. $y = \ln x^2$



49. $f(x) = \ln(-x)$



▶ **50.** $f(x) = \ln(3x)$



APPLICATIONS

▶ **51.** POPULATION GROWTH How long will it take the population of River City to double? 5.8 yr

▶ **52.** DOUBLING MONEY How long will it take \$1,000 to double if it is invested at an annual rate of 5% compounded continuously? 13.9 yr

▶ **53.** POPULATION GROWTH A population growing continuously at an annual rate r will triple in a time t given by the formula

$$t = \frac{\ln 3}{r}$$

How long will it take the population of a town to triple if it is growing at the rate of 12% per year? 9.2 yr

▶ **54.** TRIPLING MONEY Find the length of time for \$25,000 to triple when it is invested at 6% annual interest, compounded continuously. See Exercise 53. 18.3 yr

55. FORENSIC MEDICINE To estimate the number of hours t that a murder victim had been dead, a coroner used the formula

$$t = \frac{1}{0.25} \ln \frac{98.6 - T_s}{82 - T_s}$$

where T_s is the temperature of the surroundings where the body was found. If the crime took place in an apartment where the thermostat was set at 70°F, approximately how long ago did the murder occur? about 3.5 hr

56. MAKING JELL-O After the contents of a package of JELL-O are combined with boiling water, the mixture is placed in a refrigerator whose temperature remains a constant 42°F. Estimate the number of hours t that it will take for the JELL-O to cool to 50°F using the formula

$$t = -\frac{1}{0.9} \ln \frac{50 - T_r}{200 - T_r}$$

where T_r is the temperature of the refrigerator. about 3.3 hr

WRITING

57. Explain the difference between the functions $f(x) = \log x$ and $f(x) = \ln x$.

58. How are the functions $f(x) = \ln x$ and $f(x) = e^x$ related?

REVIEW

Write an equation of the required line.

59. Parallel to $y = 5x - 8$ and passing through the origin $y = 5x$

60. Having a slope of 7 and a y-intercept of 3 $y = 7x + 3$

61. Passing through the point (3, 2) and perpendicular to the line $y = \frac{2}{3}x - 12$ $y = -\frac{3}{2}x + \frac{13}{2}$

62. Parallel to the line $3x + 2y = 9$ and passing through the point $(-3, 5)$ $y = -\frac{3}{2}x + \frac{1}{2}$

63. Vertical line through the point (2, 3) $x = 2$

64. Horizontal line through the point (2, 3) $y = 3$

SECTION 9.7
Properties of Logarithms

Objectives

1 Use the four basic properties of logarithms.
2 Use the product and quotient rules for logarithms.
3 Use the power rule for logarithms.
4 Write logarithmic expressions as a single logarithm.
5 Use the change-of-base formula.
6 Use properties of logarithms to solve application problems.

In this section, we will discuss eight properties of logarithms and use them to simplify logarithmic expressions. We will then show how to change a logarithm from one base to another. We conclude the section by solving some problems from the field of chemistry.

1 Use the four basic properties of logarithms.

Since logarithms are exponents, the properties of exponents have counterparts in the theory of logarithms. We begin with four basic properties.

Properties of Logarithms

If b represents a positive number and $b \neq 1$, then

1. $\log_b 1 = 0$ **2.** $\log_b b = 1$

3. $\log_b b^x = x$ **4.** $b^{\log_b x} = x$ where $x > 0$

Properties 1 through 4 follow directly from the definition of a logarithm.

1. $\log_b 1 = 0$, because $b^0 = 1$.

2. $\log_b b = 1$, because $b^1 = b$.

3. $\log_b b^x = x$, because $b^x = b^x$.

4. $b^{\log_b x} = x$, because $\log_b x$ is the exponent to which b is raised to get x.

Properties 3 and 4 also indicate that the composition of the exponential and logarithmic functions (in both directions) is the identity function. This is expected, because the exponential and logarithmic functions with the same base are inverse functions.

EXAMPLE 1 Simplify each expression:
a. $\log_5 1$ **b.** $\log_3 3$ **c.** $\ln e^3$ **d.** $9^{\log_9 7}$

Strategy We compare each logarithmic expression to the left side of the 4 basic properties of logarithms.

WHY When we get a match, the property will provide the answer.

Solution

a. By property 1, $\log_5 1 = 0$, because $5^0 = 1$.

b. By property 2, $\log_3 3 = 1$, because $3^1 = 3$.

c. By property 3, $\ln e^3 = 3$, because $e^3 = e^3$.

d. By property 4, $9^{\log_9 7} = 7$, because $\log_9 7$ is the power to which 9 is raised to get 7.

Self Check 1

Simplify:
a. $\log_4 1$ 0 **b.** $\log_4 4$ 1
c. $\log_2 2^4$ 4 **d.** $5^{\log_5 2}$ 2

Now Try Problems 24 and 28

Teaching Example 1 Simplify:
a. $\log_\pi \pi$ **b.** $\log_7 7^9$ **c.** $6^{\log_6 5}$
d. $\log_3 1$
Answers:
a. 1 **b.** 9 **c.** 5 **d.** 0

2 Use the product and quotient rules for logarithms.

The next two properties state that

The logarithm of a product is the sum of the logarithms.
The logarithm of a quotient is the difference of the logarithms.

The Product Rule and Quotient Rule for Logarithms

For all positive real numbers M, N, and b, where $b \neq 1$,

5. $\log_b MN = \log_b M + \log_b N$ **6.** $\log_b \frac{M}{N} = \log_b M - \log_b N$

EXAMPLE 2 Use the product rule for logarithms to rewrite each of the following: **a.** $\log_2 (2 \cdot 7)$ **b.** $\log (100x)$

Strategy We will use the product rule for logarithms.

WHY Each of the logarithmic expressions has the form $\log_b MN$.

Solution

a. $\log_2 (2 \cdot 7) = \log_2 2 + \log_2 7$ *The log of a product is the sum of the logs.*
$= 1 + \log_2 7$ $\log_2 2 = 1$

b. $\log (100x) = \log 100 + \log x$ *The log of a product is the sum of the logs.*
$= 2 + \log x$ $\log 100 = 2$

Self Check 2

Use the product rule for logarithms to rewrite each of the following:
a. $\log_3 (4 \cdot 3)$ $\log_3 4 + 1$
b. $\log (1{,}000y)$ $3 + \log y$

Now Try Problems 33 and 35

Teaching Example 2 Use the product rule for logarithms to rewrite each of the following:
a. $\log_5 (5 \cdot 3)$ **b.** $\log (10{,}000x)$
Answers:
a. $1 + \log_5 3$ **b.** $4 + \log x$

EXAMPLE 3 Use the quotient rule for logarithms to rewrite each of the following: **a.** $\ln \frac{10}{7}$ **b.** $\log_4 \frac{x}{64}$

Strategy We will use the quotient rule for logarithms.

WHY Each of the logarithmic expressions has the form $\log_b \frac{M}{N}$.

Self Check 3

Use the quotient rule to rewrite each of the following:

a. $\log_6 \frac{6}{5}$ $1 - \log_6 5$

b. $\ln \frac{y}{100}$ $\ln y - \ln 100$

Now Try Problems 37 and 39

Teaching Example 3 Use the quotient rule for logarithms to rewrite each of the following:

a. $\ln \frac{5}{2}$ **b.** $\log \frac{100}{x}$

Answers:

a. $\ln 5 - \ln 2$ **b.** $2 - \log x$

Solution

a. $\ln \frac{10}{7} = \ln 10 - \ln 7$ — The log of a quotient is the difference of the logs.

b. $\log_4 \frac{x}{64} = \log_4 x - \log_4 64$ — The log of a quotient is the difference of the logs.

$= \log_4 x - 3$ — $\log_4 64 = 3$

Self Check 4

Use logarithm properties to rewrite the expression: $\log_b \frac{x}{yz}$

Now Try Problem 44

Self Check 4 Answer

$\log_b x - \log_b y - \log_b z$

Teaching Example 4 Use logarithm properties to rewrite the expression: $\log \frac{xy}{100z}$

Answer:

$\log x + \log y - 2 - \log z$

EXAMPLE 4

Use logarithm properties to rewrite the expression: $\log \frac{xy}{z}$

Strategy We will use the quotient rule for logarithms and then the product rule.

WHY We will use the quotient rule first because $\log \frac{xy}{z}$ has the form $\log_b \frac{M}{N}$. We then use the product rule because the numerator of $\frac{xy}{z}$ contains a product.

Solution

In the expression $\log \frac{xy}{z}$, we have the logarithm of a quotient.

$\log \frac{xy}{z} = \log (xy) - \log z$ — The log of a quotient is the difference of the logs.

$= (\log x + \log y) - \log z$ — The log of a product is the sum of the logs.

$= \log x + \log y - \log z$ — Remove parentheses.

Caution! By property 5 of logarithms, the logarithm of a *product* is equal to the *sum* of the logarithms. The logarithm of a sum or a difference usually does not simplify. In general,

$$\log_b (M + N) \neq \log_b M + \log_b N \quad \text{and} \quad \log_b (M - N) \neq \log_b M - \log_b N$$

By property 6, the logarithm of a *quotient* is equal to the *difference* of the logarithms. The logarithm of a quotient is not the quotient of the logarithms:

$$\log_b \frac{M}{N} \neq \frac{\log_b M}{\log_b N}$$

Using Your CALCULATOR Verifying Properties of Logarithms

We can use a calculator to illustrate property 5 of logarithms by showing that

$$\ln [(3.7)(15.9)] = \ln 3.7 + \ln 15.9$$

We calculate the left- and right-hand sides of the equation separately and compare the results. To use a scientific calculator to find $\ln [(3.7)(15.9)]$, we enter these numbers and press these keys:

3.7 $\boxed{\times}$ 15.9 $\boxed{=}$ $\boxed{\text{LN}}$ 4.074651929

To find $\ln 3.7 + \ln 15.9$, we enter these numbers and press these keys:

3.7 $\boxed{\text{LN}}$ $\boxed{+}$ 15.9 $\boxed{\text{LN}}$ $\boxed{=}$ 4.074651929

Since the left- and right-hand sides are equal, the equation $\ln [(3.7)(15.9)] = \ln 3.7 + \ln 15.9$ is true.

3 Use the power rule for logarithms.

Two more properties of logarithms state that

The logarithm of a power is the power times the logarithm.

If the logarithms of two numbers are equal, the numbers are equal.

Properties of Logarithms

If M, p, and b represent positive numbers and $b \neq 1$, then

7. $\log_b M^p = p \log_b M$ **8.** If $\log_b x = \log_b y$, then $x = y$.

PROOF To prove property 7, we let $x = \log_b M$, write the expression in exponential form, and raise both sides to the pth power:

$$M = b^x$$

$$(M)^p = (b^x)^p \quad \text{Raise both sides to the } p\text{th power.}$$

$$M^p = b^{px} \quad \text{Keep the base and multiply the exponents.}$$

Using the definition of logarithms gives

$$\log_b M^p = px$$

Substituting the value for x completes the proof.

$$\log_b M^p = p \log_b M$$

Property 8 follows from the fact that the logarithmic function is a one-to-one function. Property 8 will be used in the next section, when we solve logarithmic equations.

EXAMPLE 5 Use the power rule for logarithms to rewrite each of the following: **a.** $\log_5 6^2$ **b.** $\log\sqrt{10}$

Strategy In each case, we will use the power rule for logarithms.

WHY We use the power rule because $\log_5 6^2$ has the form $\log_b M^p$, as will $\log\sqrt{10}$ once we write $\sqrt{10}$ as $10^{1/2}$.

Solution

a. $\log_5 6^2 = 2\log_5 6$ The log of a power is the power times the log.

b. $\log\sqrt{10} = \log(10)^{1/2}$ Write $\sqrt{10}$ using a fractional exponent: $\sqrt{10} = (10)^{1/2}$.

$= \dfrac{1}{2}\log 10$ The log of a power is the power times the log.

$= \dfrac{1}{2}$ Simplify: $\log 10 = 1$.

Self Check 5

Use the power rule for logarithms to rewrite each of the following:

a. $\ln x^4$ $4 \ln x$

b. $\log_2 \sqrt[3]{3}$ $\frac{1}{3}\log_2 3$

Now Try **Problems 45 and 48**

Teaching Example 5 Use the power rule for logarithms to rewrite each of the following:

a. $\log_3 7^4$ **b.** $\log\sqrt[4]{5}$ **c.** $\log x^9$

Answers:

a. $4\log_3 7$ **b.** $\frac{1}{4}\log 5$ **c.** $9\log x$

EXAMPLE 6 Use properties of logarithm to rewrite each expression as the sum and/or difference of logarithms of a single quantity:

a. $\log_b(x^2y^3z)$ **b.** $\ln\dfrac{y^3\sqrt{x}}{z}$

Strategy In each case, we will use the appropriate product, quotient, and/or power rule for logarithms.

Self Check 6

Use properties of logarithm to rewrite each expression as the sum and/or difference of logarithms of a single quantity:

$\log\sqrt[4]{\dfrac{x^3y}{z}}$ $\frac{1}{4}(3\log x + \log y - \log z)$

Now Try **Problems 50 and 55**

Teaching Example 6 Use properties of logarithm to rewrite each expression as the sum and/or difference of logarithms of a single quantity:

a. $\ln \frac{5\sqrt[3]{x}}{y^2}$

b. $\log_2 \sqrt[3]{\frac{x^2 y}{z}}$

Answers:

a. $\ln 5 + \frac{1}{3}\ln x - 2\ln y$

b. $\frac{2}{3}\log_2 x + \frac{1}{3}\log_2 y - \frac{1}{3}\log_2 z$

WHY Each case is a logarithm of a combination of products, quotients and/or powers.

Solution

a. We begin by recognizing that $\log_b (x^2y^3z)$ is the logarithm of a product.

$$\log_b (x^2y^3z) = \log_b x^2 + \log_b y^3 + \log_b z$$ The log of a product is the sum of the logs.

$$= 2\log_b x + 3\log_b y + \log_b z$$ The log of a power is the power times the log.

b. The expression $\ln \frac{y^3\sqrt{x}}{z}$ is the logarithm of a quotient.

$$\ln \frac{y^3\sqrt{x}}{z} = \ln (y^3\sqrt{x}) - \ln z$$ The log of a quotient is the difference of the logs.

$$= \ln y^3 + \ln\sqrt{x} - \ln z$$ The log of a product is the sum of the logs.

$$= \ln y^3 + \ln x^{1/2} - \ln z$$ Write $\sqrt{x}$ as $x^{1/2}$.

$$= 3\ln y + \frac{1}{2}\ln x - \ln z$$ The log of a power is the power times the log.

4 Write logarithmic expressions as a single logarithm.

We can use the properties of logarithms to combine several logarithms into one logarithm.

Self Check 7

Write the expression as one logarithm:

$$2\log_a x + \frac{1}{2}\log_a y - 2\log_a (x - y)$$

Now Try **Problem 58**

Self Check 7 Answer

$\log_a \frac{x^2\sqrt{y}}{(x - y)^2}$

Teaching Example 7 Write the expression as one logarithm:

$\frac{2}{3}\log_b x - 5\log_b y + 3\log_b (x + 4)$

Answer:

$\log_b \frac{(x + 4)^3\sqrt[3]{x^2}}{y^5}$

EXAMPLE 7 Write each of the given expressions as one logarithm:

a. $3\log_a x + \frac{1}{2}\log_a y$ **b.** $\frac{1}{2}\log_b (x - 2) - \log_b y + 3\log_b z$

Strategy In each case, we will use the appropriate product, quotient, and/or power rule for logarithms in reverse.

WHY We will use the power rule when we see expressions of the form $p\log_b M$. The $+$ symbol between logarithmic terms suggests that we use the product rule, and the $-$ symbol between logarithmic terms suggests that we use the quotient rule.

Solution

a. We begin by applying the power rule to each term of the expression.

$$3\log_a x + \frac{1}{2}\log_a y$$

$$= \log_a x^3 + \log_a y^{1/2}$$ A power times a log is the log of the power.

$$= \log_a (x^3 \cdot y^{1/2})$$ The sum of two logs is the log of the product.

b. The first and third terms of this expression can be rewritten using the power rule of logarithms.

$$\frac{1}{2}\log_b (x - 2) - \log_b y + 3\log_b z$$

$$= \log_b (x - 2)^{1/2} - \log_b y + \log_b z^3$$ A power times a log is the log of the power.

$$= \log_b \frac{(x - 2)^{1/2}}{y} + \log_b z^3$$ The difference of two logs is the log of the quotient.

$$= \log_b \left(\frac{\sqrt{x - 2}}{y} \cdot z^3\right)$$

$$= \log_b \frac{z^3\sqrt{x - 2}}{y}$$ The sum of two logs is the log of the product. Write $(x - 2)^{1/2}$ as $\sqrt{x - 2}$.

We summarize the properties of logarithms as follows.

Properties of Logarithms

If b, M, and N represent positive numbers and $b \neq 1$, and p is any real number,

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b b^x = x$
4. $b^{\log_b x} = x$
5. $\log_b MN = \log_b M + \log_b N$
6. $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
7. $\log_b M^p = p \log_b M$
8. If $\log_b x = \log_b y$, then $x = y$.

EXAMPLE 8 Given that $\log 2 \approx 0.3010$ and $\log 3 \approx 0.4771$, find approximations for **a.** $\log 6$ **b.** $\log 18$

Strategy We will express 6 and 18 using factors of 2 and 3 and then use the properties of logarithms to simplify each resulting expression.

WHY We express 6 and 8 using factors of 2 and 3 because we are given values of log 2 and log 3.

Solution

a.

$$\begin{aligned} \log 6 &= \log(2 \cdot 3) && \text{Write 6 using the factors 2 and 3.} \\ &= \log 2 + \log 3 && \text{The log of a product is the sum of the logs.} \\ &\approx 0.3010 + 0.4771 && \text{Substitute the value of each logarithm.} \\ &\approx 0.7781 \end{aligned}$$

b.

$$\begin{aligned} \log 18 &= \log(2 \cdot 3^2) && \text{Write 18 using the factors 2 and 3.} \\ &= \log 2 + \log 3^2 && \text{The log of a product is the sum of the logs.} \\ &= \log 2 + 2 \log 3 && \text{The log of a power is the power times the log.} \\ &\approx 0.3010 + 2(0.4771) && \text{Substitute the value of each logarithm.} \\ &\approx 1.2552 \end{aligned}$$

Self Check 8

Given $\log_3 \approx 0.4771$ and $\log_5 \approx 0.6990$, find approximations for:

a. $\log 1.5$ 0.1761

b. $\log 0.75$ -0.1249

Now Try **Problems 61 and 64**

Teaching Example 8 Given $\log_b 5 \approx 1.462$ and $\log_b 7 \approx 1.768$, find approximations for

a. $\log_b 35$ **b.** $\log_b \dfrac{7}{5}$

c. $\log_b 49$

Answers:

a. 3.230 **b.** 0.306 **c.** 3.536

5 Use the change-of-base formula.

Most calculators can find common logarithms and natural logarithms. If we need to find a logarithm with some other base, we use a conversion formula.

If we know the base-a logarithm of a number, we can find its logarithm to some other base b by using a formula called the **change-of-base formula.**

Change-of-Base Formula

For any positive real numbers a, b, and x, with $a \neq 1$ and $b \neq 1$,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

PROOF To prove this formula, we begin with the equation $\log_b x = y$.

(1) $y = \log_b x$

$x = b^y$ Change the equation from logarithmic to exponential form.

$\log_a x = \log_a b^y$ Take the base-a logarithm of both sides.

$\log_a x = y \log_a b$ The log of a power is the power times the log.

$y = \dfrac{\log_a x}{\log_a b}$ Divide both sides by $\log_a b$.

$\log_b x = \dfrac{\log_a x}{\log_a b}$ Refer to Equation 1 and substitute $\log_b x$ for y.

If we know logarithms to base a (for example, $a = 10$), we can find the logarithm of x to a new base b. We simply divide the base-a logarithm of x by the base-a logarithm of b.

Self Check 9

Find $\log_5 3$ to four decimal places. 0.6826

Now Try **Problem 74**

Teaching Example 9
Find $\log_9 12$ to four decimal places.
Answer:
1.1309

EXAMPLE 9 Find: $\log_3 5$

Strategy To evaluate this base-3 logarithm, we will substitute into the change-of-base formula.

WHY We assume that the reader does not have a calculator that evaluates base-3 logarithms directly. Thus, the only alternative is to change the base.

Solution

We can use base-10 logarithms to find a base-3 logarithm. To do this, we substitute 3 for b, 10 for a, and 5 for x in the change-of-base formula:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} \quad b = 3,\ x = 5,\ \text{and } a = 10.$$

$$\approx 1.464973521 \quad \text{Use a calculator.}$$

To four decimal places, $\log_3 5 = 1.4650$.

We can also use the natural logarithm function (base e) in the change-of-base formula to find a base-3 logarithm.

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$\log_3 5 = \frac{\ln 5}{\ln 3} \quad b = 3,\ x = 5,\ \text{and } a = e.\ \log_e 5 = \ln 5 \text{ and } \log_e 3 = \ln 3.$$

$$\approx 1.464973521 \quad \text{Use a calculator.}$$

We obtain the same result.

Caution! Don't misapply the quotient rule: $\frac{\log_{10} 5}{\log_{10} 3}$ means $\log_{10} 5 \div \log_{10} 3$. It is the expression $\log_{10} \frac{5}{3}$ that means $\log_{10} 5 - \log_{10} 3$.

6 Use properties of logarithms to solve application problems.

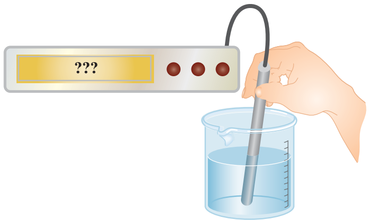
In chemistry, common logarithms are used to express the acidity of solutions. The more acidic a solution, the greater the concentration of hydrogen ions. This concentration is indicated indirectly by the *pH scale,* or *hydrogen ion index.* The pH of a solution is defined as follows.

pH of a Solution

If $[H^+]$ represents the hydrogen ion concentration in gram-ions per liter, then

$$pH = -\log [H^+]$$

EXAMPLE 10 ***pH Meters*** One of the most accurate ways to measure pH is with a probe and meter. What reading should the meter give for pure water if water has a hydrogen ion concentration $[H^+]$ of approximately 10^{-7} gram-ions per liter?



Strategy We will substitute into the formula for pH and use the power rule for logarithms to simplify the right side.

WHY After substituting 10^{-7} for H^+ in $-\log[H^+]$, the resulting expression will have the form $\log_b M^p$.

Solution

Since pure water has approximately 10^{-7} gram-ions per liter, its pH is

$$\begin{aligned} pH &= -\log [\mathbf{H^+}] & \\ pH &= -\log \mathbf{10^{-7}} & \text{Substitute } 10^{-7} \text{ for } H^+. \\ &= -(-7)\log 10 & \text{The log of a power is the power times the log.} \\ &= -(-7)\cdot 1 & \text{Simplify: } \log 10 = 1. \\ &= 7 & \end{aligned}$$

The meter should give a reading of 7.

Self Check 10

pH METERS The hydrogen ion concentration range for a freshwater aquarium has a low-end value of 2.5×10^{-8}. Find the pH level that corresponds to this value. 7.6

Now Try **Problem 95**

Teaching Example 10 pH METERS The hydrogen ion concentration range for a freshwater aquarium has a high-end value of 1.6×10^{-7}. Find the pH level that corresponds to this value.
Answer:
6.8

EXAMPLE 11 ***Hydrogen Ion Concentration*** Find the hydrogen ion concentration of seawater if its pH is 8.5.

Strategy To find the hydrogen ion concentration, we will substitute 8.5 for pH in the formula $pH = -\log[H^+]$ and solve the resulting equation for $[H^+]$.

WHY After substituting for pH, the resulting logarithmic equation can be solved by solving an equivalent exponential equation.

Solution

To find its hydrogen ion concentration, we solve the following equation for $[H^+]$.

$$\begin{aligned} \mathbf{pH} &= -\log [H^+] & \\ \mathbf{8.5} &= -\log [H^+] & \text{Substitute 8.5 for pH.} \\ -8.5 &= \log [H^+] & \text{Multiply both sides by } -1. \\ [H^+] &= 10^{-8.5} & \text{Change the equation to exponential form.} \end{aligned}$$

We can use a calculator to find that

$$[H^+] \approx 3.2 \times 10^{-9} \text{ gram-ions per liter}$$

Self Check 11

HYDROGEN ION CONCENTRATION Find the hydrogen ion concentration of a solution with a pH value of 4.8. 1.58×10^{-5}

Now Try **Problem 96**

Teaching Example 11 HYDROGEN ION CONCENTRATION Find the hydrogen ion concentration of pickles if their pH is 3.2.
Answer:
6.31×10^{-4}

ANSWERS TO SELF CHECKS

1. a. 0 **b.** 1 **c.** 4 **d.** 2 **2. a.** $\log_3 4 + 1$ **b.** $3 + \log y$ **3. a.** $1 - \log_6 5$ **b.** $\ln y - \ln 100$ **4.** $\log_b x - \log_b y - \log_b z$ **5. a.** $4 \ln x$ **b.** $\frac{1}{3}\log_2 3$ **6.** $\frac{1}{4}(3\log x + \log y - \log z)$ **7.** $\log_a \frac{x^2\sqrt{y}}{(x-y)^2}$ **8. a.** 0.1761 **b.** -0.1249 **9.** 0.6826 **10.** 7.6 **11.** 1.58×10^{-5}

SECTION 9.7 STUDY SET

VOCABULARY

Fill in the blanks.

1. The expression $\log_3 (4x)$ is the logarithm of a product.
2. The expression $\log_2 \frac{5}{x}$ is the logarithm of a quotient.
3. The expression $\log 4^x$ is the logarithm of a power.
4. ▶ In the expression $\log_5 4$, the number 5 is the base of the logarithm.

CONCEPTS

Fill in the blanks.

5. $\log_b 1 = 0$
6. $\log_b b = 1$
7. $\log_b MN = \log_b M + \log_b N$
8. $b^{\log_b x} = x$
9. If $\log_b x = \log_b y$, then $x = y$
10. ▶ $\log_b \frac{M}{N} = \log_b M - \log_b N$
11. $\log_b x^p = p \cdot \log_b x$
12. ▶ $\log_b b^x = x$
13. $\log_b (A + B) \neq \log_b A + \log_b B$
14. $\log_b A + \log_b B = \log_b AB$
15. $\log_b x = \dfrac{\log_a x}{\log_a b}$
16. $pH = -\log[H^+]$
17. Three logarithmic expressions have been evaluated, and the results are shown on the calculator display on the right. Show that each result is correct by writing the equivalent base-10 exponential statement. $10^0 = 1, 10^1 = 10, 10^2 = 10^2$

```
log(1)
                0
log(10)
                1
log(10²)
                2
```

18. Three logarithmic expressions have been evaluated, and the results are shown on the calculator display on the right. Show that each result is correct by writing the equivalent base-e exponential statement. (The notation ln(e^(2)) means $\ln e^2$.) $e^2 = e^2, e^3 = e^3, e^4 = e^4$

```
ln(e^(2))
                2
ln(e^(3))
                3
ln(e^(4))
                4
```

NOTATION

Complete each solution.

19. $\log_b rst = \log_b (rs)t$
$= \log_b (rs) + \log_b t$
$= \log_b r + \log_b s + \log b\, t$

20. ▶ $\log \frac{r}{st} = \log r - \log(st)$
$= \log r - (\log s + \log t)$
$= \log r - \log s - \log t$

GUIDED PRACTICE

Evaluate each expression. See Example 1.

21. $\log_4 1$ 0
22. ▶ $\log_4 4$ 1
23. $\log_4 4^7$ 7
24. $\ln e^8$ 8
25. ▶ $5^{\log_5 10}$ 10
26. ▶ $8^{\log_8 10}$ 10
27. $\log_5 5^2$ 2
28. ▶ $\log_4 4^2$ 2
29. $\ln e$ 1
30. $\log_7 1$ 0
31. $\log_3 3^7$ 7
32. ▶ $5^{\log_5 8}$ 8

Use the product rule for logarithms to rewrite each expression. Assume that $x > 0$. See Example 2.

33. $\log_2(4 \cdot 5)$ $2 + \log_2 5$
34. ▶ $\log_3(27 \cdot 5)$ $3 + \log_3 5$
35. $\log(10x)$ $1 + \log x$
36. $\log(1{,}000x)$ $3 + \log x$

▶ Selected exercises available online at **www.webassign.net/brookscole**

Use the quotient rule for logarithms to rewrite each expression. Assume that $x > 0$. See Example 3.

37. $\ln \frac{12}{5}$ $\ln 12 - \ln 5$

38. $\log_4 \frac{8}{3}$ $\log_4 8 - \log_4 3$

▶ **39.** $\log_6 \frac{x}{36}$ $\log_6 x - 2$

40. $\log_8 \frac{x}{8}$ $\log_8 x - 1$

Use properties of logarithms to rewrite each expression. Assume that all variables are positive. See Example 4.

41. $\log xyz$
$\log x + \log y + \log z$

▶ **42.** $\log 4xz$
$\log 4 + \log x + \log z$

43. $\log_2 \frac{2x}{y}$
$1 + \log_2 x - \log_2 y$

44. $\log_3 \frac{x}{yz}$
$\log_3 x - \log_3 y - \log_3 z$

Use the power rule for logarithms to rewrite each expression. Assume that all variables are positive. See Example 5.

45. $\log_4 5^2$ $2 \log_4 5$

46. $\log_3 z^9$ $9 \log_3 z$

47. $\log \sqrt{5}$ $\frac{1}{2} \log 5$

▶ **48.** $\log \sqrt[3]{7}$ $\frac{1}{3} \log 7$

Use properties of logarithms to rewrite each expression. Assume that all variables are positive. See Example 6.

49. $\log(x^3y^2)$
$3 \log x + 2 \log y$

▶ **50.** $\log(xy^2x^3)$
$\log x + 2 \log y + 3 \log z$

51. $\log_b (xy)^{1/2}$
$\frac{1}{2}(\log_b x + \log_b y)$

52. $\log_b x^3y^{1/2}$
$3 \log_b x + \frac{1}{2} \log_b y$

53. $\log_b \sqrt{xy}$
$\frac{1}{2}(\log_b x + \log_b y)$

54. $\log_b x^3\sqrt{y}$
$3 \log_b x + \frac{1}{2} \log_b y$

55. $\ln \frac{x^2\sqrt{y}}{z}$
$2 \ln x + \frac{1}{2} \ln y - \ln z$

56. $\ln \frac{x^3y^2}{\sqrt{z}}$
$3 \ln x + 2 \ln y - \frac{1}{2} \ln z$

Write each expression as one logarithm. Assume that all variables are positive. See Example 7.

57. $3 \log_a x + \frac{1}{3} \log_a y$
$\log_a(x^3y^{1/3})$

58. $\frac{1}{2} \log_b x + 3 \log_b y$
$\log_b (x^{1/2}y^3)$

59. $-3 \log_b x - 2 \log_b y + \frac{1}{2} \log_b z$ $\log_b \frac{z^{1/2}}{x^3y^2}$

▶ **60.** $3 \log_b (x + 1) - 2 \log_b (x + 2) + \log_b x$ $\log_b \frac{x(x + 1)^3}{(x + 2)^2}$

Assume that $\log_b 4 \approx 0.6021$, $\log_b 7 \approx 0.8451$, and $\log_b 9 \approx 0.9542$. Use these values and the properties of logarithms to approximate each value. See Example 8.

61. $\log_b 28$ 1.4472

▶ **62.** $\log_b \frac{7}{4}$ 0.2430

63. $\log_b \frac{4}{63}$ -1.1972

▶ **64.** $\log_b 36$ 1.5563

65. $\log_b \frac{63}{4}$ 1.1972

▶ **66.** $\log_b 2.25$ 0.3521

67. $\log_b 64$ 1.8063

▶ **68.** $\log_b 49$ 1.6902

Use the change-of-base formula to find each logarithm to four decimal places. See Example 9.

69. $\log_3 7$ 1.7712

▶ **70.** $\log_7 3$ 0.5646

▶ **71.** $\log_{1/3} 3$ -1.0000

▶ **72.** $\log_{1/2} 6$ -2.5850

▶ **73.** $\log_3 8$ 1.8928

74. $\log_5 10$ 1.4307

75. $\log_{\sqrt{2}} \sqrt{5}$ 2.3219

76. $\log_\pi e$ 0.8736

TRY IT YOURSELF

Use a calculator to verify each equation.

77. $\log [(2.5)(3.7)] = \log 2.5 + \log 3.7$

78. $\log 45.37 = \frac{\ln 45.37}{\ln 10}$

79. $\ln(2.25)^4 = 4 \ln 2.25$

80. $\ln \frac{11.3}{6.1} = \ln 11.3 - \ln 6.1$

81. $\log \sqrt{24.3} = \frac{1}{2} \log 24.3$

82. $\ln 8.75 = \frac{\log 8.75}{\log e}$

Determine whether the given statement is true. If a statement is false, explain why.

83. $\log xy = (\log x)(\log y)$ false

84. $\log ab = \log a + 1$ false

85. $\log_b (A - B) = \frac{\log_b A}{\log_b B}$ false

86. $\frac{\log_b A}{\log_b B} = \log_b A - \log_b B$ false

87. $\log_b \frac{A}{B} = \log_b A - \log_b B$ true

88. $\log_b 2 = \log_2 b$ false

Use properties of logarithms to rewrite each expression. Assume that all variables are positive.

89. $\log_a \frac{\sqrt[3]{x}}{\sqrt[4]{yz}}$
$\frac{1}{3} \log_a x - \frac{1}{4} \log_a y - \frac{1}{4} \log_a z$

90. $\log_b \sqrt[4]{\frac{x^3y^2}{z^4}}$
$\frac{3}{4} \log_b x + \frac{1}{2} \log_b y - \log_b z$

91. $\ln\left(\frac{x}{z} + x\right) - \ln\left(\frac{y}{z} + y\right)$ $\ln \frac{\frac{x}{z} + x}{\frac{y}{z} + y} = \ln \frac{x}{y}$

▶ **92.** $\ln(xy + y^2) - \ln(xz + yz) + \ln z$ $\ln y$

93. $\ln x\sqrt{z}$ $\ln x + \frac{1}{2} \ln z$

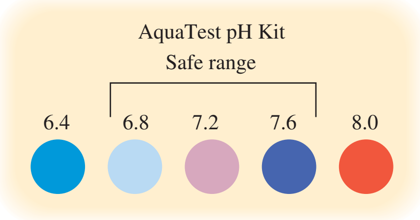
94. $\ln \sqrt{xy}$ $\frac{1}{2}(\ln x + \ln y)$

APPLICATIONS

▶ **95.** PH OF A SOLUTION Find the pH of a solution with a hydrogen ion concentration of 1.7×10^{-5} gram-ions per liter. 4.8

▶ **96.** HYDROGEN ION CONCENTRATION Find the hydrogen ion concentration of a saturated solution of calcium hydroxide whose pH is 13.2. 6.3×10^{-14} gram-ions per liter

97. AQUARIUMS To test for safe pH levels in a freshwater aquarium, a test strip is compared with the scale shown. Find the corresponding range in the hydrogen ion concentration.
from 2.5×10^{-8} to 1.6×10^{-7}



98. pH OF SOUR PICKLES The hydrogen ion concentration of sour pickles is 6.31×10^{-4}. Find the pH. 3.2

WRITING

99. Explain the difference between a logarithm of a product and the product of logarithms.

100. How can the $\boxed{\text{LOG}}$ key on a calculator be used to find $\log_2 7$?

REVIEW

Consider the line that passes through P(−2, 3) and Q(4, −4).

101. Find the slope of line PQ. $-\frac{7}{6}$

102. Find the distance PQ. $\sqrt{85}$

103. Find the midpoint of segment PQ. $\left(1, -\frac{1}{2}\right)$

104. Write the equation of line PQ. $y = -\frac{7}{6}x + \frac{2}{3}$

Objectives

1. Solve exponential equations.
2. Solve logarithmic equations.
3. Solve radioactive decay problems.
4. Solve population growth problems.

SECTION 9.8
Exponential and Logarithmic Equations

An **exponential equation** is an equation that contains a variable in one of its exponents. Some examples of exponential equations are

$$3^x = 5, \qquad 6^{x-3} = 2^x, \qquad \text{and} \qquad 2^{x^2+2x} = \frac{1}{2}$$

A **logarithmic equation** is an equation with a logarithmic expression that contains a variable. Some examples of logarithmic equations are

$$\log 5x = 1, \qquad \log(3x + 2) - \log(2x - 3) = 0, \qquad \text{and} \qquad \frac{\log_2(5x - 6)}{\log_2 x} = 2$$

In this section, we will learn how to solve many of these equations.

1 Solve exponential equations.

If both sides of an exponential equation can be expressed as a power of the same base, we can use the following property to solve the equation.

Exponent Property of Equality

If two exponential expressions with the same base are equal, their exponents are equal.

For any real number b, where $b \neq -1, 0$, or 1.

$b^x = b^y$ is equivalent to $x = y$

EXAMPLE 1 Solve: $2^{x^2+2x} = \frac{1}{2}$

Strategy We will express the right-hand side as a power of 2.

WHY We can then use the exponent property of equality to set the exponents equal and solve for x.

Solution

Since $\frac{1}{2} = 2^{-1}$, we can write the equation in the form

$2^{x^2+2x} = 2^{-1}$ Each side of the equation can be written as an exponential expression with base 2.

Since equal quantities with equal bases have equal exponents, we have

$$x^2 + 2x = -1 \quad \text{Equate the exponents.}$$
$$x^2 + 2x + 1 = 0 \quad \text{Add 1 to both sides.}$$
$$(x + 1)(x + 1) = 0 \quad \text{Factor the trinomial.}$$
$$x + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$x = -1 \quad \mid \quad x = -1$$

Verify that -1 satisfies the original equation.

Self Check 1

Solve: $3^{x^2-2x} = \frac{1}{3}$ 1, 1

Now Try Problem 26

Teaching Example 1 Solve: $5^{x^2-4x} = \frac{1}{125}$

Answer: 1, 3

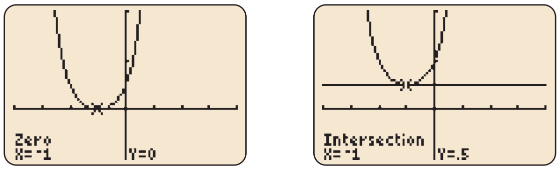
Using Your CALCULATOR **Solving Exponential Equations Graphically**

To use a graphing calculator to approximate the solutions of $2^{x^2+2x} = \frac{1}{2}$ (see Example 1), we can subtract $\frac{1}{2}$ from both sides of the equation to get

$2^{x^2+2x} - \frac{1}{2} = 0$ and graph the corresponding function $y = 2^{x^2+2x} - \frac{1}{2}$

If we use window settings of $[-4, 4]$ for x and $[-1, 2]$ for y, we obtain the graph shown in figure (a) below.

The solutions of $2^{x^2+2x} - \frac{1}{2} = 0$ are the x-coordinates of the x-intercepts of the graph of $y = 2^{x^2+2x} - \frac{1}{2}$. Using the ZERO feature, we see that the graph has only one x-intercept, $(-1, 0)$. Therefore, -1 is the only solution of $2^{x^2+2x} - \frac{1}{2} = 0$.



(a) (b)

We can also solve $2^{x^2+2x} = \frac{1}{2}$ using the INTERSECT feature found on most graphing calculators. After graphing $Y_1 = 2^{x^2+2x}$ and $Y_2 = \frac{1}{2}$, we select INTERSECT, which approximates the coordinates of the point of intersection of the two graphs. From the display shown in figure (b), we can conclude that the solution is -1. Verify this by checking.

When it is difficult or impossible to write each side of an exponential equation as a power of the same base, we can often use property 8 of logarithms to solve the equation. Recall this property states that if two positive numbers are equal, the logarithmic base-b of the numbers are equal.

Self Check 2

Solve $5^x = 4$. Give the answer to four decimal places.

Now Try **Problem 28**

Self Check 2 Answer

$\frac{\log 4}{\log 5} = 0.8614$

Teaching Example 2 Solve $7^x = 2$. Give the answer to four decimal places.

Answer:

$\frac{\log 2}{\log 7} \approx 0.3562$

EXAMPLE 2 Solve $3^x = 5$. Give the answer to four decimal places.

Strategy We will take the base-10 logarithm of both sides of the equation.

WHY We can then use the power rule of logarithms to move the variable x from its current position as an exponent to a position as a factor.

Solution

Since the logarithms of equal numbers are equal, we can take the common logarithm of each side of the equation. The power rule of logarithms then provides a way of moving the variable x from its position as an exponent to a position as a coefficient.

$$3^x = 5$$

$$\log 3^x = \log 5 \quad \text{Take the common logarithm of each side.}$$

$$x \log 3 = \log 5 \quad \text{The log of a power is the power times the log: } \log 3^x = x \log 3.$$

$$\textbf{(1)} \qquad x = \frac{\log 5}{\log 3} \quad \text{Divide both sides by log 3.}$$

$$x \approx 1.464973521 \quad \text{Use a calculator.}$$

The exact solution is $\frac{\log 5}{\log 3}$. To four decimal places, the solution is 1.4650.

We can also take the natural logarithm of each side of the equation to solve for x.

$$3^x = 5$$

$$\ln 3^x = \ln 5 \quad \text{Take the natural logarithm of each side.}$$

$$x \ln 3 = \ln 5 \quad \text{Use the power rule of logarithms: } \ln 3^x = x \ln 3.$$

$$x = \frac{\ln 5}{\ln 3} \quad \text{Divide both sides by ln 3.}$$

$$x \approx 1.464973521 \quad \text{Use a calculator.}$$

To check the solution, we substitute 1.4650 for x in 3^x and see if the result is close to 5. We can compute $3^{1.4650}$ by entering 3 $\boxed{y^x}$ 1.4650 $\boxed{=}$ on a scientific calculator. The result of 5.000145454 verifies that $x \approx 1.4650$ is an approximate solution of $3^x = 5$.

Caution! A careless reading of Equation 1 in the Example 2 solution leads to a common error. The right-hand side of Equation 1 calls for a division, not a subtraction.

$$\frac{\log 5}{\log 3} \quad \text{means} \quad (\log 5) \div (\log 3)$$

It is the expression $\log \frac{5}{3}$ that means $\log 5 - \log 3$.

EXAMPLE 3 Solve: $6^{x-3} = 2^x$

Strategy We will take the base 10 logarithm of both sides of the equation.

WHY We can then use the power rule of logarithms to move the variable expressions from their current position as an exponent to a position as a factor.

Solution

$$
\begin{aligned}
6^{x-3} &= 2^x \\
\log 6^{x-3} &= \log 2^x && \text{Take the common logarithm of each side.} \\
(x-3)\log 6 &= x \log 2 && \text{The log of a power is the power times the log.} \\
x\log 6 - 3\log 6 &= x\log 2 && \text{Use the distributive property.} \\
x\log 6 - x\log 2 &= 3\log 6 && \text{On both sides, add 3 log 6 and subtract x log 2.} \\
x(\log 6 - \log 2) &= 3\log 6 && \text{Factor out x on the left-hand side.} \\
x &= \frac{3\log 6}{\log 6 - \log 2} && \text{Divide both sides by log 6 − log 2.} \\
x &\approx 4.892789261 && \text{Use a calculator.}
\end{aligned}
$$

To four decimal places, the solution is 4.8928.

Self Check 3

Solve: $5^{x-2} = 3^x$

Now Try Problem 32

Self Check 3 Answer
$\frac{2\log 5}{\log 5 - \log 3} \approx 6.3013$

Teaching Example 3 Solve: $3^{x+4} = 2^x$
Answer:
$\frac{-4\ln 3}{\ln 3 - \ln 2} \approx -10.8380$

EXAMPLE 4 Solve: $e^{0.9t} = 8$

Strategy We will take the natural logarithm of both sides of the equation.

WHY We can then use the power rule of logarithms to move the variable expression $0.9t$ from its current position as an exponent to a position as a factor.

Solution
The exponential expression on the left-hand side has base e. In such cases, the computations are somewhat simpler if we take the natural logarithm of each side.

$$
\begin{aligned}
e^{0.9t} &= 8 \\
\ln e^{0.9t} &= \ln 8 && \text{Take the natural logarithm of each side.} \\
0.9t \ln e &= \ln 8 && \text{Use the power rule of logarithms: } \ln e^{0.9t} = 0.9t \ln e. \\
0.9t \cdot 1 &= \ln 8 && \text{Simplify: } \ln e = 1. \\
0.9t &= \ln 8 \\
t &= \frac{\ln 8}{0.9} && \text{Divide both sides by 0.9.} \\
t &\approx 2.310490602 && \text{Use a calculator.}
\end{aligned}
$$

To four decimal places, the solution is 2.3105.

Self Check 4

Solve: $e^{2.1t} = 35$

Now Try Problem 36

Self Check 4 Answer
$\frac{\ln 35}{2.1} \approx 1.6930$

Teaching Example 4 Solve: $e^{3.5t} = 42$
Answer:
$\frac{\ln 42}{3.5} \approx 1.0679$

2 Solve logarithmic equations.

We can solve many logarithmic equations using properties of logarithms.

EXAMPLE 5 Solve: $\log 5x = 3$

Strategy Recall that $\log 5x = \log_{10} 5x$. To solve $\log 5x = 3$, we will instead write and solve an equivalent base-10 exponential equation.

WHY The resulting exponential equation is easier to solve because the variable term is isolated on one side.

Self Check 5

Solve: $\log_2 (x-3) = -1\frac{7}{2}$

Now Try Problem 40

Teaching Example 5 Solve: $\log 25x = 2$
Answer:
4

Solution

Recall that $\log 5x = \log_{10} 5x$. We can change the equation $\log 5x = 3$ into the equivalent base-10 exponential equation $10^3 = 5x$ and solve for x.

$$\log 5x = 3$$
$$10^3 = 5x$$
$$1{,}000 = 5x \quad \text{Simplify: } 10^3 = 1{,}000.$$
$$200 = x \quad \text{Divide both sides by 5.}$$

The solution is 200. Check the result.

Self Check 6

Solve:
$\log(5x + 2) - \log(7x - 2) = 0$ 2

Now Try **Problem 44**

Teaching Example 6 Solve:
$\log(3x - 2) - \log(x + 4) = 0$
Answer:
3

EXAMPLE 6

Solve: $\log(3x + 2) - \log(2x - 3) = 0$

Strategy We will isolate each logarithmic expression on one side of the equation. Then we will use the logarithmic property of equality to see that $3x + 2 = 2x - 3$.

WHY We can use the logarithmic property of equality because once we isolate each logarithmic expression, the equivalent equation has the form $\log_b x = \log_b y$.

Solution

We isolate each logarithmic expression on one side of the equation.

$$\log(3x + 2) - \log(2x - 3) = 0 \quad \text{This is the equation to solve.}$$
$$\log(3x + 2) = \log(2x - 3) \quad \text{Add } \log(2x - 3) \text{ to both sides.}$$
$$(3x + 2) = (2x - 3) \quad \text{If the logarithms of two numbers are equal, the numbers are equal.}$$
$$x = -5 \quad \text{Subtract } 2x \text{ and 2 from both sides.}$$

Check:

$$\log(3x + 2) - \log(2x - 3) = 0$$
$$\log[3(-5) + 2] - \log[2(-5) - 3] \stackrel{?}{=} 0$$
$$\log(-13) - \log(-13) \stackrel{?}{=} 0$$

Since the logarithm of a negative number does not exist, the proposed solution -5 must be discarded. This equation has no solutions.

Self Check 7

Solve: $\log x + \log(x + 3) = 1$ 2

Now Try **Problem 48**

Teaching Example 7
Solve: $\log_x + \log(x - 21) = 2$
Answer:
25

EXAMPLE 7

Solve: $\log x + \log(x - 3) = 1$

Strategy We will use the product rule for logarithms in reverse: The sum of two logarithms is equal to the logarithm of a product. Then we will write and solve an equivalent exponential equation.

WHY We use the product rule for logarithms because the left side of the equation $\log x + \log(x - 3)$ has the form $\log_b M + \log_b N$.

Solution

$$\log x + \log(x - 3) = 1 \quad \text{This is the equation to solve.}$$
$$\log x(x - 3) = 1 \quad \text{Use the product rule of logarithms.}$$
$$\log_{10} x(x - 3) = 1 \quad \text{The base is 10.}$$
$$x(x - 3) = 10^1 \quad \text{Use the definition of logarithms to change the equation to exponential form: } \log x(x - 3) = \log_{10} x(x - 3).$$
$$x^2 - 3x - 10 = 0 \quad \text{Distribute the multiplication by } x \text{ and subtract 10 from both sides.}$$
$$(x + 2)(x - 5) = 0 \quad \text{Factor the trinomial.}$$

$$x + 2 = 0 \quad \text{or} \quad x - 5 = 0 \qquad \text{Set each factor equal to 0.}$$
$$x = -2 \quad \Big| \quad x = 5$$

Check: The number -2 is not a solution, because it does not satisfy the equation (a negative number does not have a logarithm). We will check the remaining number, 5.

$$\log x + \log(x - 3) = 1$$
$$\log 5 + \log(5 - 3) \stackrel{?}{=} 1 \qquad \text{Substitute 5 for } x.$$
$$\log 5 + \log 2 \stackrel{?}{=} 1$$
$$\log 10 \stackrel{?}{=} 1 \qquad \text{Use the product rule of logarithms: } \log 5 + \log 2 = \log(5 \cdot 2) = \log 10.$$
$$1 = 1 \qquad \text{Simplify: } \log 10 = 1.$$

Since 5 satisfies the equation, it is the solution.

Caution! Examples 6 and 7 illustrate that we must check the solutions of a logarithmic equation.

Using Your CALCULATOR Solving Logarithmic Equations Graphically

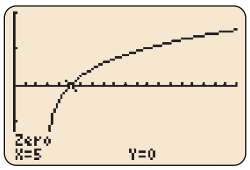
To use a graphing calculator to approximate the solutions of the logarithmic equation $\log x + \log(x - 3) = 1$ (see Example 7), we can subtract 1 from both sides of the equation to get

$$\log x + \log(x - 3) - 1 = 0$$

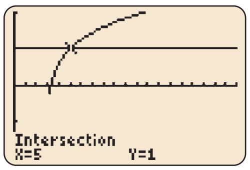
and graph the corresponding function

$$y = \log x + \log(x - 3) - 1$$

If we use window settings of [0, 20] for x and $[-2, 2]$ for y, we obtain the graph shown in figure (a). Since the solution of the equation is the x-coordinate of the x-intercept, we can find the solution using the ZERO feature. The solution is $x = 5$.



(a)



(b)

We can also solve $\log x + \log(x - 3) = 1$ using the INTERSECT feature. After graphing $Y_1 = \log x + \log(x - 3)$ and $Y_2 = 1$, we select INTERSECT, which approximates the coordinates of the point of intersection of the two graphs. From the display shown in figure (b), we can conclude that the solution is $x = 5$. Verify this by checking.

Self Check 8

Solve: $\dfrac{\log_3 (5x + 6)}{\log_3 x} = 2$ 6

Now Try **Problem 51**

Teaching Example 8 Solve: $\dfrac{\log_4 (3x - 2)}{\log_4 x} = 2$
Answer:
2

EXAMPLE 8 Solve: $\dfrac{\log_2 (5x - 6)}{\log_2 x} = 2$

Strategy We will multiply both sides by $\log_2 x$ and apply the power rule of logarithms.

WHY We will then have an equivalent equation of the form $\log_b x = \log_b y$ and we can use the property of equality of logarithms.

Solution

We can multiply both sides of the equation by $\log_2 x$ to get

$$\log_2 (5x - 6) = 2 \log_2 x$$

and apply the power rule of logarithms to get

$$\mathbf{\log_2} (5x - 6) = \mathbf{\log_2} x^2$$

By property 8 of logarithms, $5x - 6 = x^2$, because they have equal logarithms. Thus,

$$\begin{aligned} 5x - 6 &= x^2 \\ 0 &= x^2 - 5x + 6 \\ 0 &= (x - 3)(x - 2) \end{aligned}$$

$$x - 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = 3 \quad\quad\quad\quad x = 2$$

Verify that both 2 and 3 satisfy the equation.

3 Solve radioactive decay problems.

Experiments have determined the time it takes for half of a sample of a given radioactive material to decompose. This time is a constant, called the material's **half-life.**

When living organisms die, the oxygen/carbon dioxide cycle common to all living things ceases, and carbon-14, a radioactive isotope with a half-life of 5,700 years, is no longer absorbed. By measuring the amount of carbon-14 present in an ancient object, archaeologists can estimate the object's age by using the radioactive decay formula.

Radioactive Decay Formula

If A is the amount of radioactive material present at time t, A_0 was the amount present at $t = 0$, and h is the material's half-life, then

$$A = A_0 2^{-t/h}$$

Self Check 9

CARBON-14 DATING In Example 9, how old is a statue that retains 25% of its original carbon-14 content?

Now Try **Problem 103**

Self Check 9 Answer
about 11,400 years

EXAMPLE 9 ***Carbon-14 Dating*** How old is a wooden statue that retains only one-third of its original carbon-14 content?

Strategy If A_0 is the original carbon-14 content, then today's content is $A = \frac{1}{3} A_0$. We will substitute $\frac{1}{3} A_0$ for A and 5,700 for h in the radioactive decay formula and solve for t.

WHY The value of t is the estimated age.

Solution

To find the time t when $A = \frac{1}{3}A_0$, we substitute $\frac{A_0}{3}$ for A and 5,700 for h in the radioactive decay formula and solve for t:

$$A = A_0 2^{-t/h}$$

$$\frac{A_0}{3} = A_0 2^{-t/5{,}700} \qquad \text{The half-life of carbon-14 is 5,700 years.}$$

$$1 = 3(2^{-t/5{,}700}) \qquad \text{Divide both sides by } A_0 \text{ and multiply both sides by 3.}$$

$$\log 1 = \log 3(2^{-t/5{,}700}) \qquad \text{Take the common logarithm of each side.}$$

$$\log 1 = \log 3 + \log 2^{-t/5{,}700} \qquad \text{The logarithm of a product is the sum of the logarithms.}$$

$$-\log 3 = -\frac{t}{5{,}700}\log 2 \qquad \text{Subtract log 3 from both sides and use the power rule of logarithms.}$$

$$5{,}700\left(\frac{\log 3}{\log 2}\right) = t \qquad \text{Multiply both sides by } -\frac{5{,}700}{\log 2}.$$

$$t \approx 9{,}034.286254 \qquad \text{Use a calculator.}$$

The statue is approximately 9,000 years old.

Teaching Example 9 CARBON-14 DATING In Example 9, how old is a statue that retains 60% of its original carbon-14 content?
Answer:
about 4,200 years

4 Solve population growth problems.

Recall that when there is sufficient food and space, populations of living organisms tend to increase exponentially according to the Malthusian growth model.

Exponential Growth Model

If P is the population at some time t, P_0 is the initial population at $t = 0$, and k depends on the rate of growth, then

$$P = P_0 e^{kt}$$

EXAMPLE 10 ***Population Growth*** The bacteria in a laboratory culture increased from an initial population of 500 to 1,500 in 3 hours. How long will it take for the population to reach 10,000?

Strategy We will substitute 500 for P_0 and 1,500 for P into the exponential growth model and solve for k.

WHY Once we know the value of k, we can substitute 10,000 for P, 500 for P_0, and the value of k into the exponential growth model and solve for the time t.

Solution

We substitute 500 for P_0, 1,500 for P, and 3 for t and simplify to find k:

$$P = P_0 e^{kt} \qquad \text{This is the population growth formula.}$$

$$1{,}500 = 500(e^{k3}) \qquad \text{Substitute 1,500 for } P\text{, 500 for } P_0\text{, and 3 for } t.$$

$$3 = e^{3k} \qquad \text{Divide both sides by 500.}$$

$$3k = \ln 3 \qquad \text{Change the equation from exponential to logarithmic form.}$$

$$k = \frac{\ln 3}{3} \qquad \text{Divide both sides by 3.}$$

Self Check 10

POPULATION GROWTH In Example 10, how long will it take the population to reach 20,000? about 10 hours

Now Try **Problem 111**

Teaching Example 10 POPULATION GROWTH In Example 10, how long will it take the population to reach 15,000?
Answer:
about 9 hours

To find when the population will reach 10,000, we substitute 10,000 for P, 500 for P_0, and $\frac{\ln 3}{3}$ for k in the equation $P = P_0e^{kt}$ and solve for t:

$$P = P_0e^{kt}$$
$$10{,}000 = 500e^{[(\ln 3)/3]t}$$
$$20 = e^{[(\ln 3)/3]t} \quad \text{Divide both sides by 500.}$$
$$\left(\frac{\ln 3}{3}\right)t = \ln 20 \quad \text{Change the equation to logarithmic form.}$$
$$t = \frac{3 \ln 20}{\ln 3} \quad \text{Multiply both sides by } \frac{3}{\ln 3}.$$
$$\approx 8.180499084 \quad \text{Use a calculator.}$$

The culture will reach 10,000 bacteria in about 8 hours.

ANSWERS TO SELF CHECKS

1. 1, 1 **2.** 0.8614 **3.** $\frac{2 \log 5}{\log 5 - \log 3} \approx 6.3013$ **4.** $\frac{\ln 35}{2.1} \approx 1.6930$ **5.** $\frac{7}{2}$ **6.** 2 **7.** 2 **8.** 6 **9.** about 11,400 years **10.** about 10 hours

SECTION 9.8 STUDY SET

VOCABULARY

Fill in the blanks.

1. An equation with a variable in its exponent, such as $3^{2x} = 8$, is called a(n) exponential equation.

2. An equation with a logarithmic expression that contains a variable, such as $\log_5 (2x - 3) = \log_5 (x + 4)$, is a(n) logarithmic equation.

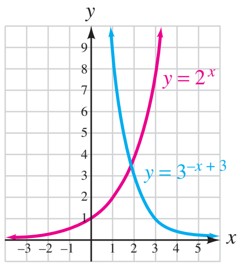
CONCEPTS

Fill in the blanks.

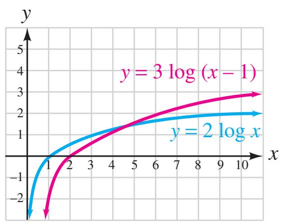
3. The formula for radioactive decay is $A = A_0 2^{-t/h}$.

4. The formula for population growth is $P = P_0 e^{kt}$.

5. Use the graphs in the illustration to estimate the solution of $2^x = 3^{-x+3}$. about 1.8



6. Use the graphs in the illustration to estimate the solution of $3 \log (x - 1) = 2 \log x$. about 4.6



7. Fill in the blanks to make the statements true. To solve $5^x = 21$, we can take the logarithm of each side of the equation to get

$$\log 5^x = \log 21$$

The power rule for logarithms then provides a way of moving the variable x from its position as an exponent to a position as a coefficient.

8. Use a calculator to determine whether $x \approx 2.5646$ is a solution of $2^{2x+1} = 70$. yes

9. Find $\frac{\log 8}{\log 5}$. Round to four decimal places. 1.2920

10. Find $\frac{2 \ln 12}{\ln 9}$. Round to four decimal places. 2.2619

11. Simplify: $\ln e$ 1

12. Does $\frac{\log 7}{\log 3} = \log 7 - \log 3$? no

13. Write the corresponding base-10 exponential equation for $\log(x + 1) = 2$. $10^2 = x + 1$

14. Write the corresponding base-e exponential equation for $\ln(x + 1) = 2$. $e^2 = x + 1$

Solve each equation. Round to the nearest hundredth.

15. $x^2 = 12$ ± 3.46

16. $2^x = 12$ 3.58

Solve each equation.

17. $\log(x - 1) = 3$ $1{,}001$

18. $\log(x - 1) = \log 3$ 4

19. Check to see whether -4 is a solution of $\log_5(x + 3) = \frac{1}{5}$. no

20. Check to see whether -2 is a solution of $5^{2x+3} = \frac{1}{5}$. yes

NOTATION

Complete each solution to solve the equation.

21. Solve:
$$\begin{aligned} 2^x &= 7 \\ \boxed{\log}\ 2^x &= \log 7 \\ x\ \boxed{\log 2} &= \log 7 \\ x &= \frac{\log 7}{\boxed{\log 2}} \end{aligned}$$

22. Solve:
$$\begin{aligned} \log_2(2x - 3) &= \log_2(x + 4) \\ \boxed{2x - 3} &= x + 4 \\ x &= \boxed{7} \end{aligned}$$

GUIDED PRACTICE

Solve each exponential equation. See Example 1.

23. $2^{x-2} = 64$ 8

24. $3^{-3x+1} = 243$ $-\frac{4}{3}$

25. $2^{x^2-2x} = 8$ $3, -1$

26. $3^{x^2-3x} = 81$ $4, -1$

Solve each exponential equation. Give answers to four decimal places when necessary. See Example 2.

27. $4^x = 5$ 1.1610

28. $7^x = 12$ 1.2770

29. $5^x = 7$ 1.2091

30. $8^x = 16$ 1.3333

Solve each exponential equation. Give answers to four decimal places when necessary. See Example 3.

31. $2^{x+1} = 3^x$ 1.7095

32. $5^{x-3} = 3^{2x}$ -8.2144

33. $2^x = 3^x$ 0

34. $3^{2x} = 4^x$ 0

Solve each exponential equation. Give answers to four decimal places when necessary. See Example 4.

35. $e^{3x} = 9$ 0.7324

36. $e^{4x} = 60$ 1.0236

37. $e^{-0.2t} = 14.2$ -13.2662

38. $e^{0.3t} = 9.1$ 7.3609

Solve each logarithmic equation. See Example 5.

39. $\log 5x = 2$ 20

40. $\log 20x = 3$ 50

41. $\log 10x = 3$ 100

42. $\log 25 = 2$ 4

Solve each logarithmic equation. See Example 6.

43. $\log(3 - 2x) - \log(x + 24) = 0$ -7

44. $\log(3x + 5) - \log(2x + 6) = 0$ 1

45. $\ln(3x + 1) = \ln(x + 7)$ 3

46. $\ln(x^2 + 4x) - \ln(x^2 + 16) = 0$ 4

Solve each logarithmic equation. See Example 7.

47. $\log x + \log(x - 48) = 2$ 50

48. $\log x + \log(x + 9) = 1$ 1

49. $\log x + \log(x - 15) = 2$ 20

50. $\log x + \log(x + 21) = 2$ 4

Solve each logarithmic equation. See Example 8.

51. $\dfrac{\log_2(6x - 8)}{\log_2 x} = 2$ $2, 4$

52. $\dfrac{\log_2(7x - 12)}{\log_2 x} = 2$ $3, 4$

53. $\dfrac{\log(8x - 7)}{\log x} = 2$ 7

54. $\dfrac{\log(5x + 6)}{2} = \log x$ 6

TRY IT YOURSELF

Solve each equation.

55. $13^{x-1} = 2$ 1.2702

56. $5^{x+1} = 3$ -0.3174

57. $5^{4x} = \dfrac{1}{125}$ $-\frac{3}{4}$

58. $8^{-x+1} = \dfrac{1}{64}$ 3

59. $7^{x^2} = 10$ ± 1.0878

60. $8^{x^2} = 11$ ± 1.0738

61. $8^{x^2} = 9^x$ $0, 1.0566$

62. $5^{x^2} = 2^{5x}$ $0, 2.1534$

63. $3^{x^2+4x} = \dfrac{1}{81}$ $-2, -2$

64. $7^{x^2+3x} = \dfrac{1}{49}$ $-2, -1$

65. $\log(x + 2) = 4$ $9{,}998$

66. $\log 5x = 4$ $2{,}000$

67. $\log(7 - x) = 2$ -93

68. $\log(2 - x) = 3$ -998

69. $\ln x = 1$ $e \approx 2.7183$

70. $\ln x = 5$ $e^5 \approx 148.4132$

71. $\ln(x + 1) = 3$ 19.0855

72. $\ln 2x = 5$ 74.2066

73. $\log 2x = \log 4$ 2

74. $\log 3x = \log 9$ 3

75. $\log \dfrac{4x + 1}{2x + 9} = 0$ 4

76. $\log \dfrac{2 - 5x}{2(x + 8)} = 0$ -2

77. $\log x^2 = 2$ $10, -10$

78. $\log x^3 = 3$ 10

79. $\log(x + 90) = 3 - \log x$ 10

80. $\log(x - 90) = 3 - \log x$ 100

81. $\log(x - 6) - \log(x - 2) = \log \dfrac{5}{x}$ 10

82. $\log(3 - 2x) - \log(x + 9) = 0$ -2

83. $\dfrac{\log(3x - 4)}{\log x} = 2$ no solution

84. $\dfrac{1}{2}\log(4x + 5) = \log x$ 5

85. $\log_3 x = \log_3\left(\dfrac{1}{x}\right) + 4$ 9

86. $\log_5(7 + x) + \log_5(8 - x) - \log_5 2 = 2$ $3, -2$

87. $2 \log_2 x = 3 + \log_2 (x - 2)$ 4

88. $2 \log_3 x - \log_3 (x - 4) = 2 + \log_3 2$ 6, 12

89. $\log (7y + 1) = 2 \log (y + 3) - \log 2$ 1, 7

90. $2 \log (y + 2) = \log (y + 2) - \log 12$ $-\frac{23}{12}$

Use a graphing calculator to solve each equation. If an answer is not exact, round to the nearest tenth.

91. $2^{x+1} = 7$ 1.8

92. $3^{x-1} = 2^x$ 2.7

93. $4(2^{x^2}) = 8^{3x}$ 8.8, 0.2

94. $3^x - 10 = 3^{-x}$ 2.1

95. $\log x + \log (x - 15) = 2$ 20

96. $\log x + \log (x + 3) = 1$ 2

97. $\ln (2x + 5) - \ln 3 = \ln (x - 1)$ 8

98. $2 \log (x^2 + 4x) = 1$ $-4.7, 0.7$

APPLICATIONS

99. TRITIUM DECAY The half-life of tritium is 12.4 years. How long will it take for 25% of a sample of tritium to decompose? 5.1 yr

▶ **100.** RADIOACTIVE DECAY In two years, 20% of a radioactive element decays. Find its half-life. 6.2 yr

101. THORIUM DECAY An isotope of thorium, ^{227}Th, has a half-life of 18.4 days. How long will it take for 80% of the sample to decompose? 42.7 days

▶ **102.** LEAD DECAY An isotope of lead, ^{201}Pb, has a half-life of 8.4 hours. How many hours ago was there 30% more of the substance? 3.2 hr

103. CARBON-14 DATING A bone fragment analyzed by archaeologists contains 60% of the carbon-14 that it is assumed to have had initially. How old is it? about 4,200 yr

104. CARBON-14 DATING Only 10% of the carbon-14 in a small wooden bowl remains. How old is the bowl? about 19,000 yr

105. COMPOUND INTEREST If \$500 is deposited in an account paying 8.5% annual interest, compounded semiannually, how long will it take for the account to increase to \$800? 5.6 yr

106. CONTINUOUS COMPOUND INTEREST In Exercise 105, how long will it take if the interest is compounded continuously? 5.5 yr

107. COMPOUND INTEREST If \$1,300 is deposited in a savings account paying 9% interest, compounded quarterly, how long will it take the account to increase to \$2,100? 5.4 yr

▶ **108.** COMPOUND INTEREST A sum of \$5,000 deposited in an account grows to \$7,000 in 5 years. Assuming annual compounding, what interest rate is being paid? 6.96%

109. RULE OF SEVENTY A rule of thumb for finding how long it takes an investment earning continuously compounded interest to double is called the **rule of seventy.** To apply the rule, divide 70 by the interest rate written as a percent. At 5%, doubling requires $\frac{70}{5}$ years to double an investment. At 7%, it takes $\frac{70}{7}$ years. Explain why this formula works. because $\ln 2 \approx 0.7$

▶ **110.** BACTERIAL GROWTH A bacterial culture grows according to the formula

$$P = P_0 a^t$$

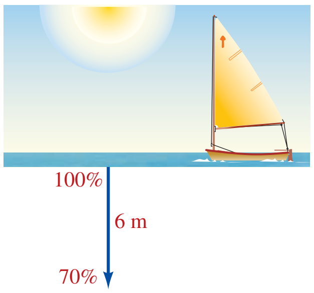
If it takes 5 days for the culture to triple in size, how long will it take to double in size? 3.2 days

111. RODENT CONTROL The rodent population in a city is currently estimated at 30,000. If it is expected to double every 5 years, when will the population reach 1 million? 25.3 yr

▶ **112.** POPULATION GROWTH The population of a city is expected to triple every 15 years. When can the city planners expect the present population of 140 persons to double? 9.5 yr

113. BACTERIAL CULTURES A bacterial culture doubles in size every 24 hours. By how much will it have increased in 36 hours? 2.828 times larger

▶ **114.** OCEANOGRAPHY The intensity I of a light a distance x meters beneath the surface of a lake decreases exponentially. From the illustration, find the depth at which the intensity will be 20%. 27 m



115. MEDICINE If a medium is inoculated with a bacterial culture containing 500 cells per milliliter, how many generations will have passed by the time the culture contains 5×10^6 cells per milliliter? 13.3

▶ **116.** MEDICINE If a medium is inoculated with a bacterial culture containing 800 cells per milliliter, how many generations will have passed by the time the culture contains 6×10^7 cells per milliliter? 16.2

117. NEWTON'S LAW OF COOLING Water initially at 100°C is left to cool in a room at temperature 60°C. After 3 minutes, the water temperature is 90°. The water temperature T is a function of time t given by

$$T = 60 + 40e^{kt}$$

Find k. $\frac{1}{3} \ln 0.75$

118. NEWTON'S LAW OF COOLING Refer to Exercise 117 and find the time for the water temperature to reach 70°C. 14.5 min

WRITING

119. Explain how to solve $2^{x+1} = 31$.

120. Explain how to solve $2^{x+1} = 32$.

REVIEW

Solve each equation.

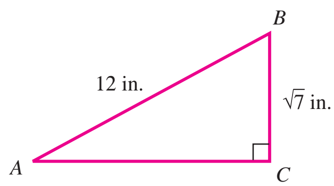
121. $5x^2 - 25x = 0$ $0, 5$

122. $4y^2 - 25 = 0$ $\frac{5}{2}, -\frac{5}{2}$

123. $3p^2 + 10p = 8$ $\frac{2}{3}, -4$

124. $4t^2 + 1 = -6t$ $\frac{-3 \pm \sqrt{5}}{4}$

125. Find the length of leg AC in the triangle. $\sqrt{137}$ in.



126. MEDICATIONS The amount of medicine a patient should take is often proportional to his or her weight. If a patient weighing 83 kilograms needs 150 milligrams of medicine, how much will be needed by a person weighing 99.6 kilograms? 180 mg

STUDY SKILLS CHECKLIST

Preparing for the Chapter 9 Test

The Chapter 9 material covers operations on functions, inverse functions, exponential functions, and logarithmic functions. As you prepare for the test over this material, be sure to also review the following checklist.

☐ $(f \cdot g)(x)$ means to form a new function by multiplying $f(x)$ and $g(x)$.

$(f \circ g)(x)$ means to form a new function by finding $f(g(x))$.

For $f(x) = x^2 + 3$ and $g(x) = 4x + 5$:

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x)\\ &= (x^2 + 3)(4x + 5)\\ &= 4x^3 + 5x^2 + 12x + 15\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Change from } \circ \text{ notation to nested parentheses.}\\ &= f(4x + 5) && \text{We are given } g(x) = 4x + 5.\\ &= (4x + 5)^2 + 3 && \text{Find } f(4x + 5) \text{ using } f(x) = x^2 + 3.\\ &= 16x^2 + 40x + 25 + 3 && \text{Remember: } (4x + 5)^2 = (4x + 5)(4x + 5).\\ &= 16x^2 + 40x + 28 && \text{Combine like terms.}\end{aligned}$$

☐ The logarithm of a number is an exponent. $\log_b x$ is the exponent to which b is raised to get x.

$$\log_3 9 = 2 \text{ because } 3^2 = 9 \qquad \log_5 \frac{1}{125} = -3 \text{ because } 5^{-3} = \frac{1}{125}$$

☐ Use properties of logarithms to simplify and expand logarithmic expressions.

Write the expression as a sum of logarithms and simplify.

$$\begin{aligned}\log_7 49x^5 &= \log_7 49 + \log_7 x^5\\ &= 2 + 5 \log_7 x\end{aligned}$$

☐ Use properties of logarithms to write expressions as one logarithm.

$$\begin{aligned}3 \log x + 5 \log y - \frac{1}{2} \log z &= \log x^3 + \log y^5 - \log z^{1/2}\\ &= \log x^3y^5 - \log z^{1/2}\\ &= \log \frac{x^3y^5}{\sqrt{z}}\end{aligned}$$

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

CHAPTER 9 SUMMARY AND REVIEW

SECTION 9.1 Algebra and Composition of Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
Just as it is possible to perform arithmetic operations on real numbers, it is possible to perform those operations on functions. The **sum, difference, product,** and **quotient functions** are defined as: $(f+g)(x) = f(x) + g(x)$ $(f-g)(x) = f(x) - g(x)$ $(f \cdot g)(x) = f(x)g(x)$ $(f/g)(x) = \dfrac{f(x)}{g(x)}$, with $g(x) \neq 0$	Let $f(x) = 2x + 1$ and $g(x) = x^2$. $(f+g)(x) = \mathbf{f(x)} + \mathbf{g(x)} = \mathbf{2x+1} + \mathbf{x^2} = x^2 + 2x + 1$ $(f-g)(x) = \mathbf{f(x)} - \mathbf{g(x)} = \mathbf{2x+1} - \mathbf{x^2} = -x^2 + 2x + 1$ $(f \cdot g)(x) = \mathbf{f(x)} \cdot \mathbf{g(x)} = \mathbf{(2x+1)}\mathbf{x^2} = 2x^3 + x^2$ $(f/g)(x) = \dfrac{\mathbf{f(x)}}{\mathbf{g(x)}} = \dfrac{\mathbf{2x+1}}{\mathbf{x^2}}$
Often one quantity is a function of a second quantity that depends, in turn, on a third quantity. Such chains of dependence can be modeled by a **composition of functions.** **Composition of functions:** $(f \circ g)(x) = f(g(x))$	Let $f(x) = 4x - 9$ and $g(x) = x^3$. Find $(f \circ g)(2)$ and $(f \circ g)(x)$. $(f \circ g)(2) = f(\mathbf{g(2)})$ Change to nested parentheses notation. $= f(\mathbf{8})$ Evaluate: $g(2) = 2^3 = 8$. $= 4(\mathbf{8}) - 9$ Evaluate $f(8)$ using $f(x) = 4x - 9$. $= 23$ $(f \circ g)(x) = f(\mathbf{g(x)}) = f(\mathbf{x^3}) = 4\mathbf{x^3} - 9$

REVIEW EXERCISES

Let $f(x) = 2x$ and $g(x) = x + 1$. Find each function and its domain.

1. $f + g$ $(f+g)(x) = 3x + 1, (-\infty, \infty)$
2. $f - g$ $(f-g)(x) = x - 1, (-\infty, \infty)$
3. $f \cdot g$ $(f \cdot g)(x) = 2x^2 + 2x, (-\infty, \infty)$
4. f/g $(f/g)(x) = \frac{2x}{x+1}, (-\infty, -1) \cup (-1, \infty)$

Let $f(x) = x^2 + 2$ and $g(x) = 2x + 1$. Find each of the following.

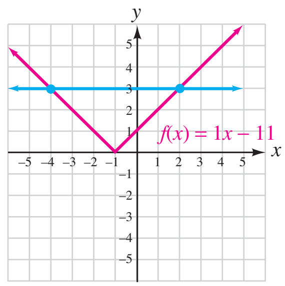
5. $(f \circ g)(-1)$ 3
6. $(g \circ f)(0)$ 5
7. $(f \circ g)(x)$ $(f \circ g)(x) = 4x^2 + 4x + 3$
8. $(g \circ f)(x)$ $(g \circ f)(x) = 2x^2 + 5$

SECTION 9.2 Inverse Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
A function is called a **one-to-one function** if different inputs determine different outputs.	The function $f(x) = 3x - 5$ is a one-to-one function because different inputs have different outputs. Since two different inputs, -2 and 2, have the same output 16, the function $f(x) = x^4$ is not one-to-one.

Horizontal line test: A function is one-to-one if every horizontal line intersects the graph of the function at most once.

The function $f(x) = |x + 1|$ is not a one-to-one function because we can draw a horizontal line that intersects its graph twice.



To find the inverse of a function, replace $f(x)$ with y, interchange the variables x and y, solve for y, and replace y with $f^{-1}(x)$.

To find the inverse of the one-to-one function $f(x) = 2x + 1$, we proceed as follows:

$$f(x) = 2x + 1$$

$$y = 2x + 1 \qquad \text{Replace } f(x) \text{ with } y.$$

$$x = 2y + 1 \qquad \text{Interchange the variables } x \text{ and } y.$$

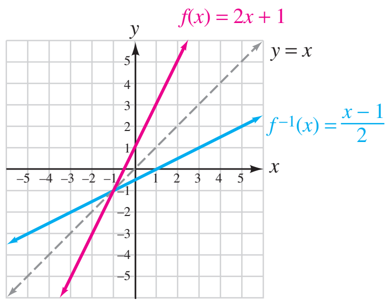
$$\frac{x-1}{2} = y \qquad \text{Solve for } y.$$

$$f^{-1}(x) = \frac{x-1}{2} \qquad \text{Replace } y \text{ with } f^{-1}(x).$$

If a point (a, b) is on the graph of function f, it follows that the point (b, a) is on the graph of f^{-1}, and vice versa.

The graph of a function and its inverse are **symmetric about the line $y = x$.**

The graphs of $f(x) = 2x + 1$ and $f^{-1}(x) = \frac{x-1}{2}$ are symmetric about the line $y = x$ as shown in the illustration.



REVIEW EXERCISES

In Exercises 11–16, determine whether the function is one-to-one.

9. $f(x) = x^2 + 3$ no

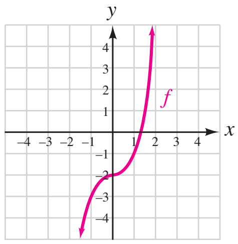
10. $f(x) = \frac{1}{3}x - 8$ yes

11. $\{(3, 4), (5, 10), (10, -1), (6, 6)\}$ yes

12. no

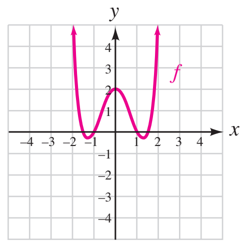
x	$f(x)$
0	−5
2	10
4	−5
6	15

13.



yes

14.



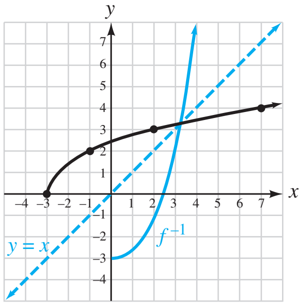
no

15. Use the table of values of the one-to-one function f to complete a table of values for f^{-1}.

x	$f(x)$
−6	−6
−1	−3
7	12
20	3

x	$f^{-1}(x)$
−6	−6
−3	−1
12	7
3	20

16. Given the graph of function f, graph f^{-1} on the same coordinate axes. Label the axis of symmetry.



Find the inverse of each function.

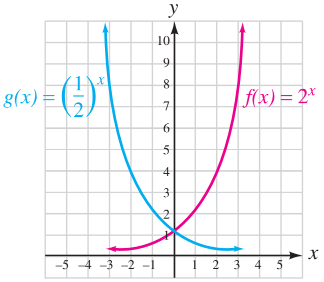
17. $f(x) = 6x - 3$ $f^{-1}(x) = \frac{x+3}{6}$

18. $f(x) = \dfrac{4}{x-1}$ $f^{-1}(x) = \frac{4}{x} + 1$

19. $f(x) = (x+2)^3$ $f^{-1}(x) = \sqrt[3]{x} - 2$

20. $f(x) = \dfrac{x}{6} - \dfrac{1}{6}$ $f^{-1}(x) = 6x + 1$

SECTION 9.3 Exponential Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
An **exponential function** with base b is defined by the equation $\boldsymbol{f(x) = b^x}$, with $b > 0, b \neq 1$ Properties of an exponential function $f(x) = b^x$: The **domain** is the interval $(-\infty, \infty)$. The **range** is the interval $(0, \infty)$. Its graph has a ***y*-intercept** of $(0, 1)$. The x-axis is an **asymptote** of its graph. The graph **passes through** the point $(1, b)$. If $b > 1$, then $f(x) = b^x$ is an **increasing function.** If $0 < b < 1$, then $f(x) = b^x$ is a **decreasing function.**	The graphs of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ are shown below.  Since the base 2 is greater than 1, the function $f(x) = 2^x$ is an increasing function. Since the base $\frac{1}{2}$ is such that $0 < \frac{1}{2} < 1$, the function $g(x) = \left(\frac{1}{2}\right)^x$ is a decreasing function.
Exponential functions are used to model many situations, such as population **growth,** the spread of an epidemic, the temperature of a heated object as it cools, and radioactive **decay.** Exponential functions are suitable models for describing **compound interest:** If \$$P$ is the deposit, and interest is paid k times a year at an annual rate r, the amount A in the account after t years is given by $A = P\left(1 + \frac{r}{k}\right)^{kt}$	If \$15,000 is deposited in an account paying an annual interest rate of 7.5%, compounded monthly, how much will be in the account in 60 years? $A(t) = \mathbf{15{,}000}\left(1 + \frac{\mathbf{0.075}}{\mathbf{12}}\right)^{\mathbf{12}t}$ *To write the formula in function notation, substitute for P, r, and k.* $A(\mathbf{60}) = 15{,}000\left(1 + \frac{0.075}{12}\right)^{12(\mathbf{60})}$ *Substitute 60 for t.* $= 15{,}000\left(1 + \frac{0.075}{12}\right)^{720}$ $\approx 1{,}331{,}479.52$ *Use a calculator.* In 60 years, the account will contain \$1,331,479.52.

REVIEW EXERCISES

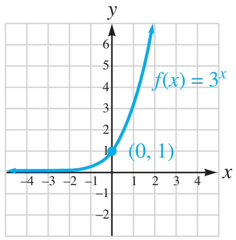
Use properties of exponents to simplify each expression.

21. $5^{\sqrt{6}} \cdot 5^{3\sqrt{6}}$ $5^{4\sqrt{6}}$

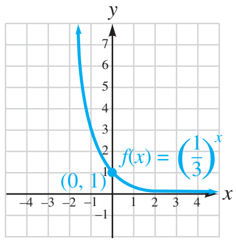
22. $\left(2^{\sqrt{14}}\right)^{\sqrt{2}}$ $2^{2\sqrt{7}}$

Graph each function and give the domain and the range. Label the y-intercept.

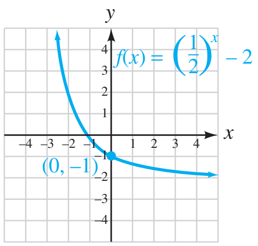
23. $f(x) = 3^x$ D: $(-\infty, \infty)$, R: $(0, \infty)$



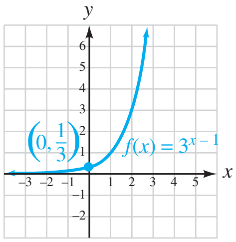
24. $f(x) = \left(\frac{1}{3}\right)^x$ D: $(-\infty, \infty)$, R: $(0, \infty)$



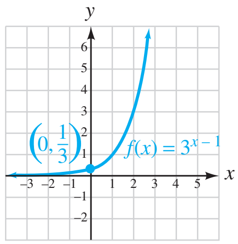
25. $f(x) = \left(\frac{1}{2}\right)^x - 2$ D: $(-\infty, \infty)$, R: $(-2, \infty)$



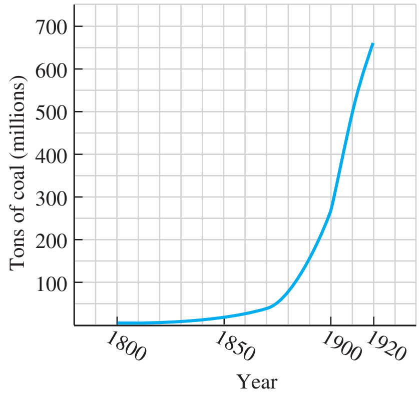
26. $f(x) = 3^{x-1}$ D: $(-\infty, \infty)$, R: $(0, \infty)$



27. In Exercise 26, what is the asymptote of the graph of $f(x) = 3^{x-1}$? the x-axis ($y = 0$)



28. COAL PRODUCTION The table gives the number of tons of coal produced in the United States for the years 1800–1920. Graph the data. What type of function does it appear could be used to model coal production over this period?
an exponential function



Year	Tons	Year	Tons
1800	108,000	1870	40,429,000
1810	178,000	1880	79,407,000
1820	881,000	1890	157,771,000
1830	1,334,000	1900	269,684,000
1840	2,474,000	1910	501,596,000
1850	8,356,000	1920	658,265,000
1860	20,041,000		

Source: *World Book Encyclopedia*

29. COMPOUND INTEREST How much will \$10,500 become if it earns 9% annual interest, compounded quarterly, for 60 years? \$2,189,703.45

30. DEPRECIATION The value (in dollars) of a certain model car is given by the function $V(t) = 12{,}000\left(10^{-0.155t}\right)$, where t is the number of years from the present. Find the value of the car in 5 years. about \$2,015

SECTION 9.4 Base-e Exponential Functions

DEFINITIONS AND CONCEPTS

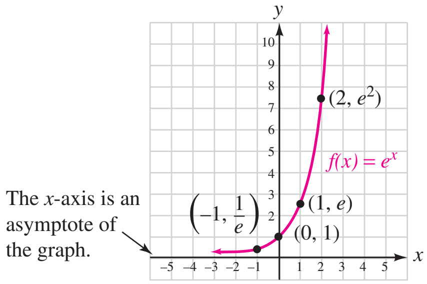
Of all possible bases for an exponential function, $\boldsymbol{e}$ is the most convenient for problems involving growth or decay.

$$e = 2.718281828459\ldots$$

The function defined by $\boldsymbol{f(x) = e^x}$ is the **natural exponential function.**

EXAMPLES

From the graph, we see that the domain of the natural exponential function is $(-\infty, \infty)$ and the range is $(0, \infty)$.



DEFINITIONS AND CONCEPTS

Exponential growth/decay: If a quantity increases or decreases at an annual rate r, **compounded continuously,** the amount A after t years is given by

$$A = Pe^{rt}$$

If r is negative, the amount decreases.

EXAMPLES

If \$30,000 accumulates interest at an annual rate of 9%, compounded continuously, find the amount in the account after 25 years.

$$\begin{aligned} A &= \boldsymbol{Pe^{rt}} && \text{This is the formula for continuous compound interest.} \\ &= \boldsymbol{30{,}000}e^{\boldsymbol{0.09 \cdot 25}} && \text{Substitute 30,000 for } P\text{, 0.09 for } r\text{, and 25 for } t. \\ &= 30{,}000e^{2.25} \\ &\approx 284{,}632.08 && \text{Use a calculator.} \end{aligned}$$

In 30 years, the account will contain \$284,632.08.

Suppose the population of a city of 50,000 people is decreasing exponentially according to the function $P(t) = 50{,}000e^{-0.003t}$, where t is measured in years from the present date. Find the expected population of the city in 20 years.

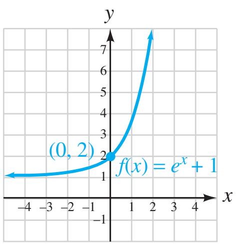
$$\begin{aligned} P(\boldsymbol{t}) &= 50{,}000e^{-0.003\boldsymbol{t}} && \text{Since } r \text{ is negative, this is the exponential decay model.} \\ P(\boldsymbol{20}) &= 50{,}000e^{-0.003(\boldsymbol{20})} && \text{Substitute 20 for } t. \\ &= 50{,}000e^{-0.06} \\ &\approx 47{,}088 && \text{Use a calculator.} \end{aligned}$$

After 20 years, the expected population will be about 47,088 people.

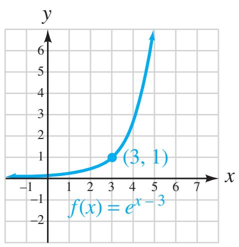
REVIEW EXERCISES

Graph each function, and give the domain and the range.

31. $f(x) = e^x + 1$ D: $(-\infty, \infty)$, R: $(1, \infty)$



32. $f(x) = e^{x-3}$ D: $(-\infty, \infty)$, R: $(0, \infty)$



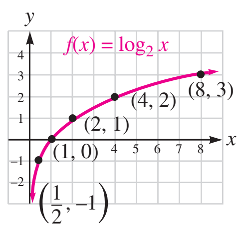
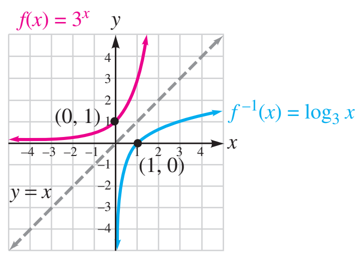
33. INTEREST COMPOUNDED CONTINUOUSLY If \$10,500 accumulates interest at an annual rate of 9%, compounded continuously, how much will be in the account in 60 years? \$2,324,767.37

34. THE GRAND CANYON STATE In 2006, Arizona ended Nevada's 19-year reign as the nation's fastest growing state. The population of Arizona at the time was 6,166,318 with an annual growth rate of 3.6%. Predict the population of Arizona in 2016, assuming the growth rate remains the same. 8,838,365

35. MORTGAGE RATES There was the housing boom in the 1980s as the baby boomers (those born from 1946–1964) bought their homes. The average annual interest rate in percent on a 30-year fixed-rate home mortgage for the years 1980–1996 can be approximated by the function $r(t) = 13.9e^{-0.035t}$, where t is the number of years since 1980. To the nearest hundredth of a percent, what does this model predict was the 30-year fixed rate in 1980? In 1985? In 1990? 13.9%, 11.67%, 9.80%

36. MEDICAL TESTS A radioactive dye is injected into a patient as part of a test to detect heart disease. The amount of dye remaining in his bloodstream t hours after the injection is given by the function $f(t) = 10e^{-0.27t}$. How can you determine from the function that the amount of dye in the bloodstream is decreasing?
The exponent on the base e is negative.

SECTION 9.5 Logarithmic Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
Definition of logarithm: If $b > 0$, $b \neq 1$, and x is positive, then $y = \log_b x$ means $x = b^y$	***Logarithmic form*** ***Exponential form*** $\log_5 125 = 3$ is equivalent to $5^3 = 125$ $\log_2 \frac{1}{8} = -3$ is equivalent to $2^{-3} = \frac{1}{8}$
$\log_b x$ is the exponent to which b is raised to get x.	To evaluate $\log_4 16$ we ask: "To what power must we raise 4 to get 16?" Since $4^{\mathbf{2}} = 16$, the answer is: the 2nd power. Thus, $\log_4 16 = \mathbf{2}$
For computational purposes and in many applications, we use base-10 logarithms, called **common logarithms.** $\log x$ means $\log_{10} x$	$\log \frac{1}{1{,}000} = \mathbf{-3}$ because $10^{\mathbf{-3}} = \frac{1}{1{,}000}$
If $b > 0$ and $b \neq 1$, the **logarithmic function with base b** is defined by $\boldsymbol{f(x) = \log_b x}$. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$. If $b > 1$, then $f(x) = \log_b x$ is an increasing function. If $0 < b < 1$, then $f(x) = \log_b x$ is a decreasing function.	The graph of the logarithmic function $f(x) = \log_2 x$. From the graph, we see that $f(x) = \log_2 x$ is an increasing function. 
The exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$ are inverses of each other.	$f(x) = 3^x$ and $f^{-1}(x) = \log_3 x$ are inverses of each other. Their graphs are symmetric about the line $y = x$. 

Logarithmic functions, like exponential functions, can be used to **model** certain types of growth and decay.

Decibel voltage gain:

$$\text{db gain} = 20 \log \frac{E_O}{E_I}$$

The Richter scale:

$$R = \log \frac{A}{P}$$

If the input to an amplifier is 0.4 volt and the output is 30 volts, find the decibel voltage gain.

$$\text{db gain} = 20 \log \frac{E_O}{E_I}$$

$$= 20 \log \frac{30}{0.4} \quad \text{Substitute 30 for } E_O \text{ and 0.4 for } E_I.$$

$$\approx 37.50122527 \quad \text{Use a calculator.}$$

The db gain is about 38 decibels.

REVIEW EXERCISES

37. Give the domain and range of $f(x) = \log x$.
D: $(0, \infty)$, R: $(-\infty, \infty)$

38. Explain why a student got the following message when she used a calculator to evaluate log 0.

$\boxed{\text{Error}}$ Since there is no real number such that $10^? = 0$, log 0 is undefined.

39. Write the statement $\log_4 64 = 3$ in exponential form.
$4^3 = 64$

40. Write the statement $7^{-1} = \frac{1}{7}$ in logarithmic form.
$\log_7 \frac{1}{7} = -1$

Evaluate, if possible.

41. $\log_3 9$ 2

42. $\log_9 \frac{1}{81}$ -2

43. $\log_{1/2} 1$ 0

44. $\log_5 (-25)$ undefined

45. $\log_6 \sqrt{6}$ $\frac{1}{2}$

46. $\log 1{,}000$ 3

Solve for x.

47. $\log_2 x = 5$ 32

48. $\log_3 x = -4$ $\frac{1}{81}$

49. $\log_x 16 = 2$ 4

50. $\log_x \frac{1}{100} = -2$ 10

51. $\log_9 3 = x$ $\frac{1}{2}$

52. $\log_{27} 3 = x$ $\frac{1}{3}$

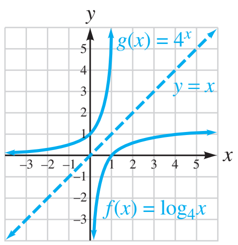
Use a calculator to find the value of x to four decimal places.

53. $\log 4.51 = x$ 0.6542

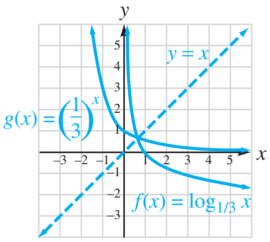
54. $\log x = 1.43$ 26.9153

Graph each function and its inverse on the same coordinate system. Draw the axis of symmetry.

55. $f(x) = \log_4 x$ and $g(x) = 4^x$

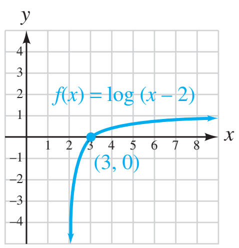


56. $f(x) = \log_{1/3} x$ and $g(x) = \left(\frac{1}{3}\right)^x$

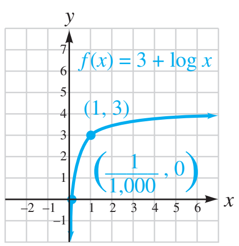


Graph each function. Label the x-intercept.

57. $f(x) = \log (x - 2)$



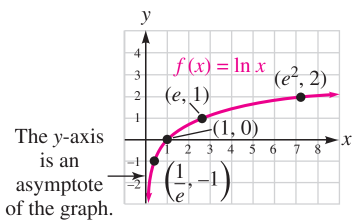
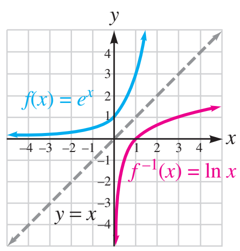
58. $f(x) = 3 + \log x$



59. ELECTRICAL ENGINEERING Find the db gain of an amplifier with an output of 18 volts and an input of 0.04 volt. about 53

60. EARTHQUAKES An earthquake had a period of 0.3 second and an amplitude of 7,500 micrometers. Find its measure on the Richter scale. about 4.4

SECTION 9.6 Base-*e* Logarithmic Functions

DEFINITIONS AND CONCEPTS	EXAMPLES
Of all possible bases for a logarithmic function, e is the most convenient for problems involving growth or decay. Since these situations occur often in natural settings, base-e logarithms are called **natural logarithms:** $\ln x$ means $\log_e x$	$\ln 5.7$ means $\log_e 5.7$
$\ln x$ is the exponent to which e is raised to get x.	To evaluate $\ln \frac{1}{e^4}$ we ask: "To what power must we raise e to get $\frac{1}{e^4}$?" Since $e^{-4} = \frac{1}{e^4}$, the answer is: the $-4th$ power. Thus, $\ln \frac{1}{e^4} = -4$
The **natural logarithmic function** with base e is defined by $\boldsymbol{f(x) = \ln x}$ The domain is the interval $(0, \infty)$ and the range is the interval $(-\infty, \infty)$.	The graph of the natural logarithmic function $f(x) = \ln x$ From the graph, we see that $f(x) = \ln x$ is an increasing function. 
The natural exponential function $f(x) = e^x$ and the natural logarithmic function $f^{-1}(x) = \ln x$ are **inverses** of each other.	The graphs are symmetric about the line $y = x$. 
If a population grows exponentially at a certain annual rate r, the time required for the population to double is called the **doubling time.** It is given by the formula: $t = \frac{\ln 2}{r}$	The population of a town is growing at a rate of 3% per year. If this rate continues, how long will it take the population to double? We substitute 0.03 for r and use a calculator to perform the computation. $t = \frac{\ln 2}{r} = \frac{\ln 2}{0.03} \approx 23.10490602$ The population will double in about 23.1 years.

REVIEW EXERCISES

Evaluate each expression, if possible. Do not use a calculator.

61. $\ln e$ 1
62. $\ln e^2$ 2
63. $\ln \frac{1}{e^5}$ -5
64. $\ln \sqrt{e}$ $\frac{1}{2}$
65. $\ln(-e)$ undefined
66. $\ln 0$ undefined
67. $\ln 1$ 0
68. $\ln e^{-7}$ -7

Use a calculator to evaluate each expression. Express all answers to four decimal places.

69. $\ln 452$ 6.1137
70. $\ln 0.85$ -0.1625

Solve each equation. Express all answers to four decimal places.

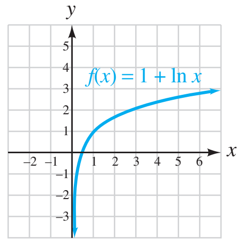
71. $\ln x = 2.336$ 10.3398
72. $\ln x = -8.8$ 0.0002

73. Explain the difference between the functions $f(x) = \log x$ and $g(x) = \ln x$.
They have different bases: $\log x = \log_{10} x$ and $\ln x = \log_e x$.

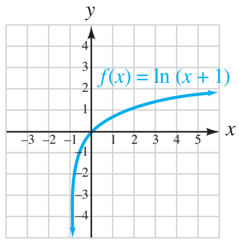
74. What function is the inverse of $f(x) = \ln x$?
$f^{-1}(x) = e^x$

Graph each function.

75. $f(x) = 1 + \ln x$



76. $f(x) = \ln(x + 1)$



77. POPULATION GROWTH How long will it take the population of Mexico to double if the growth rate is currently about 1.153%?
about 60 yr

78. BOTANY The height (in inches) of a certain plant is approximated by the function $H(a) = 13 + 20.03 \ln a$, where a is its age in years. How tall will it be when it is 19 years old?
about 72 in. (6 ft)

SECTION 9.7 Properties of Logarithms

DEFINITIONS AND CONCEPTS	EXAMPLES
Properties of logarithms: If M, N, and b are positive real numbers, $b \neq 1$ **1.** $\log_b 1 = 0$ **2.** $\log_b b = 1$ **3.** $\log_b b^x = x$ **4.** $b^{\log_b x} = x$ **5.** ***Product rule for logarithms:*** $\log_b MN = \log_b M + \log_b N$ **6.** ***Quotient rule for logarithms:*** $\log_b \frac{M}{N} = \log_b M - \log_b N$ **7.** ***Power rule for logarithms:*** $\log_b M^p = p \log_b M$	Apply a property of logarithms and then simplify, if possible. **1.** $\log_3 1 = 0$ **2.** $\log_7 7 = 1$ **3.** $\log_5 5^3 = 3$ **4.** $9^{\log_9 10} = 10$ **5.** $\log_2(6 \cdot 8) = \log_2 6 + \log_2 8$ $= \log_2 6 + 3$ **6.** $\log_2 \frac{8}{6} = \log_2 8 - \log_2 6$ $= 3 - \log_2 6$ **7.** $\log_2 7^3 = 3 \log_2 7$
Properties of logarithms can be used to **expand** logarithmic expressions.	Write $\log_3 (x^2y^3)$ as the sum and/or difference of logarithms of a single quantity. $\log_3 (x^2y^3) = \log_3 x^2 + \log_3 y^3$ The log of a product is the sum of the logs. $= 2 \log_3 x + 3 \log_3 y$ The log of a power is the power times the log.
Properties of logarithms can be used to **condense** certain logarithmic expressions.	Write $3 \ln x - \frac{1}{2} \ln y$ as a single logarithm. $3 \ln x - \frac{1}{2} \ln y = \ln x^3 - \ln y^{1/2}$ A power times a log is the log of the power. $= \ln \frac{x^3}{y^{1/2}}$ The difference of two logs is the log of the quotient. $= \ln \frac{x^3}{\sqrt{y}}$ Write $y^{1/2}$ as $\sqrt{y}$.

If we need to find a logarithm with some base other than 10 or e, we can use a conversion formula. **Change-of-base formula:** $\log_b x = \dfrac{\log_a x}{\log_a b}$	Find $\log_7 6$ to four decimal places. $\log_7 6 = \dfrac{\log 6}{\log 7} \approx 0.920782221$ To four decimal places, $\log_7 6 = 0.9208$. To check, verify that $7^{0.9208}$ is approximately 6.
In chemistry, common logarithms are used to express the acidity of solutions. **pH scale:** $\text{pH} = -\log[H^+]$	Find the pH of a liquid with a hydrogen ion concentration of 10^{-8} gram-ions per liter. $\text{pH} = -\log\,[\mathbf{H^+}]$ $= -\log \mathbf{10^{-8}}$ Substitute 10^{-8} for $[H^+]$. $= -(-8)\log 10$ The log of a power is the power times the log. $= 8$ Simplify: $\log 10 = 1$.

REVIEW EXERCISES

Simplify each expression.

79. $\log_2 1$ 0

80. $\log_9 9$ 1

81. $\log 10^3$ 3

82. $7^{\log_7 4}$ 4

Write each logarithm as the sum and/or difference of logarithms of a single quantity. Then simplify, if possible.

83. $\log_3 27x$ $3 + \log_3 x$

84. $\log \dfrac{100}{x}$ $2 - \log x$

85. $\log_5 \sqrt{27}$ $\frac{1}{2}\log_5 27$

86. $\log_b 10ab$ $\log_b 10 + \log_b a + 1$

Write each logarithm as the sum and/or difference of logarithms of a single quantity.

87. $\log_b \dfrac{x^2y^3}{z}$ $2\log_b x + 3\log_b y - \log_b z$

88. $\ln \sqrt{\dfrac{x}{yz^2}}$ $\frac{1}{2}(\ln x - \ln y - 2\ln z)$

Write each logarithmic expression as one logarithm.

89. $3\log_2 x - 5\log_2 y + 7\log_2 z$ $\log_2 \frac{x^3z^7}{y^5}$

90. $-3\log_b y - 7\log_b z + \frac{1}{2}\log_b (x + 2)$ $\log_b \frac{\sqrt{x+2}}{y^3z^7}$

Assume that $\log_b 5 = 1.1609$ *and* $\log_b 8 = 1.5000$ *and find each value to four decimal places.*

91. $\log_b 40$ 2.6609

92. $\log_b 64$ 3.0000

93. Find $\log_5 17$ to four decimal places. 1.7604

94. pH OF GRAPEFRUIT The pH of grapefruit juice is about 3.1. Find its hydrogen ion concentration. about 7.9×10^{-4} gram-ions/liter

SECTION 9.8 Exponential and Logarithmic Equations

DEFINITIONS AND CONCEPTS	EXAMPLES
An **exponential equation** contains a variable in one of its exponents. If both sides of an exponential equation can be expressed as a power of the same base, we can use the following property to solve it: $b^x = b^y$ is equivalent to $x = y$	Solve: $3^{x+2} = 27$ We express the right side of the equation as a power of 3. $\mathbf{3^{x+2}} = \mathbf{3^3}$ Write 27 as 3^3. $x + 2 = 3$ If two exponential expressions with the same base are equal, their exponents are equal. $x = 1$ The solution is 1. Check it in the original equation.

When it is difficult to write each side of an exponential equation as a power of the same base, **take the logarithm of each side.**	Solve $4^x = 7$ and give the answer to four decimal places. We take the base-10 logarithm of both sides of the equation. $\log 4^x = \log 7$ $x \log 4 = \log 7$ The log of a power is the power times the log. $x = \frac{\log 7}{\log 4}$ To isolate x, divide both sides by log 4. $x \approx 1.4037$ Use a calculator. To four decimal places, the solution is 1.4037. To check the approximate solution, we substitute 1.4037 for x in $4^x = 7$ and use a calculator to evaluate the left side.
A **logarithmic equation** is an equation containing a variable in a logarithmic expression. Certain logarithmic equations can be solved using the following property: $\log_b x = \log_b y$ is equivalent to $x = y$	Solve: $\log(4x - 3) = \log(2x + 7)$ $\log(4x - 3) = \log(2x + 7)$ $4x - 3 = 2x + 7$ If the logarithms of two numbers are equal, the numbers are equal. $2x = 10$ $x = 5$ The solution is 5. Check it in the original equation.
To solve some logarithmic equations, we instead write and solve an equivalent exponential equation.	Solve: $\log_4(x + 1) = 2$ We will write the equivalent base-4 exponential equation. $\log_4(x + 1) = 2$ $x + 1 = 4^2$ $x + 1 = 16$ $x = 15$ The solution is 15. Check it in the original equation.
When there is sufficient food and space available, populations of living organisms tend to increase exponentially according to the following **growth model.** ***Population growth*** $P = P_0 e^{kt}$	Find the number of bacteria in a culture of 1,000 bacteria if they are allowed to reproduce for 5 hours. Assume $k = \frac{\ln 3}{3}$. $P = P_0 e^{kt}$ This is the population growth model. $= 1{,}000 e^{\frac{\ln 3}{3} \cdot 5}$ Substitute. $\approx 6{,}240$ Use a calculator. In 5 hours, there will be approximately 6,240 bacteria.

REVIEW EXERCISES

Solve each equation. Give approximate answers to four decimal places.

95. $5^{x+6} = 25$ -4

96. $2^{x^2+4x} = \frac{1}{8}$ $-3, -1$

97. $3^x = 7$ 1.7712

98. $2^x = 3^{x-1}$ 2.7095

99. $e^x = 7$ 1.9459

100. $e^{-0.4t} = 25$ -8.0472

Solve each equation.

101. $\log(x - 4) = 2$ 104

102. $\ln(2x - 3) = \ln 15$ 9

103. $\log x + \log(29 - x) = 2$ 25, 4

104. $\log_2 x + \log_2(x - 2) = 3$ 4, -2 is extraneous

105. $\frac{\log(7x - 12)}{\log x} = 2$ 4, 3

106. $\log_2(x + 2) + \log_2(x - 1) = 2$ 2

107. $\log x + \log(x - 5) = \log 6$ 6

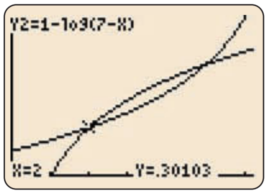
108. $\log 3 - \log(x - 1) = -1$ 31

109. Evaluate both sides of the statement $\frac{\log 8}{\log 15} \neq \log 8 - \log 15$ to show that the sides are indeed not equal. $0.76787 \neq -0.27300$

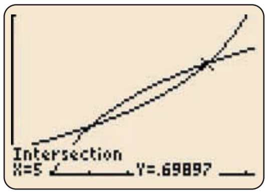
110. CARBON-14 DATING A wooden statue found in Egypt has a carbon-14 content that is two-thirds of that found in living wood. If the half-life of carbon-14 is 5,700 years, how old is the statue? about 3,300 yr

111. ANTS The number of ants in a colony is estimated to be 800. If the ant population is expected to triple every 14 days, how long will it take for the population to reach one million? about 91 days

112. The approximate coordinates of the points of intersection of the graphs of $f(x) = \log x$ and $g(x) = 1 - \log(7 - x)$ are shown in figures (a) and (b) below. Use the graphs to estimate the solutions of the logarithmic equation $\log x = 1 - \log(7 - x)$. Then check your answers. 2, 5



(a)



(b)

CHAPTER 9 TEST

Let $f(x) = 4x$ and $g(x) = x - 1$. Find each function or value.

1. $g + f$ $(g + f)(x) = 5x - 1$

2. $g \cdot f$ $(g \cdot f)(x) = 4x^2 - 4x$

3. Find: $(g \circ f)(1)$ 3

4. Find: $f(g(x))$ $4(x - 1)$

Find the inverse of each function.

5. $f(x) = \frac{3}{2}x + 6$

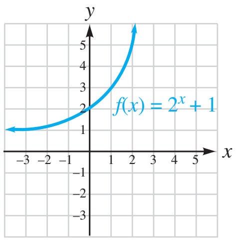
$f^{-1}(x) = \frac{2x - 12}{3}$

6. $f(x) = 3x^2 + 4 (x \geq 0)$

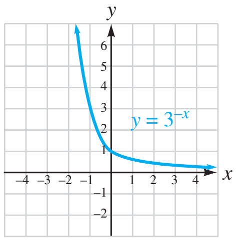
$f^{-1}(x) = \sqrt{\frac{x - 4}{3}}$

Graph each function.

7. $f(x) = 2^x + 1$



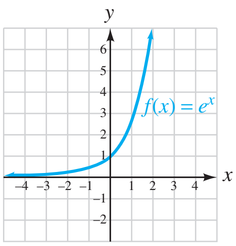
8. $y = 3^{-x}$



9. RADIOACTIVE DECAY A radioactive material decays according to the formula $A = A_0(2)^{-t}$. How much of a 3-gram sample will be left in 6 years? $\frac{3}{64}g = 0.046875g$

10. COMPOUND INTEREST An initial deposit of \$1,000 earns 6% interest, compounded twice a year. How much will be in the account in one year? \$1,060.90

11. Graph: $f(x) = e^x$



12. CONTINUOUS COMPOUNDING An account contains \$2,000 and has been earning 8% interest, compounded continuously. How much will be in the account in 10 years? \$4,451.08

Find x.

13. $\log_4 16 = x$ 2

14. $\log_x 81 = 4$ 3

15. $\log_3 x = -3$ $\frac{1}{27}$

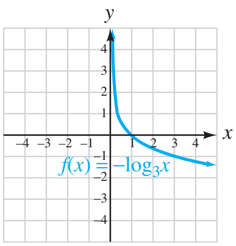
16. $\ln x = 1$ e

17. Write the statement $\log_6 \frac{1}{36} = -2$ in exponential form: $6^{-2} = \frac{1}{36}$

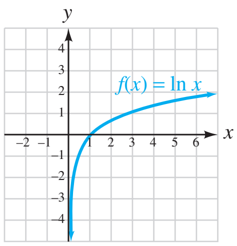
18. Give the domain and range of the function $f(x) = \log x$. D: $(0, \infty)$, R: $(-\infty, \infty)$

Graph each function.

19. $f(x) = -\log_3 x$



20. $f(x) = \ln x$



21. Write the expression $\log a^2bc^3$ in terms of the logarithms of a, b, and c. $2 \log a + \log b + 3 \log c$

22. Write the expression $\frac{1}{2} \ln (a + 2) + \ln b - 3 \ln c$ as a logarithm of a single quantity. $\ln \frac{b\sqrt{a + 2}}{c^3}$

23. Use the change-of-base formula to find $\log_7 3$ to four decimal places. 0.5646

24. What function is the inverse of $y = 10^x$? $y = \log x$

25. pH Find the pH of a solution with a hydrogen ion concentration of 3.7×10^{-7}. (*Hint:* $\text{pH} = -\log\,[\text{H}^+]$.) 6.4

26. ELECTRONICS Find the db gain of an amplifier when $E_O = 60$ volts and $E_I = 0.3$ volt. $\left(\textit{Hint}\text{: db gain} = 20\log\left(\frac{E_O}{E_I}\right).\right)$ 46

Solve each equation. Round to four decimal places when necessary.

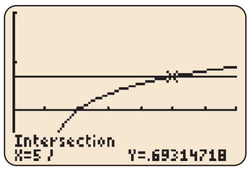
27. $5^x = 3$ 0.6826

28. $3^{x-1} = 27$ 4

29. $\ln(5x + 2) = \ln(2x + 5)$ 1

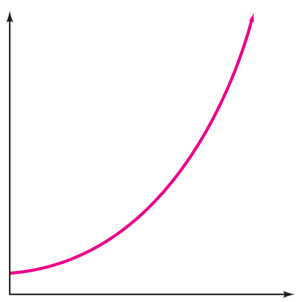
30. $\log x + \log(x - 9) = 1$ 10

31. The illustration shows the graphs of $y = \frac{1}{2}\ln(x - 1)$ and $y = \ln 2$ and the approximate coordinates of their point of intersection. Estimate the solution of the logarithmic equation $\frac{1}{2}\ln(x - 1) = \ln 2$. 5



32. Show a check of your answer to Problem 31.
$\frac{1}{2}\ln(5 - 1) = \frac{1}{2}\ln 4 = 0.69314718\ldots$, which is ln 2.

33. Give an example of a situation studied in this chapter that is modeled by a function with a graph that has the shape shown. Label the axes. You do not have to scale the axes. answers may vary

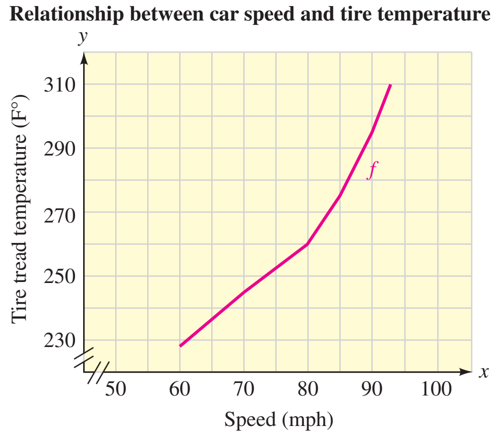


Consider the graph shown below.

34. Is it the graph of a function? yes

35. Is its inverse a function? yes

36. What is $f^{-1}(260)$? What information about temperature and tire tread does it give?
80; when the temperature of the tire tread is 260°, the vehicle is traveling 80 mph



37. POPULATION GROWTH As of July 2003, the population of India was estimated to be 1,050,000,000, with an annual growth rate of 1.47%. If the growth rate remains the same, how large will the population be in 30 years? about 1,631,973,737

38. INSECTS The number of insects attracted to a bright light is currently 5. If the number is expected to quadruple every 6 minutes, how long will it take for the number to reach 500? about 20 min.

CHAPTERS 1–9 CUMULATIVE REVIEW

Write the formula associated with each concept.

1. Perimeter of a rectangle [Section 1.6] $P = 2l + 2w$

2. Area of a circle [Section 1.6] $A = \pi r^2$

3. Area of a triangle [Section 1.6] $A = \frac{1}{2}bh$

4. Volume of a cube [Section 1.6] $V = s^3$

5. Simple interest [Section 1.8] $I = Prt$

6. Distance (uniform motion) [Section 1.8] $d = rt$

7. Midpoint of a line segment $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ [Section 2.1]

8. Slope–intercept form of the equation of a line [Section 2.4] $y = mx + b$

9. Point–slope form of the equation of a line [Section 2.4] $y - y_1 = m(x - x_1)$

10. Slope of a line [Section 2.3] $m = \frac{y_2 - y_1}{x_2 - x_1}$

11. Pythagorean theorem [Section 7.6] $a^2 + b^2 = c^2$

12. Distance between two points [Section 7.6] $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

13. Direct variation [Section 6.9] $y = kx$

14. Inverse variation [Section 6.9] $y = \frac{k}{x}$

15. Quadratic formula [Section 8.2] $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

16. Exponential growth [Section 9.4] $A = Pe^{rt}$

17. Change-of-base formula for logarithms [Section 9.7] $\log_b x = \frac{\log_a x}{\log_a b}$

Fill in the blanks to complete the rules for exponents. [Section 5.1]

18. $x^1 = x$

19. $x^m x^n = x^{m+n}$

20. $(x^m)^n = x^{mn}$

21. $(xy)^n = x^n y^n$

22. $x^0 = 1$

23. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

24. $\frac{x^m}{x^n} = x^{m-n}$

25. $x^{-n} = \frac{1}{x^n}$

26. $\frac{1}{x^{-n}} = x^n$

27. $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$

28. $x^{1/n} = \sqrt[n]{x}$

29. $x^{m/n} = \left(\sqrt[n]{x}\right)^m$

Complete each factorization or product formula.

30. $x^2 - y^2 = (x + y)(x - y)$ [Section 5.6]

31. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ [Section 5.6]

32. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ [Section 5.6]

33. $(x + y)^2 = x^2 + 2xy + y^2$ [Section 5.4]

34. $(x - y)^2 = x^2 - 2xy + y^2$ [Section 5.4]

35. $(x + y)(x - y) = x^2 - y^2$ [Section 5.4]

Complete each property of radicals. [Section 7.2]

36. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

37. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$

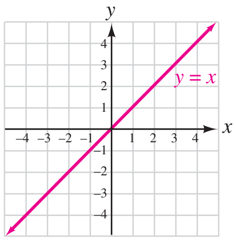
Approximate each irrational number to the nearest hundredth. [Section 1.2]

38. π 3.14

39. $\sqrt{2}$ 1.41

40. e 2.72

41. To graph $y \geq x$, we first graph the boundary line $y = x$ as in the illustration. Explain how we determine which side of the boundary to shade. [Section 4.4]
Pick a test point on one side of the boundary line. In $y \geq x$, replace x and y with the coordinates of that point. If the inequality is satisfied, shade the side that contains that point. If the inequality is not satisfied, shade the other side.



Complete each property of logarithms. [Section 9.7]

42. $\log_b 1 = 0$

43. $\log_b b = 1$

44. $\log_b b^x = x$

45. $b^{\log_b x} = x$

46. $\log_b MN = \log_b M + \log_b N$

47. $\log_b \frac{M}{N} = \log_b M - \log_b N$

48. $\log_b M^p = p \log_b M$

49. Fill in the blanks to complete the fundamental property of fractions. If a, b, and k represent real numbers, and $b \neq 0$ and $k \neq 0$, then $\frac{a}{b} = \frac{a \cdot \boxed{k}}{b \cdot \boxed{k}}$

[Section 6.1]

The following partial solutions show three important applications of the fundamental property of fractions. In each case, explain why it was used.

50. $\frac{5a}{24b} + \frac{11a}{18b^2} = \frac{5a \cdot \mathbf{3b}}{24b \cdot \mathbf{3b}} + \frac{11a \cdot \mathbf{4}}{18b^2 \cdot \mathbf{4}}$ [Section 6.3]
to build up the fractions so they have the same denominator

51. $\frac{\sqrt{70}}{\sqrt{3}} = \frac{\sqrt{70} \cdot \sqrt{\mathbf{3}}}{\sqrt{3} \cdot \sqrt{\mathbf{3}}}$ [Section 7.3]
to rationalize the denominator

52. $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} = \frac{\mathbf{xy}\left(\frac{1}{x} + \frac{1}{y}\right)}{\mathbf{xy}\left(\frac{1}{x} - \frac{1}{y}\right)}$ [Section 6.4]
to simplify a complex fraction

53. The following partial solution shows an important application of the fundamental property of fractions. Explain why it was used and what the slashes and 1's mean. [Section 6.1]

$$\frac{6a^2 - 13a + 6}{3a^2 + a - 2} = \frac{\overset{1}{\cancel{(3a - 2)}}(2a - 3)}{\underset{1}{\cancel{(3a - 2)}}(a + 1)}$$

It was used to simplify a rational expression. The slashes and 1's show a common factor of $3a - 2$ being divided out.

54. Fill in the blanks to complete the fundamental property of proportions. [Section 6.9]

If $\frac{a}{b} = \frac{c}{d}$, then $ad = \boxed{bc}$

55. What is a quadratic equation? [Section 8.1]
an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$

56. Consider the equation $(x + 1)(x - 7) = 0$. How is the zero-factor property used to solve this equation? [Section 5.9] $x + 1 = 0$ or $x - 7 = 0$

57. What is the first step used to solve the following equation? [Section 6.7]

$$\frac{x-3}{x-2} - \frac{1}{x} = \frac{x-3}{x}$$

Multiply both sides of the equation by the LCD, which is $x(x - 2)$

58. Which method would be the most efficient to solve the following system? [Section 3.2] the addition method

$$\begin{cases} 3x - 2y = -10 \\ 6x + 5y = 25 \end{cases}$$

Fill in the blanks to complete each definition.

59. For any real number x,

$$\begin{cases} \text{if } x \geq 0, \text{ then } |x| = \boxed{x} \\ \text{if } x < 0, \text{ then } |x| = \boxed{-x} \end{cases}$$

[Section 4.3]

60. The number b is a square root of a if $\boxed{b^2} = a$.
[Section 7.1]

61. If x can be any real number, then $\sqrt{x^2} = \boxed{|x|}$.
[Section 7.1]

62. If x is any real number, then $\sqrt[3]{x^3} = \boxed{x}$. [Section 7.1]

63. If a and b are real numbers, $a - b = a \boxed{+} (-b)$.
[Section 1.3]

64. $i = \boxed{\sqrt{-1}}$ [Section 7.7]

Fill in the blanks.

65. A function is a correspondence between a set of input values x (called the _domain_) and a set of output values y (called the _range_), in which exactly _one_ y-value in the range is assigned to each number x in the domain. [Section 2.5]

66. The solution set of a compound inequality containing the word *and* consists of all the numbers that make _both_ inequalities true. [Section 4.2]

67. The solution set of a compound inequality containing the word *or* consists of all the numbers that make _one_ or the other or _both_ inequalities true.
[Section 4.2]

Conic Sections; More Graphing

10

from Campus to Careers

Traffic Engineer

Traffic engineers design roads, streets, and highways for the safe and efficient movement of people and goods. They use traffic flow formulas to determine what kinds of roads are needed and then find economical ways to construct and operate them. During the planning stages, traffic engineers make detailed drawings and graphs of the project. Because highway and street construction is often publicly funded, they make budgets, submit bid proposals, and perform cost analysis studies to make sure that highway tax money is spent wisely.

In **Problem 85** of **Study Set 10.1,** you will design two sections of a freeway that are to be joined with a curve that is one-quarter of a circle.

JOB TITLE:
Traffic Engineer
EDUCATION: A bachelor's degree in civil engineering is required.
JOB OUTLOOK: Good; It is expected to increase 9% to 17% through 2014.
ANNUAL EARNINGS: The median salary in 2007 was $95,300.
FOR MORE INFORMATION:
http://www.ite.org/career/index.asp

Objectives

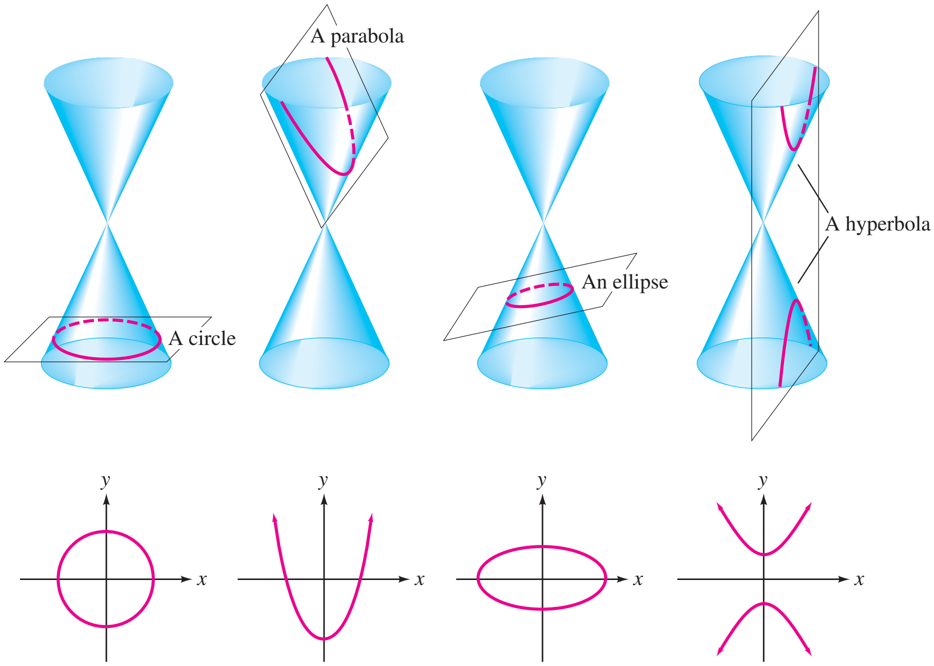
1. Identify conic sections and some of their applications.
2. Graph equations of circles written in standard form.
3. Write the equation of a circle, given its center and radius.
4. Convert the general form of the equation of a circle to standard form.
5. Solve problems involving circles.
6. Convert the general form of the equation of a parabola to standard form to graph it.

SECTION 10.1 The Circle and the Parabola

We have previously graphed first-degree equations in two variables such as $y = 3x + 8$ and $4x - 3y = 12$. Their graphs are lines. In this section, we will graph second-degree equations in two variables such as $x^2 + y^2 = 25$ and $x = -3y^2 - 12y - 13$. The graphs of these equations are *conic sections.*

1 Identify conic sections and some of their applications.

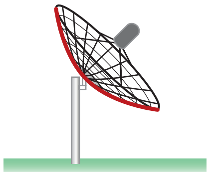
The curves formed by the intersection of a plane with an infinite right-circular cone are called **conic sections.** Those curves have four basic shapes, called **circles, parabolas, ellipses,** and **hyperbolas,** as shown below.



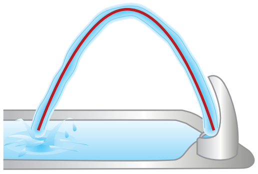
Conic sections have many applications. For example, everyone is familiar with circular wheels and gears, pizza cutters, and hula hoops.

Parabolas can be rotated to generate dish-shaped surfaces called **paraboloids.** Any light or sound placed at the **focus** of a paraboloid is reflected outward in parallel paths. This property makes parabolic surfaces ideal for flashlight and headlight reflectors. It also makes parabolic surfaces good antennas, because signals captured by such antennas are concentrated at the focus. Parabolic mirrors are capable of concentrating the rays of the sun at a single point, thereby generating tremendous heat. This property is used in the design of solar furnaces.

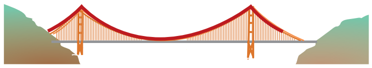
Any object thrown upward and outward travels in a parabolic path. An example of this is a stream of water flowing from a drinking fountain. In architecture, many arches are parabolic in shape, because this gives them strength. Cables that support suspension bridges hang in the shape of a parabola.



Radar dish

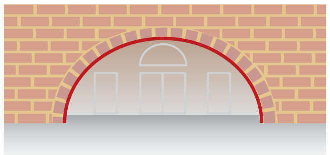


Stream of water

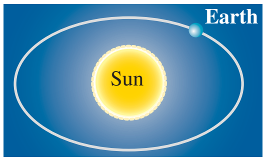


Support cables

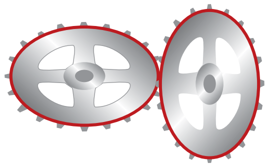
Ellipses have optical and acoustical properties that are useful in architecture and engineering. Many arches are portions of an ellipse, because the shape is pleasing to the eye. The planets and many comets have elliptical orbits. Certain gears have elliptical shapes to provide nonuniform motion.



Arches



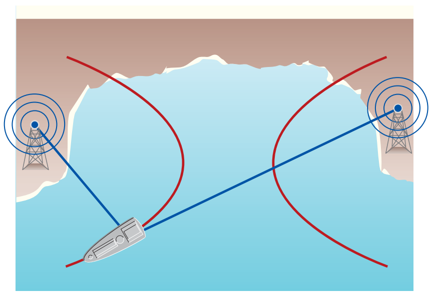
Earth's orbit



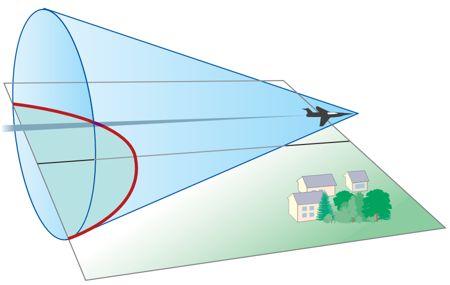
Gears

Hyperbolas serve as the basis of a navigational system known as LORAN (LOng RAnge Navigation). They are also used to find the source of a distress signal, are the basis for the design of hypoid gears, and describe the orbits of some comets.

A sonic shock wave created by a jet aircraft has the shape of a cone. In level flight, the sound wave intersects the ground as one branch of a hyperbola, as shown below. People in different places along the curve on the ground hear and feel the sonic boom at the same time.



Navigation



Sonic boom

2 Graph equations of circles written in standard form.

Every conic section can be represented by a second-degree equation in x and y. To find the equation of a circle, we use the following definition.

Definition of a Circle

A **circle** is the set of all points in a plane that are a fixed distance from a fixed point called its **center.** The fixed distance is called the **radius** of the circle.

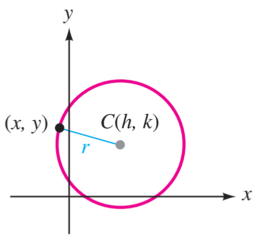
If we let (h, k) be the center of a circle and (x, y) be some point on a circle that is graphed on a rectangular coordinate system, the distance from (h, k) to (x, y) is the radius r of the circle. We can use the distance formula to find r.

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

We can square both sides to eliminate the radical and obtain

$$r^2 = (x - h)^2 + (y - k)^2$$

This result is called the *standard form of the equation of a circle* with radius r and center at (h, k).



Equation of a Circle

The **standard form of the equation of a circle** with radius r and center at (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

Success Tip An equation of the form

$$x^2 + y^2 = r^2$$

has a graph that is a circle with radius r and center at (0, 0).

EXAMPLE 1

Find the center and the radius of each circle and then graph it:

a. $(x - 4)^2 + (y - 1)^2 = 9$ **b.** $x^2 + y^2 = 25$ **c.** $(x + 3)^2 + y^2 = 12$

Strategy We will compare each equation to the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$, and identify h, k, and r.

WHY The center of the circle is the point with coordinates (h, k) and the radius of the circle is r.

Solution

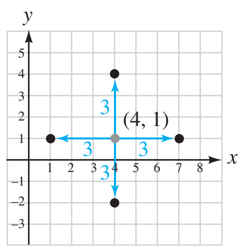
a. The color highlighting shows how to compare the given equation to the standard form to find h, k, and r.

$$(x - \mathbf{4})^2 + (y - \mathbf{1})^2 = \mathbf{9}$$
$$\uparrow \qquad \uparrow \qquad \uparrow$$
$$(x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \boldsymbol{r^2}$$

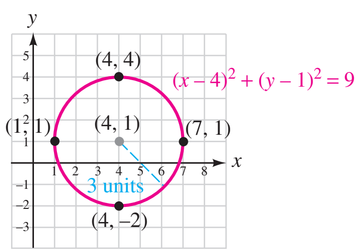
$h = 4$, $k = 1$, and $r^2 = 9$. Since the radius of a circle must be positive, $r = 3$.

The center of the circle is $(h, k) = (4, 1)$ and the radius is 3.

To plot four points on the circle, we move up, down, left, and right 3 units from the center, as shown in figure (a). Then we draw a circle through the points to get the graph of $(x - 4)^2 + (y - 1)^2 = 9$, as shown in figure (b).



(a)

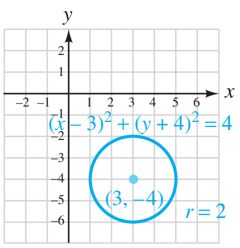


(b)

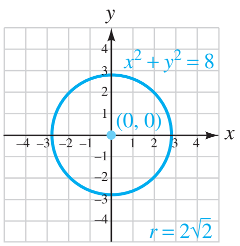
Self Check 1

Find the center and the radius of each circle and then graph it:

a. $(x - 3)^2 + (y + 4)^2 = 4$



b. $x^2 + y^2 = 8$



Now Try **Problems 15, 19, and 21**

Self Check 1 Answers

a. $(3, -4)$, $r = 2$

b. $(0, 0)$, $r = 2\sqrt{2} \approx 2.8$

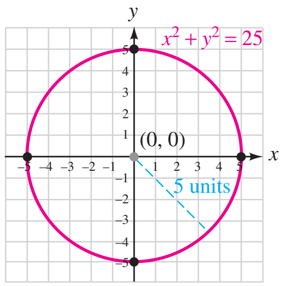
b. To find h and k, we will write $x^2 + y^2 = 25$ in the following way:

$$\begin{array}{c} (x - \mathbf{0})^2 + (y - \mathbf{0})^2 = \mathbf{25} \\ \uparrow \qquad\quad \uparrow \qquad \uparrow \\ (x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \boldsymbol{r^2} \end{array}$$

$h = 0$, $k = 0$, and $r^2 = 25$. Since the radius must be positive, $r = 5$.

The center of the circle is at (0, 0) and the radius is 5.

To plot four points on the circle, we move up, down, left, and right 5 units from the center. Then we draw a circle through the points to get the graph of $x^2 + y^2 = 25$, as shown.



c. To find h, we will write $x + 3$ as $x - (-3)$.

Standard form requires a minus symbol here.

$$\begin{array}{c} [x - (\mathbf{-3})]^2 + (y - \mathbf{0})^2 = \mathbf{12} \\ \uparrow \qquad\quad \uparrow \qquad \uparrow \\ (x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \boldsymbol{r^2} \end{array}$$

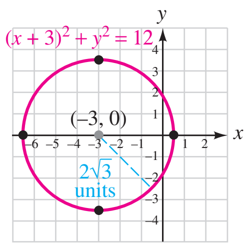
$h = -3$, $k = 0$, and $r^2 = 12$.

Since $r^2 = 12$, we have

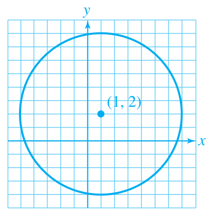
$$r = \pm\sqrt{12} = \pm 2\sqrt{3} \qquad \text{Use the square root property.}$$

Since the radius can't be negative, $r = 2\sqrt{3}$. The center of the circle is at $(-3, 0)$ and the radius is $2\sqrt{3}$.

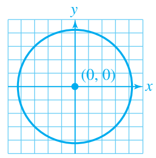
To plot four points on the circle, we move up, down, left, and right $2\sqrt{3} \approx 3.5$ units from the center. We then draw a circle through the points to get the graph of $(x + 3)^2 + y^2 = 12$, as shown on the right.



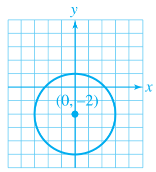
Teaching Example 1 Find the center and the radius of each circle and then graph it:
a. $(x - 1)^2 + (y - 2)^2 = 36$
b. $x^2 + y^2 = 18$
c. $x^2 + (y + 2)^2 = 9$
Answers:
a. $(1, 2)$, $r = 6$



b. $(0, 0)$, $r = 3\sqrt{2}$



c. $(0, -2)$, $r = 3$



3 Write the equation of a circle, given its center and radius.

Because a circle is determined by its center and radius, that information is all we need to know to write its equation.

EXAMPLE 2 Write the equation of the circle with radius 9 and center at $(6, -5)$.

Strategy We substitute 9 for r, 6 for h, and -5 for k in the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$.

WHY When writing the standard form, the center is represented by the ordered pair (h, k) and the radius as r.

Solution

$$(x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \boldsymbol{r}^2$$

$$(x - \mathbf{6})^2 + [y - (\mathbf{-5})]^2 = \mathbf{9}^2 \qquad \text{Substitute 6 for } h\text{, } -5 \text{ for } k\text{, and 9 for } r.$$

$$(x - 6)^2 + (y + 5)^2 = 9^2 \qquad \text{Write } y - (-5) \text{ as } y + 5.$$

If we express 9^2 as 81, we have

$$(x - 6)^2 + (y + 5)^2 = 81$$

Self Check 2

Write the equation of the circle with radius 10 and center at $(-7, 1)$. $(x + 7)^2 + (y - 1)^2 = 100$

Now Try **Problems 23, 27, and 31**

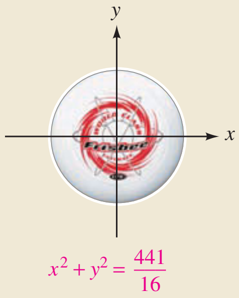
Teaching Example 2 Write the equation of the circle with radius 5 and center at $(-3, -2)$.
Answer:
$(x + 3)^2 + (y + 2)^2 = 25$

THINK IT THROUGH *Ultimate Frisbee*

"Ultimate Frisbee combines speed, grace, and powerful hurling with a grueling pace."

The Wall Street Journal

College students have tossed the plastic discs called Frisbees on campuses since the WHAM-O toy company first introduced them in 1958. Recently, the popularity of the Frisbee has been growing dramatically with a game called Ultimate Frisbee. Ultimate, as the players call it, is a high-endurance sport with few basic rules that combines the nonstop movement of soccer, the defensive strategies of basketball, and the passing of football.



The illustration shows a drawing of an official ultimate Frisbee centered on a rectangular coordinate system. Determine the diameter of the disc from the given equation. The units are inches. $\frac{21}{2}$ in. $= 10.5$ in.

4 Convert the general form of the equation of a circle to standard form.

In Example 2, the result was written in standard form: $(x - 6)^2 + (y + 5)^2 = 81$. If we square $x - 6$ and $y + 5$, we obtain a different form for the equation of the circle.

$$(x - 6)^2 + (y + 5)^2 = 9^2$$

$$x^2 - 12x + 36 + y^2 + 10y + 25 = 81 \quad \text{Square each binomial.}$$

$$x^2 - 12x + y^2 + 10y - 20 = 0 \quad \text{Subtract 81 from both sides. Combine like terms.}$$

$$x^2 + y^2 - 12x + 10y - 20 = 0 \quad \text{Rearrange the terms, writing the squared terms first.}$$

Success Tip This example illustrates an important fact: The equation of a circle contains both x^2 and y^2 terms on the same side of the equation with equal coefficients.

This result is written in the *general form of the equation of a circle.*

Equation of a Circle

The **general form of the equation of a circle** is

$$x^2 + y^2 + Dx + Ey + F = 0$$

We can convert from the general form to the standard form of the equation of a circle by completing the square.

EXAMPLE 3 Write the equation $x^2 + y^2 - 4x + 2y - 11 = 0$ in standard form and graph it.

Strategy We will rearrange the terms to write the equation in the form $x^2 - 4x + y^2 + 2y = 11$ and complete the square on x and y.

WHY Standard form contains the expressions $(x - h)^2$ and $(y - k)^2$. We can obtain a perfect-square trinomial that factors as $(x - 2)^2$ by completing the square on $x^2 - 4x$. We can complete the square on $y^2 + 2y$ to obtain an expression of the form $(y + 1)^2$.

Solution

To write the equation in standard form, we complete the square twice.

$$x^2 + y^2 - 4x + 2y - 11 = 0$$

$$x^2 - 4x \quad + y^2 + 2y \qquad = 11 \qquad \text{Write the x-terms together, the y-terms together, and add 11 to both sides.}$$

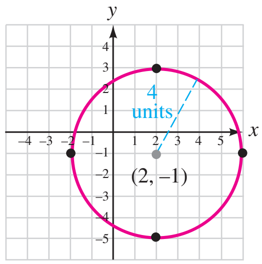
To complete the square on $x^2 - 4x$, we note that $\frac{1}{2}(-4) = -2$ and $(-2)^2 = \mathbf{4}$. To complete the square on $y^2 + 2y$, we note that $\frac{1}{2}(2) = 1$ and $1^2 = \mathbf{1}$. We add **4** and **1** to both sides of the equation.

$$x^2 - 4x + \mathbf{4} + y^2 + 2y + \mathbf{1} = 11 + \mathbf{4} + \mathbf{1}$$

$$(x - 2)^2 + (y + 1)^2 = 16 \qquad \text{Factor } x^2 - 4x + 4 \text{ and } y^2 + 2y + 1.$$

The equation can also be written as $(x - 2)^2 + (y + 1)^2 = 4^2$.

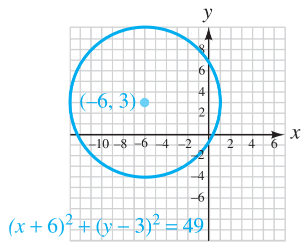
We can determine the circle's center and radius by comparing this equation to the standard form of the equation of a circle, $(x - h)^2 + (y - k)^2 = r^2$. We see that $h = 2$, $k = -1$, and $r = 4$. We can use the center, $(h, k) = (2, -1)$ and the radius $r = 4$, to graph the circle as shown on the right.



$x^2 + y^2 - 4x + 2y - 11 = 0$
or
$(x - 2)^2 + (y + 1)^2 = 16$

Self Check 3

Write the equation $x^2 + y^2 + 12x - 6y - 4 = 0$ in standard form and graph it.



Now Try **Problem 35**

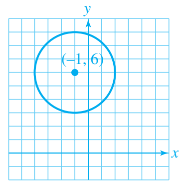
Self Check 3 Answer

$(x + 6)^2 + (y - 3)^2 = 49$

Teaching Example 3 Write the equation $x^2 + y^2 + 2x - 12y = -28$ in standard form and graph it.

Answer:

$(x + 1)^2 + (y - 6)^2 = 9$



Using Your CALCULATOR Graphing Circles

Since the graphs of circles fail the vertical line test, their equations do not represent functions. It is more difficult to use a graphing calculator to graph equations that are not functions. For example, to graph the circle described by $(x - 1)^2 + (y - 2)^2 = 4$, we must split the equation into two functions and graph each one separately. We begin by solving the equation for y.

$$(x - 1)^2 + (y - 2)^2 = 4$$

$$(y - 2)^2 = 4 - (x - 1)^2 \qquad \text{Subtract } (x - 1)^2 \text{ from both sides.}$$

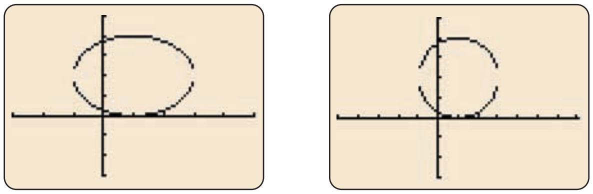
$$y - 2 = \pm\sqrt{4 - (x - 1)^2} \qquad \text{Use the square root property.}$$

$$y = 2 \pm \sqrt{4 - (x - 1)^2} \qquad \text{Add 2 to both sides.}$$

This equation defines two functions. If we graph

$$y = 2 + \sqrt{4 - (x - 1)^2} \quad \text{and} \quad y = 2 - \sqrt{4 - (x - 1)^2}$$

we get the distorted circle shown in figure (a) on the next page. To get a better circle, we can use the graphing calculator's square window feature, which gives an equal unit distance on both the x- and y-axes. (Press ZOOM, 5, ENTER.) Using this feature, we get the circle shown in figure (b). Sometimes the two arcs will not connect because of approximations made by the calculator at each endpoint.



(a) (b)

The graph of $y = 2 + \sqrt{4 - (x - 1)^2}$ is the top half of the circle.

The graph of $y = 2 - \sqrt{4 - (x - 1)^2}$ is the bottom half of the circle.

5 Solve problems involving circles.

Self Check 4

A landscape architect is designing a circular flower bed bounded by the circle $x^2 + y^2 = 36$ where x and y are measured in feet. Another circular flower bed in his design is bounded by

$$(x - 3)^2 + (y + 4)^2 = 16$$

Find the length of the sidewalk from the center of the first circle to the furthest edge of the second circle. 9 ft

Now Try **Problem 83**

Teaching Example 4 The broadcast area of a television station is bounded by the circle $x^2 + y^2 = 3{,}600$ where x and y are measured in miles. A translator station picks up the signal and transmits it from the center of a circular area bounded by

$$(x + 30)^2 + (y + 40)^2 = 900$$

Find the location of the translator and the greatest distance from the main transmitter that the signal can be received.
Answer:
50 miles from the station, 80 mi

EXAMPLE 4 ***Radio Translators*** The broadcast area of a television station is bounded by the circle $x^2 + y^2 = 3{,}600$, where x and y are measured in miles. A translator station picks up the signal and retransmits it from the center of a circular area bounded by

$$(x + 30)^2 + (y - 40)^2 = 1{,}600$$

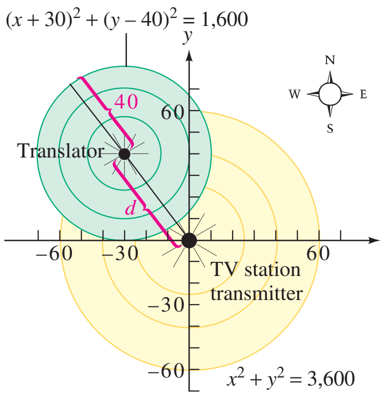
Find the location of the translator and the greatest distance from the main transmitter that the signal can be received.

Strategy Refer to the figure below. We will find two distances: the distance from the TV station transmitter to the translator and the distance from the translator to the outer edge of its coverage.

WHY The greatest distance of reception from the main transmitter is the sum of those two distances.

Solution

The coverage of the TV station is bounded by $x^2 + y^2 = 60^2$, a circle centered at the origin with a radius of 60 miles, as shown in yellow in the figure. Because the translator is at the center of the circle $(x + 30)^2 + (y - 40)^2 = 1{,}600$, it is located at $(-30, 40)$, a point 30 miles west and 40 miles north of the TV station. The radius of the translator's coverage is $\sqrt{1{,}600}$, or 40 miles.



As shown in the figure, the greatest distance of reception is the sum of d, the distance from the translator to the television station, and 40 miles, the radius of the translator's coverage.

To find d, we use the distance formula to find the distance between the origin, $(x_1, y_1) = (0, 0)$, and $(x_2, y_2) = (-30, 40)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula was introduced in Section 7.6.

$$d = \sqrt{(-30 - 0)^2 + (40 - 0)^2}$$

$$
\begin{aligned}
d &= \sqrt{(-30)^2 + 40^2} \\
&= \sqrt{900 + 1{,}600} \\
&= \sqrt{2{,}500} \\
&= 50
\end{aligned}
$$

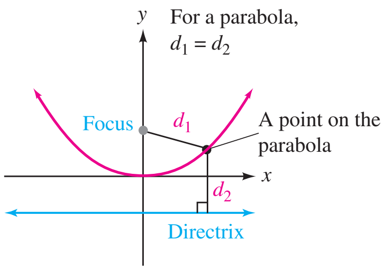
The translator is located 50 miles from the television station, and it broadcasts the signal 40 miles. The greatest reception distance from the main transmitter signal is, therefore, 50 + 40, or 90 miles.

6 Convert the general form of the equation of a parabola to standard form to graph it.

Another type of conic section is the parabola.

Definition of a Parabola

A **parabola** is the set of all points in a plane that are equidistant from a fixed point, called the **focus,** and a fixed line, called the **directrix.**



We have previously discussed parabolas whose graphs open upward or downward. Parabolas can also open to the right and to the left, but they do not define functions because their graphs fail the vertical line test.

The two general forms of the equation of a parabola are similar.

Equation of a Parabola

The **general forms of the equation of a parabola** are:

1. $y = ax^2 + bx + c$ The graph opens upward if $a > 0$ and downward if $a < 0$.

2. $x = ay^2 + by + c$ The graph opens to the right if $a > 0$ and to the left if $a < 0$.

Recall from Chapter 8 that equations written in the standard form $y = a(x - h)^2 + k$ represent parabolas with vertex at (h, k) and axis of symmetry $x = h$. They open upward when $a > 0$ and downward when $a < 0$.

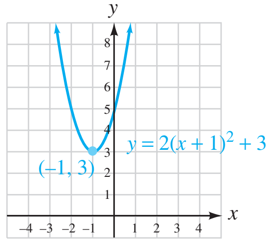
EXAMPLE 5 Write $y = -2x^2 + 12x - 15$ in standard form and graph it.

Strategy We will complete the square on x to write the equation in standard form, $y = a(x - h)^2 + k$.

WHY Standard form contains the expression $(x - h)^2$. We can obtain a perfect-square trinomial that factors into that form by completing the square on x.

Self Check 5

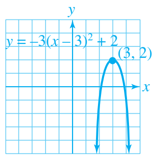
Write $y = 2x^2 + 4x + 5$ in standard form and graph it.



Now Try **Problem 39**

Self Check 5 Answer
$y = 2(x + 1)^2 + 3$

Teaching Example 5 Write $y = -3x^2 + 18x - 25$ in standard form and graph it.
Answer:
$y = -3(x - 3)^2 + 2$



Solution

Because the equation is not in standard form, the coordinates of the vertex are not obvious. To write the equation in standard form, we complete the square on x.

$$y = -2x^2 + 12x - 15$$

$$y = -2(x^2 - 6x \qquad) - 15 \qquad \text{Factor out } -2 \text{ from } -2x^2 + 12x.$$

This step adds $-2 \cdot 9$ or -18 to this side.

Add 18 to counteract the addition of -18.

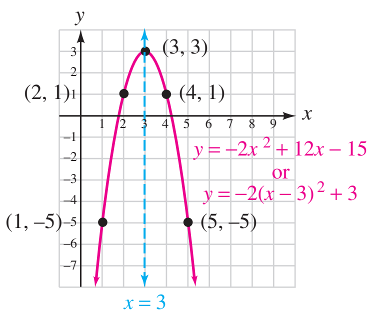
$$y = -2(x^2 - 6x + 9) - 15 + 18 \qquad \text{Complete the square on } x^2 - 6x.$$

$$y = -2(x - 3)^2 + 3 \qquad \text{Factor } x^2 - 6x + 9 \text{ and combine like terms.}$$

This equation is written in the form $y = a(x - h)^2 + k$, where $a = -2$, $h = 3$, and $k = 3$. Thus, the graph of the equation is a parabola that opens downward with vertex at (3, 3) and an axis of symmetry $x = 3$. We can construct a table of solutions and use symmetry to plot several points on the parabola. Then we draw a smooth curve through the points to get the graph of $y = -2x^2 + 12x - 15$, as shown below.

$y = -2x^2 + 12x - 15$		
x	y	
1	-5	→ $(1, -5)$
2	1	→ $(2, 1)$

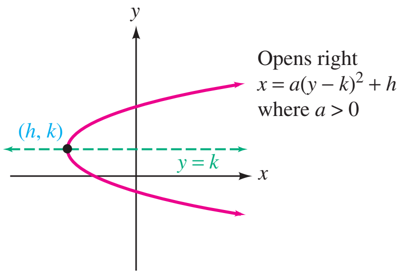
Because the x-coordinate of the vertex is 3, choose values for x that are close to 3 on the same side of the axis of symmetry.

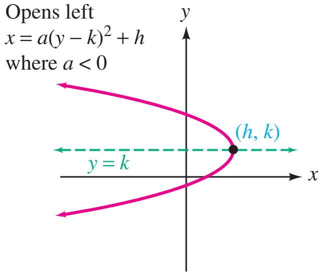


> ***Success Tip*** Recall that we can find the x-coordinate of the vertex using
>
> $$x = -\frac{b}{2a} = -\frac{12}{2(-2)} = 3$$
>
> To find the y-coordinate, substitute:
>
> $$\begin{aligned} y &= -2(3)^2 + 12(3) - 15 \\ &= 3 \end{aligned}$$
>
> The vertex is at (3, 3).

The *standard form* for the equation of a parabola that opens to the right or left is similar to $y = a(x - h)^2 + k$, except that the variables, x and y, exchange positions as do the constants, h and k.

Standard Form of the Equation of a Parabola





EXAMPLE 6

Graph: $x = \frac{1}{2}y^2$

Strategy We will compare the equation to the standard form of the equation of a parabola to find a, h, and k.

WHY Once we know these values, we can locate the vertex of the graph. We also know whether the parabola will open to the left or to the right.

Solution

This equation is written in the form $x = a(y - k)^2 + h$, where $a = \frac{1}{2}$, $k = 0$, and $h = 0$. The graph of the equation is a parabola that opens to the right with vertex at (0, 0) and an axis of symmetry $y = 0$.

To construct a table of solutions, we choose values of y and find their corresponding values of x. For example, if $y = 1$, we have

$$x = \frac{1}{2}y^2$$

$$x = \frac{1}{2}(1)^2 \qquad \text{Substitute 1 for } y.$$

$$x = \frac{1}{2}$$

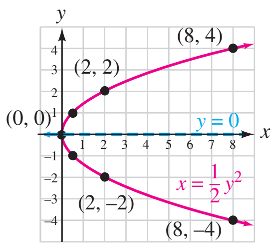
The point $\left(\frac{1}{2}, 1\right)$ is on the parabola.

We plot the ordered pairs from the table and use symmetry to plot three more points on the parabola. Then we draw a smooth curve through the points to get the graph of $x = \frac{1}{2}y^2$, as shown below.

$x = \frac{1}{2}y^2$	
x	y
$\frac{1}{2}$	1
2	2
8	4

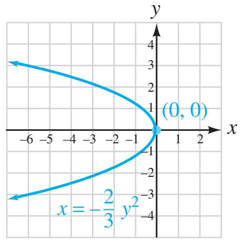
→ $\left(\frac{1}{2}, 1\right)$
→ (2, 2)
→ (8, 4)

Because the y-coordinate of the vertex is 0, choose values for y that are close to 0 on the same side of the axis of symmetry.



Self Check 6

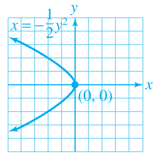
Graph: $x = -\frac{2}{3}y^2$



Now Try **Problem 43**

Teaching Example 6 Graph: $x = -\frac{1}{2}y^2$

Answer:



EXAMPLE 7

Write $x = -3y^2 - 12y - 13$ in standard form and graph it.

Strategy We will complete the square on y to write the equation in standard form, $x = a(y - k)^2 + h$.

WHY Standard form contains the expression $(y - k)^2$. We can obtain a perfect-square trinomial that factors into that form by completing the square on y.

Solution

$$x = -3y^2 - 12y - 13$$

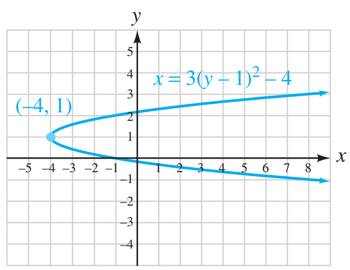
$$x = -3(y^2 + 4y \qquad) - 13 \qquad \text{Factor out } -3 \text{ from } -3y^2 - 12y.$$

$$x = -3(y^2 + 4y + 4) - 13 + 12 \qquad \text{Complete the square on } y^2 + 4y. \text{ Then add 12 to the right side to counteract } -3 \cdot 4 = -12.$$

$$x = -3(y + 2)^2 - 1 \qquad \text{Factor } y^2 + 4y + 4 \text{ and combine like terms.}$$

Self Check 7

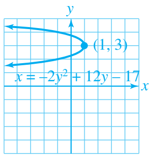
Write $x = 3y^2 - 6y - 1$ in standard form and graph it.



Now Try **Problem 49**

Self Check 7 Answer
$x = 3(y - 1)^2 - 4$

Teaching Example 7 Write $x = -2y^2 + 12y - 17$ in standard form and graph it.
Answer:
$x = -2(y - 3)^2 + 1$



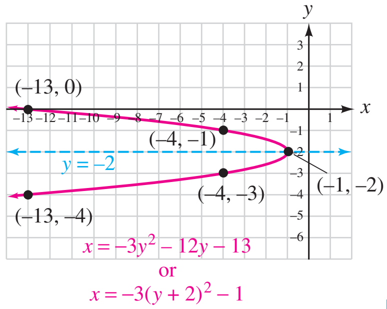
This equation is in the standard form $x = a(y - k)^2 + h$, where $a = -3$, $k = -2$, and $h = -1$. The graph of the equation is a parabola that opens to the left with vertex at $(-1, -2)$ and an axis of symmetry $y = -2$.

We can construct a table of solutions and use symmetry to plot several points on the parabola. Then we draw a smooth curve through the points to get the graph of $x = -3y^2 - 12y - 13$, as shown below.

$x = -3y^2 - 12y - 13$
or
$x = -3(y + 2)^2 - 1$

x	y	
-4	-1	$\rightarrow (-4, -1)$
-13	0	$\rightarrow (-13, 0)$

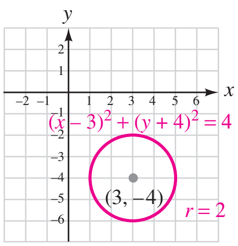
Choose values for *y*, and find the corresponding *x*-values.



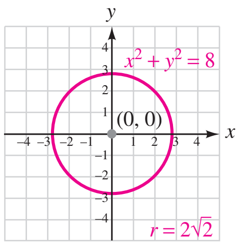
Success Tip The equation of a circle contains an x^2 and a y^2 term. The equation of a parabola has either an x^2 term or a y^2 term, but not both.

ANSWERS TO SELF CHECKS

1. a.

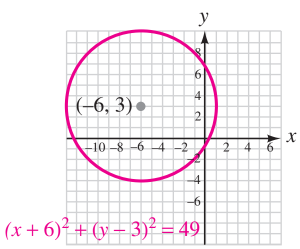


b.

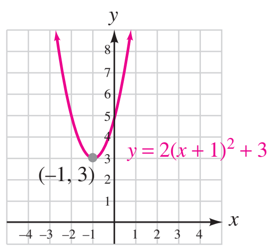


2. $(x + 7)^2 + (y - 1)^2 = 100$

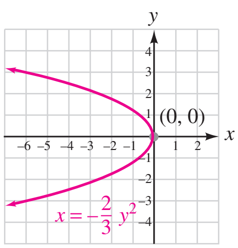
3.



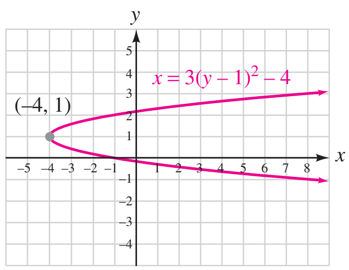
4. 9 ft **5.**



6.



7.



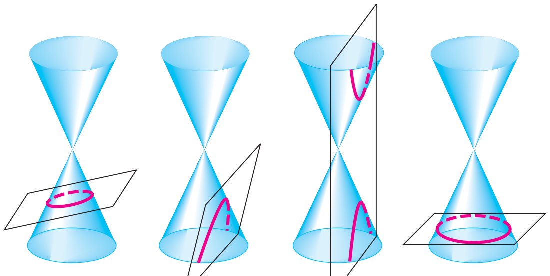
SECTION 10.1 STUDY SET

VOCABULARY

Fill in the blanks.

▶ **1.** The curves formed by the intersection of a plane with an infinite right-circular cone are called conic sections.

▶ **2.** Give the name of each curve shown below.



ellipse parabola hyperbola circle

▶ **3.** A circle is the set of all points in a plane that are a fixed distance from a fixed point called its center. The fixed distance is called the radius.

4. A parabola is the set of all points in a plane that are equidistant from a fixed point and a fixed line.

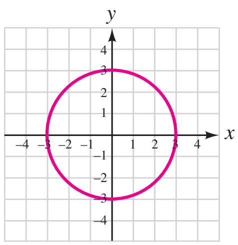
CONCEPTS

5. a. Write the standard form of the equation of a circle. $(x - h)^2 + (y - k)^2 = r^2$

b. Write the standard form of the equation of a circle with the center at the origin. $x^2 + y^2 = r^2$

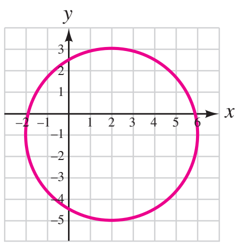
6. a. Find the center and the radius of the circle graphed on the right. $(0, 0)$, $r = 3$

b. Write the equation of the circle. $x^2 + y^2 = 9$



7. a. Find the center and the radius of the circle graphed on the right. $(2, -1)$, $r = 4$

b. Write the equation of the circle. $(x - 2)^2 + (y + 1)^2 = 16$



▶ **8.** Fill in the blanks. To complete the square on $x^2 + 2x$ and on $y^2 - 6y$, what numbers must be added to each side of the equation?

$$x^2 + 2x + y^2 - 6y = 2$$
$$x^2 + 2x + \boxed{1} + y^2 - 6y + \boxed{9} = 2 + \boxed{1} + \boxed{9}$$

9. a. What is the standard form of the equation of a parabola opening upward or downward?
$y = a(x - h)^2 + k$

b. What is the standard form of the equation of a parabola opening to the right or left?
$x = a(y - k)^2 + h$

▶ **10.** Fill in the blanks.

a. To complete the square on the right side, what should be factored from the first two terms?

$$x = 4y^2 + 16y + 9$$
$$x = \boxed{4}(y^2 + 4y \qquad) + 9$$

b. To complete the square on $y^2 + 4y$, what should be added within the parentheses, and what should be subtracted outside the parentheses?

$$x = 4\left(y^2 + 4y + \boxed{4}\right) + 9 - \boxed{16}$$

11. Determine whether the graph of each equation is a circle or a parabola.

a. $x^2 + y^2 - 6x + 8y - 10 = 0$
circle

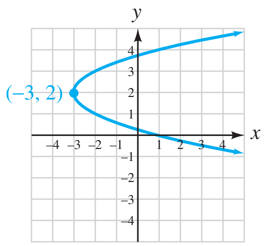
b. $y^2 - 2x + 3y - 9 = 0$
parabola

c. $x^2 + 5x - y = 0$
parabola

d. $x^2 + 12x + y^2 = 0$
circle

12. Draw a parabola using the given facts.

- Opens right
- Vertex $(-3, 2)$
- Passes through $(-2, 1)$
- x-intercept $(1, 0)$



NOTATION

▶ **13.** Find h, k, and r: $(x - 6)^2 + (y + 2)^2 = 9$ $6, -2, 3$

▶ **14. a.** Find a, h, and k: $y = 6(x - 5)^2 - 9$ $6, 5, -9$

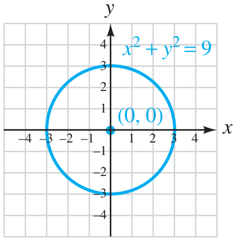
b. Find a, h, and k: $x = -3(y + 2)^2 + 1$ $-3, 1, -2$

▶ Selected exercises available online at
www.webassign.net/brookscole

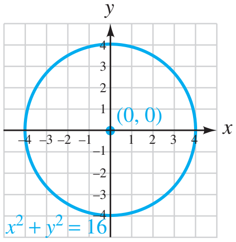
GUIDED PRACTICE

Find the center and radius of each circle and graph it. **See Example 1.**

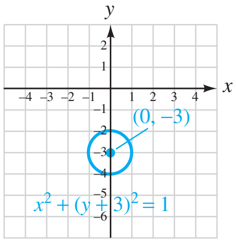
15. $x^2 + y^2 = 9$
$(0, 0)$, $r = 3$



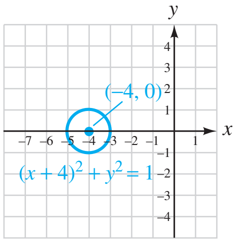
16. $x^2 + y^2 = 16$
$(0, 0)$, $r = 4$



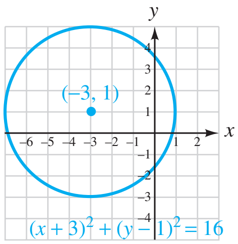
▶ **17.** $x^2 + (y + 3)^2 = 1$
$(0, -3)$, $r = 1$



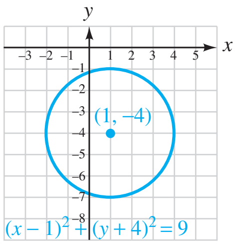
▶ **18.** $(x + 4)^2 + y^2 = 1$
$(-4, 0)$, $r = 1$



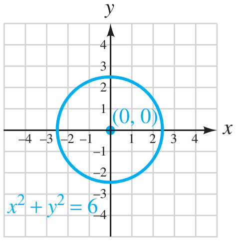
▶ **19.** $(x + 3)^2 + (y - 1)^2 = 16$
$(-3, 1)$, $r = 4$



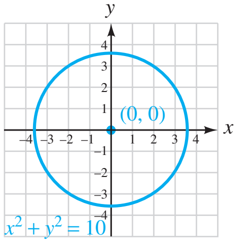
▶ **20.** $(x - 1)^2 + (y + 4)^2 = 9$
$(1, -4)$, $r = 3$



21. $x^2 + y^2 = 6$
$(0, 0)$, $r = \sqrt{6} \approx 2.4$



22. $x^2 + y^2 = 10$
$(0, 0)$, $r = \sqrt{10} \approx 3.2$



Write the equation of a circle in standard form with the following properties. **See Example 2.**

23. Center at the origin, radius 1
$x^2 + y^2 = 1$

▶ **24.** Center at the origin, radius 4
$x^2 + y^2 = 16$

▶ **25.** Center at (6, 8), radius 5
$(x - 6)^2 + (y - 8)^2 = 25$

▶ **26.** Center at (5, 3), radius 2
$(x - 5)^2 + (y - 3)^2 = 4$

27. Center at $(-2, 6)$, radius 12
$(x + 2)^2 + (y - 6)^2 = 144$

▶ **28.** Center at $(5, -4)$, radius 6
$(x - 5)^2 + (y + 4)^2 = 36$

▶ **29.** Center at (0, 0), radius $\frac{1}{4}$
$x^2 + y^2 = \frac{1}{16}$

▶ **30.** Center at (0, 0), radius $\frac{1}{3}$
$x^2 + y^2 = \frac{1}{9}$

31. Center at $\left(\frac{2}{3}, -\frac{7}{8}\right)$, radius $\sqrt{2}$
$\left(x - \frac{2}{3}\right)^2 + \left(y + \frac{7}{8}\right)^2 = 2$

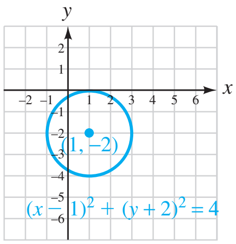
32. Center at $(-0.7, -0.2)$, radius $\sqrt{11}$
$(x + 0.7)^2 + (y + 0.2)^2 = 11$

33. Center at the origin, diameter $4\sqrt{2}$
$x^2 + y^2 = 8$

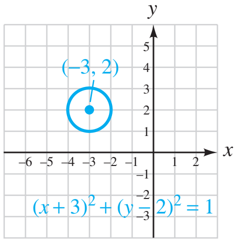
▶ **34.** Center at the origin, diameter $8\sqrt{3}$
$x^2 + y^2 = 48$

Write each equation of a circle in standard form and graph it. Give the coordinates of its center and give the radius. **See Example 3.**

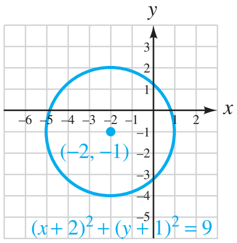
▶ **35.** $x^2 + y^2 - 2x + 4y = -1$
$(x - 1)^2 + (y + 2)^2 = 4$; $(1, -2)$, $r = 2$



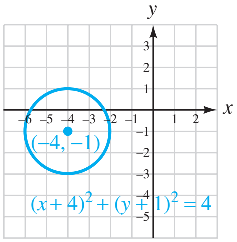
▶ **36.** $x^2 + y^2 + 6x - 4y = -12$
$(x + 3)^2 + (y - 2)^2 = 1$; $(-3, 2)$, $r = 1$



▶ **37.** $x^2 + y^2 + 4x + 2y = 4$
$(x + 2)^2 + (y + 1)^2 = 9$; $(-2, -1)$, $r = 3$

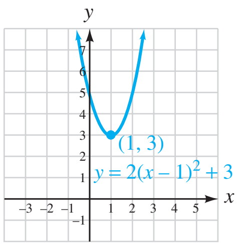


▶ **38.** $x^2 + y^2 + 8x + 2y = -13$
$(x + 4)^2 + (y + 1)^2 = 4$; $(-4, -1)$, $r = 2$

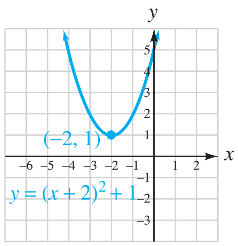


Write each equation of a parabola in standard form and graph it. Give the coordinates of the vertex. See Example 5.

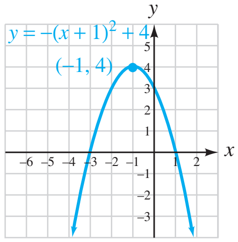
39. $y = 2x^2 - 4x + 5$
$y = 2(x - 1)^2 + 3$,
vertex: $(1, 3)$



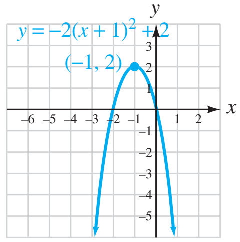
▶ **40.** $y = x^2 + 4x + 5$
$y = (x + 2)^2 + 1$,
vertex: $(-2, 1)$



▶ **41.** $y = -x^2 - 2x + 3$
$y = -(x + 1)^2 + 4$,
vertex: $(-1, 4)$

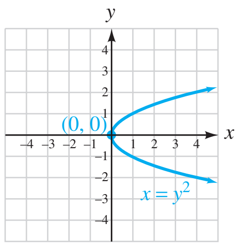


42. $y = -2x^2 - 4x$
$y = -2(x + 1)^2 + 2$,
vertex: $(-1, 2)$

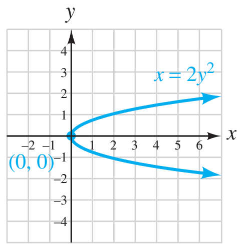


Graph each equation of a parabola. Give the coordinates of the vertex. See Example 6.

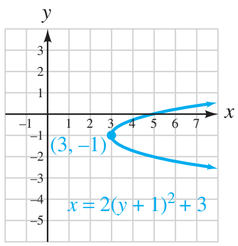
43. $x = y^2$
vertex: $(0, 0)$



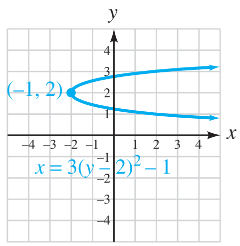
▶ **44.** $x = 2y^2$
vertex: $(0, 0)$



▶ **45.** $x = 2(y + 1)^2 + 3$
vertex: $(3, -1)$

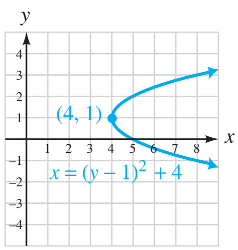


46. $x = 3(y - 2)^2 - 1$
vertex: $(-1, 2)$

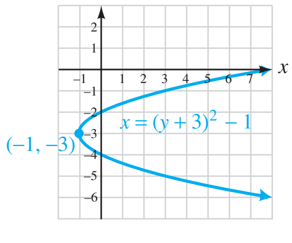


Write each equation of a parabola in standard form and graph it. Give the coordinates of the vertex. See Example 7.

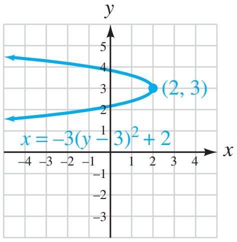
47. $x = y^2 - 2y + 5$
$x = (y - 1)^2 + 4$,
vertex: $(4, 1)$



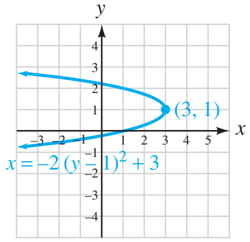
▶ **48.** $x = y^2 + 6y + 8$
$x = (y + 3)^2 - 1$,
vertex: $(-1, -3)$



▶ **49.** $x = -3y^2 + 18y - 25$
$x = -3(y - 3)^2 + 2$,
vertex: (2, 3)

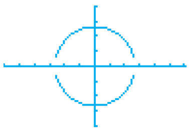


50. $x = -2y^2 + 4y + 1$
$x = -2(y - 1)^2 + 3$,
vertex: (3, 1)

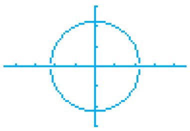


*Use a graphing calculator to graph each equation. (***Hint:** *Solve for y and graph two functions.)* **See Using Your Calculator: Graphing Circles.**

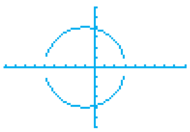
51. $x^2 + y^2 = 7$



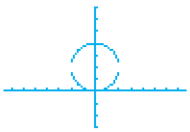
▶ **52.** $x^2 + y^2 = 5$



53. $(x + 1)^2 + y^2 = 16$

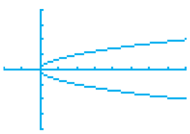


54. $x^2 + (y - 2)^2 = 4$

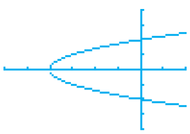


*Use a graphing calculator to graph each equation. (***Hint:** *Solve for y and graph two functions when necessary.)*

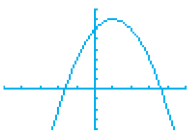
55. $x = 2y^2$



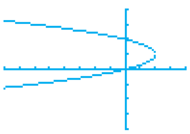
56. $x = y^2 - 4$



57. $x^2 - 2x + y = 6$



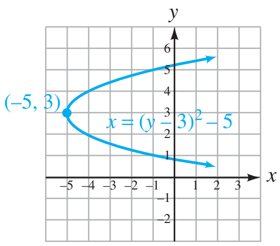
58. $x = -2(y - 1)^2 + 2$



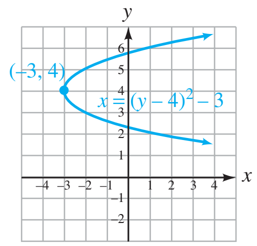
TRY IT YOURSELF

Write each equation in standard form, if it is not already so, and graph it. If the graph is a circle, give the coordinates of its center and its radius. If the graph is a parabola, give the coordinates of its vertex.

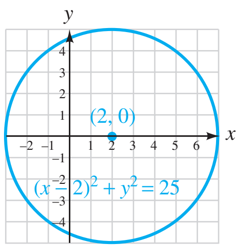
59. $x = y^2 - 6y + 4$
$x = (y - 3)^2 - 5$,
vertex: $(-5, 3)$



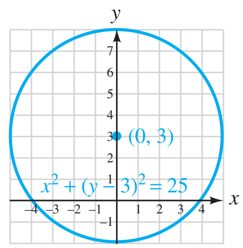
60. $x = y^2 - 8y + 13$
$x = (y - 4)^2 - 3$,
vertex: $(-3, 4)$



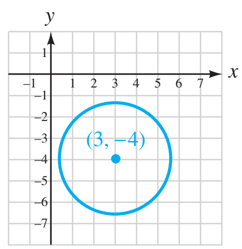
61. $(x - 2)^2 + y^2 = 25$
(2, 0), $r = 5$



62. $x^2 + (y - 3)^2 = 25$
(0, 3), $r = 5$

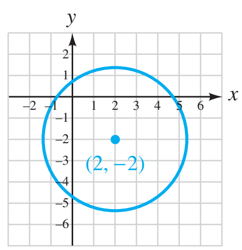


63. $x^2 + y^2 - 6x + 8y + 18 = 0$
$(x - 3)^2 + (y + 4)^2 = 7$, $(3, -4)$, $r = \sqrt{7} \approx 2.6$



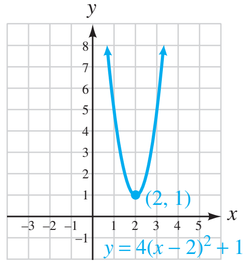
$(x - 3)^2 + (y + 4)^2 = 7$

▶ **64.** $x^2 + y^2 - 4x + 4y - 3 = 0$
$(x - 2)^2 + (y + 2)^2 = 11$, $(2, -2)$, $r = \sqrt{11} \approx 3.3$

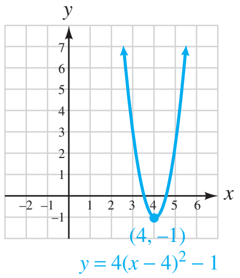


$(x - 2)^2 + (y + 2)^2 = 11$

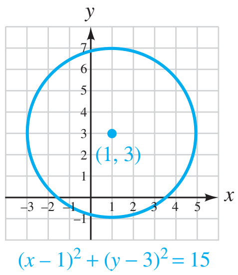
65. $y = 4x^2 - 16x + 17$
$y = 4(x-2)^2 + 1,$
vertex: $(2, 1)$



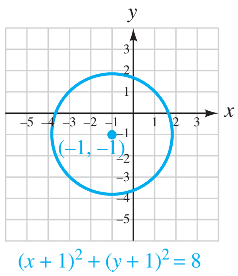
66. $y = 4x^2 - 32x + 63$
$y = 4(x-4)^2 - 1,$
vertex: $(4, -1)$



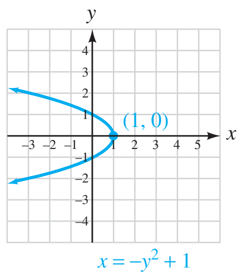
67. $(x-1)^2 + (y-3)^2 = 15$
$(1, 3), r = \sqrt{15} \approx 3.9$



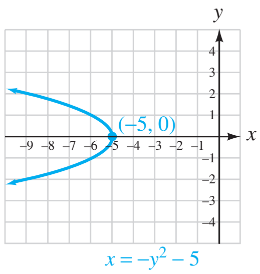
68. $(x+1)^2 + (y+1)^2 = 8$
$(-1, -1), r = 2\sqrt{2} \approx 2.8$



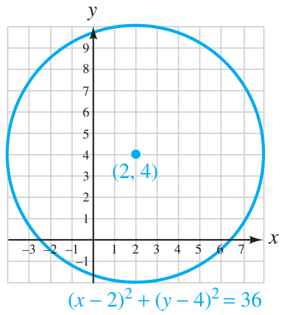
69. $x = -y^2 + 1$
vertex: $(1, 0)$



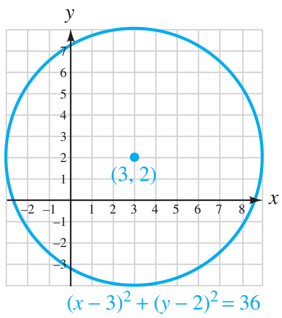
70. $x = -y^2 - 5$
vertex: $(-5, 0)$



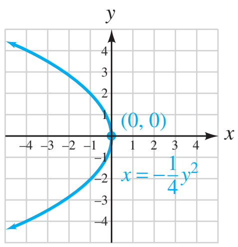
71. $(x-2)^2 + (y-4)^2 = 36$
$(2, 4), r = 6$



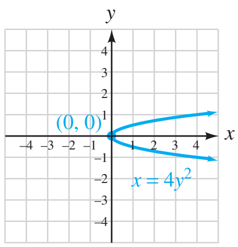
72. $(x-3)^2 + (y-2)^2 = 36$
$(3, 2), r = 6$



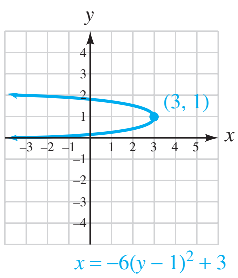
▶ **73.** $x = -\frac{1}{4}y^2$
vertex: $(0, 0)$



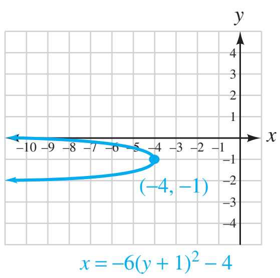
74. $x = 4y^2$
vertex: $(0, 0)$



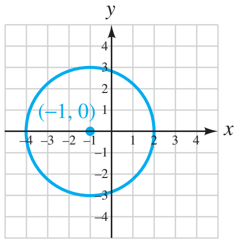
75. $x = -6(y-1)^2 + 3$
vertex: $(3, 1)$



▶ **76.** $x = -6(y+1)^2 - 4$
vertex: $(-4, -1)$

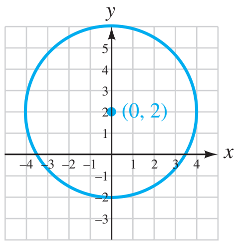


77. $x^2 + y^2 + 2x - 8 = 0$
$(x + 1)^2 + y^2 = 9, (-1, 0), r = 3$



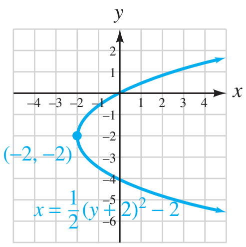
$(x+1)^2 + y^2 = 9$

78. $x^2 + y^2 - 4y = 12$
$x^2 + (y - 2)^2 = 16, (0, 2), r = 4$



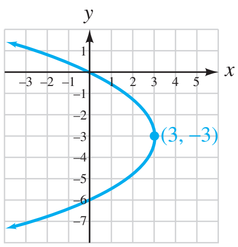
$x^2 + (y - 2)^2 = 16$

79. $x = \frac{1}{2}y^2 + 2y$
$x = \frac{1}{2}(y + 2)^2 - 2,$
vertex: $(-2, -2)$



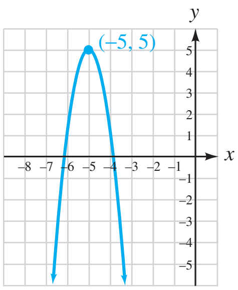
$x = \frac{1}{2}(y+2)^2 - 2$

80. $x = -\frac{1}{3}y^2 - 2y$
$x = -\frac{1}{3}(y + 3)^2 + 3,$
vertex: $(3, -3)$



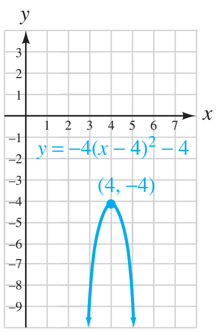
$x = -\frac{1}{3}(y + 3)^2 + 3$

81. $y = -4(x + 5)^2 + 5$
vertex: $(-5, 5)$



$y = -4(x + 5)^2 + 5$

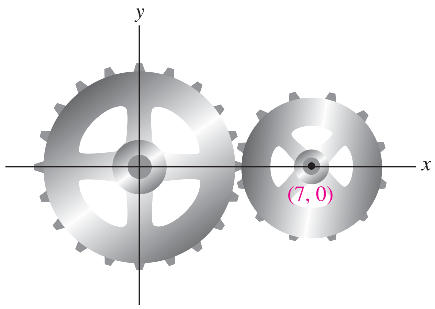
82. $y = -4(x - 4)^2 - 4$
vertex: $(4, -4)$



APPLICATIONS

83. BROADCAST RANGES Radio stations applying for licensing may not use the same frequency if their broadcast areas overlap. One station's coverage is bounded by $x^2 + y^2 - 8x - 20y + 16 = 0$, and the other's by $x^2 + y^2 + 2x + 4y - 11 = 0$. May they be licensed for the same frequency? no

84. MESHING GEARS For design purposes, the large gear is described by the circle $x^2 + y^2 = 16$. The smaller gear is a circle centered at (7, 0) and tangent to the larger circle. Find the equation of the smaller gear. $(x - 7)^2 + y^2 = 9$

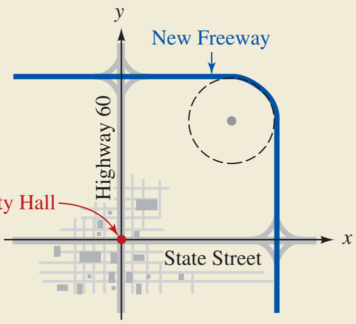


85. Suppose you are a traffic engineer and you are designing two sections of a new freeway so that they join with a curve that is one-quarter of a circle, as shown. The equation of the circle is $x^2 + y^2 - 10x - 12y + 52 = 0$, where distances are measured in miles.

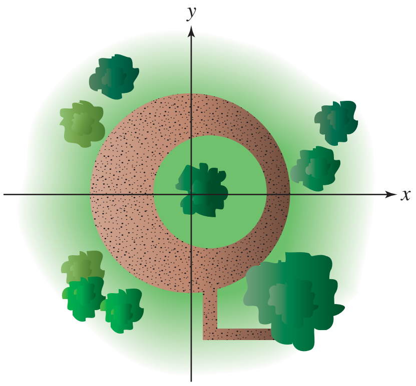
from Campus to Careers
Traffic Engineer

a. How far from City Hall will the new freeway intersect State Street? 8 mi

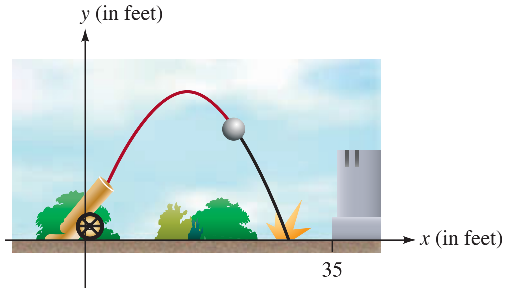
b. How far from City Hall will the new freeway intersect Highway 60? 9 mi



86. WALKWAYS The walkway shown is bounded by the two circles $x^2 + y^2 = 2{,}500$ and $(x - 10)^2 + y^2 = 900$, measured in feet. Find the largest and the smallest width of the walkway. 30 ft and 10 ft

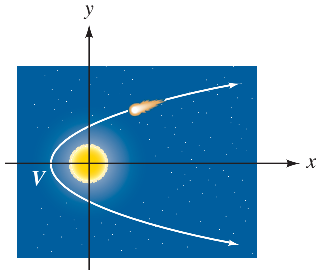


87. PROJECTILES The cannonball in the illustration follows the parabolic path $y = 30x - x^2$. How far short of the castle does it land? 5 ft

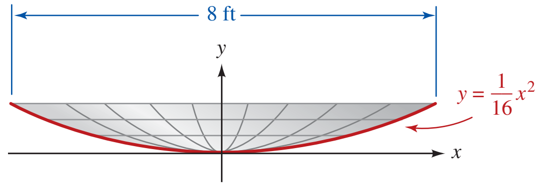


88. PROJECTILES In Exercise 87, how high does the cannonball get? 225 ft

89. COMETS If the orbit of the comet is approximated by the equation $2y^2 - 9x = 18$, how far is it from the sun at the vertex V of the orbit? Distances are measured in astronomical units (AU). 2 AU



90. SATELLITE ANTENNAS The cross section of the satellite antenna in the illustration is a parabola given by the equation $y = \frac{1}{16}x^2$, with distances measured in feet. If the dish is 8 feet wide, how deep is it? 1 ft



WRITING

91. Explain how to decide from its equation whether the graph of a parabola opens up, down, right, or left.

92. From the equation of a circle, explain how to determine the radius and the coordinates of the center.

93. On the day of an election, the following warning was posted in front of a school. Explain what it means.

> *No electioneering within a 1,000-foot radius of this polling place.*

94. What is meant by the *turning radius* of a truck?

REVIEW

Solve each equation.

95. $|3x - 4| = 11$ $\quad 5, -\frac{7}{3}$

96. $\left|\frac{4 - 3x}{5}\right| = 12$ $\quad \frac{64}{3}, -\frac{56}{3}$

97. $|3x + 4| = |5x - 2|$ $\quad 3, -\frac{1}{4}$

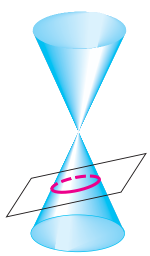
98. $|6 - 4x| = |x + 2|$ $\quad \frac{4}{5}, \frac{8}{3}$

SECTION 10.2
The Ellipse

Objectives

1. Define an ellipse.
2. Graph ellipses centered at the origin.
3. Graph ellipses centered at (h, k).
4. Solve problems involving ellipses.

A third conic section is an oval-shaped curve called an *ellipse.* Ellipses can be nearly round and look almost like a circle, or they can be long and narrow. In this section, we will learn how to construct ellipses and how to graph equations that represent ellipses.



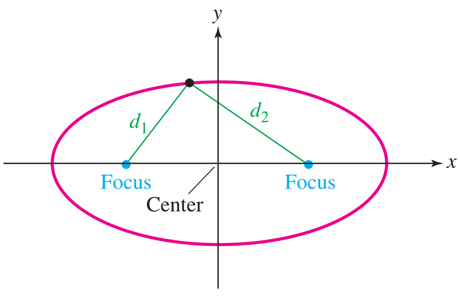
1 Define an ellipse.

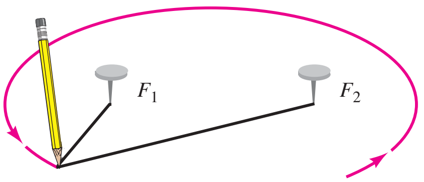
To define a circle, we considered a fixed distance from a fixed point. The definition of an ellipse involves *two* distances from *two* fixed points.

Definition of an Ellipse

An **ellipse** is the set of all points in a plane for which the sum of the distances from two fixed points is a constant.

The figure below illustrates that any point on an ellipse is a constant distance $d_1 + d_2$ from two fixed points, each of which is called a **focus.** Midway between the **foci** is the **center** of the ellipse.





We can construct an ellipse by placing two thumbtacks fairly close together to serve as foci. We then tie each end of a piece of string to a thumbtack, catch the loop with the point of a pencil, and (keeping the string taut) draw the ellipse.

2 Graph ellipses centered at the origin.

The definition of an ellipse can be used to develop the standard equation of an ellipse.

Equation of an Ellipse Centered at the Origin

The **standard form of the equation of an ellipse** that is symmetric with respect to both axes and centered at (0, 0) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a > 0 \text{ and } b > 0$$

To graph an ellipse centered at the origin, it is helpful to know the intercepts of the graph. To find the x-intercepts of the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we let $y = 0$ and solve for x.

$$\frac{x^2}{a^2} + \frac{0^2}{b^2} = 1 \qquad \text{Substitute 0 for } y.$$

$$\frac{x^2}{a^2} + 0 = 1 \qquad \text{Simplify: } \frac{0^2}{b^2} = 0.$$

$$x^2 = a^2 \qquad \text{Simplify and multiply both sides by } a^2.$$

$$x = \pm a \qquad \text{Use the square root property.}$$

The x-intercepts are $(a, 0)$ and $(-a, 0)$.

To find the y-intercepts of the graph, we can let $x = 0$ and solve for y.

$$\frac{0^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Substitute 0 for } x.$$

$$0 + \frac{y^2}{b^2} = 1 \qquad \text{Simplify: } \frac{0^2}{a^2} = 0.$$

$$y^2 = b^2 \qquad \text{Simplify and multiply both sides by } b^2.$$

$$y = \pm b \qquad \text{Use the square root property.}$$

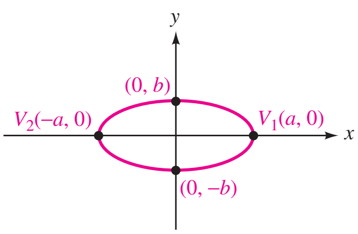
The y-intercepts are $(0, b)$ and $(0, -b)$.

In general, we have the following results.

The Intercepts of an Ellipse

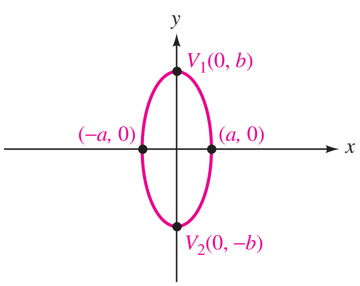
The graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, centered at the origin, with x-intercepts $(a, 0)$ and $(-a, 0)$ and y-intercepts $(0, b)$ and $(0, -b)$.

For $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$, the ellipse is horizontal, as shown in figure (a). If $b > a$, the ellipse is vertical, as shown in figure (b). The points V_1 and V_2 are called the **vertices** of the ellipse. The line segment joining the vertices is called the **major axis,** and its midpoint is called the **center** of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis at the center is called the **minor axis** of the ellipse.



Horizontal ellipse

(a)



Vertical ellipse

(b)

EXAMPLE 1

Graph: $\frac{x^2}{36} + \frac{y^2}{9} = 1$

Strategy This equation is in standard $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ form. We will identify a and b.

WHY Once we know a and b, we can determine the intercepts of the graph of the ellipse.

Solution

The color highlighting shows how to compare the given equation to the standard form to find a and b.

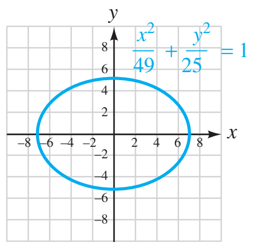
$$\frac{x^2}{36} + \frac{y^2}{9} = 1 \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $a^2 = 36$, it follows that $a = 6$.

Since $b^2 = 9$, it follows that $b = 3$.

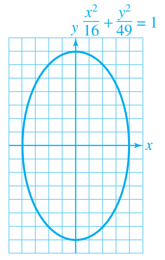
Self Check 1

Graph: $\frac{x^2}{49} + \frac{y^2}{25} = 1$

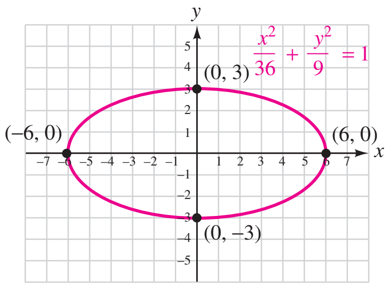


Now Try Problem 17

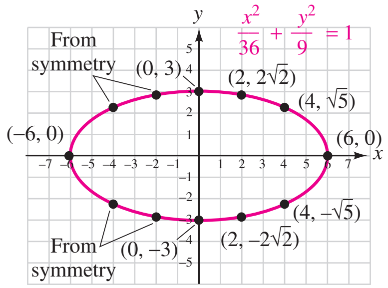
Teaching Example 1 Graph:
$\frac{x^2}{16} + \frac{y^2}{49} = 1$
Answer:



The x-intercepts are $(a, 0)$ and $(-a, 0)$, or $(6, 0)$ and $(-6, 0)$. The y-intercepts are $(0, b)$ and $(0, -b)$, or $(0, 3)$ and $(0, -3)$. Using these four points as a guide, we draw an oval-shaped curve through them, as shown in figure (a). The result is a horizontal ellipse.



(a)



(b)

To increase the accuracy of the graph, we can find additional ordered pairs that satisfy the equation and plot them. For example, if $x = 2$, we have

$$\frac{\mathbf{2}^2}{36} + \frac{y^2}{9} = 1 \quad \text{Substitute 2 for } x \text{ in the equation of the ellipse.}$$

$$\mathbf{36}\left(\frac{4}{36} + \frac{y^2}{9}\right) = \mathbf{36}(1) \quad \text{To clear the fractions, multiply both sides by the LCD, 36.}$$

$$4 + 4y^2 = 36 \quad \text{Distribute the multiplication by 36 and simplify.}$$

$$y^2 = 8 \quad \text{Subtract 4 from both sides and divide both sides by 4.}$$

$$y = \pm\sqrt{8} \quad \text{Use the square root property.}$$

$$y = \pm 2\sqrt{2} \quad \text{Simplify the radical.}$$

Since two values of y, $2\sqrt{2}$ and $-2\sqrt{2}$, correspond to the x-value 2, we have found two points on the ellipse: $\left(2, 2\sqrt{2}\right)$ and $\left(2, -2\sqrt{2}\right)$.

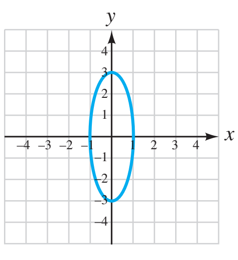
In a similar way, we can find the corresponding values of y for the x-value 4. In figure (b) we record these ordered pairs in a table, plot them, use symmetry with respect to the y-axis to plot four other points, and draw the graph of the ellipse.

$\frac{x^2}{36} + \frac{y^2}{9} = 1$		
x	y	
2	$\pm 2\sqrt{2}$	→ $(2, \pm 2\sqrt{2})$
4	$\pm\sqrt{5}$	→ $(4, \pm\sqrt{5})$

Approximate the radicals to graph.

Self Check 2

Graph: $9x^2 + y^2 = 9$



$\frac{x^2}{1} + \frac{y^2}{9} = 1$

Now Try **Problem 21**

EXAMPLE 2

Graph: $16x^2 + y^2 = 16$

Strategy We will write the equation in standard $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ form.

WHY When the equation is in standard form, we will be able to identify the center and the intercepts of the graph of the ellipse.

Solution

The given equation is not in standard form. To write it in standard form with 1 on the right side, we divide both sides by 16.

$$16x^2 + y^2 = 16$$

$$\frac{16x^2}{\mathbf{16}} + \frac{y^2}{\mathbf{16}} = \frac{16}{\mathbf{16}} \quad \text{Divide both sides by 16.}$$

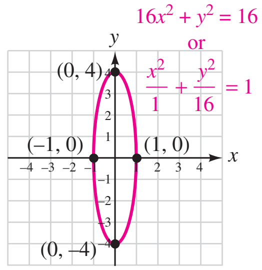
$$\frac{x^2}{1} + \frac{y^2}{16} = 1 \quad \text{Simplify: } \frac{16x^2}{16} = x^2 = \frac{x^2}{1} \text{ and } \frac{16}{16} = 1.$$

Success Tip Although the term $\frac{16x^2}{16}$ simplifies to x^2, we write it as the fraction $\frac{x^2}{1}$ so that it has the form $\frac{x^2}{a^2}$.

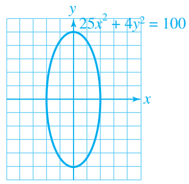
To determine a and b, we can write the equation in the form

$$\frac{x^2}{1^2} + \frac{y^2}{4^2} = 1 \quad \text{To find } a \text{, write 1 as } 1^2\text{. To find } b\text{, write 16 as } 4^2\text{.}$$

Since a^2 (the denominator of x^2) is 1^2, it follows that $a = 1$, and since b^2 (the denominator of y^2) is 4^2, it follows that $b = 4$. Thus, the x-intercepts of the graph are $(1, 0)$ and $(-1, 0)$ and the y-intercepts are $(0, 4)$ and $(0, -4)$. We use these four points as guides to sketch the graph of the ellipse, as shown. The result is a vertical ellipse.



Teaching Example 2 Graph: $25x^2 + 4y^2 = 100$
Answer:



3 Graph ellipses centered at (h, k).

Not all ellipses are centered at the origin. As with the graphs of circles and parabolas, the graph of an ellipse can be translated horizontally and vertically.

The Equation of an Ellipse Centered at (h, k)

The **standard form of the equation of a horizontal or vertical ellipse** centered at (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{where } a > 0 \text{ and } b > 0$$

For a horizontal ellipse, a is the distance from the center to a vertex. For a vertical ellipse, b is the distance from the center to a vertex.

EXAMPLE 3

Graph: $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{25} = 1$

Strategy The equation is in standard $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ form. We will identify h, k, a, and b.

WHY If we know h, k, a, and b, we can graph the ellipse.

Solution

To determine h, k, a, and b, we write the equation in the form

$$\frac{(x-2)^2}{4^2} + \frac{[y-(-3)]^2}{5^2} = 1$$

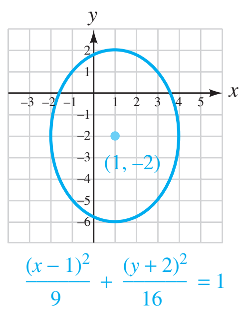
To find k, write $y + 3$ as $y - (-3)$.
To find a, write 16 as 4^2. To find b, write 25 as 5^2.

We find the center of the ellipse in the same way we would find the center of a circle, by examining $(x - 2)^2$ and $(y + 3)^2$. Since $h = 2$ and $k = -3$, this is the equation of an ellipse centered at $(h, k) = (2, -3)$. From the denominators, 4^2 and 5^2, we find that $a = 4$ and $b = 5$. Because $b > a$, it is a vertical ellipse.

We first plot the center, as shown on the next page. Since b is the distance from the center to a vertex for a vertical ellipse, we can locate the vertices by counting 5 units above and 5 units below the center. The vertices are the points $(2, 2)$ and $(2, -8)$.

Self Check 3

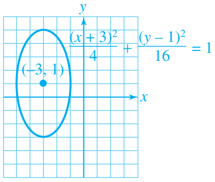
Graph: $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{16} = 1$



Now Try Problem 25

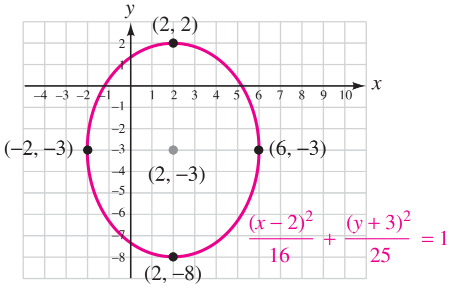
Teaching Example 3 Graph: $\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16}=1$

Answer:



To locate two more points on the ellipse, we use the fact that a is 4 and count 4 units to the left and to the right of the center. We see that the points $(-2, -3)$ and $(6, -3)$ are also on the graph.

Using these four points as guides, we draw the graph shown below.



Using Your CALCULATOR **Graphing Ellipses**

To use a graphing calculator to graph the equation from Example 3,

$$\frac{(x-2)^2}{16}+\frac{(y+3)^2}{25}=1$$

we clear the equation of fractions and solve for y.

$$25(x-2)^2+16(y+3)^2=400$$ Multiply both sides by 400.

$$16(y+3)^2=400-25(x-2)^2$$ Subtract $25(x-2)^2$ from both sides.

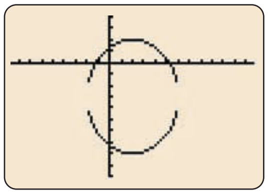
$$(y+3)^2=\frac{400-25(x-2)^2}{16}$$ Divide both sides by 16.

$$y+3=\pm\frac{\sqrt{400-25(x-2)^2}}{4}$$ Use the square root property.

$$y=-3\pm\frac{\sqrt{400-25(x-2)^2}}{4}$$ Subtract 3 from both sides.

The previous equation represents two functions. On a calculator, we can graph them in a square window to get the ellipse shown below.

$$y=-3+\frac{\sqrt{400-25(x-2)^2}}{4} \quad \text{and} \quad y=-3-\frac{\sqrt{400-25(x-2)^2}}{4}$$



As we saw with circles, the two portions of the ellipse do not quite connect. This is because the graphs are nearly vertical there.

EXAMPLE 4 Graph: $4(x-2)^2+9(y-1)^2=36$

Strategy We will write the equation in standard $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ form. Then we will identify h, k, a, and b.

WHY If we know h, k, a, and b, we can graph the ellipse.

Solution

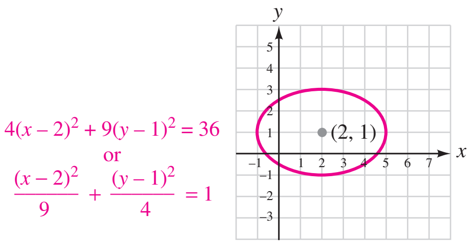
This equation is not in standard form. To write it in standard form with 1 on the right side, we divide both sides by 36.

$$4(x-2)^2 + 9(y-1)^2 = 36$$

$$\frac{4(x-2)^2}{36} + \frac{9(y-1)^2}{36} = \frac{36}{36} \qquad \text{Divide both sides by 36.}$$

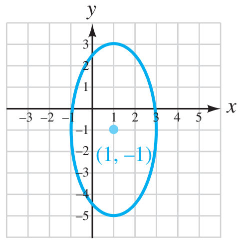
$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1 \qquad \text{Simplify: } \frac{4}{36} = \frac{1}{9}, \frac{9}{36} = \frac{1}{4}, \text{ and } \frac{36}{36} = 1.$$

This is the standard form of the equation of a horizontal ellipse, centered at (2, 1), with $a = 3$ and $b = 2$. The graph of the ellipse is shown on the right.



Self Check 4

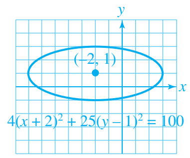
Graph:
$12(x-1)^2 + 3(y+1)^2 = 48$



$$\frac{(x-1)^2}{4} + \frac{(y+1)^2}{16} = 1$$

Now Try **Problem 29**

Teaching Example 4 Graph:
$4(x+2)^2 + 25(y-1)^2 = 100$
Answer:



4 Solve problems involving ellipses.

EXAMPLE 5 ***Landscape Design*** A landscape architect is designing an elliptical pool that will fit in the center of a 20-by-30-foot rectangular garden, leaving 5 feet of clearance on all sides, as shown in the illustration below. Find the equation of the ellipse.

Strategy We will establish a coordinate system with its origin at the center of the garden. Then we will determine the x- and y-intercepts of the edge of the pool.

WHY If we know the x- and y-intercepts of the graph of the edge of the elliptical pool, we can use that information to write its equation.

Solution

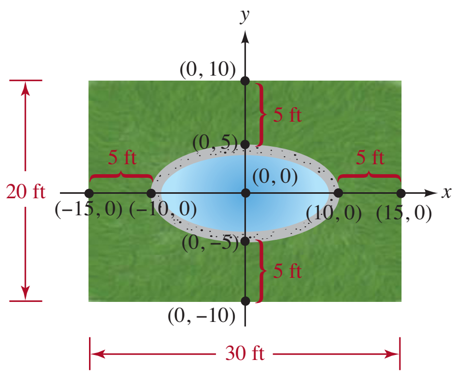
We place the rectangular garden in the coordinate system shown below. To maintain 5 feet of clearance at the ends of the ellipse, the x-intercepts must be the points (10, 0) and (−10, 0). Similarly, the y-intercepts are the points (0, 5) and (0, −5).

Since the ellipse is centered at the origin, its equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a = 10$ and $b = 5$. Thus, the equation of the boundary of the pool is

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$



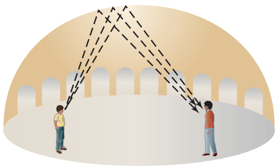
Self Check 5

An interior decorator is designing an elliptical-shaped mirror that will fit in the center of a 52-by-40 in. rectangular panel, leaving 2 in. of clearance on all sides. Find the equation of the ellipse. $\frac{x^2}{324} + \frac{y^2}{576} = 1$

Now Try **Problem 53**

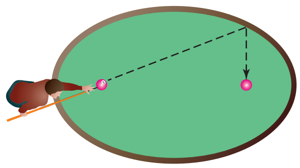
Teaching Example 5 A cabinetmaker is designing an elliptical-shaped pattern in the center of a 22-by-50 in. rectangular coffee table, leaving 3 in. of clearance on all sides. If he establishes a coordinate system with the length of the table being along the x-axis, find the equation of the ellipse.
Answer:
$\frac{x^2}{484} + \frac{y^2}{64} = 1$

Ellipses, like parabolas, have reflective properties that are used in many practical applications. For example, any light or sound originating at one focus of an ellipse is reflected by the interior of the figure to the other focus.



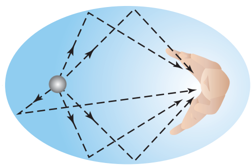
Whispering Galleries

In an elliptical dome, even the slightest whisper made by a person standing at one focus can be heard by a person standing at the other focus.



Elliptical billiards tables

When a ball is shot from one focus, it will rebound off the side of the table into a pocket located at the other focus.

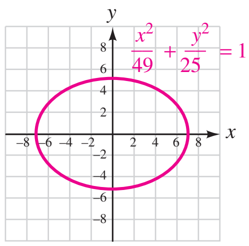


Treatment for kidney stones

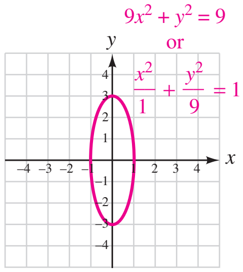
The patient is positioned in an elliptical tank of water so that the kidney stone is at one focus. High-intensity sound waves generated at another focus are reflected to the stone to shatter it.

ANSWERS TO SELF CHECKS

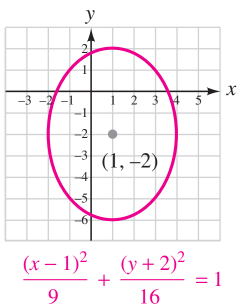
1.



2.



3.



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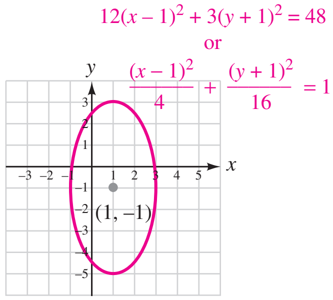
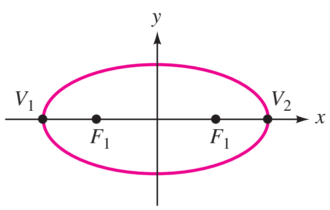


SECTION 10.2 STUDY SET

VOCABULARY

Fill in the blanks.

1. The curve graphed below is an ellipse.



▶ **2.** An ellipse is the set of all points in a plane for which the sum of the distances from two fixed points is a constant.

▶ **3.** In the graph in Exercise 1, F_1 and F_2 are the foci of the ellipse. Each one is called a focus of the ellipse.

▶ **4.** In the graph in Exercise 1, V_1 and V_2 are the vertices of the ellipse. Each one is called a vertex of the ellipse.

▶ **5.** The line segment joining the vertices of an ellipse is called the major axis of the ellipse.

▶ **6.** The midpoint of the major axis of an ellipse is the center of the ellipse.

▶ Selected exercises available online at **www.webassign.net/brookscole**

CONCEPTS

7. Write the standard form of the equation of an ellipse centered at the origin and symmetric to both axes.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

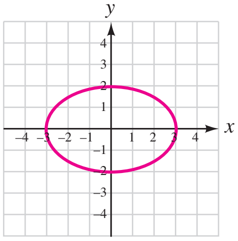
8. Write the standard form of the equation of a horizontal or vertical ellipse centered at (h, k).
$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

9. Find the x- and the y-intercepts of the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
x-intercepts: $(a, 0)$, $(-a, 0)$; y-intercepts: $(0, b)$, $(0, -b)$

10. a. Find the center of the ellipse graphed on the right. What are a and b?
$(0, 0)$, $a = 3$, $b = 2$

b. Is the ellipse horizontal or vertical?
horizontal

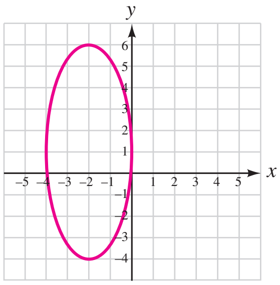
▶ **c.** Find the equation of the ellipse.
$\frac{x^2}{9} + \frac{y^2}{4} = 1$



11. a. Find the center of the ellipse graphed on the right. What are a and b?
$(-2, 1)$, $a = 2$, $b = 5$

b. Is the ellipse horizontal or vertical?
vertical

▶ **c.** Find the equation of the ellipse.
$\frac{(x + 2)^2}{4} + \frac{(y - 1)^2}{25} = 1$



▶ **12.** Find two points on the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ by letting $x = 2$ and finding the corresponding values of y.
$\left(2, \sqrt{3}\right), \left(2, -\sqrt{3}\right)$

13. Divide both sides of the equation by 64 and write the equation in standard form:

$$4(x - 1)^2 + 64(y + 5)^2 = 64 \quad \frac{(x - 1)^2}{16} + \frac{(y + 5)^2}{1} = 1$$

▶ **14.** Determine whether the graph of each equation is a circle, a parabola, or an ellipse.

a. $x = y^2 - 2y + 10$
parabola

b. $\frac{x^2}{49} + \frac{y^2}{64} = 1$
ellipse

c. $(x - 3)^2 + (y + 4)^2 = 25$
circle

d. $2(x - 1)^2 + 8(y + 5)^2 = 32$
ellipse

NOTATION

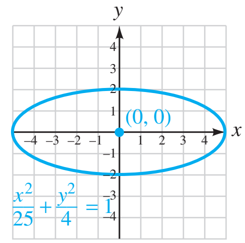
15. Find h, k, a, and b: $\frac{(x + 8)^2}{100} + \frac{(y - 6)^2}{144} = 1$
$h = -8$, $k = 6$, $a = 10$, $b = 12$

▶ **16.** Write each denominator in the equation $\frac{x^2}{81} + \frac{y^2}{49} = 1$ as the square of a number.
$\frac{x^2}{9^2} + \frac{y^2}{7^2} = 1$

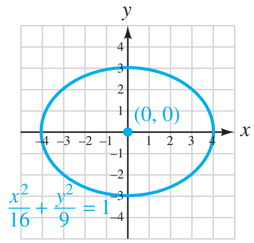
GUIDED PRACTICE

Graph each equation. See Example 1.

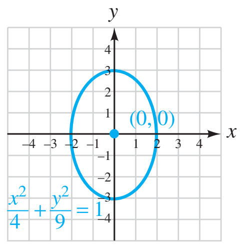
▶ **17.** $\frac{x^2}{25} + \frac{y^2}{4} = 1$



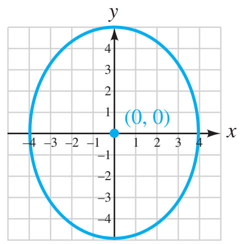
▶ **18.** $\frac{x^2}{16} + \frac{y^2}{9} = 1$



▶ **19.** $\frac{x^2}{4} + \frac{y^2}{9} = 1$



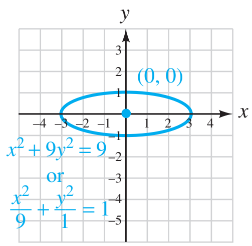
▶ **20.** $\frac{x^2}{16} + \frac{y^2}{25} = 1$



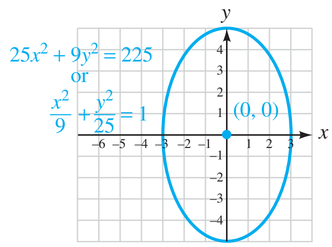
$\frac{x^2}{16} + \frac{y^2}{25} = 1$

Graph each equation. See Example 2.

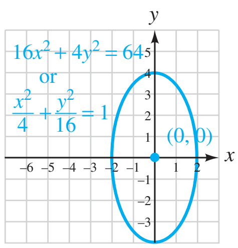
▶ **21.** $x^2 + 9y^2 = 9$



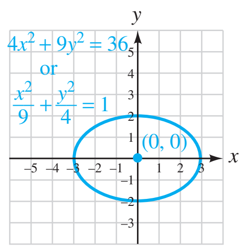
▶ **22.** $25x^2 + 9y^2 = 225$



▶ **23.** $16x^2 + 4y^2 = 64$

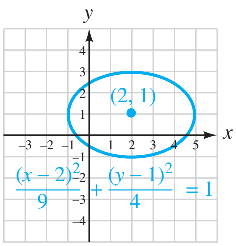


▶ **24.** $4x^2 + 9y^2 = 36$

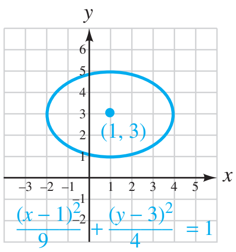


Graph each equation. **See Example 3.**

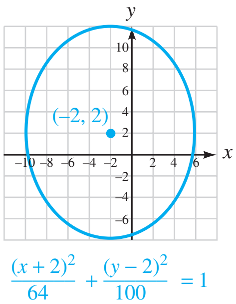
▶ **25.** $\dfrac{(x-2)^2}{9}+\dfrac{(y-1)^2}{4}=1$



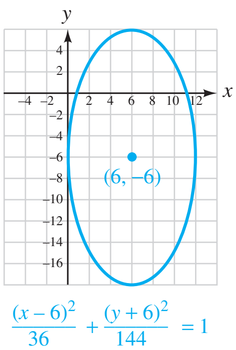
▶ **26.** $\dfrac{(x-1)^2}{9}+\dfrac{(y-3)^2}{4}=1$



▶ **27.** $\dfrac{(x+2)^2}{64}+\dfrac{(y-2)^2}{100}=1$

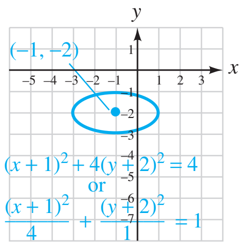


▶ **28.** $\dfrac{(x-6)^2}{36}+\dfrac{(y+6)^2}{144}=1$

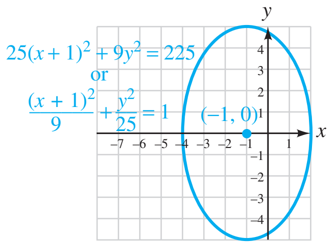


Graph each equation. **See Example 4.**

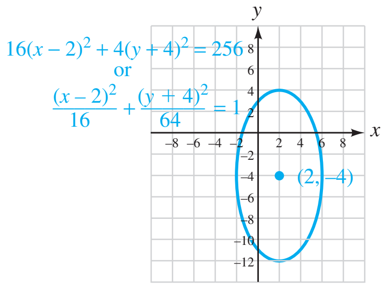
▶ **29.** $(x+1)^2+4(y+2)^2=4$



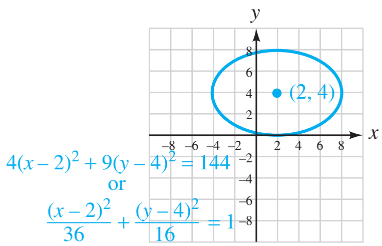
▶ **30.** $25(x+1)^2+9y^2=225$



31. $16(x-2)^2+4(y+4)^2=256$

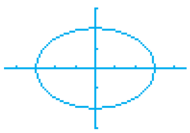


32. $4(x-2)^2+9(y-4)^2=144$

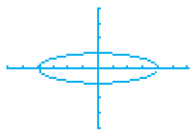


Use a graphing calculator to graph each equation. **See Using Your Calculator: Graphing Ellipses.**

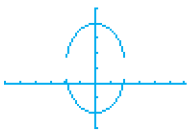
33. $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$



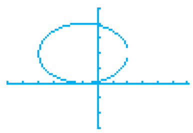
34. $x^2+16y^2=16$



35. $\dfrac{x^2}{4}+\dfrac{(y-1)^2}{9}=1$



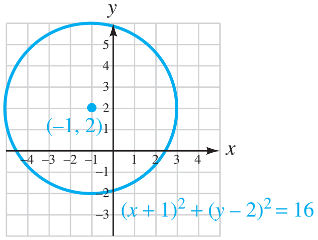
36. $\dfrac{(x+1)^2}{9}+\dfrac{(y-2)^2}{4}=1$



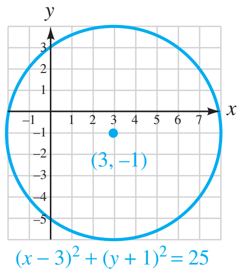
TRY IT YOURSELF

Write each equation in standard form, if it is not already so, and graph it. The problems include equations that describe circles, parabolas, and ellipses.

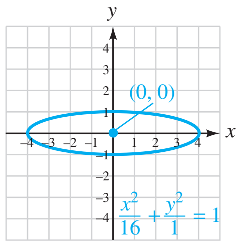
37. $(x + 1)^2 + (y - 2)^2 = 16$



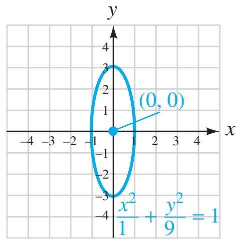
38. $(x - 3)^2 + (y + 1)^2 = 25$



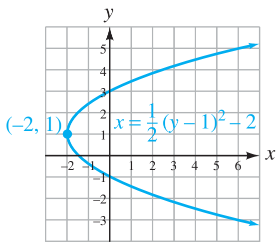
▶ **39.** $\dfrac{x^2}{16} + \dfrac{y^2}{1} = 1$



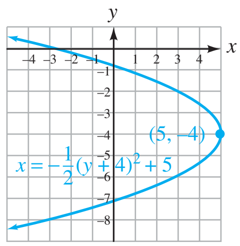
▶ **40.** $\dfrac{x^2}{1} + \dfrac{y^2}{9} = 1$



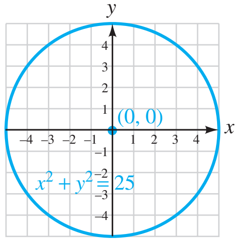
41. $x = \dfrac{1}{2}(y - 1)^2 - 2$



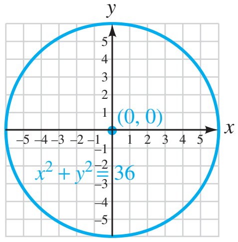
42. $x = -\dfrac{1}{2}(y + 4)^2 + 5$



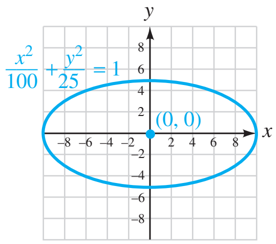
▶ **43.** $x^2 + y^2 - 25 = 0$



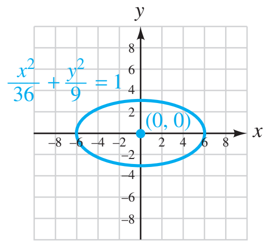
44. $x^2 = 36 - y^2$



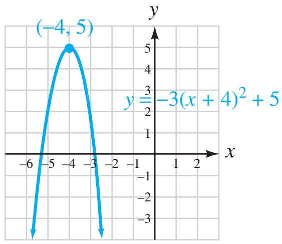
45. $x^2 = 100 - 4y^2$



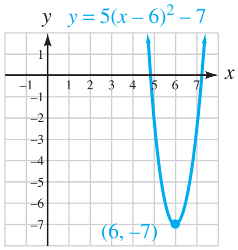
46. $x^2 = 36 - 4y^2$



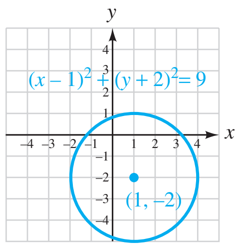
47. $y = -3x^2 - 24x - 43$



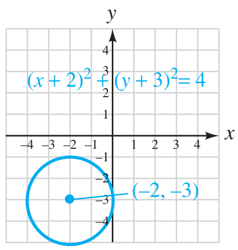
48. $y = 5x^2 - 60x + 173$



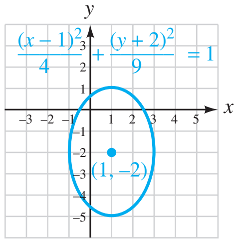
49. $x^2 + y^2 - 2x + 4y - 4 = 0$



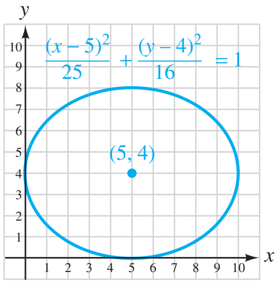
50. $x^2 + y^2 + 4x + 6y + 9 = 0$



51. $9(x-1)^2 + 4(y+2)^2 = 36$

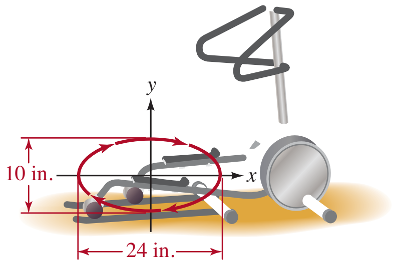


▶ **52.** $16(x-5)^2 + 25(y-4)^2 = 400$

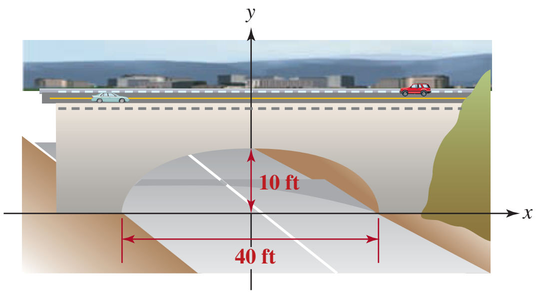


APPLICATIONS

▶ **53.** FITNESS EQUIPMENT With elliptical cross-training equipment, the feet move through the natural elliptical pattern that one experiences when walking, jogging, or running. Write the equation of the elliptical pattern shown below. $\frac{x^2}{144} + \frac{y^2}{25} = 1$

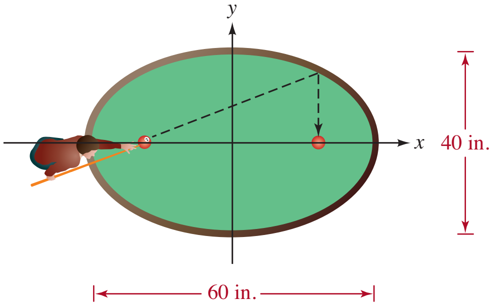


▶ **54.** DESIGNING AN UNDERPASS The arch of an underpass is a part of an ellipse. Find the equation of the ellipse. $\frac{x^2}{400} + \frac{y^2}{100} = 1$



▶ **55.** CALCULATING CLEARANCE Find the height of the elliptical arch in Exercise 54 at a point 10 feet from the center of the roadway that passes under the arch. $5\sqrt{3}$ ft ≈ 8.7 ft

▶ **56.** POOL TABLES Find the equation of the outer edge of the elliptical pool table shown below. $\frac{x^2}{900} + \frac{y^2}{400} = 1$

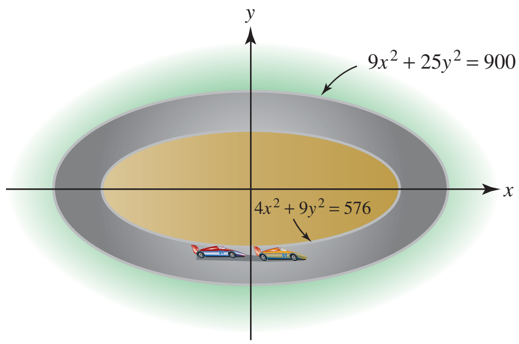


▶ **57.** AREA OF AN ELLIPSE The area A bounded by the following ellipse is given by $A = \pi ab$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the area bounded by the ellipse described by $9x^2 + 16y^2 = 144$. 12π sq. units ≈ 37.7 sq. units

▶ **58.** AREA OF A TRACK The elliptical track shown in the figure is bounded by the ellipses $4x^2 + 9y^2 = 576$ and $9x^2 + 25y^2 = 900$. Find the area of the track. (See Exercise 57.) 36π sq. units ≈ 113.1 sq. units



WRITING

59. What is an ellipse?

60. Explain the difference between the focus of an ellipse and the vertex of an ellipse.

61. Compare the graphs of $\frac{x^2}{81} + \frac{y^2}{64} = 1$ and $\frac{x^2}{64} + \frac{y^2}{81} = 1$. Do they have any similarities?

▶ **62.** What are the reflective properties of an ellipse?

REVIEW

Find each product.

63. $3x^{-2}y^2(4x^2 + 3y^{-2})$ $12y^2 + \frac{9}{x^2}$

▶ **64.** $(2a^{-2} - b^{-2})(2a^{-2} + b^{-2})$ $\frac{4}{a^4} - \frac{1}{b^4}$

Simplify each expression.

65. $\dfrac{x^{-2} + y^{-2}}{x^{-2} - y^{-2}}$ $\frac{y^2 + x^2}{y^2 - x^2}$

66. $\dfrac{2x^{-3} - 2y^{-3}}{4x^{-3} + 4y^{-3}}$ $\frac{y^3 - x^3}{2(y^3 + x^3)}$

SECTION 10.3

The Hyperbola

Objectives

1 Define a hyperbola.

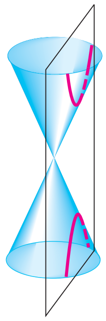
2 Graph hyperbolas centered at the origin.

3 Graph hyperbolas centered at (h, k).

4 Graph equations of the form $xy = k$.

5 Solve problems involving hyperbolas.

The final conic section that we will discuss, the *hyperbola,* is a curve that has two branches. In this section, we will learn how to graph equations that represent hyperbolas.



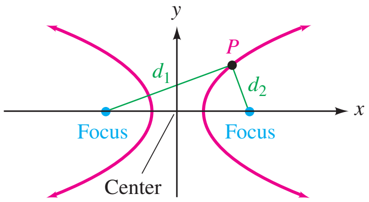
1 Define a hyperbola.

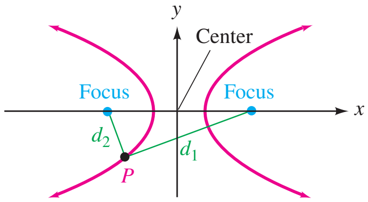
Ellipses and hyperbolas have completely different shapes, but their definitions are similar. Instead of the *sum* of distances, the definition of a hyperbola involves a *difference* of distances.

Definition of a Hyperbola

A **hyperbola** is the set of all points in a plane for which the difference of the distances from two fixed points is a constant.

The figure below illustrates that any point P on the hyperbola is a constant distance $d_1 - d_2$ from two fixed points, each of which is called a **focus.** Midway between the **foci** is the **center** of the hyperbola.





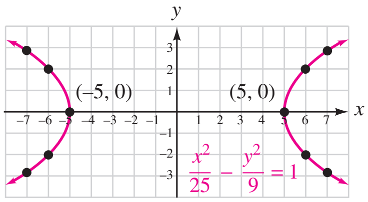
2 Graph hyperbolas centered at the origin.

The graph of the equation

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$

is a hyperbola. To graph the equation, we make a table of solutions that satisfy the equation, plot each point, and join them with a smooth curve.

$\frac{x^2}{25} - \frac{y^2}{9} = 1$		
x	y	
-7	± 2.9	→ $(-7, \pm 2.9)$
-6	± 2.0	→ $(-6, \pm 2.0)$
-5	0	→ $(-5, 0)$
5	0	→ $(5, 0)$
6	± 2.0	→ $(6, \pm 2.0)$
7	± 2.9	→ $(7, \pm 2.9)$



Caution! Although the two branches of a hyperbola look like parabolas, they are not parabolas.

This graph is centered at the origin and intersects the x-axis at $(5, 0)$ and $(-5, 0)$. We also note that the graph does not intersect the y-axis.

It is possible to draw a hyperbola without plotting points. For example, if we want to graph the hyperbola with an equation of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

we first find the x- and y-intercepts. To find the x-intercepts, we let $y = 0$ and solve for x:

$$\frac{x^2}{a^2} - \frac{0^2}{b^2} = 1$$

$$x^2 = a^2$$

$$x = \pm a \quad \text{Use the square root property.}$$

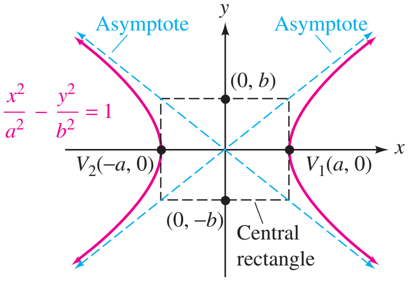
The hyperbola crosses the x-axis at the points $V_1(a, 0)$ and $V_2(-a, 0)$, called the **vertices** of the hyperbola.

To attempt to find the y-intercepts, we let $x = 0$ and solve for y:

$$\frac{0^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y^2 = -b^2$$

$$y = \pm\sqrt{-b^2}$$



Since b^2 is always positive, $\sqrt{-b^2}$ is an imaginary number. This means that the hyperbola does not intersect the y-axis.

If we construct a rectangle, called the **central rectangle,** whose sides pass horizontally through $\pm b$ on the y-axis and vertically through $\pm a$ on the x-axis, the extended diagonals of the rectangle will be **asymptotes** of the hyperbola. As the hyperbola gets farther away from the origin, its branches get closer and closer to the asymptotes. The asymptotes are not part of the hyperbola, but they serve as a guide when drawing its graph. Since the slopes of the diagonals are $\frac{b}{a}$ and $-\frac{b}{a}$, the equations of the asymptotes are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

The Language of Algebra The central rectangle is also called the *fundamental rectangle.*

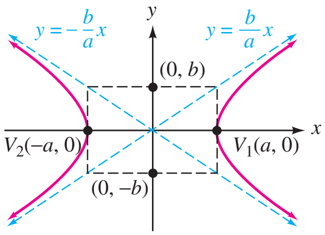
Standard Form of the Equation of a Hyperbola Centered at the Origin and Intersecting the x-Axis

Any equation that can be written in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has a graph that is a hyperbola centered at the origin. The x-intercepts are the vertices $V_1(a, 0)$ and $V_2(-a, 0)$. There are no y-intercepts.

The asymptotes of the hyperbola are the extended diagonals of the central rectangle, and their equations are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.



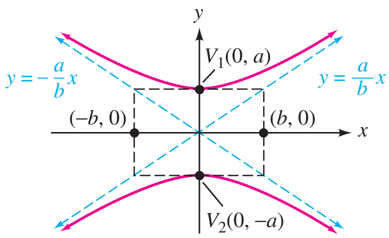
The branches of the hyperbola in previous discussions open to the left and to the right. It is possible for hyperbolas to have different orientations with respect to the x- and y-axes. For example, the branches of a hyperbola can open upward and downward. In that case, the following equation applies.

Standard Form of the Equation of a Hyperbola Centered at the Origin and Intersecting the y-Axis

Any equation that can be written in the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has a graph that is a hyperbola centered at the origin. The y-intercepts are the vertices $V_1(0, a)$ and $V_2(0, -a)$. There are no x-intercepts.



The asymptotes of the hyperbola are the extended diagonals of the central rectangle, and their equations are $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$.

EXAMPLE 1

Graph: $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

Strategy This equation is in standard $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form. We will identify a and b.

WHY We can use a and b to find the vertices of the graph of the hyperbola and the location of the central rectangle.

Solution

The color highlighting shows how to compare the given equation with the standard form to find a and b.

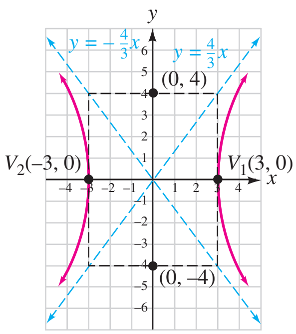
$$\frac{x^2}{\mathbf{9}} - \frac{y^2}{\mathbf{16}} = 1 \qquad \frac{x^2}{\boldsymbol{a^2}} - \frac{y^2}{\boldsymbol{b^2}} = 1$$

Since $a^2 = 9$, it follows that $a = 3$.

Since $b^2 = 16$, it follows that $b = 4$.

This is the standard form of the equation of a hyperbola, centered at the origin, that opens left and right. The x-intercepts are $(a, 0)$ and $(-a, 0)$, or $(3, 0)$ and $(-3, 0)$. They are also the vertices of the hyperbola.

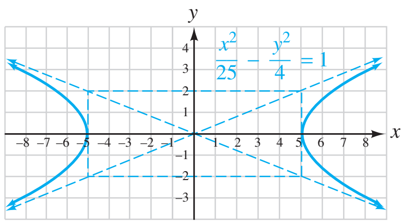
To construct the central rectangle, we use the values of $a = 3$ and $b = 4$. The rectangle passes through $(3, 0)$ and $(-3, 0)$ on the x-axis, and $(0, 4)$ and $(0, -4)$ on the y-axis. We draw extended diagonal dashed lines through the rectangle to obtain the asymptotes and write their equations: $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$. Then we draw a smooth curve through each vertex that gets close to the asymptotes.



$\frac{x^2}{9} - \frac{y^2}{16} = 1$

Self Check 1

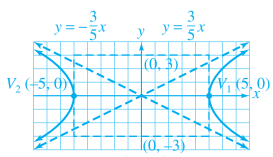
Graph:
$\dfrac{x^2}{25} - \dfrac{y^2}{4} = 1$



Now Try **Problem 17**

Teaching Example 1 Graph:
$\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

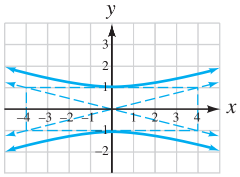
Answer:



$\frac{x^2}{25} - \frac{y^2}{9} = 1$

Self Check 2

Graph: $16y^2 - x^2 = 16$

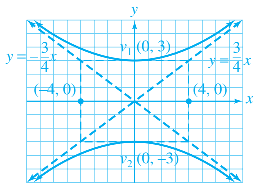


$\frac{y^2}{1} - \frac{x^2}{16} = 1$

Now Try **Problem 21**

Teaching Example 2 Graph: $16y^2 - 9x^2 = 144$

Answer:



$\frac{y^2}{9} - \frac{x^2}{16} = 1$

EXAMPLE 2

Graph: $9y^2 - 4x^2 = 36$

Strategy We will write the equation in standard $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ form.

WHY When the equation is in standard form, we will be able to identify the center and the vertices of the graph of the hyperbola and the location of the central rectangle.

Solution

To write the equation in standard form, we divide both sides by 36.

$$9y^2 - 4x^2 = 36$$

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36} \quad \text{To get a 1 on the right side, divide both sides by 36.}$$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \quad \text{Simplify each fraction.}$$

This is the standard form of the equation of a hyperbola, centered at the origin, that opens up and down. The color highlighting shows how we compare the resulting equation to the standard form to find a and b.

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Since $a^2 = 4$, it follows that $a = 2$.

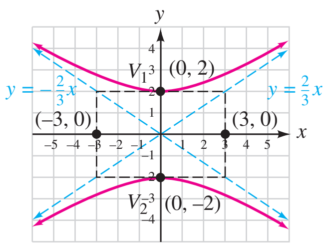
Since $b^2 = 9$, it follows that $b = 3$.

Success Tip The positive variable term in the standard form equation determines whether a hyperbola is vertical or horizontal. In this example, the positive variable term involves y, so the hyperbola is vertical.

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

The y-intercepts are $(0, a)$ and $(0, -a)$, or $(0, 2)$ and $(0, -2)$. They are also the vertices of the hyperbola.

Since $a = 2$ and $b = 3$, the central rectangle passes through $(0, 2)$ and $(0, -2)$, and $(3, 0)$ and $(-3, 0)$. We draw its extended diagonals and sketch the hyperbola.



$9y^2 - 4x^2 = 36$

or

$\frac{y^2}{4} - \frac{x^2}{9} = 1$

We can determine whether an equation, when graphed, will be a circle, a parabola, an ellipse, or a hyperbola by examining its variable terms.

$$x^2 + y^2 = 16$$ With the variable terms on the same side of the equation, we see that the coefficients of the squared terms are the same. The graph is a circle.

$$4x^2 + 9y^2 = 144$$ With the variable terms on the same side of the equation, we see that the coefficients of the squared terms are different, but have the same sign. The graph is an ellipse.

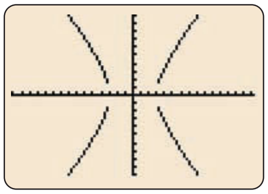
$4x^2 - 9y^2 = 144$ — *With the variable terms on the same side of the equation, we see that the coefficients of the squared terms have different signs. The graph is a hyperbola.*

$x = y^2 + y - 16$ — *Since one variable is squared and the other is not, the graph is a parabola.*

Using Your CALCULATOR **Graphing Hyperbolas**

To graph $\frac{x^2}{9} - \frac{y^2}{16} = 1$ from Example 1 using a graphing calculator, we follow the same procedure that we used for circles and ellipses. To write the equation as two functions, we solve for y to get $y = \pm\frac{\sqrt{16x^2 - 144}}{3}$. Then we graph the following two functions in a square window setting to get the graph of the hyperbola shown below.

$$y = \frac{\sqrt{16x^2 - 144}}{3} \quad \text{and} \quad y = -\frac{\sqrt{16x^2 - 144}}{3}$$



3 Graph hyperbolas centered at (h, k).

If a hyperbola is centered at a point with coordinates (h, k), the following equations apply.

Standard Form of the Equation of a Hyperbola Centered at (h, k)

Any equation that can be written in the form

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

is a hyperbola that has its center at (h, k) and opens left and right.

Any equation of the form

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

is a hyperbola that has its center at (h, k) and opens up and down.

EXAMPLE 3

Graph:

a. $\frac{(x - 3)^2}{16} - \frac{(y + 1)^2}{4} = 1$ **b.** $\frac{(y - 2)^2}{9} - \frac{(x - 1)^2}{9} = 1$

Strategy We will write each equation in a form that makes it easy to identify h, k, a, and b.

WHY If we know h, k, a, and b, we can graph the hyperbola and the central rectangle.

Solution

a. We can write the given equation as

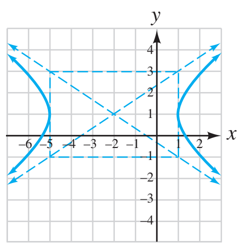
$$\frac{(x - 3)^2}{4^2} - \frac{[y - (-1)]^2}{2^2} = 1$$

To find k, write $y + 1$ as $y - (-1)$.
To find a, write 16 as 4^2. To find b, write 4 as 2^2.

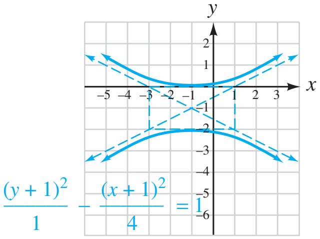
Self Check 3

Graph:

a. $\frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{4} = 1$



b. $\dfrac{(y+1)^2}{1} - \dfrac{(x+1)^2}{4} = 1$



Now Try **Problems 25 and 27**

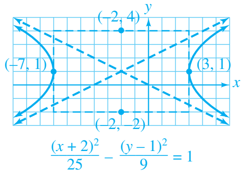
Teaching Example 3 Graph

a. $\dfrac{(x+2)^2}{25} - \dfrac{(y-1)^2}{9} = 1$

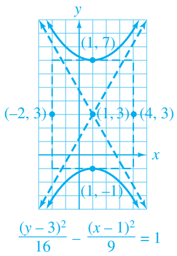
b. $\dfrac{(y-3)^2}{16} - \dfrac{(x-1)^2}{9} = 1$

Answers:

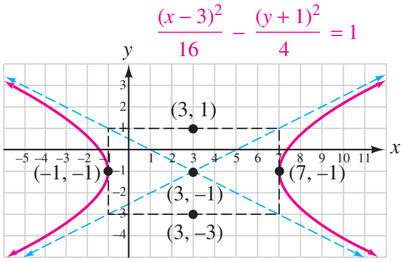
a.



b.



Because the term involving x is positive, the hyperbola opens left and right. We find the center by examining $(x-3)^2$ and $[y-(-1)]^2$. Since $h = 3$ and $k = -1$, the hyperbola is centered at $(h, k) = (3, -1)$. From the denominators, 4^2 and 2^2, we find that $a = 4$ and $b = 2$. Thus, its vertices are located 4 units to the right and left of the center, at $(7, -1)$ and $(-1, -1)$.

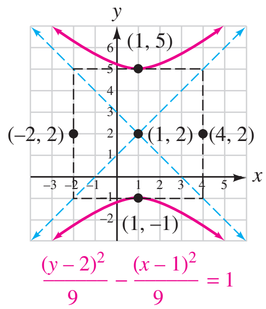


Since $b = 2$, we can count 2 units above and below the center to locate points $(3, 1)$ and $(3, -3)$. With these four points, we can draw the central rectangle along with its extended diagonals (the asymptotes). We can then sketch the hyperbola, as shown.

b. We can write the given equation as

$$\frac{(y-2)^2}{3^2} - \frac{(x-1)^2}{3^2} = 1$$

Because the term involving y is positive, the hyperbola opens up and down. We find its center by examining $(y-2)^2$ and $(x-1)^2$. Since $k = 2$ and $h = 1$, the hyperbola is centered at $(h, k) = (1, 2)$. From the denominators, 3^2 and 3^2, we find that $a = 3$ and $b = 3$, and we use that information to draw the central rectangle and its extended diagonals (the asymptotes), as shown.

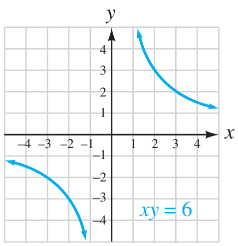


4 Graph equations of the form $xy = k$.

There is a special type of hyperbola (also centered at the origin) that does not intersect either the x- or the y-axis. These hyperbolas have equations of the form $xy = k$, where $k \neq 0$.

Self Check 4

Graph: $xy = 6$



Now Try **Problem 33**

EXAMPLE 4 Graph: $xy = -8$

Strategy We will make a table of solutions, plot the points, and connect the points with a smooth curve.

WHY Since this equation cannot be written in standard form, we cannot use the methods used in the previous examples.

Solution

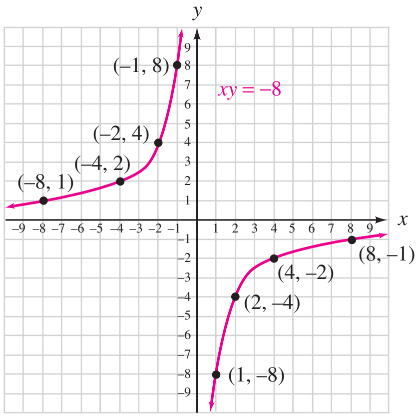
To make a table of solutions, we can solve the equation $xy = -8$ for y:

$$y = \frac{-8}{x}$$

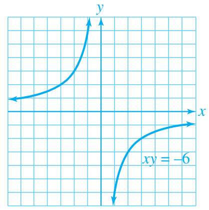
Then we choose several values for x, find the corresponding values of y, and record the results in the table on the next page. We plot the ordered pairs and join them with a smooth curve to obtain the graph of the hyperbola.

$xy = -8$ or $y = \frac{-8}{x}$

x	y	
1	−8	→ (1, −8)
2	−4	→ (2, −4)
4	−2	→ (4, −2)
8	−1	→ (8, −1)
−1	8	→ (−1, 8)
−2	4	→ (−2, 4)
−4	2	→ (−4, 2)
−8	1	→ (−8, 1)



Teaching Example 4 Graph: $xy = -6$
Answer:



The Language of Algebra The asymptotes of this hyperbola are the x- and y-axes. A hyperbola for which the asymptotes are perpendicular is called a **rectangular hyperbola.**

The result in Example 4 illustrates the following general equation.

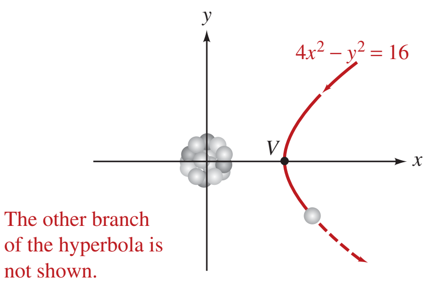
Equations of Hyperbolas of the Form $xy = k$

Any equation of the form $xy = k$, where $k \neq 0$, has a graph that is a **hyperbola,** which does not intersect either the x- or y-axis.

5 Solve problems involving hyperbolas.

EXAMPLE 5 ***Atomic Structure***

In an experiment that led to the discovery of the atomic structure of matter, Lord Rutherford (1871–1937) shot high-energy alpha particles toward a thin sheet of gold. Many of them were reflected, and Rutherford showed the existence of the nucleus of a gold atom. An alpha particle is repelled by the nucleus at the origin; it travels along the hyperbolic path given by $4x^2 - y^2 = 16$. How close does the particle come to the nucleus?



Self Check 5

Some comets have a hyperbolic orbit, with the sun as one focus and Earth at the center. For one such comet, the equation of its path is $\frac{x^2}{1 \times 10^{18}} - \frac{y^2}{2 \times 10^{18}} = 1$. How close does this comet come to Earth? 1×10^9 mi

Now Try **Problem 61**

Teaching Example 5 An alpha particle is repelled by the nucleus at the origin and it travels along the hyperbolic path given by $9x^2 - y^2 = 81$. How close does the particle come to the nucleus?
Answer:
3 units

Strategy We will write the equation in standard form and find the coordinates of point V.

WHY The distance from the origin to point V is the closest the particle comes to the nucleus.

Solution

To find the distance from the nucleus at the origin, we must find the coordinates of the vertex V. To do so, we write the equation of the particle's path in standard form:

$$4x^2 - y^2 = 16$$

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16} \qquad \text{Divide both sides by 16.}$$

$$\frac{x^2}{4} - \frac{y^2}{16} = 1 \qquad \text{Simplify.}$$

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{To determine } a \text{ and } b\text{, write 4 as } 2^2 \text{ and 16 as } 4^2.$$

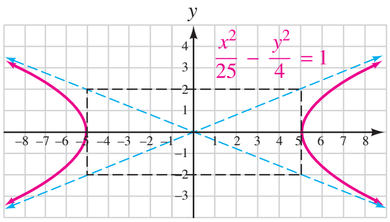
This equation is in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

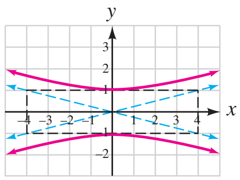
with $a = 2$. Thus, the vertex of the path is $(2, 0)$. The particle is never closer than 2 units from the nucleus.

ANSWERS TO SELF CHECKS

1.

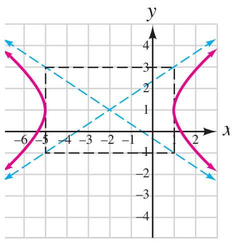


2.



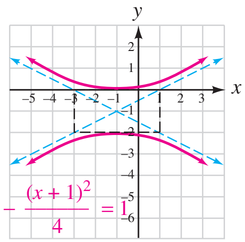
$16y^2 - x^2 = 16$ or $\frac{y^2}{1} - \frac{x^2}{16} = 1$

3. a.



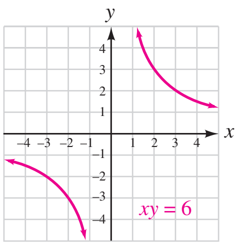
$$\frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{4} = 1$$

b.



$$\frac{(y + 1)^2}{1} - \frac{(x + 1)^2}{4} = 1$$

4.



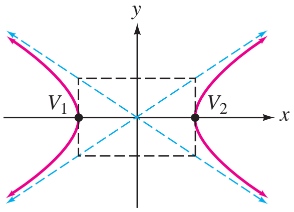
5. 1×10^9 mi

SECTION 10.3 STUDY SET

VOCABULARY

Fill in the blanks.

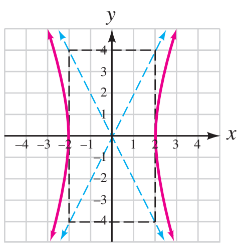
1. The two-branch curve graphed below is a hyperbola.



2. A hyperbola is the set of all points in a plane for which the difference of the distances from two fixed points is a constant.
3. In the graph in Exercise 1, V_1 and V_2 are the vertices of the hyperbola.
4. In the graph in Exercise 1, the figure drawn using dashed black lines is called the central rectangle.
5. The extended diagonals of the central rectangle are asymptotes of the hyperbola.
6. To write $9x^2 - 4y^2 = 36$ in standard form, we divide both sides by 36.

CONCEPTS

7. Write the standard form of the equation of a hyperbola centered at the origin that opens left and right.
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
8. Write the standard form of the equation of a hyperbola centered at (h, k) that opens up and down.
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
9. Write the standard form of the equation of a hyperbola centered at (h, k) that opens left and right.
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
10. **a.** Find the center of the hyperbola graphed at the right. What are a and b?
$(0, 0)$; $a = 2$, $b = 4$

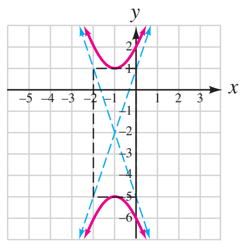


b. Find the x-intercepts of the graph. What are the y-intercepts of the graph?
$(2, 0)$, $(-2, 0)$; none

c. Find the equation of the hyperbola.
$\frac{x^2}{4} - \frac{y^2}{16} = 1$

d. Find the equations of the asymptotes.
$y = 2x$, $y = -2x$

11. **a.** Find the center of the hyperbola graphed on the right. What are a and b?
$(-1, -2)$; $a = 3$, $b = 1$



b. Find the equation of the hyperbola.
$\frac{(y+2)^2}{9} - \frac{(x+1)^2}{1} = 1$

12. **a.** Fill in the blank: An equation of the form $xy = k$, where $k \neq 0$, has a graph that is a hyperbola that does not intersect either the x-axis or the y-axis.

b. Complete the table of solutions for $xy = 10$.

x	y
-2	-5
5	2

13. Divide both sides of the equation by 100 and write the equation in standard form:

$$100(x+1)^2 - 25(y-5)^2 = 100$$

$\frac{(x+1)^2}{1} - \frac{(y-5)^2}{4} = 1$

14. Determine whether the graph of the equation will be a circle, a parabola, an ellipse, or a hyperbola.

a. $x^2 + y^2 = 10$
circle

b. $9y^2 - 16x^2 = 144$
hyperbola

c. $x = y^2 - 3y + 6$
parabola

d. $4x^2 + 25y^2 = 100$
ellipse

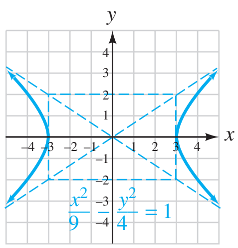
NOTATION

15. Find h, k, a, and b: $\frac{(x-5)^2}{25} - \frac{(y+11)^2}{36} = 1$
$h = 5$, $k = -11$, $a = 5$, $b = 6$
16. Write each denominator in the equation $\frac{x^2}{36} - \frac{y^2}{81} = 1$ as the square of a number. $\frac{x^2}{6^2} - \frac{y^2}{9^2} = 1$

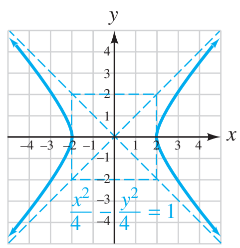
GUIDED PRACTICE

Graph each hyperbola. See Example 1.

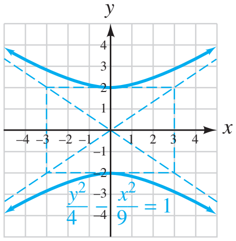
17. $\frac{x^2}{9} - \frac{y^2}{4} = 1$



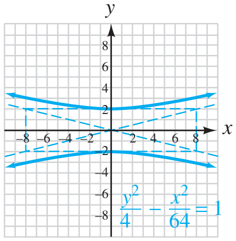
18. $\frac{x^2}{4} - \frac{y^2}{4} = 1$



▶ **19.** $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$

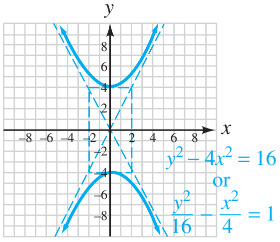


20. $\dfrac{y^2}{4} - \dfrac{x^2}{64} = 1$

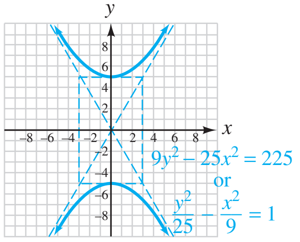


Graph each hyperbola. See Example 2.

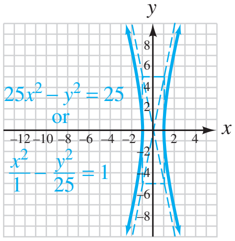
21. $y^2 - 4x^2 = 16$



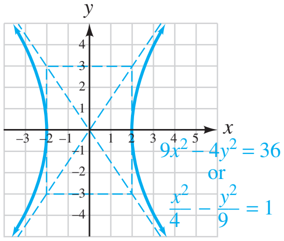
▶ **22.** $9y^2 - 25x^2 = 225$



▶ **23.** $25x^2 - y^2 = 25$

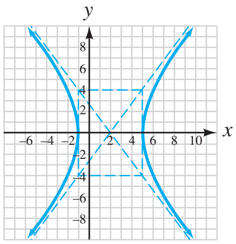


▶ **24.** $9x^2 - 4y^2 = 36$



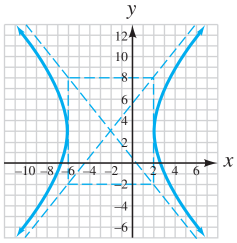
Graph each hyperbola. See Example 3.

▶ **25.** $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{16} = 1$



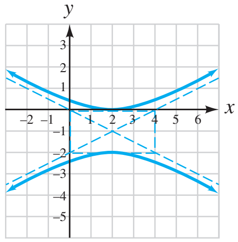
$\frac{(x-2)^2}{9} - \frac{y^2}{16} = 1$

▶ **26.** $\dfrac{(x+2)^2}{16} - \dfrac{(y-3)^2}{25} = 1$



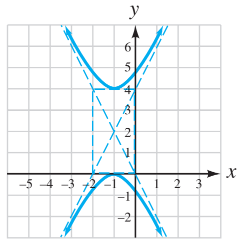
$\frac{(x+2)^2}{16} - \frac{(y-3)^2}{25} = 1$

▶ **27.** $\dfrac{(y+1)^2}{1} - \dfrac{(x-2)^2}{4} = 1$



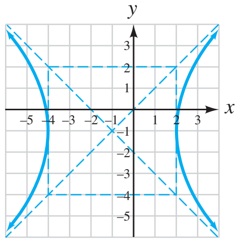
$\frac{(y+1)^2}{1} - \frac{(x-2)^2}{4} = 1$

28. $\dfrac{(y-2)^2}{4} - \dfrac{(x+1)^2}{1} = 1$



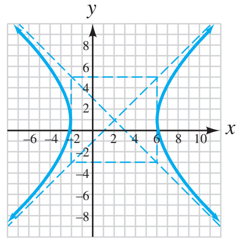
$\frac{(y-2)^2}{4} - \frac{(x+1)^2}{1} = 1$

29. $\dfrac{(x+1)^2}{9} - \dfrac{(y+1)^2}{9} = 1$



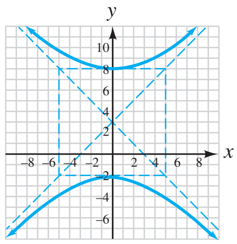
$\frac{(x+1)^2}{9} - \frac{(y+1)^2}{9} = 1$

▶ **30.** $\dfrac{(x-2)^2}{16} - \dfrac{(y-1)^2}{16} = 1$



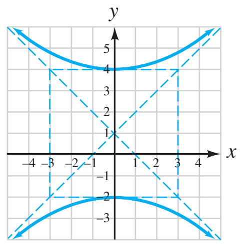
$\frac{(x-2)^2}{16} - \frac{(y-1)^2}{16} = 1$

31. $\dfrac{(y-3)^2}{25} - \dfrac{x^2}{25} = 1$



$\frac{(y-3)^2}{25} - \frac{x^2}{25} = 1$

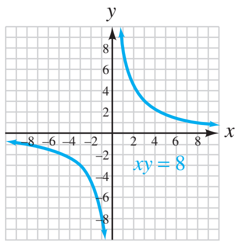
32. $\dfrac{(y-1)^2}{9} - \dfrac{x^2}{9} = 1$



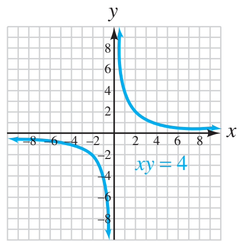
$\frac{(y-1)^2}{9} - \frac{x^2}{9} = 1$

Graph each equation. See Example 4.

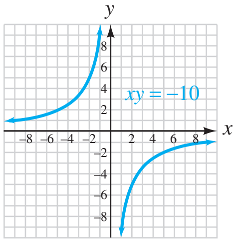
33. $xy = 8$



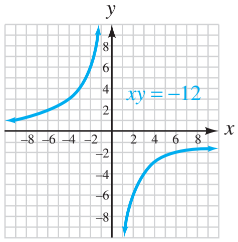
34. $xy = 4$



35. $xy = -10$

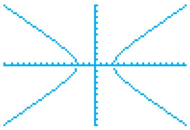


36. $xy = -12$

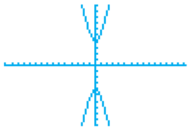


Use a graphing calculator to graph each equation. See Using Your Calculator: Graphing Hyperbolas.

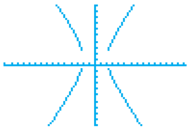
37. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$



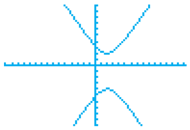
38. $y^2 - 16x^2 = 16$



39. $\dfrac{x^2}{4} - \dfrac{(y-1)^2}{9} = 1$



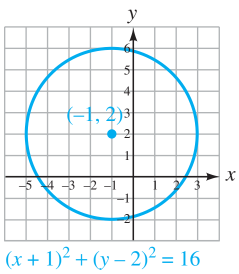
40. $\dfrac{(y+1)^2}{9} - \dfrac{(x-2)^2}{4} = 1$



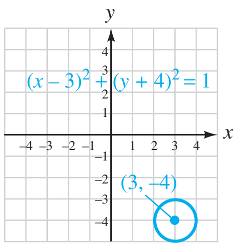
TRY IT YOURSELF

Write each equation in standard form, if it is not already so, and graph it. The problems include equations that describe circles, parabolas, ellipses, and hyperbolas.

41. $(x+1)^2 + (y-2)^2 = 16$

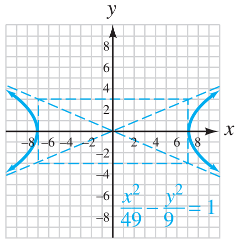


42. $(x-3)^2 + (y+4)^2 = 1$



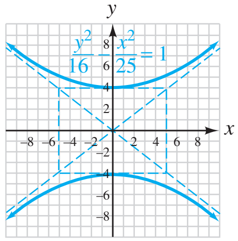
43. $9x^2 - 49y^2 = 441$

$\frac{x^2}{49} - \frac{y^2}{9} = 1$



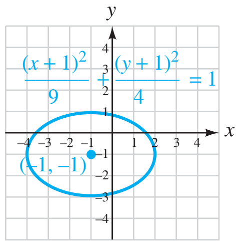
44. $25y^2 - 16x^2 = 400$

$\frac{y^2}{16} - \frac{x^2}{25} = 1$



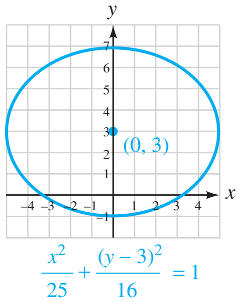
45. $4(x+1)^2 + 9(y+1)^2 = 36$

$\frac{(x+1)^2}{9} + \frac{(y+1)^2}{4} = 1$



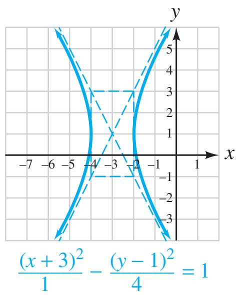
46. $16x^2 + 25(y-3)^2 = 400$

$\frac{x^2}{25} + \frac{(y-3)^2}{16} = 1$



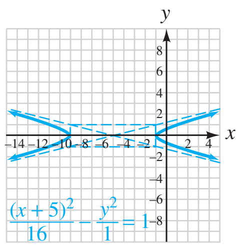
47. $4(x+3)^2 - (y-1)^2 = 4$

$\frac{(x+3)^2}{1} - \frac{(y-1)^2}{4} = 1$

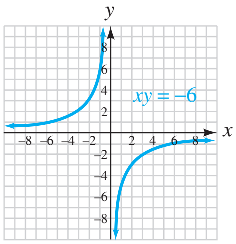


48. $(x+5)^2 - 16y^2 = 16$

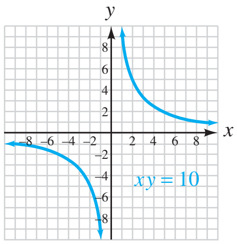
$\frac{(x+5)^2}{16} - \frac{y^2}{1} = 1$



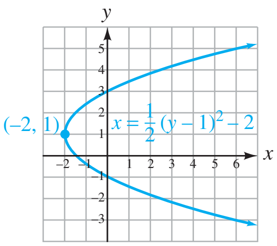
▶ **49.** $xy = -6$



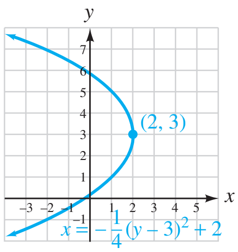
▶ **50.** $xy = 10$



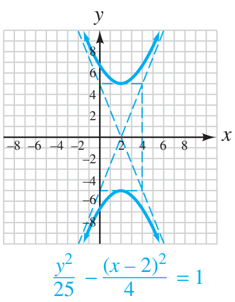
▶ **51.** $x = \frac{1}{2}(y - 1)^2 - 2$



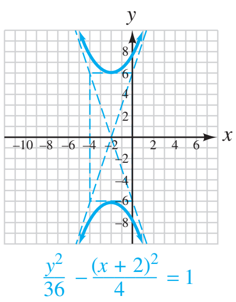
52. $x = -\frac{1}{4}(y - 3)^2 + 2$



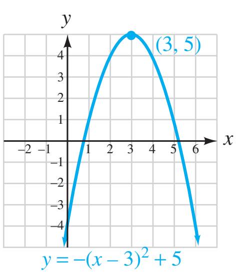
53. $\frac{y^2}{25} - \frac{(x - 2)^2}{4} = 1$



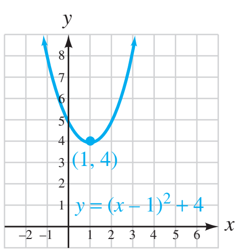
54. $\frac{y^2}{36} - \frac{(x + 2)^2}{4} = 1$



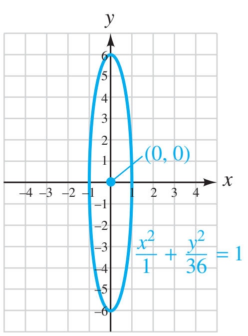
55. $y = -x^2 + 6x - 4$
$y = -(x - 3)^2 + 5$



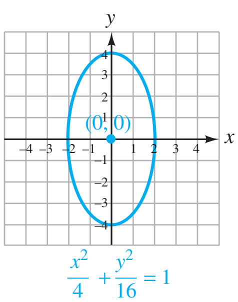
56. $y = x^2 - 2x + 5$
$y = (x - 1)^2 + 4$



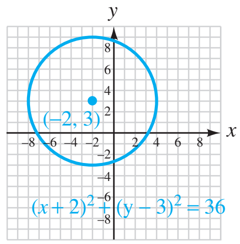
57. $\frac{x^2}{1} + \frac{y^2}{36} = 1$



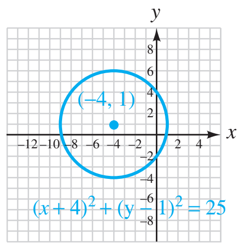
58. $\frac{x^2}{4} + \frac{y^2}{16} = 1$



▶ **59.** $x^2 + y^2 + 4x - 6y - 23 = 0$ $(x + 2)^2 + (y - 3)^2 = 36$

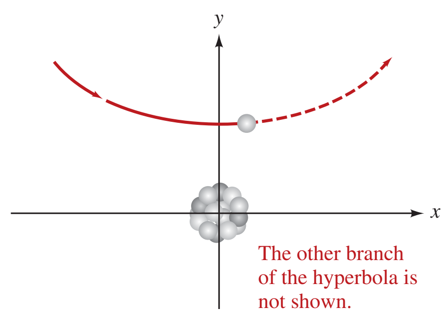


60. $x^2 + y^2 + 8x - 2y - 8 = 0$ $(x + 4)^2 + (y - 1)^2 = 25$

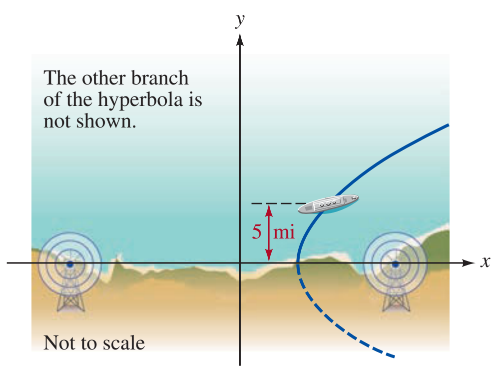


APPLICATIONS

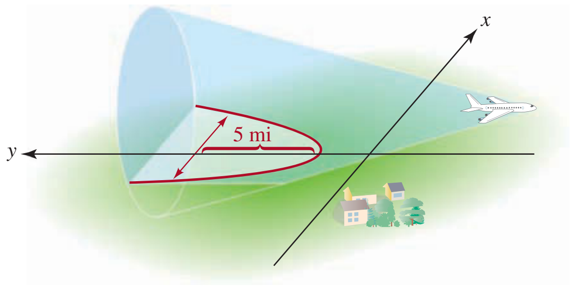
▶ **61.** ALPHA PARTICLES The particle in the illustration below approaches the nucleus at the origin along the path $9y^2 - x^2 = 81$ in the coordinate system shown. How close does the particle come to the nucleus? 3 units



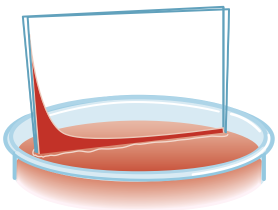
▶ **62.** LORAN By determining the difference of the distances between the ship in the illustration and two radio transmitters, the LORAN navigation system places the ship on the hyperbola $x^2 - 4y^2 = 576$ in the coordinate system shown. If the ship is 5 miles out to sea, find its coordinates. (26, 5)



63. SONIC BOOM The position of a sonic boom caused by the faster-than-sound aircraft is one branch of the hyperbola $y^2 - x^2 = 25$ in the coordinate system shown. How wide is the hyperbola 5 miles from its vertex? $10\sqrt{3}$ miles



64. FLUIDS See the illustration below. Two glass plates in contact at the left, and separated by about 5 millimeters on the right, are dipped in beet juice, which rises by capillary action to form a hyperbola. The hyperbola is modeled by an equation of the form $xy = k$. If the curve passes through the point (12, 2), what is k? 24



WRITING

65. What is a hyperbola?

66. Compare the graphs of $\frac{x^2}{81} - \frac{y^2}{64} = 1$ and $\frac{y^2}{81} - \frac{x^2}{64} = 1$. Do they have any similarities?

67. Explain how to determine the dimensions of the central rectangle that is associated with the graph of

$$\frac{x^2}{36} - \frac{y^2}{25} = 1$$

68. Explain why the graph of the following hyperbola has no y-intercept.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

REVIEW

Find each value of x.

69. $\log_8 x = 2$ 64

70. $\log_{25} x = \frac{1}{2}$ 5

71. $\log_{1/2} \frac{1}{8} = x$ 3

72. $\log_{12} x = 0$ 1

73. $\log_x \frac{9}{4} = 2$ $\frac{3}{2}$

74. $\log_6 216 = x$ 3

75. $\log_x 1{,}000 = 3$ 10

76. $\log_2 \sqrt{2} = x$ $\frac{1}{2}$

SECTION 10.4
Solving Nonlinear Systems of Equations

Objectives

1. Solve systems by graphing.
2. Solve systems by substitution.
3. Solve systems by addition (elimination).

In Chapter 3, we discussed how to solve systems of linear equations by the graphing, substitution, and addition (elimination) methods. In this section, we will use these methods to solve systems where at least one of the equations is nonlinear.

1 Solve systems by graphing.

A solution of a **nonlinear system of equations** is an ordered pair of real numbers that satisfies all of the equations in the system. The **solution set of a nonlinear system** is the set of all such ordered pairs. One way to solve a system of two equations in two variables is to graph the equations on the same rectangular coordinate system.

EXAMPLE 1

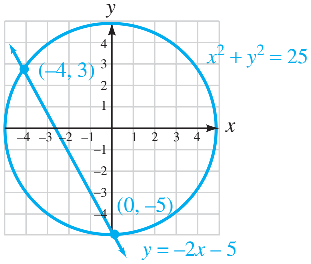
Solve $\begin{cases} x^2 + y^2 = 25 \\ 2x + y = 10 \end{cases}$ by graphing.

Strategy We will graph both equations on the same coordinate system.

WHY If the equations are graphed on the same coordinate system, we can see whether they have any common solutions.

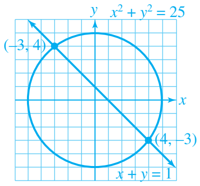
Self Check 1

Solve $\begin{cases} x^2 + y^2 = 25 \\ y = -2x - 5 \end{cases}$ by graphing. $(-4, 3), (0, -5)$



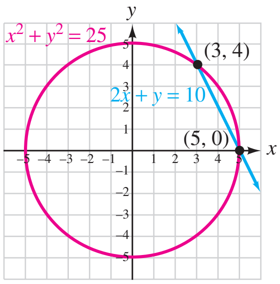
Now Try **Problem 15**

Teaching Example 1 Solve
$\begin{cases} x^2 + y^2 = 25 \\ x + y = 1 \end{cases}$
Answer:



Solution

The graph of $x^2 + y^2 = 25$ is a circle with center at the origin and radius of 5. The graph of $2x + y = 10$ is a line. Depending on whether the line is a **secant** (intersecting the circle at two points) or a **tangent** (intersecting the circle at one point) or does not intersect the circle at all, there are two, one, or no solutions to the system, respectively.



After graphing the circle and the line, it appears that the points of intersection are (5, 0) and (3, 4). To verify that they are solutions of the system, we need to check each one.

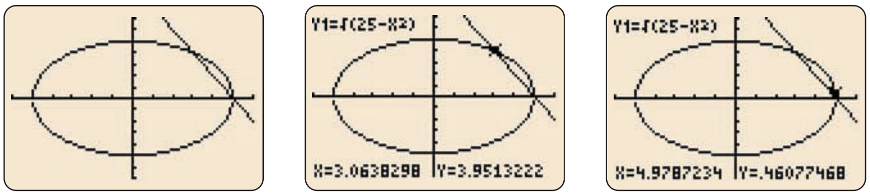
Check:	***For*** **(5, 0)**		***For*** **(3, 4)**	
	$2x + y = 10$	$x^2 + y^2 = 25$	$2x + y = 10$	$x^2 + y^2 = 25$
	$2(5) + 0 \stackrel{?}{=} 10$	$5^2 + 0^2 \stackrel{?}{=} 25$	$2(3) + 4 \stackrel{?}{=} 10$	$3^2 + 4^2 \stackrel{?}{=} 25$
	$10 = 10$	$25 = 25$	$10 = 10$	$25 = 25$
	True	True	True	True

The ordered pair (5, 0) satisfies both equations of the system, and so does (3, 4). Thus, there are two solutions, (5, 0) and (3, 4), and the solution set is {(5, 0), (3, 4)}.

Using Your CALCULATOR **Solving Systems of Equations**

To solve Example 1 with a graphing calculator, we graph the circle and the line on one set of coordinate axes. See figure (a). We then trace to find the coordinates of the intersection points of the graphs. See figures (b) and (c).

We can zoom for better results.



(a) (b) (c)

2 Solve systems by substitution.

When solving a system by graphing, it is often difficult to determine the coordinates of the intersection points. A more precise algebraic method called the **substitution method** can be used to solve certain systems involving nonlinear equations.

Self Check 2

Solve $\begin{cases} x^2 + y^2 = 10 \\ y = x + 2 \end{cases}$ by substitution. $(1, 3), (-3, -1)$

Now Try **Problem 23**

EXAMPLE 2 Solve $\begin{cases} x^2 + y^2 = 2 \\ 2x - y = 1 \end{cases}$ by substitution.

Strategy We will solve the second equation for y and substitute the result for y in the first equation.

WHY We can solve the resulting equation for x and then back substitute to find y.

Teaching Example 2 Solve $\begin{cases} x^2 + y^2 = 25 \\ x - y = 1 \end{cases}$ by substitution.
Answer:
$(-3, -4), (4, 3)$

Solution

This system has one second-degree equation and one first-degree equation. We can solve this type of system by substitution. Solving the linear equation for y gives

$$2x - y = 1$$
$$-y = -2x + 1 \quad \text{Subtract } 2x \text{ from both sides.}$$
$$y = 2x - 1 \quad \text{Multiply both sides by } -1. \text{ We call this the substitution equation.}$$

Because y and $2x - 1$ are equal, we can substitute $2x - 1$ for y in the first equation of the system.

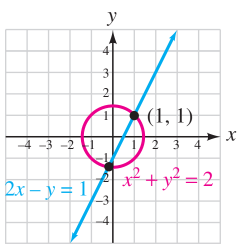
$$y = \mathbf{2x - 1} \qquad x^2 + \mathbf{y}^2 = 2$$

Then we solve the resulting quadratic equation for x.

$$x^2 + y^2 = 2$$
$$x^2 + (2x - 1)^2 = 2 \quad \text{Substitute } 2x - 1 \text{ for } y.$$
$$x^2 + 4x^2 - 4x + 1 = 2 \quad \text{Use a special-product rule to find } (2x - 1)^2.$$
$$5x^2 - 4x - 1 = 0 \quad \text{To get 0 on the right side, subtract 2 from both sides and then combine like terms.}$$
$$(5x + 1)(x - 1) = 0 \quad \text{Factor.}$$
$$5x + 1 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$x = -\frac{1}{5} \qquad\qquad x = 1$$

If we substitute $-\frac{1}{5}$ for x in the equation $y = 2x - 1$, we get $y = -\frac{7}{5}$. If we substitute 1 for x in $y = 2x - 1$, we get $y = 1$. Thus, the system has two solutions, $\left(-\frac{1}{5}, -\frac{7}{5}\right)$ and $(1, 1)$. Verify that each ordered pair satisfies both equations of the original system.

The graph on the right confirms that the system has two solutions, and that one of them is $(1, 1)$. However, it would be virtually impossible to determine from the graph that the coordinates of the second point of intersection are $\left(-\frac{1}{5}, -\frac{7}{5}\right)$.



EXAMPLE 3

Solve: $\begin{cases} 4x^2 + 9y^2 = 5 \\ y = x^2 \end{cases}$

Strategy Since $y = x^2$, we will substitute y for x^2 in the first equation.

WHY This will give an equation in one variable that we can solve for y. We can then find x by back substitution.

Solution

We can solve this system by substitution.

$$4\mathbf{x^2} + 9y^2 = 5 \qquad \mathbf{y} = x^2$$

When we substitute y for x^2 in the first equation, the result is a quadratic equation in y.

$$4x^2 + 9y^2 = 5$$
$$4y + 9y^2 = 5 \quad \text{Substitute } y \text{ for } x^2.$$

Self Check 3

Solve: $\begin{cases} x^2 + y^2 = 20 \\ y = x^2 \end{cases}$

Now Try **Problem 27**

Self Check 3 Answer
$(2, 4), (-2, 4)$

Teaching Example 3 Solve: $\begin{cases} x^2 + y^2 = 25 \\ y = -x^2 + 13 \end{cases}$
Answer:
$(3, 4), (-3, 4), (\sqrt{10}, 4), (-\sqrt{10}, 4)$

$$
\begin{aligned}
9y^2 + 4y - 5 &= 0 && \text{To get 0 on the right side, subtract 5 from both sides.}\\
(9y - 5)(y + 1) &= 0 && \text{Factor } 9y^2 + 4y - 5.
\end{aligned}
$$

$$9y - 5 = 0 \quad \text{or} \quad y + 1 = 0 \qquad \text{Set each factor equal to 0.}$$

$$y = \frac{5}{9} \qquad\qquad y = -1$$

Since $y = x^2$, the values of x are found by solving the equations

$$x^2 = \frac{5}{9} \quad \text{or} \quad \cancel{x^2 = -1}$$

Because $x^2 = -1$ has no real solutions, this possibility is discarded. The solutions of $x^2 = \frac{5}{9}$ are

$$x = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3} \quad \text{or} \quad x = -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{\sqrt{9}} = -\frac{\sqrt{5}}{3}$$

Thus, the solutions of the system are

$$\left(\frac{\sqrt{5}}{3}, \frac{5}{9}\right) \quad \text{and} \quad \left(-\frac{\sqrt{5}}{3}, \frac{5}{9}\right)$$

Caution! In this section, we are solving for only the real values of x and y.

3 Solve systems by addition (elimination).

Another method for solving nonlinear system of equations is the **addition** or **elimination method.** With this method, we combine the equations in a way that will eliminate the terms of one of the variables.

Self Check 4

Solve: $\begin{cases} x^2 + 4y^2 = 16 \\ x^2 - y^2 = 1 \end{cases}$

Now Try **Problem 31**

Self Check 4 Answer

$(2, \sqrt{3}), (2, -\sqrt{3}), (-2, \sqrt{3}), (-2, -\sqrt{3})$

Teaching Example 4 Solve:

$\begin{cases} -3x^2 + 2y^2 = 5 \\ x^2 + y^2 = 25 \end{cases}$

Answer:

$(3, 4), (-3, 4), (3, -4), (-3, -4)$

EXAMPLE 4

Solve: $\begin{cases} 3x^2 + 2y^2 = 36 \\ 4x^2 - y^2 = 4 \end{cases}$

Strategy We will multiply both sides of the second equation by 2 and add the result to the first equation.

WHY This will eliminate the y^2-terms and produce an equation that we can solve for x.

Solution

To solve this system of two second-degree equations, we can use either the substitution or the addition method. We will use the addition method because the y^2-terms can be eliminated by multiplying the second equation by 2 and adding it to the first equation.

$$\begin{cases} 3x^2 + 2y^2 = 36 \\ 4x^2 - y^2 = 4 \end{cases} \begin{matrix} \xrightarrow{\text{Unchanged}} \\ \xrightarrow[\text{Multiply by 2}]{} \end{matrix} \begin{cases} 3x^2 + 2y^2 = 36 \\ 8x^2 - 2y^2 = 8 \end{cases}$$

We add the two equations on the right to eliminate y^2 and solve the resulting equation for x:

$$
\begin{aligned}
11x^2 &= 44\\
x^2 &= 4
\end{aligned}
$$

$$x = 2 \quad \text{or} \quad x = -2$$

To find y, we can substitute 2 for x and then -2 for x into any equation containing both variables. It appears that the computations will be simplest if we use $3x^2 + 2y^2 = 36$.

For $x = 2$

$$\begin{aligned} 3x^2 + 2y^2 &= 36 \\ 3(2)^2 + 2y^2 &= 36 \\ 12 + 2y^2 &= 36 \\ 2y^2 &= 24 \\ y^2 &= 12 \end{aligned}$$

$y = \sqrt{12}$ or $y = -\sqrt{12}$

$y = 2\sqrt{3}$ | $y = -2\sqrt{3}$

For $x = -2$

$$\begin{aligned} 3x^2 + 2y^2 &= 36 \\ 3(-2)^2 + 2y^2 &= 36 \\ 12 + 2y^2 &= 36 \\ 2y^2 &= 24 \\ y^2 &= 12 \end{aligned}$$

$y = \sqrt{12}$ or $y = -\sqrt{12}$

$y = 2\sqrt{3}$ | $y = -2\sqrt{3}$

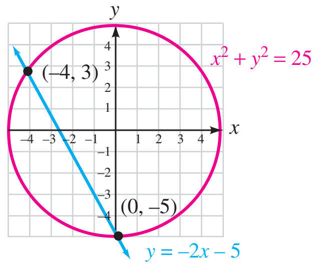
The four solutions of this system are

$$\left(2, 2\sqrt{3}\right), \quad \left(2, -2\sqrt{3}\right), \quad \left(-2, 2\sqrt{3}\right), \quad \text{and} \quad \left(-2, -2\sqrt{3}\right)$$

Success Tip The addition method is generally better than the substitution method when both equations of the system are of the form $Ax^2 + By^2 = C$.

ANSWERS TO SELF CHECKS

1. $(-4, 3), (0, -5)$



2. $(1, 3), (-3, -1)$ **3.** $(2, 4), (-2, 4)$ **4.** $\left(2, \sqrt{3}\right), \left(2, -\sqrt{3}\right), \left(-2, \sqrt{3}\right), \left(-2, -\sqrt{3}\right)$

SECTION 10.4 STUDY SET

VOCABULARY

Fill in the blanks.

1. $\begin{cases} 4x^2 + 6y^2 = 24 \\ 9x^2 - y^2 = 9 \end{cases}$ is a ___system___ of two nonlinear equations.

2. The graph of $2x + y = 10$ is a ___line___ and the graph of $x^2 + y^2 = 25$ is a ___circle___.

3. When solving a system by graphing, it is often difficult to determine the coordinates of the points of ___intersection___ of the graphs.

4. Two algebraic methods for solving systems of nonlinear equations are the ___substitution___ method and the ___addition___ method.

▶ **5.** A ___secant___ is a line that intersects a circle at two points.

▶ **6.** A ___tangent___ is a line that intersects a circle at one point.

▶ Selected exercises available online at
www.webassign.net/brookscole

CONCEPTS

7. a. A line can intersect an ellipse in at most ___two___ points.

b. An ellipse can intersect a parabola in at most ___four___ points.

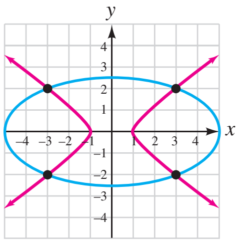
▶ **c.** An ellipse can intersect a circle in at most ___four___ points.

▶ **d.** A hyperbola can intersect a circle in at most ___four___ points.

▶ **8.** Determine whether $(1, -1)$ is a solution of the system:

$$\begin{cases} 2x + y - 1 = 0 \\ x^2 - y^2 = 3 \end{cases}$$ no

9. Find the solutions of the system $\begin{cases} x^2 + 4y^2 = 25 \\ x^2 - 2y^2 = 1 \end{cases}$ that is graphed on the right.
$(-3, 2), (3, 2), (-3, -2), (3, -2)$



10. Find a substitution equation that can be used to solve the system: $\begin{cases} x^2 + y^2 = 9 \\ 2x - y = 3 \end{cases}$ $y = 2x - 3$

11. Consider the system: $\begin{cases} 6x^2 + y^2 = 9 \\ 3x^2 + 4y^2 = 36 \end{cases}$

a. If the y^2-terms are to be eliminated, by what should the first equation be multiplied? -4

b. If the x^2-terms are to be eliminated, by what should the second equation be multiplied? -2

12. Suppose you begin to solve the system $\begin{cases} x^2 + y^2 = 10 \\ 4x^2 + y^2 = 13 \end{cases}$ and find that x is ± 1. Use the first equation to find the corresponding y-values for $x = 1$ and $x = -1$. State the solutions as ordered pairs. $(1, 3), (1, -3), (-1, 3), (-1, -3)$

NOTATION

Complete each solution to solve the system.

13. Solve: $\begin{cases} x^2 + y^2 = 5 \\ y = 2x \end{cases}$

$$x^2 + y^2 = 5 \quad \text{This is the first equation.}$$
$$x^2 + (\boxed{2x})^2 = 5$$
$$x^2 + 4x^2 = \boxed{5}$$
$$\boxed{5}x^2 = 5$$
$$x^2 = \boxed{1}$$
$$x = 1 \quad \text{or} \quad x = -1$$

If $x = 1$, then $y = 2(\boxed{1}) = 2$. Use the second equation.

If $x = -1$, then $y = 2(\boxed{-1}) = -2$.

The solutions are $(1, 2)$ and $(-1, \boxed{-2})$.

14. Solve: $\begin{cases} y = x^2 + 2 \\ y = -x^2 + 4 \end{cases}$

$$2y = \boxed{6} \quad \text{Add the equations.}$$
$$y = \boxed{3}$$

If $y = 3$, then

$$\boxed{3} = x^2 + 2 \quad \text{This is the first equation.}$$
$$1 = x^2$$
$$\boxed{\pm}1 = x$$

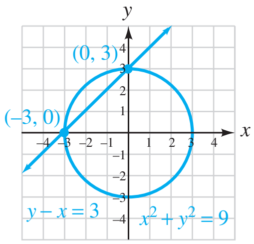
The solutions are

$(1, \boxed{3})$ and $(\boxed{-1}, 3)$

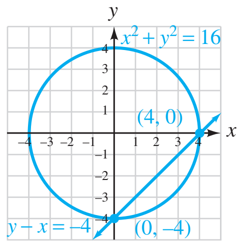
GUIDED PRACTICE

Solve each system of equations by graphing. See Example 1.

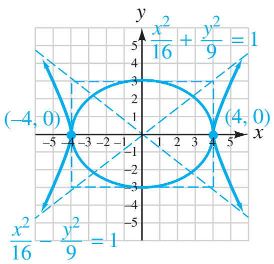
15. $\begin{cases} x^2 + y^2 = 9 \\ y - x = 3 \end{cases}$
$(0, 3), (-3, 0)$



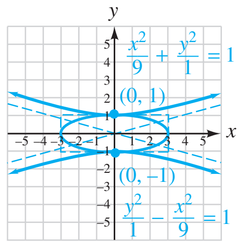
16. $\begin{cases} x^2 + y^2 = 16 \\ y - x = -4 \end{cases}$
$(0, -4), (4, 0)$



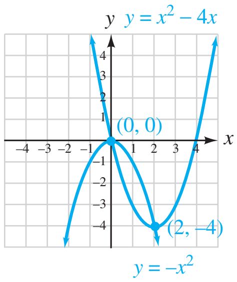
17. $\begin{cases} 9x^2 + 16y^2 = 144 \\ 9x^2 - 16y^2 = 144 \end{cases}$
$(-4, 0), (4, 0)$



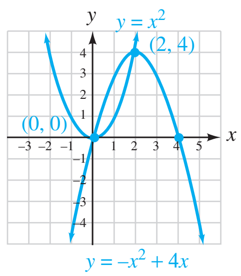
18. $\begin{cases} x^2 + 9y^2 = 9 \\ 9y^2 - x^2 = 9 \end{cases}$
$(0, -1), (0, 1)$



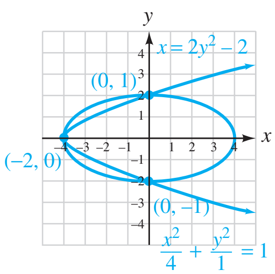
19. $\begin{cases} y = x^2 - 4x \\ x^2 + y = 0 \end{cases}$
$(0, 0), (2, -4)$



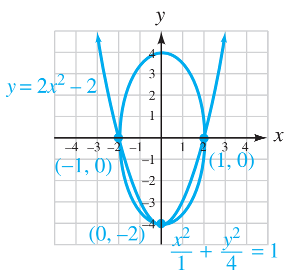
20. $\begin{cases} x^2 - y = 0 \\ y = -x^2 + 4x \end{cases}$
$(0, 0), (2, 4)$



21. $\begin{cases} x^2 + 4y^2 = 4 \\ x = 2y^2 - 2 \end{cases}$
$(-2, 0), (0, -1), (0, 1)$



22. $\begin{cases} 4x^2 + y^2 = 4 \\ y = 2x^2 - 2 \end{cases}$
$(0, -2), (1, 0), (-1, 0)$



Solve each system of equations by substitution for real values of x and y. See Examples 2 and 3.

23. $\begin{cases} x^2 + y^2 = 5 \\ x + y = 3 \end{cases}$ $(1, 2), (2, 1)$

24. $\begin{cases} x^2 - x - y = 2 \\ 4x - 3y = 0 \end{cases}$ $\left(-\frac{2}{3}, -\frac{8}{9}\right), (3, 4)$

25. $\begin{cases} y = x^2 + 6x + 7 \\ 2x + y = -5 \end{cases}$ $(-6, 7), (-2, -1)$

26. $\begin{cases} 2x + y = 1 \\ x^2 + y = 4 \end{cases}$ $(-1, 3), (3, -5)$

27. $\begin{cases} x^2 + y^2 = 13 \\ y = x^2 - 1 \end{cases}$ $(-2, 3), (2, 3)$

28. $\begin{cases} x^2 + y^2 = 10 \\ y = 3x^2 \end{cases}$ $(-1, 3), (1, 3)$

29. $\begin{cases} x^2 + y^2 = 30 \\ y = x^2 \end{cases}$ $\left(\sqrt{5}, 5\right), \left(-\sqrt{5}, 5\right)$

30. $\begin{cases} x^2 + y^2 = 20 \\ y = x^2 \end{cases}$ $(2, 4), (-2, 4)$

Solve each system of equations by addition (elimination) for real values of x and y. See Example 4.

31. $\begin{cases} x^2 + y^2 = 20 \\ x^2 - y^2 = -12 \end{cases}$ $(2, 4), (2, -4), (-2, 4), (-2, -4)$

32. $\begin{cases} x^2 + y^2 = 13 \\ x^2 - y^2 = 5 \end{cases}$ $(3, 2), (3, -2), (-3, 2), (-3, -2)$

33. $\begin{cases} 9x^2 - 7y^2 = 81 \\ x^2 + y^2 = 9 \end{cases}$ $(3, 0), (-3, 0)$

34. $\begin{cases} x^2 + y^2 = 25 \\ 2x^2 - 3y^2 = 5 \end{cases}$ $(4, 3), (-4, 3), (4, -3), (-4, -3)$

35. $\begin{cases} 2x^2 + y^2 = 6 \\ x^2 - y^2 = 3 \end{cases}$ $\left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right)$

36. $\begin{cases} x^2 + y^2 = 36 \\ 49x^2 + 36y^2 = 1{,}764 \end{cases}$ $(6, 0), (-6, 0)$

37. $\begin{cases} x^2 - y^2 = -5 \\ 3x^2 + 2y^2 = 30 \end{cases}$ $(-2, 3), (2, 3), (-2, -3), (2, -3)$

38. $\begin{cases} 6x^2 + 8y^2 = 182 \\ 8x^2 - 3y^2 = 24 \end{cases}$ $(3, 4), (3, -4), (-3, 4), (-3, -4)$

Solve each system. See Using Your Calculator: Solving Systems of Equations.

39. $\begin{cases} x^2 - 6x - y = -5 \\ x^2 - 6x + y = -5 \end{cases}$ $(1, 0), (5, 0)$

40. $\begin{cases} x^2 - y^2 = -5 \\ 3x^2 + 2y^2 = 30 \end{cases}$ $(-2, 3), (2, 3), (-2, -3), (2, -3)$

TRY IT YOURSELF

Solve each system of equations for real values of x and y.

41. $\begin{cases} 2x^2 - 3y^2 = 5 \\ 3x^2 + 4y^2 = 16 \end{cases}$ $(2, 1), (-2, 1), (2, -1), (-2, -1)$

42. $\begin{cases} 2x^2 - y^2 + 2 = 0 \\ 3x^2 - 2y^2 + 5 = 0 \end{cases}$ $(1, 2), (-1, 2), (1, -2), (-1, -2)$

43. $\begin{cases} y = x^2 - 4 \\ x^2 - y^2 = -16 \end{cases}$ $(0, -4), (-3, 5), (3, 5)$

44. $\begin{cases} y - x = 0 \\ 4x^2 + y^2 = 10 \end{cases}$ $\left(-\sqrt{2}, -\sqrt{2}\right), \left(\sqrt{2}, \sqrt{2}\right)$

45. $\begin{cases} 3y^2 = xy \\ 2x^2 + xy - 84 = 0 \end{cases}$ $(6, 2), (-6, -2), \left(-\sqrt{42}, 0\right), \left(\sqrt{42}, 0\right)$

46. $\begin{cases} x^2 + y^2 = 10 \\ 2x^2 - 3y^2 = 5 \end{cases}$ $\left(\sqrt{7}, \sqrt{3}\right), \left(\sqrt{7}, -\sqrt{3}\right), \left(-\sqrt{7}, \sqrt{3}\right), \left(-\sqrt{7}, -\sqrt{3}\right)$

47. $\begin{cases} y^2 = 40 - x^2 \\ y = x^2 - 10 \end{cases}$ $\left(-\sqrt{15}, 5\right), \left(\sqrt{15}, 5\right), (-2, -6), (2, -6)$

48. $\begin{cases} 25x^2 + 9y^2 = 225 \\ 5x + 3y = 15 \end{cases}$ $(3, 0), (0, 5)$

49. $\begin{cases} 3x - y = -3 \\ 25y^2 - 9x^2 = 225 \end{cases}$ $(0, 3), \left(-\frac{25}{12}, -\frac{13}{4}\right)$

50. $\begin{cases} x - 2y = 2 \\ 9x^2 - 4y^2 = 36 \end{cases}$ $(2, 0), \left(-\frac{5}{2}, -\frac{9}{4}\right)$

51. $\begin{cases} x^2 - y = 0 \\ x^2 - 4x + y = 0 \end{cases}$ $(0, 0), (2, 4)$

52. $\begin{cases} xy = -\dfrac{9}{2} \\ 3x + 2y = 6 \end{cases}$ $\left(-1, \frac{9}{2}\right), \left(3, -\frac{3}{2}\right)$

53. $\begin{cases} x^2 - 2y^2 = 6 \\ x^2 + 2y^2 = 2 \end{cases}$ no solution, $\varnothing$

54. $\begin{cases} x^2 + 9y^2 = 1 \\ x^2 - 9y^2 = 3 \end{cases}$ no solution, $\varnothing$

55. $\begin{cases} y = x^2 - 4 \\ 6x - y = 13 \end{cases}$ $(3, 5)$

56. $\begin{cases} y = x + 1 \\ x^2 - y^2 = 1 \end{cases}$ $(-1, 0)$

57. $\begin{cases} x^2 + y^2 = 4 \\ 9x^2 + y^2 = 9 \end{cases}$ $\left(\frac{\sqrt{10}}{4}, \frac{3\sqrt{6}}{4}\right)\left(\frac{\sqrt{10}}{4}, -\frac{3\sqrt{6}}{4}\right), \left(-\frac{\sqrt{10}}{4}, \frac{3\sqrt{6}}{4}\right), \left(-\frac{\sqrt{10}}{4}, -\frac{3\sqrt{6}}{4}\right)$

58. $\begin{cases} 2x^2 - 6y^2 + 3 = 0 \\ 4x^2 + 3y^2 = 4 \end{cases}$ $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right)$

59. $\begin{cases} xy = \dfrac{1}{6} \\ y + x = 5xy \end{cases}$ $\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right)$

60. $\begin{cases} xy = \dfrac{1}{12} \\ y + x = 7xy \end{cases}$ $\left(\frac{1}{4}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{4}\right)$

61. $\begin{cases} x^2 = 4 - y \\ y = x^2 + 2 \end{cases}$ $(-1, 3), (1, 3)$

62. $\begin{cases} 3x + 2y = 10 \\ y = x^2 - 5 \end{cases}$ $\left(\frac{5}{2}, \frac{5}{4}\right), (-4, 11)$

63. $\begin{cases} x^2 - y^2 = 4 \\ x + y = 4 \end{cases}$ $\left(\frac{5}{2}, \frac{3}{2}\right)$

64. $\begin{cases} x - y = -1 \\ y^2 - 4x = 0 \end{cases}$ $(1, 2)$

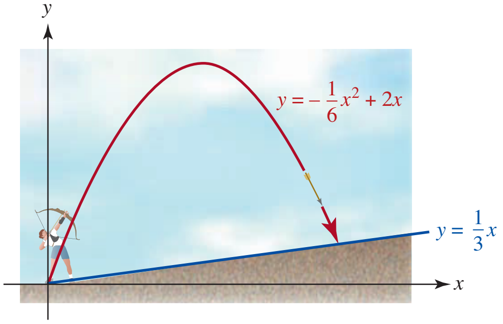
APPLICATIONS

Use a nonlinear system of equations to solve each problem.

65. INTEGER PROBLEM The product of two integers is 32, and their sum is 12. Find the integers. 4, 8

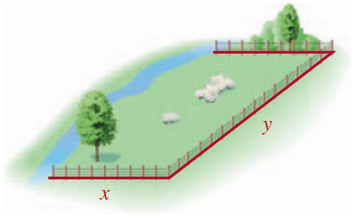
66. NUMBER PROBLEM The sum of the squares of two numbers is 221, and the sum of the numbers is 9. Find the numbers. 14, −5

▶ **67.** ARCHERY See the illustration. An arrow shot from the base of a hill follows the parabolic path $y = -\frac{1}{6}x^2 + 2x$, with distances measured in meters. The inclined hill has a slope of $\frac{1}{3}$ and can therefore be modeled by the equation $y = \frac{1}{3}x$. Find the coordinates of the point of impact of the arrow and then its distance from the archer. $\left(10, \frac{10}{3}\right), \frac{10}{3}\sqrt{10}$ m



▶ **68.** GEOMETRY The area of a rectangle is 63 square centimeters, and its perimeter is 32 centimeters. Find the dimensions of the rectangle. 7 cm by 9 cm

▶ **69.** FENCING PASTURES The rectangular pasture shown here is to be fenced in along a riverbank. If 260 feet of fencing is to enclose an area of 8,000 square feet, find the dimensions of the pasture. 80 ft by 100 ft or 50 ft by 160 ft



▶ **70.** DRIVING RATES Jim drove 306 miles. Jim's brother made the same trip at a speed 17 mph slower than Jim did and required an extra $1\frac{1}{2}$ hours. What was Jim's rate and time? 68 mph, 4.5 hr

▶ **71.** INVESTING Grant receives \$225 annual income from one investment. Jeff invested \$500 more than Grant, but at an annual rate of 1% less. Jeff's annual income is \$240. What are the amount and rate of Grant's investment? \$2,500 at 9%

▶ **72.** INVESTING Carol receives \$67.50 annual income from one investment. John invested \$150 more than Carol at an annual rate of $1\frac{1}{2}$% more. John's annual income is \$94.50. What are the amount and rate of Carol's investment? (*Hint:* There are two answers.) Either \$750 at 9% or \$900 at 7.5%

WRITING

▶ **73.** **a.** Describe the benefits of the graphical method for solving a system of equations.

b. Describe the drawbacks of the graphical method.

▶ **74.** Explain why the elimination method, not the substitution method, is the better method to solve the system

$$\begin{cases} 4x^2 + 9y^2 = 52 \\ 9x^2 + 4y^2 = 52 \end{cases}$$

REVIEW

Solve each equation.

75. $\log 5x = 4$ 2,000

▶ **76.** $\log 3x = \log 9$ 3

77. $\dfrac{\log (8x - 7)}{\log x} = 2$ 7

78. $\log x + \log (x + 9) = 1$ 1

STUDY SKILLS CHECKLIST

Preparing for the Chapter 10 Test

The material in Chapter 10 on conic sections is very closely related and can be easily confused. It is important to compare and contrast the forms of the equations to be successful with this material. As you prepare for the exam over this material, make sure you also review the following checklist.

- ☐ The equation of a circle contains both x^2 and y^2 terms on the same side of the equation with equal coefficients.

  $3x^2 + 3y^2 - 12x + 24y = 6$ is the equation of a circle

  $5x^2 - 5y^2 + 10x + 20y = 5$ is not the equation of a circle

- ☐ The equation of a parabola contains only one variable squared term.

  $2x^2 + y = 12x - 15$ is the equation of a parabola

  $x - y + x^2 = 2 - y^2$ is not the equation of a parabola

- ☐ The equation of an ellipse contains both x^2 and y^2 on the same side of the equation with coefficients that have the same sign.

  $3x^2 + 4y^2 - 12x + 24y = 6$ is the equation of an ellipse

  $5x^2 - y^2 + 10x + 20y = 5$ is not the equation of an ellipse

- ☐ The equation of a hyperbola contains both x^2 and y^2 on the same side of the equation with coefficients that have opposite signs.

  $5x^2 - y^2 + 10x + 20y = 5$ is the equation of a hyperbola

  $-5x^2 - y^2 + 10x + 20y = -5$ is not the equation of a hyperbola

- ☐ To find the standard form of the equation of an ellipse, complete the square on each of the variables.

  Find the standard form of:

  $$4x^2 + 8x + 9y^2 - 18y = 23$$
  $$4(x^2 + 2x \quad) + 9(y^2 - 2y \quad) = 23$$
  $$4(x^2 + 2x + 1) + 9(y^2 - 2y + 1) = 23 + 4 + 9$$
  $$4(x + 1)^2 + 9(y - 1)^2 = 36$$
  $$\frac{(x + 1)^2}{9} + \frac{(y - 1)^2}{4} = 1$$

- ☐ With a hyperbola, the positive variable term in the standard-form equation determines whether a hyperbola is vertical or horizontal. In this example, the positive variable term involves y, so the hyperbola is vertical.

  $$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

- ☐ When solving a system of equations that involves a nonlinear equation and a linear equation, use the linear equation as the substitution equation.

- ☐ When solving a system of equations that involves two equations of the form $Ax^2 + By^2 = C$, the addition method is generally better to use than the substitution method.

Teaching Guide: Refer to the Instructor's Resource Binder to find activities, worksheets on key concepts, more examples, instruction tips, overheads, and assessments.

SECTION 10.1 The Circle and the Parabola

DEFINITIONS AND CONCEPTS

A **circle** is the set of all points in a plane that are a fixed distance from a fixed point called its **center.** The fixed distance is called the **radius** of the circle.

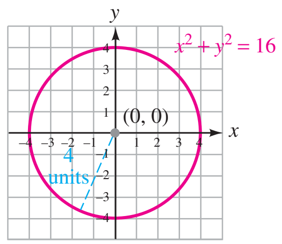
Standard forms of the equation of a circle:

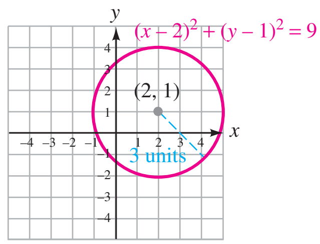
$x^2 + y^2 = r^2$ Center (0, 0), radius r

$(x - h)^2 + (y - k)^2 = r^2$ Center (h, k), radius r

EXAMPLES

The graph of the equation $x^2 + y^2 = 16$, which can be written $x^2 + y^2 = 4^2$, is a circle with center at (0, 0) and a radius of **4.**





The graph of the equation $(x - 2)^2 + (y - 1)^2 = 9$, which can be written $(x - \mathbf{2})^2 + (y - \mathbf{1})^2 = \mathbf{3}^2$, is a circle with center at (**2**, **1**) and a radius of **3.**

DEFINITIONS AND CONCEPTS

Because a circle is determined by its center and radius, that information is all we need to know to write its equation.

EXAMPLES

Write the equation of a circle centered at (4, −3) and with a radius of 5.

In this problem, $h = 4$, $k = -3$, and $r = 5$. We substitute these values into the standard form of the equation of a circle and simplify.

$$(x - \mathbf{h})^2 + (y - \mathbf{k})^2 = \mathbf{r}^2$$
$$(x - \mathbf{4})^2 + [y - (\mathbf{-3})]^2 = \mathbf{5}^2$$
$$(x - 4)^2 + (y + 3)^2 = 25$$

DEFINITIONS AND CONCEPTS

A **parabola** is the set of all points in a plane that are equidistant from a fixed point, called the **focus,** and a fixed line, called the **directrix.**

General forms of the equation of a parabola:

$y = ax^2 + bx + c$ $a > 0$: up; $a < 0$: down

$x = ay^2 + by + c$ $a > 0$: right; $a < 0$: left

Standard forms of the equation of a parabola:

$y = a(x - h)^2 + k$ $a > 0$: up; $a < 0$: down

Vertex at (h, k) Axis of symmetry is $x = h$

$x = a(y - k)^2 + h$ $a > 0$: right; $a < 0$: left

Vertex at (h, k) Axis of symmetry is $y = k$

EXAMPLES

The equation $x = 2y^2 - 4y + 5$ is the equation of a parabola that opens to the right. To find its vertex and axis of symmetry, we complete the square on y and write the equation in standard form.

$x = 2y^2 - 4y + 5$

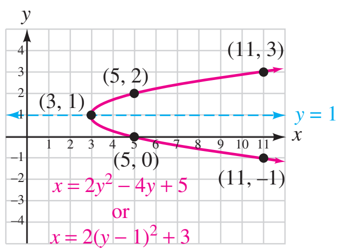
$x = 2(y^2 - 2y \qquad) + 5$ Factor out 2.

$x = 2(y^2 - 2y + \mathbf{1}) + 5 - 2$ Complete the square.

$x = 2(y - 1)^2 + 3$ Factor and simplify.

From the standard form, we see that $h = 3$ and $k = 1$. Thus, the vertex is at (3, 1) and the axis of symmetry is $y = 1$. To construct a table of solutions, we choose values of y and find their corresponding values of x.

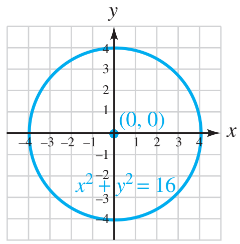
$x = 2(y - 1)^2 + 3$		
x	y	
5	2	→ (5, 2)
11	3	→ (11, 3)



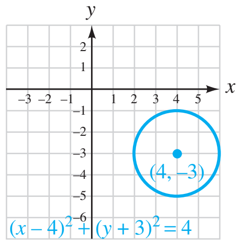
REVIEW EXERCISES

Graph each equation.

1. $x^2 + y^2 = 16$

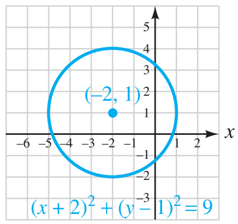


2. $(x - 4)^2 + (y + 3)^2 = 4$

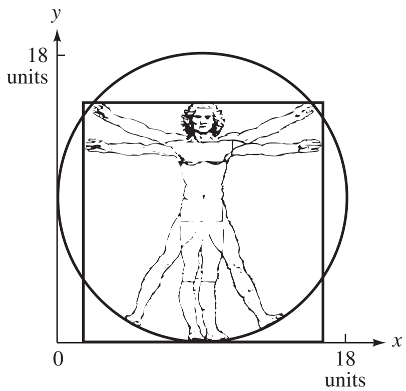


3. Write the equation in standard form and graph it.

$$x^2 + y^2 + 4x - 2y = 4 \qquad (x + 2)^2 + (y - 1)^2 = 9$$



4. ART HISTORY Leonardo da Vinci's *Vitruvian Man* (1492) is one of the most famous pen-and-ink drawings of all time. Use the coordinate system that is superimposed on the drawing to write the equation of the circle in standard form.
$(x - 9)^2 + (y - 9)^2 = 9^2$ or $(x - 9)^2 + (y - 9)^2 = 81$

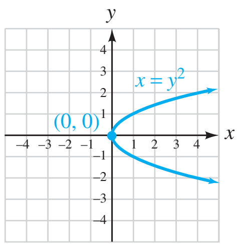


5. Find the center and the radius of the circle whose equation is $(x + 6)^2 + y^2 = 24$. $(-6, 0), r = 2\sqrt{6}$

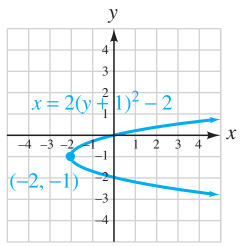
6. Fill in the blanks: A circle is the set of all points in a plane that are a fixed distance from a point called its _center_. The fixed distance is called the _radius_ of the circle.

Graph each parabola and give the coordinates of the vertex.

7. $x = y^2$
$(0, 0)$

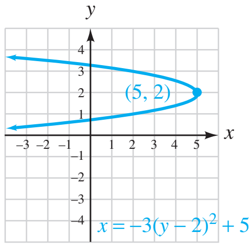


8. $x = 2(y + 1)^2 - 2$
$(-2, -1)$

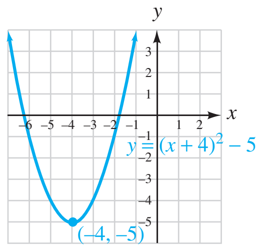


Write each equation in standard form and graph it.

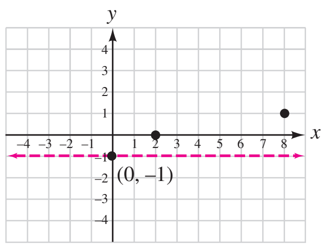
9. $x = -3y^2 + 12y - 7$
$x = -3(y - 2)^2 + 5$



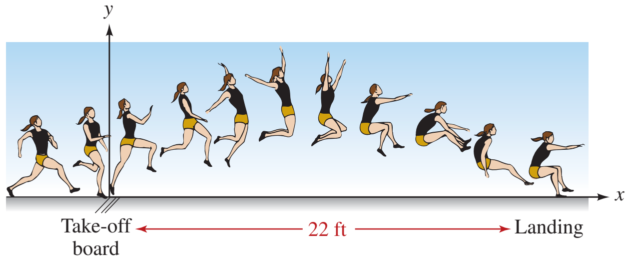
10. $y = x^2 + 8x + 11$
$y = (x + 4)^2 - 5$



11. The axis of symmetry, the vertex, and two additional points on the graph of a parabola are shown. Find the coordinates of two other points on the parabola.
$(2, -2), (8, -3)$



12. LONG JUMP The equation describing the flight path of the long jumper is $y = -\frac{5}{121}(x - 11)^2 + 5$. Show that she will land at a point 22 feet away from the take-off board.
When $x = 22$, $y = 0$: $-\frac{5}{121}(22 - 11)^2 + 5 = 0$



SECTION 10.2 The Ellipse

DEFINITIONS AND CONCEPTS

An **ellipse** is the set of all points in a plane for which the sum of the distances from two fixed points is a constant.

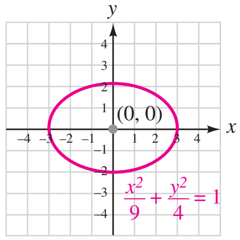
Standard forms of the equation of an ellipse:

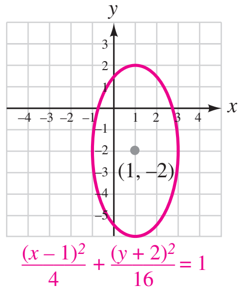
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Center } (0, 0)$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad \text{Center } (h, k)$$

EXAMPLES

The equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$, which can be written $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$, represents an ellipse that is centered at the origin. Here, $a = 3$ and $b = 2.$





The equation $\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} = 1$, which can be written $\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{4^2} = 1$, represents an ellipse that is centered at $(1, -2)$. Here, $a = 2$ and $b = 4.$

To write the equation $25x^2 + 16y^2 = 400$ in standard form, divide both sides by 400 and simplify.

$$25x^2 + 16y^2 = 400$$

$$\frac{25x^2}{400} + \frac{16y^2}{400} = \frac{400}{400} \qquad \text{To get 1 on the right side, divide both sides by 400.}$$

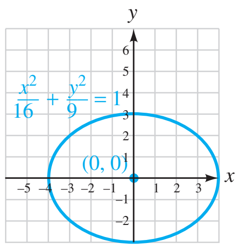
$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \qquad \text{Simplify each fraction.}$$

This result represents an ellipse that is centered at $(0, 0)$, with $a = 4$ and $b = 5$.

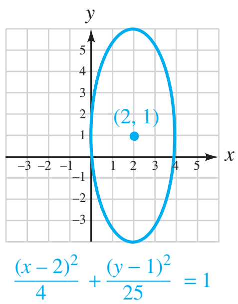
REVIEW EXERCISES

Graph each ellipse.

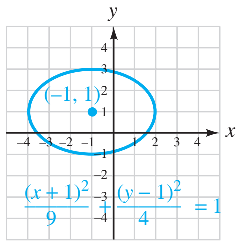
13. $\frac{x^2}{16} + \frac{y^2}{9} = 1$



14. $\frac{(x - 2)^2}{4} + \frac{(y - 1)^2}{25} = 1$



15. $4(x + 1)^2 + 9(y - 1)^2 = 36$



16. Consider the equation $\frac{x^2}{144} + \frac{y^2}{1} = 1$. Write each term on the left side with a denominator that is the square of a number. $\frac{x^2}{12^2} + \frac{y^2}{1^2} = 1$

17. Determine whether the graph of each equation is a circle, a parabola, or an ellipse.

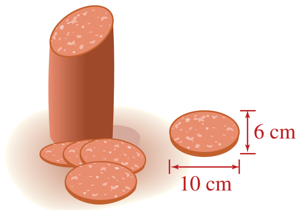
a. $(x - 1)^2 + (y + 9)^2 = 100$ circle

b. $\dfrac{x^2}{49} + \dfrac{y^2}{121} = 1$ ellipse

c. $x = y^2 - 2y + 6$ parabola

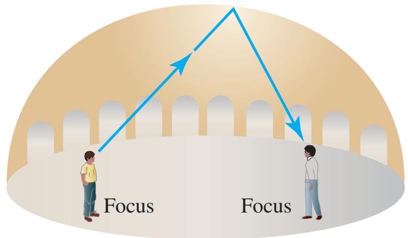
d. $16(x - 4)^2 + 4(y + 8)^2 = 16$ ellipse

18. SALAMI When a delicatessen slices a cylindrical salami at an angle, the results are elliptical pieces that are larger than circular pieces. Write the equation of the shape of the slice of salami shown if it was centered at the origin of a coordinate system. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

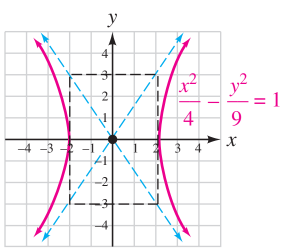
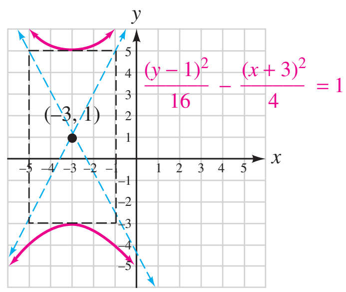


19. Fill in the blanks: An ellipse is the set of all points in a plane for which the sum of the distances from two fixed points is a constant. Each of the fixed points is called a focus.

20. CONSTRUCTION Sketch the path of the sound when a person, standing at one focus, whispers something in the whispering gallery dome shown below. Answers may vary.



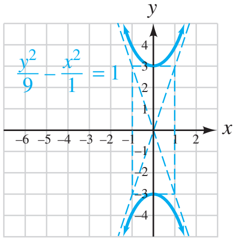
SECTION 10.3 The Hyperbola

DEFINITIONS AND CONCEPTS	EXAMPLES
A **hyperbola** is the set of all points in a plane for which the difference of the distances from two fixed points is a constant. ***Standard forms of the equation of a hyperbola:*** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ Center (0, 0), opens left and right $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ Center (0, 0), opens up and down $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ Center (h, k), opens left and right $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ Center (h, k), opens up and down	The equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$, which can be written $\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$, represents a hyperbola, centered at (0, 0), that opens left and right. Here, $a = \mathbf{2}$ and $b = \mathbf{3}$.   The equation $\frac{(y-1)^2}{16} - \frac{(x+3)^2}{4} = 1$, which can be written $\frac{(y-1)^2}{4^2} - \frac{(x+3)^2}{2^2} = 1$, represents a hyperbola, centered at $(-3, 1)$, that opens up and down. Here, $a = \mathbf{4}$ and $b = \mathbf{2}$.
	To write the equation $25y^2 - 9x^2 = 225$ in standard form, divide both sides by 225 and simplify. $25y^2 - 9x^2 = 225$ $\dfrac{25y^2}{\mathbf{225}} - \dfrac{9x^2}{\mathbf{225}} = \dfrac{225}{\mathbf{225}}$ To get 1 on the right side, divide both sides by 225. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$ Simplify each fraction. This result represents a hyperbola centered at the origin that opens up and down. Here, $a = 3$ and $b = 5$.

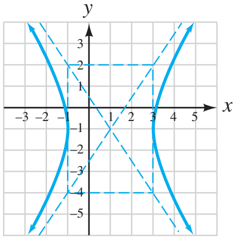
REVIEW EXERCISES

Graph each hyperbola.

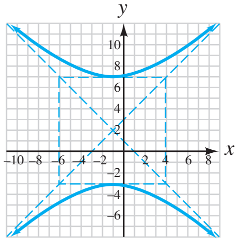
21. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$



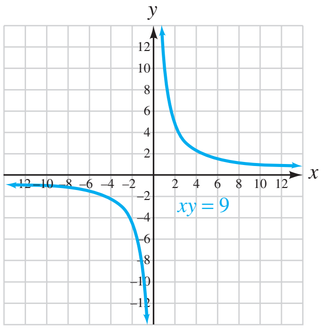
22. $9(x - 1)^2 - 4(y + 1)^2 = 36$



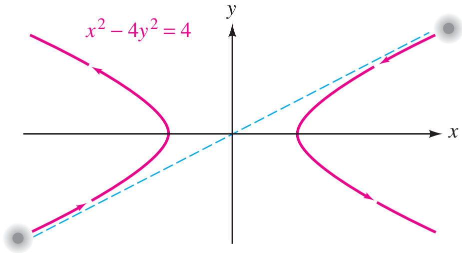
23. $\dfrac{(y - 2)^2}{25} - \dfrac{(x + 1)^2}{25} = 1$



24. $xy = 9$



25. ELECTROSTATIC REPULSION Two similarly charged particles are shot together for an almost head-on collision, as in the illustration. They repel each other and travel the two branches of the hyperbola given by $x^2 - 4y^2 = 4$ on the given coordinate system. How close do they get? 4 units



26. Determine whether the graph of each equation will be a circle, parabola, ellipse, or hyperbola.

a. $\dfrac{(x - 4)^2}{16} + \dfrac{y^2}{49} = 1$ ellipse

b. $16(x + 3)^2 - 4(y - 1)^2 = 64$ hyperbola

c. $x = -4y^2 - y + 1$ parabola

d. $x^2 + 2x + y^2 - 4y = 40$ circle

SECTION 10.4 Solving Nonlinear Systems of Equations

DEFINITIONS AND CONCEPTS	EXAMPLES

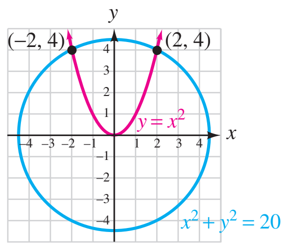
DEFINITIONS AND CONCEPTS

A **nonlinear system of equations** is a system that contains at least one nonlinear equation.

Systems of nonlinear equations are solved by **graphing,** by **substitution,** or by **addition (elimination).**

EXAMPLES

To solve the nonlinear system $\begin{cases} x^2 + y^2 = 20 \\ y = x^2 \end{cases}$ by graphing, we graph the equations on the same rectangular coordinate system, and determine the coordinates of the points of intersection of the graphs.



Since the points of intersection of the graphs are $(-2, 4)$ and $(2, 4)$, the solutions of the system are $(-2, 4)$ and $(2, 4)$.

DEFINITIONS AND CONCEPTS

With the **addition (elimination),** the objective is to use an appropriate substitution to obtain *one* equation in *one* variable.

EXAMPLES

To use substitution to solve the nonlinear system $\begin{cases} y = 3x - 5 \\ x^2 + y^2 = 5 \end{cases}$, we substitute $3x - 5$ for y in the second equation and solve for x.

$$\begin{aligned} x^2 + \mathbf{y}^2 &= 5 \\ x^2 + (\mathbf{3x - 5})^2 &= 5 && \text{This is a quadratic equation in x.} \\ x^2 + 9x^2 - 30x + 25 &= 5 \\ 10x^2 - 30x + 20 &= 0 && \text{Combine terms and subtract 5 from both sides.} \\ x^2 - 3x + 2 &= 0 && \text{Divide both sides by 10.} \\ (x - 2)(x - 1) &= 0 && \text{Factor.} \end{aligned}$$

$$x - 2 = 0 \quad \text{or} \quad x - 1 = 0 \qquad \text{Set each factor equal to 0.}$$
$$x = 2 \quad | \quad x = 1$$

If $x = 2$, then $y = 3\mathbf{x} - 5 = 3(\mathbf{2}) - 5 = 1$.

If $x = 1$, then $y = 3\mathbf{x} - 5 = 3(\mathbf{1}) - 5 = -2$.

The two solutions of the system are $(2, 1)$ and $(1, -2)$.

DEFINITIONS AND CONCEPTS

With the **addition (elimination) method,** we combine the equations in a way that will eliminate the terms of one of the variables.

EXAMPLES

To use addition to solve the nonlinear system $\begin{cases} x^2 - y = 0 \\ x + y = 0 \end{cases}$, we add the equations to get $x^2 + x = 0$. Then we factor this result to get $x = 0$ or $x = -1$. We can substitute these values into the second equation to find y.

$$\begin{aligned} \text{If } x = 0\text{: } \mathbf{x} + y &= 0 && \text{This is the second equation.} \\ \mathbf{0} + y &= 0 && \text{Substitute 0 for x.} \\ y &= 0 \end{aligned}$$

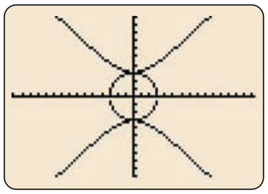
$$\begin{aligned} \text{If } x = -1\text{: } \mathbf{x} + y &= 0 && \text{This is the second equation.} \\ \mathbf{-1} + y &= 0 && \text{Substitute −1 for x.} \\ y &= 1 \end{aligned}$$

The two solutions of the system are $(0, 0)$ and $(-1, 1)$.

REVIEW EXERCISES

27. Determine whether $\left(-\sqrt{11}, -3\right)$ is a solution of the system: $\begin{cases} x^2 + y^2 = 20 \\ x^2 - y^2 = 2 \end{cases}$ yes

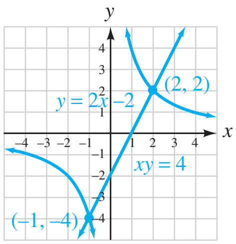
28. The graphs of $y^2 - x^2 = 9$ and $x^2 + y^2 = 9$ are shown. Estimate the solutions of the system



$$\begin{cases} y^2 - x^2 = 9 \\ x^2 + y^2 = 9 \end{cases}$$

$(0, 3), (0, -3)$

29. Solve the system $\begin{cases} xy = 4 \\ y = 2x - 2 \end{cases}$ by graphing.
$(2, 2), (-1, -4)$



30. Determine the maximum number of solutions there could be for a system of equations consisting of the given curves.

a. A line and an ellipse 2

b. Two hyperbolas 4

c. An ellipse and a circle 4

d. A parabola and a circle 4

31. Suppose the x-coordinate of both points of intersection of the circle, represented by $x^2 + y^2 = 1$, and the hyperbola, defined by $4y^2 - x^2 = 4$, is 0. Without graphing, determine the y-coordinates of both points of intersection. Express the answers as ordered-pair solutions.
$(0, 1), (0, -1)$

32. Find a substitution equation that can be used to solve $\begin{cases} x^2 + y^2 = 16 \\ 3x - y = 1 \end{cases}$. Do not solve the system.
$y = 3x - 1$

Solve each system for real values of x and y.

33. $\begin{cases} y^2 - x^2 = 16 \\ y + 4 = x^2 \end{cases}$ $(0, -4), (-3, 5), (3, 5)$

34. $\begin{cases} y = -x^2 + 2 \\ x^2 - y - 2 = 0 \end{cases}$ $\left(\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$

35. $\begin{cases} x^2 + 2y^2 = 12 \\ 2x - y = 2 \end{cases}$ $(2, 2), \left(-\frac{2}{9}, -\frac{22}{9}\right)$

36. $\begin{cases} 3x^2 + y^2 = 52 \\ x^2 - y^2 = 12 \end{cases}$ $(4, 2), (4, -2), (-4, 2), (-4, -2)$

37. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{12} = 1 \\ \dfrac{x^2}{1} - \dfrac{y^2}{3} = 1 \end{cases}$ $(2, 3), (2, -3), (-2, 3), (-2, -3)$

38. $\begin{cases} xy = 4 \\ \dfrac{x^2}{1} + \dfrac{y^2}{2} = 9 \end{cases}$ $\left(2\sqrt{2}, \sqrt{2}\right), \left(-2\sqrt{2}, -\sqrt{2}\right), (1, 4), (-1, -4)$

39. $\begin{cases} y = -x^2 + 1 \\ x + y = 5 \end{cases}$ no solution; $\varnothing$

40. $\begin{cases} x = y^2 - 3 \\ x = y^2 - 3y \end{cases}$ $(-2, 1)$

CHAPTER 10 TEST

1. Fill in the blanks.

a. The curves formed by the intersection of a plane with an infinite right-circular cone are called conic sections.

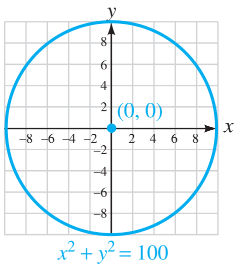
b. A circle is the set of all points in a plane that are a fixed distance from a point called its center. The fixed distance is called the radius of the circle.

c. The standard form for the equation of a(n) hyperbola centered at the origin that opens left and right is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

d. $\begin{cases} y = x^2 + x - 4 \\ x^2 + y^2 = 36 \end{cases}$ is a(n) nonlinear system of equations.

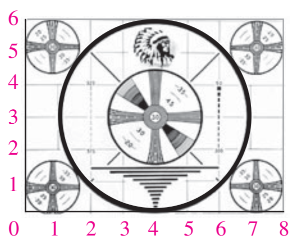
e. The standard form for the equation of a(n) ellipse centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2. Find the center and the radius of the circle represented by the equation $x^2 + y^2 = 100$ and graph it. $(0, 0)$, $r = 10$



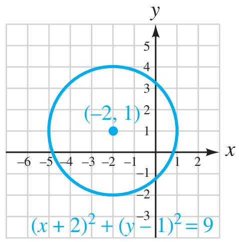
3. Find the center and the radius of the circle represented by the equation $x^2 + y^2 + 4x - 6y = 5$. $(-2, 3)$, $r = 3\sqrt{2}$

4. TV HISTORY In the early days of television, stations broadcast a black-and-white test pattern like that shown here during the early morning hours. Use the given coordinate system to write an equation of the large, bold circle in the center of the pattern. $(x - 4)^2 + (y - 3)^2 = 9$

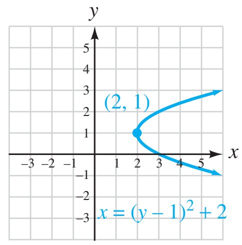


Graph each equation.

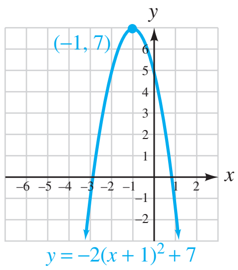
5. $(x + 2)^2 + (y - 1)^2 = 9$



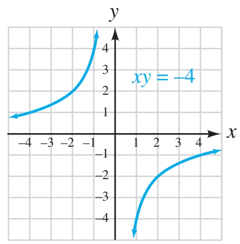
6. $x = y^2 - 2y + 3$



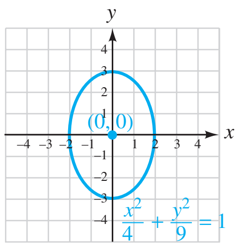
7. $y = -2x^2 - 4x + 5$



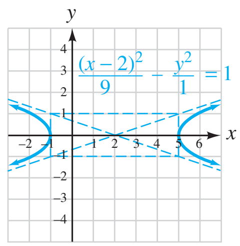
8. $xy = -4$



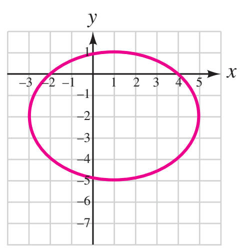
9. $9x^2 + 4y^2 = 36$



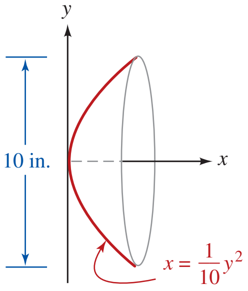
10. $\frac{(x - 2)^2}{9} - \frac{y^2}{1} = 1$



11. Write the equation in standard form of the ellipse graphed here. $\frac{(x - 1)^2}{16} + \frac{(y + 2)^2}{9} = 1$



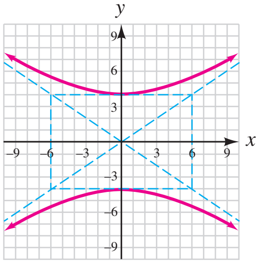
12. LIGHT The cross section of a parabolic mirror is given by the equation $x = \frac{1}{10}y^2$, with distances measured in inches. If the dish is 10 inches wide, how deep is it?
2.5 in.



13. Give an example of the reflective properties of an ellipse. Include a drawing and label it completely.
answers may vary.

14. Find the center and the length and width of the central rectangle of the graph of $(x + 1)^2 - (y - 1)^2 = 4$.
$(-1, 1)$; length: 4 units, width: 4 units

15. Find the equation in standard form of the hyperbola graphed here.
$\frac{y^2}{16} - \frac{x^2}{36} = 1$



16. Determine whether the graph of each equation will be a circle, a parabola, an ellipse, or a hyperbola.

a. $25x^2 + 100y^2 = 400$ ellipse

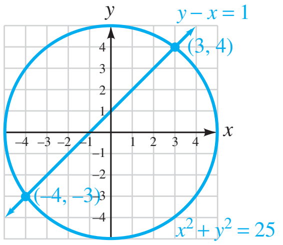
b. $9x^2 - y^2 = 9$ hyperbola

c. $x^2 + 8x + y^2 - 16y - 1 = 0$ circle

d. $x = 8y^2 - 9y + 4$ parabola

Solve each system graphically.

17. $\begin{cases} x^2 + y^2 = 25 \\ y - x = 1 \end{cases}$
$(-4, -3), (3, 4)$



Solve each system for real values of x and y.

18. $\begin{cases} 2x - y = -2 \\ x^2 + y^2 = 16 + 4y \end{cases}$ $(2, 6), (-2, -2)$

19. $\begin{cases} 5x^2 - y^2 - 3 = 0 \\ x^2 + 2y^2 = 5 \end{cases}$ $(1, \sqrt{2}), (1, -\sqrt{2}), (-1, \sqrt{2}), (-1, -\sqrt{2})$

20. $\begin{cases} xy = -\dfrac{9}{2} \\ 3x + 2y = 6 \end{cases}$ $\left(-1, \frac{9}{2}\right), \left(3, -\frac{3}{2}\right)$

21. $\begin{cases} y = x + 1 \\ x^2 - y^2 = 1 \end{cases}$ $(-1, 0)$

22. $\begin{cases} x^2 + 3y^2 = 6 \\ x^2 + y = 8 \end{cases}$ no solution; $\varnothing$

CHAPTERS 1–10 CUMULATIVE REVIEW

Consider the set $\left\{-\frac{4}{3}, \pi, 5.6, \sqrt{2}, 0, -23, e, 7i\right\}$. *List the elements in the set that are* [Section 1.2]

1. whole numbers 0

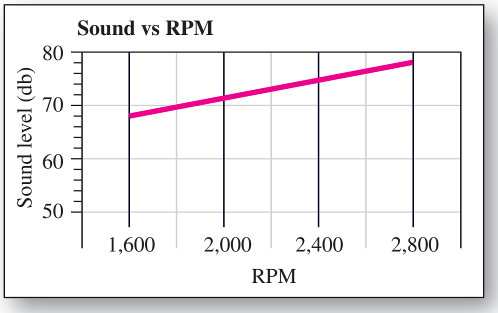
2. rational numbers $-\frac{4}{3}, 5.6, 0, -23$

3. irrational numbers $\pi, \sqrt{2}, e$

4. real numbers $-\frac{4}{3}, \pi, 5.6, \sqrt{2}, 0, -23, e$

5. FINANCIAL PLANNING Ana has some money to invest. Her financial planner tells her that if she can come up with \$3,000 more, she will qualify for an 11% annual interest rate. Otherwise, she will have to invest the money at 7.5% annual interest. The financial planner urges her to invest the larger amount, because the 11% investment would yield twice as much annual income as the 7.5% investment. How much does she originally have on hand to invest? [Section 1.8] \$8,250

6. BOATING Use the graph below to determine the average rate of change in the sound level of the engine of a boat in relation to rpm of the engine. [Section 2.1] $\frac{1}{120}$ db/rpm



Determine whether the graphs of the equations are parallel or perpendicular. [Section 2.3]

7. $3x - 4y = 12, y = \frac{3}{4}x - 5$ parallel

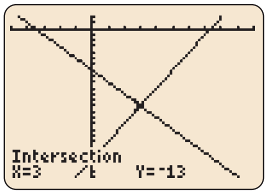
8. $y = 3x + 4, x = -3y + 4$ perpendicular

Write an equation of the line with the given properties in slope–intercept form. [Section 2.4]

9. $m = -2$, passing through $(0, 5)$ $y = -2x + 5$

10. Passing through $(8, -5)$ and $(-5, 4)$ $y = -\frac{9}{13}x + \frac{7}{13}$

11. The graphs of $y = 4(x - 5) - x - 2$ and $y = -(2x + 6) - 1$ are shown. Use the information in the display to solve $4(x - 5) - x - 2 = -(2x + 6) - 1$ graphically. [Section 2.6] 3



12. Use substitution to solve: $\begin{cases} 3x + y = 4 \\ 2x - 3y = -1 \end{cases}$

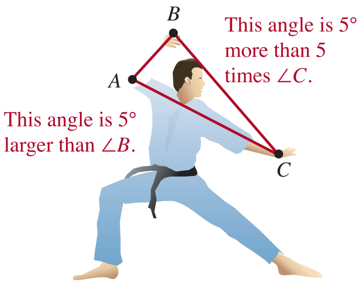
[Section 3.2] $(1, 1)$

13. Use addition to solve: $\begin{cases} x + 2y = -2 \\ 2x - y = 6 \end{cases}$

[Section 3.2] $(2, -2)$

14. Solve: $\begin{cases} b + 2c = 7 - a \\ a + c = 8 - 2b \\ 2a + b + c = 9 \end{cases}$ [Section 3.4] $(3, 2, 1)$

15. MARTIAL ARTS Find the measure of each angle of the triangle shown. [Section 3.5] $85°, 80°, 15°$



16. Solve using Cramer's rule: $\begin{cases} 4x - 3y = -1 \\ 3x + 4y = -7 \end{cases}$

[Section 3.7] $(-1, -1)$

17. Evaluate: $\begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix}$ [Section 3.7] -1

18. Solve: $4.5x - 1 < -10$ or $6 - 2x \geq 12$

[Section 4.2] $(-\infty, -2)$

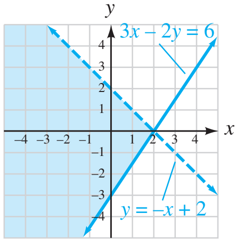
Give the solution set in interval notation and then graph it.

19. Solve: $|5 - 3x| \leq 14$

[Section 4.3] $\left[-3, \frac{19}{3}\right]$

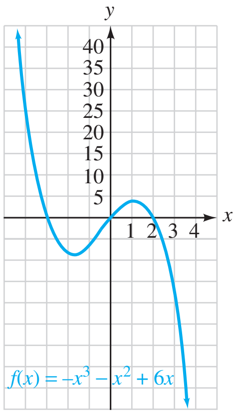
20. Solve: $\begin{cases} 3x - 2y \le 6 \\ y < -x + 2 \end{cases}$

[Section 4.5]



21. Complete the table of values for $f(x) = -x^3 - x^2 + 6x$ and then graph the function. What are the x- and y-intercepts of the graph? [Section 5.3]
$(-3, 0), (0, 0), (2, 0); (0, 0)$

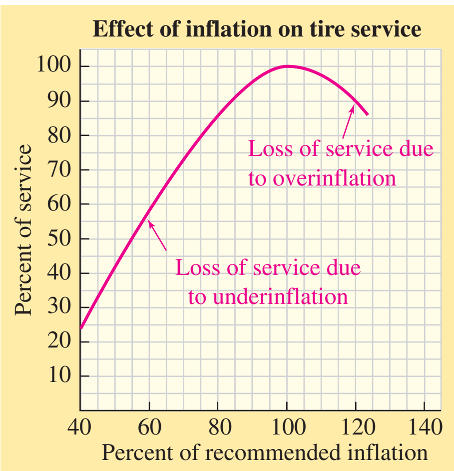
x	$f(x)$
−4	24
−3	0
−2	−8
−1	−6
0	0
1	4
2	0
3	−18



22. TIRE WEAR See the illustration below. [Section 5.3]

a. What type of function does it appear would model the relationship between the inflation of a tire and the percent of service it gives?
a quadratic function

b. At what percent(s) of inflation will a tire offer only 90% of its possible service?
at about 85% and 120% of the suggested inflation



Perform the operations. [Section 5.4]

23. $(4x - 3y)(3x + y)$
$12x^2 - 5xy - 3y^2$

24. $(-2x^2y^3 + 6xy + 5y^2) - (-4x^2y^3 - 7xy + 2y^2)$
$2x^2y^3 + 13xy + 3y^2$

25. $(a - 2b)^2$ $a^2 - 4ab + 4b^2$

26. $(a + 2)(3a^2 + 4a - 2)$ $3a^3 + 10a^2 + 6a - 4$

Factor the expression completely. [Section 5.8]

27. $3x^3y - 4x^2y^2 - 6x^2y + 8xy^2$ $xy(3x - 4y)(x - 2)$

28. $256x^4y^4 - z^8$ $(16x^2y^2 + z^4)(4xy + z^2)(4xy - z^2)$

29. Solve for λ: $\dfrac{A\lambda}{2} + 1 = 2d + 3\lambda$. [Section 5.9] $\lambda = \frac{4d - 2}{A - 6}$

Simplify.

30. $\left(\dfrac{4a^{-2}b}{3ab^{-3}}\right)^3$ [Section 6.1] $\frac{64b^{12}}{27a^9}$

31. $\dfrac{6x^2 + 13x + 6}{6 - 5x - 6x^2}$ [Section 6.2] $-\frac{3x + 2}{3x - 2}$

32. $\dfrac{p^3 - q^3}{q^2 - p^2} \cdot \dfrac{q^2 + pq}{p^3 + p^2q + pq^2}$ [Section 6.2] $-\frac{q}{p}$

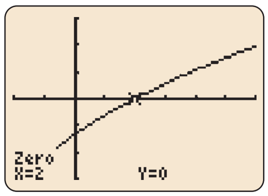
33. $\dfrac{2}{a - 2} + \dfrac{3}{a + 2} - \dfrac{a - 1}{a^2 - 4}$ [Section 6.3] $\frac{4a - 1}{(a + 2)(a - 2)}$

34. Solve: $\dfrac{x - 4}{x - 3} + \dfrac{x - 2}{x - 3} = x - 3$
[Section 6.5] 5; 3 is extraneous

35. Solve $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$ for R.
[Section 6.5] $R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

36. Use the long division method to find $(2x^2 + 4x - x^3 + 3) \div (x - 1)$
[Section 6.6] $-x^2 + x + 5 + \frac{8}{x - 1}$

37. The graph of $f(x) = \sqrt{2x + 5} + \sqrt{x + 2} - 5$ is shown. Use the information in the display to determine the solution of $\sqrt{2x + 5} = -\sqrt{x + 2} + 5$.
[Section 7.1] 2



Simplify each expression. [Section 7.2]

38. $\sqrt{98} + \sqrt{8} - \sqrt{32}$ $5\sqrt{2}$

39. $12\sqrt[3]{648x^4} + 3\sqrt[3]{81x^4}$ $81x\sqrt[3]{3x}$

40. Rationalize the denominator: $\dfrac{3t - 1}{\sqrt{3t} + 1}$
[Section 7.3] $\sqrt{3t} - 1$

Solve each equation.

41. $\sqrt{3a + 1} = a - 1$ [Section 7.4] 5, 0 is extraneous

42. $\sqrt{x + 3} - \sqrt{3} = \sqrt{x}$ [Section 7.4] 0

43. Evaluate: $\left(\dfrac{25}{49}\right)^{-3/2}$ [Section 7.5] $\frac{343}{125}$

44. CHANGING DIAPERS The illustration shows how to put a diaper on a baby. If the diaper is a square with sides 16 inches long, what is the largest waist size that this diaper can wrap around, assuming an overlap of 1 inch to pin the diaper? [Section 7.6] about $21\frac{1}{2}$ in.

Simplify: Write the result in the form $a + bi$. [Section 7.7]

45. $\left(-7 + \sqrt{-81}\right) - \left(-2 - \sqrt{-64}\right)$ $-5 + 17i$

46. $\dfrac{2 - 5i}{2 + 5i}$ $-\frac{21}{29} - \frac{20}{29}i$

47. $6a^2 + 5a - 6 = 0$ [Section 8.3] $\frac{2}{3}, -\frac{3}{2}$

48. $4w^2 + 6w + 1 = 0$ [Section 8.3] $\frac{-3 \pm \sqrt{5}}{4}$

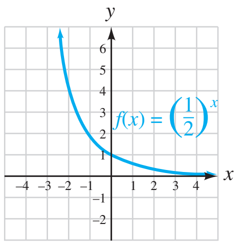
49. $3x^2 - 4x = -2$ [Section 8.3] $\frac{2 \pm i\sqrt{2}}{3}$

50. $2(2x + 1)^2 - 7(2x + 1) + 6 = 0$ [Section 8.5] $\frac{1}{4}, \frac{1}{2}$

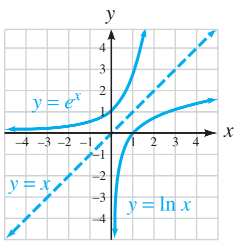
51. If $f(x) = x^2 - 2$ and $g(x) = 2x + 1$, find $(f \circ g)(x)$. [Section 9.1] $(f \circ g)(x) = 4x^2 + 4x - 1$

52. Find the inverse function of $f(x) = 2x^3 - 1$. [Section 9.2] $f^{-1}(x) = \sqrt[3]{\frac{x + 1}{2}}$

53. Graph: $f(x) = \left(\dfrac{1}{2}\right)^x$ [Section 9.3]



54. Graph $y = e^x$ and its inverse on the same coordinate system. Label the axis of symmetry. [Section 9.4]



55. Write $y = \log_2 x$ as an exponential equation. [Section 9.5] $2^y = x$

56. Find the inverse of $f(x) = \log_2 x$. [Section 9.5] $f^{-1}(x) = 2^x$

57. Apply properties of logarithms to simplify $\log_6 \frac{x}{36}$. [Section 9.7] $(\log_6 x) - 2$

58. If $\log_{10} 10^x = y$, then y equals what quantity? [Section 9.7] x

Let $\log 7 = 0.8451$ *and* $\log 14 = 1.1461$. *Evaluate each expression without using a calculator or tables.* [Section 9.7]

59. $\log 98$ 1.9912

60. $\log 2$ 0.301

61. Find: $\log_6 8$ [Section 9.7] 1.16056

Find x. [Section 9.8]

62. $\log_x 25 = 2$ 5

63. $\log_5 125 = x$ 3

64. $\log_3 x = -3$ $\frac{1}{27}$

65. $\ln e = x$ 1

Solve each equation. Round to four decimal places when necessary. [Section 9.8]

66. $2^{x+2} = 3^x$ 3.4190

67. $\log x + \log(x + 9) = 1$ 1, -10 is extraneous

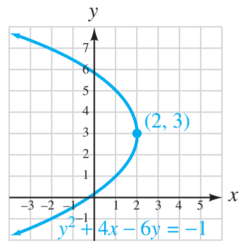
68. $5^{4x} = \dfrac{1}{125}$ $-\frac{3}{4}$

69. $\log_3 x = \log_3\left(\dfrac{1}{x}\right) + 4$ 9

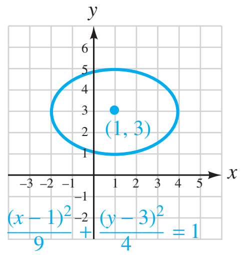
70. BOAT DEPRECIATION How much will a \$9,000 boat be worth after 9 years if it depreciates 12% per year? [Section 9.8] \$2,848.31

71. Write the equation of the circle that has its center at (1, 3) and passes through $(-2, -1)$. [Section 10.1] $x^2 + y^2 - 2x - 6y - 15 = 0$

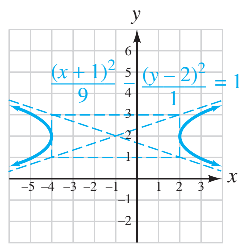
72. Graph: $y^2 + 4x - 6y = -1$ [Section 10.1]



73. Graph:
$$\frac{(x - 1)^2}{9} + \frac{(y - 3)^2}{4} = 1$$
[Section 10.2]



74. Write the equation in standard form and then graph it. [Section 10.3]
$$x^2 - 9y^2 + 2x + 36y = 44$$



APPENDIX I

Roots and Powers

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$
1	1	1.000	1	1.000	51	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	52	2,704	7.211	140,608	3.733
3	9	1.732	27	1.442	53	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	54	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	55	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	56	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	57	3,249	7.550	185,193	3.849
8	64	2.828	512	2.000	58	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	59	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	60	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	61	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	62	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	63	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	64	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	65	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	66	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	67	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	68	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	69	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	70	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	71	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	72	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	73	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	74	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	75	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	76	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	77	5,929	8.775	456,533	4.254
28	784	5.292	21,952	3.037	78	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	79	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	80	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	81	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	82	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	83	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	84	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	85	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	86	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	87	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	88	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	89	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	90	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	91	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	92	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	93	8,649	9.644	804,357	4.531
44	1,936	6.633	85,184	3.530	94	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	95	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	96	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	97	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	98	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	99	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	100	10,000	10.000	1,000,000	4.642

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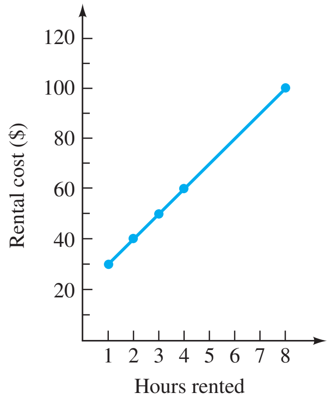
Answers to Selected Exercises

APPENDIX II

Study Set Section 1.1 (page 7)

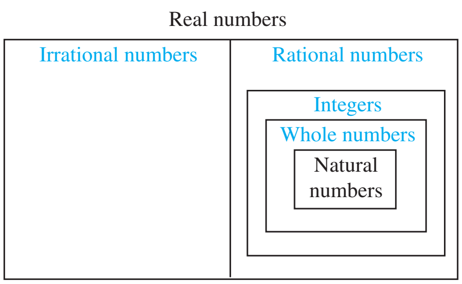
1. variable **3.** equation **5.** addition **7. a.** expression **b.** equation **c.** expression **d.** expression **9.** $b = t - 10$; the height of the base is 10 ft less than the height of the tower **11. a.** $7d = h$ **b.** $t = 2{,}500 - d$ **13.** $c = 13u + 24$ **15.** $w = \dfrac{c}{75}$ **17.** $A = t + 15$ **19.** $c = 12b$ **21.** The amount (in dollars) that the husband will keep is the quotient of the amount of the couple's refund and 2, decreased by 75. $h = \dfrac{t}{2} - 75$ (answers may vary) **23.** The number of bottles of water left on his truck is the difference of 300 and the product of 6 and the number of stops that he has made. $b = 300 - 6s$ (answers may vary) **25.** 2, 6, 15 **27.** 22.44, 21.43, 0 **29. a.** a bar graph **b.** 1 year; \$100 **c.** 2000, \$700; 2005, \$860 **31.** r, 2, r, 4 **33. a.** The rental cost is the product of 10 and the number of hours it is rented, increased by 20. **b.** $C = 10h + 20$ **c.** 30, 40, 50, 60, 100



Study Set Section 1.2 (page 20)

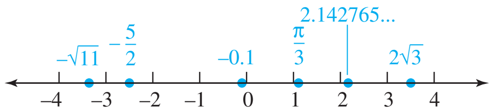
1. whole, natural, integers **3.** prime, composite **5.** irrational **7.** inequality **9.** $-8, 4$ **11.** nonrepeating, irrational **13.** repeating, rational

15.



17. a. $12 < 19$ **b.** $-5 \geq -6$ **19.** is less than; is greater than or equal to **21.** braces **23.** $\left\{\dfrac{a}{b} \,\middle|\, a \text{ and } b \text{ are integers, with } b \neq 0.\right\}$ **25. a.** rational numbers **b.** irrational numbers **c.** real numbers **27.** true **29.** false **31.** false **33.** true **35.** 0.6, terminating **37.** $-0.7\overline{3}$, repeating **39.** $1.2\overline{27}$, repeating **41.** $0.01\overline{6}$, terminating **43.** 1, 2, 9 **45.** $-3, 0, 1, 2, 9$ **47.** $\sqrt{3}, \pi$ **49.** 2 **51.** 2 **53.** 9

55.



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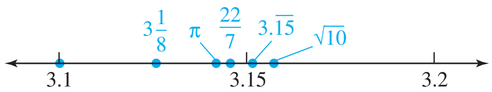

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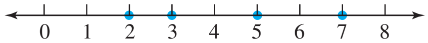

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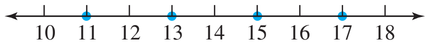

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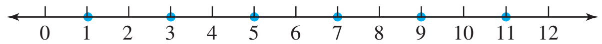

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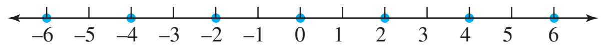


67. $<$ **69.** $>$ **71.** $>$ **73.** $<$ **75.** 20 **77.** 5.9 **79.** -6 **81.** $-\dfrac{9}{4}$ **83.** $3\dfrac{2}{25} = 3.0800$, $\dfrac{77}{50} = 1.5400$, $\dfrac{15}{16} = 0.9375$, $2\dfrac{5}{8} = 2.6250$, $\dfrac{\pi}{4} \approx 0.7854$, $\sqrt{8} \approx 2.8284$ **85.** π in. (3.141592654 . . . in.); 3.14 in. **91.** expression **93.** 2.2, 8.5, 29.1

Study Set Section 1.3 (page 34)

1. sum, difference **3.** reciprocal **5.** squared, cubed **7.** parentheses, outermost **9. a.** multiplication **b.** opposite **c.** negative, positive **11. a.** addition first or multiplication first **b.** 12; multiplication is performed before addition **13. a.** $(-4)^2 = 16$ **b.** $-4^2 = -16$ **15.** -8 **17.** -4.3 **19.** -13 **21.** $\dfrac{1}{6}$ **23.** -7 **25.** 0 **27.** $\dfrac{11}{10}$ **29.** -2 **31.** -12 **33.** -1.5 **35.** 60 **37.** $-\dfrac{6}{7}$ **39.** -2 **41.** -14 **43.** 9 **45.** $\dfrac{24}{25}$ **47.** 1,296 **49.** 62.41 **51.** -25 **53.** $-\dfrac{27}{125}$ **55.** 8 **57.** -9 **59.** $-\dfrac{3}{4}$ **61.** 0.2 **63.** -17 **65.** 32 **67.** -8 **69.** -32 **71.** 10,000 **73.** 64 **75.** -8 **77.** 13 **79.** -114 **81.** -32 **83.** $-\dfrac{1}{18}$ **85.** $\dfrac{1}{2}$ **87.** 5 **89.** $-\dfrac{1}{2}$ **91.** -24 **93.** -2 **95.** 61 **97.** 1 **99.** 10 **101.** 4 **103.** There was a net inflow of \$390 billion. **105.** New York, Atlanta, Boise, Omaha, Helena **107.** 9 **111.** -7 and 3 **113.** $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ **115.** true

Study Set Section 1.4 (page 46)

1. term **3.** constant **5.** commutative, associative **7.** Like **9. a.** $(x + y) + z = x + (y + z)$ **b.** $xy = yx$ **c.** $r(s + t) = rs + rt$ **11. a.** a, commutative property of multiplication **b.** $a(bc)$, associative property of multiplication **c.** 0, multiplicative property of 0 **d.** a, identity property of multiplication **e.** 1, multiplicative inverse property **13. a.** 0 **b.** 1 **c.** $-x$ **d.** $\frac{1}{x}$ **15. a.** 5 **b.** $\frac{1}{5}$ **17.** multiplication by -1 **19.** $3x^3, 11x^2, -x, 9; 3, 11, -1, 9$ **21.** $\frac{11}{12}a^4, -\frac{3}{4}b^2, 25b; \frac{11}{12}, -\frac{3}{4}, 25$ **23.** $7 + 3$ **25.** $3 \cdot 2 + 3d$ **27.** c **29.** 1 **31.** $(8 + 7) + a$ **33.** $2(x + y)$ **35.** $72m$ **37.** $-45q$ **39.** $-49x$ **41.** $64ry$ **43.** $81x + 18$ **45.** $12t - 12$ **47.** $-24 + d$ or $d - 24$ **49.** $2s^2 - 6$ **51.** $0.7m + 1.4n$ **53.** $9x + 2y$ **55.** $45t^2 - 60t - 15$ **57.** $4x - 5y + 1$ **59.** $2t + 3$ **61.** $-3y + 6$ **63.** $18x$ **65.** $-3.1h$ **67.** $0.8x^2$ **69.** $-2x$ **71.** $\frac{9}{10}ab$ **73.** $\frac{14}{15}t$ **75.** $12ad - 44a$ **77.** $6m + 2t$ **79.** $-2x^2 + 15x$ **81.** $-6p + 17$ **83.** $8x - 9$ **85.** $17y - 27$ **87.** $-4x + 87$ **89.** $56b + 6$ **91.** $-2a - A - 3$ **93.** $14cd + 62c$ **95.** $6.4a^2 + 2.6a + 5.7$ **97.** $-\frac{19}{16}x$ **99.** $14z - 5$ **101.** $18h^2 - 8h$ **103.** $-3.8y + 38.7$ **105.** 0 **107. a.** $20(x + 6)$ m^2 **b.** $(20x + 120)$ m^2 **c.** $20(x + 6) = 20x + 120$; distributive property **113.** $-\frac{7}{8}$ **115.** 988

Study Set Section 1.5 (page 58)

1. equation **3.** satisfies **5.** identity **7.** c, c, adding, both **9. a.** 3 **b.** 9 **c.** 2 **d.** 18 **11. a.** all real numbers, $\mathbb{R}$ **b.** no solution, $\varnothing$ **13.** $-2x, 14, 14, -2, -2, -17, -10, \stackrel{?}{=}, 20, -17$ **15.** yes **17.** no **19.** 6 **21.** $\frac{15}{8}$ **23.** 28 **25.** -30 **27.** 2.52 **29.** -0.25 **31.** 29 **33.** 7 **35.** 15 **37.** $-\frac{5}{2}$ **39.** -16 **41.** 18 **43.** -11 **45.** 1 **47.** 1.7 **49.** $\frac{17}{4}$ **51.** $\frac{21}{5}$ **53.** 0 **55.** -8 **57.** -11 **59.** 4 **61.** 2 **63.** 30 **65.** $-\frac{1}{2}$ **67.** 63 **69.** 24 **71.** 0 **73.** $\frac{21}{19}$ **75.** 3 **77.** 24 **79.** all real numbers, $\mathbb{R}$: identity **81.** no solution, $\varnothing$; contradiction **83.** all real numbers, $\mathbb{R}$: identity **85.** no solution, $\varnothing$; contradiction **87.** 13 **89.** 6 **91.** -15 **93.** 1,000 **95.** $\frac{9}{13}$ **97.** no solution, $\varnothing$; contradiction **99.** -1.2 **101.** -5 **103.** $\frac{28}{57}$ **105.** $\frac{5}{2}$ **111. a.** $a + b = b + a$ **b.** $(ab)c = a(bc)$ **c.** $a(b + c) = ab + ac$ **113. a.** $0 + a = a$ **b.** $1 \cdot a = a$

Study Set Section 1.6 (page 68)

1. formula **3.** volume **5. a.** area, ft^2 **b.** volume, ft^3 **c.** circumference, ft **d.** perimeter, ft **7.** t, t, a **9.** $ad, ad, bc, b, b, c, \frac{t - ad}{b}$ **11.** 8 yd **13.** 37 in. **15.** 10.2 ft^2 **17.** 295.84 mi^2 **19.** 23.56 in. **21.** 15.71 ft **23.** 102.1 in.^{2} **25.** 86.6 ft^2 **27.** 95.08 ft^3 **29.** 808.86 m^3 **31.** $t = \frac{d}{r}$ **33.** $h = \frac{V}{lw}$ **35.** $h = \frac{3V}{\pi r^2}$ **37.** $W = T - ma$ **39.** $a = \frac{2h - 96t}{t^2}$ or $a = \frac{2(h - 48t)}{t^2}$ **41.** $b_2 = \frac{2A - b_1h}{h}$ or $b_2 = \frac{2A}{h} - b_1$ **43.** $n = \frac{l - a + d}{d}$ or $n = \frac{l - a}{d} + 1$ **45.** $w = \frac{P - 2h - 2l}{2}$ or $w = \frac{P}{2} - h - l$ **47.** $A = \frac{\lambda}{x + B}$ **49.** $T_a = \frac{T_f}{1 - F}$ **51.** $d = \frac{l - a}{n - 1}$ **53.** $t = \frac{d_1 - d_2}{v}$ **55.** $y = \frac{2}{5}x - 4$ **57.** $y = -\frac{4}{3}x - 4$ **59.** $x = \frac{y - b}{m}$ **61.** $R = \frac{L - 2d - 3.25r}{3.25}$ **63.** $g = \frac{2(s - vt)}{t^2}$ **65.** $x = \frac{y - y_1 + mx_1}{m}$ **67.** $S = \frac{U + pV - G}{T}$ **69.** $r = \frac{PV}{nt}$ **71.** $R = \frac{E - Ir}{I}$ or $R = \frac{E}{I} - r$ **73.** $s_3 = 3A - s_1 - s_2$ **75.** $d = \frac{2S - 2an}{n(n - 1)}$ **77.** $h = \frac{3d}{4\pi}$ **79.** perimeter, 216 in. **81.** 1st term: area of bottom flap; 2nd term: area of left and right flaps; 3rd term: area of top flap; 4th term: area of face; 42.5 in.^{2} **83.** $C = \frac{5}{9}(F - 32)$ or $C = \frac{5(F - 32)}{9}$; 432, -179; 58, -89; 17, -66 **85.** $d = \frac{360A}{\pi(r_1^2 - r_2^2)}$; 140, 160 **87.** $n = \frac{PV}{R(T + 273)}$; 0.008, 0.090 **89.** $n = \frac{C - 6.50}{0.07}$; 621, 1,000, about 1,692.9 kwh **91.** $h = \frac{A - 2\pi r^2}{2\pi r}$ **97.** $26r + 132t + 1$ **99.** $-12a + 101$ **101.** $-0.6pt + 11p$

Study Set Section 1.7 (page 79)

1. acute **3.** complementary **5.** right **7.** angles **9.** $d + 15, 2d - 10, 2d + 20, \frac{d}{2} - 10, 2d$ **11. a.** $5x, 6x, 10(x - 2), 5x$ **b.** $5x + 6x + 10(x - 2) + 5x$ **c.** $5x + 6x + 10(x - 2) + 5x = 110$ **13.** Cheerios: \$689 million, Frosted Flakes: \$250 million **15.** pedestal: 154 ft; statue: 151 ft **17.** 42 min **19.** 6 in. **21.** 20 **23.** 310 mi **25.** 5,000 shares of stock funds, 7,000 shares of bond funds **27.** 35 \$12 calculators, 50 \$99 calculators **29.** 30°, 150°

31. a. 5.6° **b.** 1.4° **33.** ∠1: 50°; ∠2: 60°; ∠3: 70° **35.** 50° **37.** 10° **39.** 10 ft **41.** 156 ft by 312 ft **43.** 8 ft, 11 ft **47.** repeating **49.** $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ **51.** 0

Think It Through (page 86)

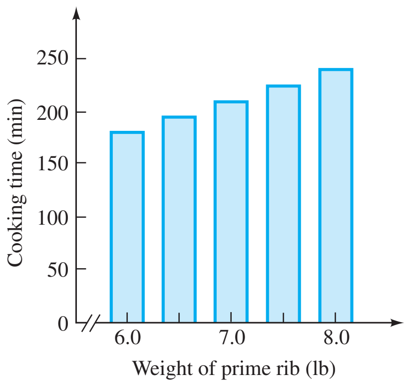
medical assistant, about 59%

Study Set Section 1.8 (page 92)

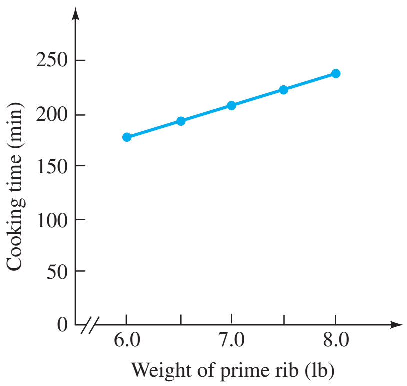
1. amount, base **3.** mean, mode, median **5.** ☐ is ☐ % of ☐? **7. a.** 51,824 **b.** 51,824, what, 3,734,536
9. a. $0.055x$, $0.07(10{,}850 - x)$
b. $0.055x + 0.07(10{,}850 - x) = 1{,}205$
11. a. 1, $0.15x$, x, 1, $0.18x$ **b.** $0.15x + 0.18x = 3{,}300$
13. a. $7.45p$, $50 - p$, $8.25(50 - p)$, $7.75(50)$
b. $7.45p + 8.25(50 - p) = 7.75(50)$
15. a. 0.025 **b.** 6% **17.** $x = 0.05 \cdot 10.56$
19. $32.5 = 0.74x$ **21.** 448 quadrillion Btu **23.** 597
25. 20% **27.** \$50 **29.** 9.3% **31.** −0.7%, 4.1%
33. city: mean 37.2, median 32.5, mode 32; hwy: mean 41, median 40, mode 40 **35.** 94 **37.** CD: \$10,000; money market: \$2,000 **39. a.** \$15,000 at 7%, \$30,000 at 10%
b. \$45,000 **41.** \$100,000 **43.** $\frac{1}{4}$ hr = 15 min **45.** $\frac{2}{3}$ hr
47. 3:30 P.M. **49.** $1\frac{1}{2}$ hr **51.** 10 lb **53.** 1.8 lb of each
55. 4,000 ft^3 of the premium mix, 2,000 ft^3 of sawdust
57. 28 lb **59.** 2 gal **61.** 10 oz **67.** 0 **69.** 8

Chapter 1 Review (page 98)

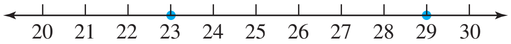
1. a. $C = 2t + 15$ **b.** $l = \frac{25}{w}$ **c.** $P = u - 3$
2. 180, 195, 210, 225, 240
3. a.



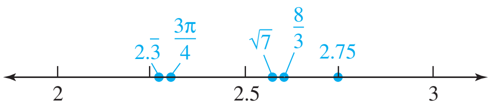
b.



4. An equation such as $2x + 3 = 1$ contains an equal symbol. An expression such as $2x + 3$ does not. **5. a.** 7 **b.** 0, 7
6. a. −5, 0, 7 **b.** $-5, 0, 2.4, 7, -\frac{2}{3}, -3.\overline{6}, \frac{15}{4}$
7. a. $-\sqrt{3}, \pi, 0.13242368\ldots$ **b.** all
8. a. $-5, -\sqrt{3}, -\frac{2}{3}, -3.\overline{6}$ **b.** $2.4, 7, \pi, \frac{15}{4}, 0.13242368\ldots$
9. a. 7 **b.** none **10. a.** 0 **b.** −5, 7
11.



12.



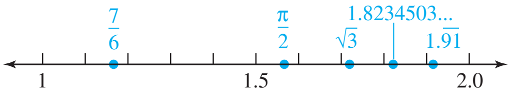
13. a. false **b.** true **14. a.** false **b.** true **15. a.** > **b.** < **16. a.** false **b.** true **17.** 18 **18.** −6.26 **19.** −27
20. 10.1 **21.** $-\frac{3}{4}$ **22.** 2 **23.** 12.6 **24.** $-\frac{1}{32}$ **25.** 0.2
26. $-\frac{3}{56}$ **27.** −33 **28.** −5.7 **29.** −40 **30.** 1
31. −243 **32.** $\frac{4}{81}$ **33.** 0.064 **34.** −25 **35.** 2 **36.** −10
37. $\frac{3}{5}$ **38.** 0.8 **39.** 44 **40.** 1 **41.** −12 **42.** 58 **43.** 8
44. 3 **45.** 3,000 **46.** −16 **47.** 56 **48.** $-\frac{1}{2}$ **49.** $3x + 21$
50. $5t$ **51.** 0 **52.** $27 + (1 + 99)$ **53.** 1 **54.** m **55.** 1
56. 0 **57.** $-3(5 \cdot 2)$ **58.** $(z + t) \cdot t$ **59. a.** 1 **b.** −25
60. a. 0 **b.** undefined **61.** $72x + 48$ **62.** $-6y + 2$
63. $3.6x - 2.4y$ **64.** $6c^2 - 3c + \frac{3}{4}$ **65.** $48k$ **66.** $75xy$
67. $-189p$ **68.** $45a + 16$ **69.** 0 **70.** $3m - 40n$
71. $\frac{13}{20}x$ **72.** $-24.54l$ **73.** $40a^3 - 16a^2$ **74.** $\frac{1}{4}h + 8$
75. yes **76.** no **77.** −225 **78.** 7.9 **79.** 0.014 **80.** −4
81. $\frac{12}{5}$ **82.** −9 **83.** −6 **84.** $\frac{11}{7}$ **85.** $\frac{88}{17}$ **86.** 12
87. 0.06 **88.** −8 **89.** 0 **90.** 3 **91.** no solution, ∅; contradiction **92.** all real numbers, ℝ: identity **93.** 31 ft
94. 53.41 cm; 226.98 cm^2 **95.** 1,767.15 m^3 **96. a.** 100 in.^{2}
b. $80\pi \approx 251.3$ in.^{3} **97.** $h = \frac{3V}{\pi r^2}$ **98.** $M = \frac{2K - Iw^2}{v_0^2}$
99. $d = \frac{l - a}{n - 1}$ **100.** $y = \frac{9}{5}x - 7$ **101.** O'Hare: 76.2 million; Atlanta: 84.8 million **102.** $245 - 5c$ **103.** 600 **104.** 42 ft, 45 ft, 48 ft, 51 ft **105.** 50°, 130° **106.** 27 in. by 40 in.
107. 120 **108.** 32% **109. a.** 3.4% **b.** 4.0%
110. 2, 1.5, 0 **111.** \$18,000 at 10%; \$7,000 at 9%
112. 2 min after the photographer leaves **113.** $6\frac{2}{3}$ gal
114. mild: 50 lb, robust: 40 lb

Chapter 1 Test (page 112)

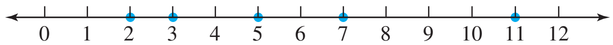
1. a. undefined **b.** inequality **c.** like terms **d.** solve
e. addition, equality **2. a.** $s = T + 10$ **b.** $A = \frac{1}{2}bh$
3. a. 200 calories **b.** 1,200 **c.** $3\frac{1}{2}$ **4. a.** −2, 0, 5

b. $-2, 0, -3\frac{3}{4}, 9.2, \frac{14}{5}, 5$ **c.** $\pi, -\sqrt{7}$ **d.** all **5. a.** true **b.** false **c.** true **d.** true

6.



7.

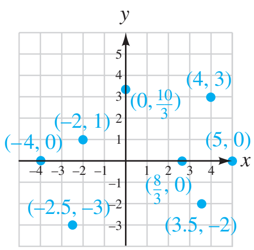


8. a. false **b.** false **9.** $\frac{4}{15}$ **10.** $\frac{8}{9}$ **11.** -209 **12.** -3 **13.** 100 mg **14. a.** commutative property of addition **b.** associative property of multiplication **c.** additive inverse property **d.** multiplicative identity property **15.** $11.1n^2 - 7.8n - 9.8$ **16.** $90st$ **17.** $-12c + 108$ **18.** $-\frac{1}{36}xy + 16x$ **19.** 15 **20.** 6 **21.** no solution, $\varnothing$; contradiction **22.** 12 **23.** yes **24.** $i = \frac{f(P-L)}{s}$ **25.** $x_1 = \frac{y_1 + mx - y}{m}$ **26.** 1,018 m^2 **27.** 8 **28.** 25 **29.** 85°, 85°, 10° **30.** 4 cm by 9 cm **31.** 28% **32. a.** 2.2 **b.** 2 **c.** 2 **33.** \$4,000 **34.** 400 mi **35.** 10 oz **36.** Skin Soother: 3 oz, Cool Sport: 5 oz

Study Set Section 2.1 (page 123)

1. ordered **3.** rectangular **5.** origin **7.** midpoint **9.** origin, right, down **11.** II **13.** $-3, -1, 1, 3$ **15.** one **17.** t **19.** x sub 1

21–27.

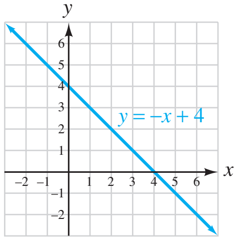


29. (2, 4) **31.** (−2.5, −1.5) **33.** (3, 0) **35.** (0, 0) **37. a.** on the surface **b.** diving **c.** 1,000 ft **d.** 500 ft **39. a.** 1993 **b.** Imports exceeded production by about 3.5 million barrels per day. **41. a.** \$2 **b.** \$9 **c.** 3 days **43.** (3, 4) **45.** (9, 12) **47.** $\left(\frac{7}{2}, 6\right)$ **49.** $\left(\frac{1}{2}, -2\right)$ **51.** $(-4, 0)$ **53.** $\left(-\frac{3}{2}, \frac{5}{2}\right)$ **55.** (4, 1) **57.** (−20, −3) **59.** Jonesville (5, B), Easley (1, B), Hodges (2, E), Union (6, C) **61. a.** (2, −1) **b.** no **c.** yes **63. a.** 6 strokes **b.** 7 strokes **c.** 16th **d.** 18th **65.** 4, 3, 1, 5, 2 **67.** tip of tail, front of engine, tip of wing **71.** 20 **73.** $\frac{1}{2}$ **75.** 5 **77.** 0.7

Study Set Section 2.2 (page 138)

1. ordered pair **3.** linear **5.** vertical **7. a.** yes **b.** no **9.** x-intercept: $(-6, 0)$; y-intercept: $(0, 3)$ **11. a.** $(-3, 0), (0, 4)$ **b.** false **13.** $y = -4x - 1$ **15.** $-3, -2$ **17.** the y-axis **19.** 5, 4, 2 **21.** 0, −1, −2

23.



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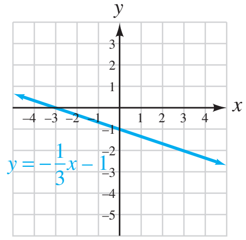

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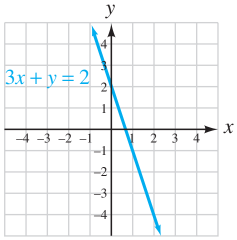

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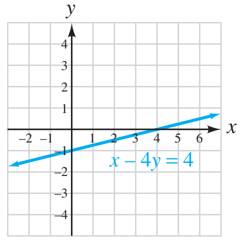
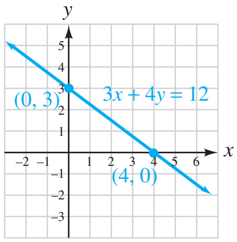


31.



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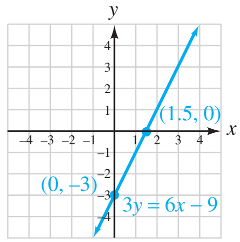

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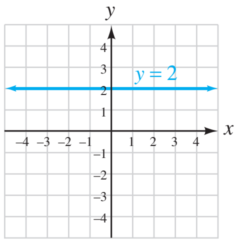

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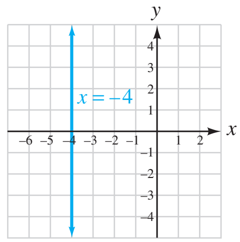

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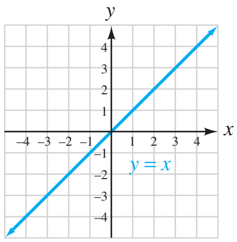
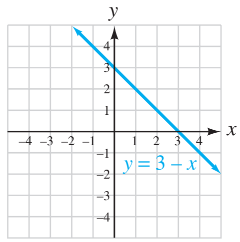


41.



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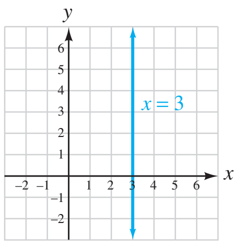

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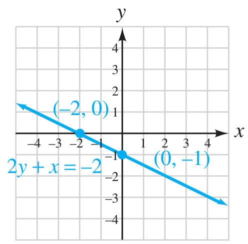

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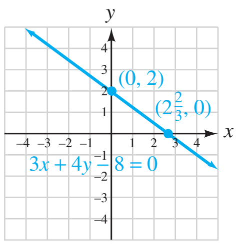
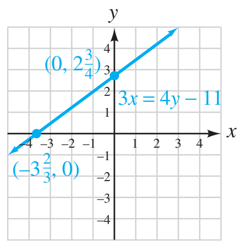
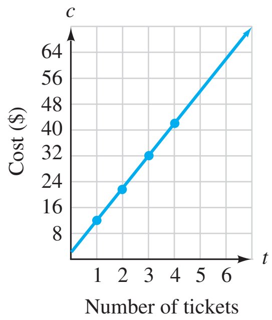


49.



51. 1.22 **53.** 4.67 **55.** \$60 **57. a.** In 1990, there were 65.5 million swimmers. **b.** about 58.2 million **59.** \$162,500 **61.** 12.5 yr **63. a.** $c = 10t + 2$

b.



c. \$62; 12, 22, 32, 42 **67.** 11, 13, 17, 23, 29 **69.** III **71.** 80*s* **73.** $3x + 8$

Think It Through (page 148)

1960–1970, an increase of about 50 community colleges per yr

Study Set Section 2.3 (page 151)

1. Slope **3.** change **5.** reciprocals **7. a.** l_3, 0 **b.** l_2, undefined **c.** l_1, 2 **d.** l_4, -3 **9. a.** an increase of 73 million units/yr **b.** a decrease of 35 million units/yr **11. a.** $-\frac{4}{3}$ **b.** $-\frac{2}{3}$ **13.** $m = \frac{y_2 - y_1}{x_2 - x_1}$ **15. a.** 6 **b.** 8 **c.** $\frac{3}{4}$ **17.** $\frac{6}{7}$ **19.** $-\frac{8}{3}$ **21.** -3 **23.** $\frac{1}{2}$ **25.** 3 **27.** -1 **29.** $-\frac{1}{3}$ **31.** 0 **33.** undefined **35.** -1 **37.** parallel **39.** neither **41.** perpendicular **43.** neither **45.** parallel **47.** perpendicular **49.** neither **51.** parallel **53.** $-\frac{3}{2}$ **55.** $\frac{3}{4}$ **57.** $\frac{1}{2}$ **59.** 0 **61.** $\frac{3}{140}, \frac{1}{15}, \frac{1}{20}$, part 2 **63.** $\frac{1}{10}, \frac{1}{4}$ **65.** $\frac{1}{25}$, 4% **67.** brace: $\frac{1}{2}$; support 1: -2; support 2: -1; yes, to support 1 **73.** 40 lb licorice, 20 lb gumdrops **75.** 4 hr

Study Set Section 2.4 (page 165)

1. $y - y_1 = m(x - x_1)$ **3.** perpendicular **5.** no **7.** $m = \frac{2}{3}$, $y + 3 = \frac{2}{3}(x + 2)$ **9.** $m = -\frac{2}{3}$, $(0, 1)$ **11.** yes **13. a.** $(0, 0)$ **b.** none **15.** No; the slopes are not negative reciprocals. Their product is not -1: $1(-0.9) = -0.9$. **17.** $\frac{1}{3}x$, 2, 2, 1, $\frac{1}{3}$, -1 **19.** $y = 5x + 7$ **21.** $y = -3x + 6$ **23.** $y = x$ **25.** $y = \frac{7}{3}x - 3$ **27.** $y = \frac{3}{4}x - 3$ **29.** $y = \frac{2}{3}x + \frac{11}{3}$ **31.** $y = 3x + 17$ **33.** $y = -7x + 54$ **35.** $y = -4$ **37.** $y = -\frac{1}{2}x + 11$ **39.** $\frac{3}{2}$, $(0, -4)$ **41.** $-\frac{1}{3}$, $\left(0, -\frac{5}{6}\right)$ **43.** 1, $(0, -1)$ **45.** $\frac{2}{3}$, $(0, 2)$ **47.** parallel **49.** perpendicular **51.** neither **53.** perpendicular **55.** $y = 4x$ **57.** $y = 4x - 3$ **59.** $y = \frac{4}{5}x - \frac{26}{5}$ **61.** $y = \frac{2}{3}x + \frac{13}{3}$ **63.** $y = -\frac{1}{4}x$ **65.** $y = \frac{1}{4}x + \frac{9}{2}$ **67.** $y = -\frac{5}{4}x + 3$ **69.** $y = -3x + 1$ **71.** $y = \frac{7}{3}x - 3$ **73.** $y = \frac{4}{3}x + \frac{7}{3}$ **75.** $y = -\frac{1}{4}x + \frac{11}{2}$ **77.** $y = x$ **79.** $y = 4x$ **81.** $y = \frac{1}{3}x + \frac{17}{36}$ **83.** $y = \frac{4}{3}x + 4$ **85.** $y = -\frac{950}{3}x + 1{,}750$ **87.** $y = -1{,}811{,}250x + 36{,}225{,}000$ **89. a.** $B = \frac{1}{100}p - 195$ **b.** 905 **91. a.** $c = 514t + 8{,}850$ **b.** \$44,830 **93. a.** $c = 7.8m + 220$ **b.** About $10\frac{1}{4}$ min **95.** not quite **101.** \$29,100 **103.** 0

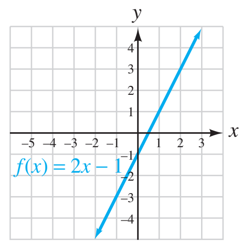
Think It Through (page 177)

\$28,000, \$61,250

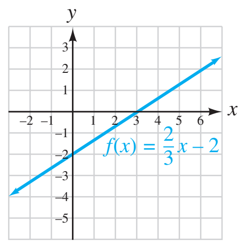
Study Set Section 2.5 (page 182)

1. relations **3.** range **5.** domain, range **7.** function, equals **9.** identity function **11.** horizontal, vertical **13.** $f(-1)$ **15.** D: all real numbers greater than or equal to 0, R: all real numbers greater than or equal to 2 **17.** -5, 25 **19.** of **21.** D: $\{-2, 0, 2\}$, R: $\{1, 4, 5\}$ **23.** D: $\{-23, 0, 7\}$, R: $\{1, 35\}$ **25.** yes **27.** no; $(4, 2)$, $(4, 4)$, $(4, 6)$ **29.** no; $(3, 4)$, $(3, -4)$ or $(4, 3)$, $(4, -3)$ **31.** yes **33.** yes **35.** no; $(-1, 0)$, $(-1, 2)$ **37.** yes **39.** yes **41.** no; $(1, 1)$, $(1, -1)$ **43.** no; $(1, 1)$, $(1, -1)$ **45.** 9, -3 **47.** 3, -5 **49.** 22, 2 **51.** 3, 11 **53.** $g(w) = 2w$, $g(w + 1) = 2w + 2$ **55.** $g(w) = 3w - 5$, $g(w + 1) = 3w - 2$ **57.** $\frac{1}{4}$, 0.09 **59.** $-\frac{7}{8}$, -0.973 **61.** $\frac{9}{4}$, 1.69 **63.** 0, -0.12

65.



67.



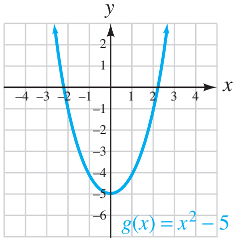
69. D: $\{-2, 4, 6\}$, R: $\{3, 5, 7\}$ **71.** D; the set of all real numbers except 4, R: the set of all real numbers except 0 **73.** not a function **75.** a function **77.** a function **79.** a function **81.** 3.7, 1.1, 3.4 **83.** $\frac{-27}{64}, \frac{1}{216}, \frac{125}{8}$ **85.** D: the set of all real numbers except 6 **87.** D: the set of all real numbers **89.** D: the set of all real numbers **91.** D: the set of all real numbers **93.** no **95.** yes **97.** 4, 4 **99.** 2, 2 **101.** $\frac{1}{5}$, 1 **103.** -2, $\frac{2}{5}$ **105.** between 20°C and 25°C **107. a.** $I(b) = 1.75b - 50$ **b.** \$142.50 **109. a.** $(200, 25)$, $(200, 90)$, $(200, 105)$ **b.** It doesn't pass the vertical line test.

111. a. 3,400; the tax on an income of \$25,000 is \$3,400. **b.** $T(a) = 3{,}910 + 0.25(a - 28{,}400)$ **113. a.** 624 ft **b.** 0; the rocket strikes the ground 16 seconds after being shot **117.** $-\frac{15}{4}$ **119.** $\frac{1}{3}$

Study Set Section 2.6 (page 196)

1. nonlinear **3.** cubing **5.** vertical, translation **7.** 4, left **9.** 5, up **11. a.** 2 **b.** 0 **13.** -6 **15.** $-3, 1, 2$ **17. a.** 3 **b.** 0 **c.** 1.5 **19.** D; the set of real numbers, R: all real numbers greater than or equal to -3 **21.** D; the set of real numbers, R: the set of real numbers **23.** D; the set of real numbers, R: the set of all real numbers greater than or equal to -2 **25.** D: the set of real numbers, R: the set of real numbers greater than or equal to 0

27.



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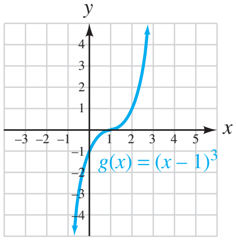

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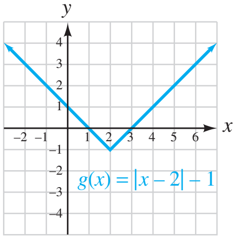

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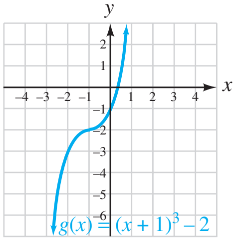

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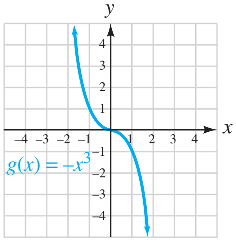
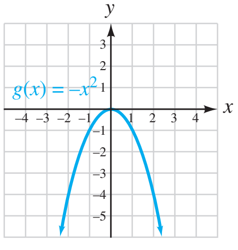
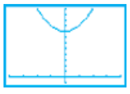


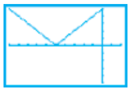
37.



39. 4 **41.** -3 **43.**



45.



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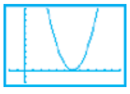

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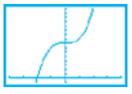
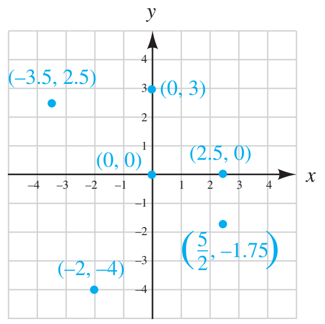


51. $f(x) = |x|$ **53.** a parabola **59.** $W = T - ma$ **61.** $g = \frac{2(s - vt)}{t^2}$ **63.** \$5.4 million

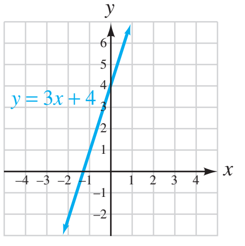
Chapter 2 Review Exercises (page 201)

1–6.

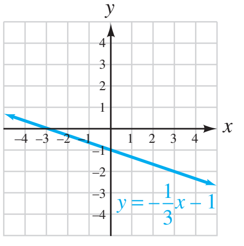


7. $-5, -5, 0, 0, -1, -1, 4, 4, 3, 3, 2, 2, 3, 3$ **8.** 1 ft below its normal level **9.** decreased by 3 ft **10.** from day 3 to the beginning of day 4 **11.** \$10 increments **12.** \$800 **13.** (2, 2) **14.** $(2, -6)$

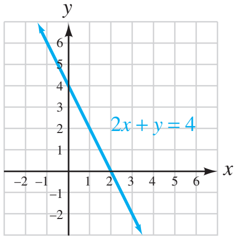
15.



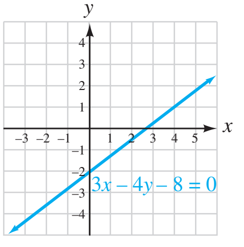
16.



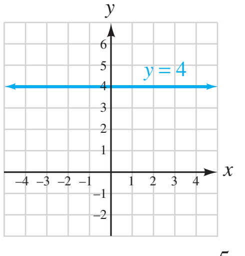
17.



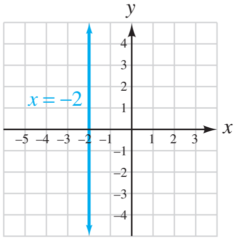
18.



19.



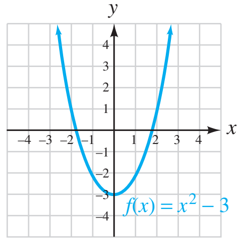
20.



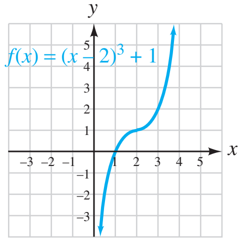
21. $9, 0, -9$ **22.** $-4, -\frac{5}{2}, -1$ **23.** Slope of $l_1 = \frac{4}{5}$, slope of $l_2 = -\frac{8}{5}$ **24.** 1.37% per yr **25.** 1 **26.** $-\frac{14}{9}$ **27.** 0 **28.** undefined **29.** $\frac{2}{3}$ **30.** -2 **31.** undefined **32.** 0 **33.** perpendicular **34.** parallel **35.** $y = 3x + 29$ **36.** $y = -\frac{13}{8}x + \frac{3}{4}$ **37.** $3x - 2y = 1$ **38.** $2x + 3y = -21$ **39.** $y = -\frac{3}{4}x - 3$; $m = -\frac{3}{4}$, $(0, -3)$ **40.** $y = -1{,}720x + 8{,}700$ **41.** yes **42.** yes **43.** no **44.** no **45.** -7 **46.** 18 **47.** 8 **48.** $3t + 2$ **49.** D: the set of real numbers, R: the set of real numbers

50. D: the set of real numbers, R: the set of all real numbers greater than or equal to 1 **51.** D: the set of all real numbers except 2, R: the set of all real numbers except 0 **52.** D; the set of real numbers, R: the set of nonpositive real numbers **53.** function **54.** not a function **55.** $c(x) = 105x + 175$ **56.** \$805 **57.** yes **58.** no

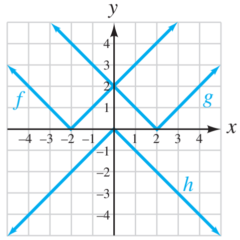
59.



60.



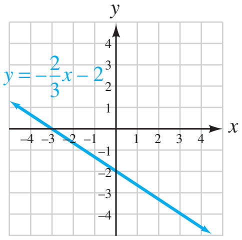
61.



62. a. -4 **b.** 3 **63.** 4 **64.** -1 **65.** 2

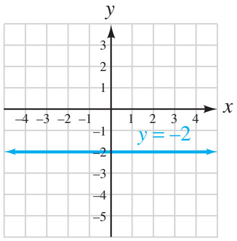
Chapter 2 Test (page 211)

1. 240 ft **2.** 1 sec and 7 sec **3.** about 260 ft **4.** 8 sec **5.** $\left(\frac{5}{2}, \frac{3}{2}\right)$ **6.**



7. $(5, 0), (0, -2)$

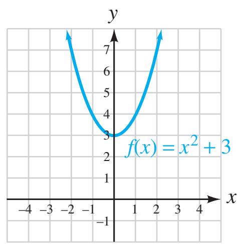
8.



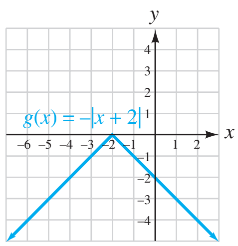
9. 3 **10.** -1.5 degree/hr **11.** $\frac{1}{2}$ **12.** $\frac{2}{3}$ **13.** undefined **14.** 0 **15.** $y = \frac{2}{3}x - \frac{23}{3}$ **16.** $8x - y = -22$ **17.** $m = -\frac{1}{3}, \left(0, -\frac{3}{2}\right)$

18. neither **19.** $y = \frac{3}{2}x$ **20. a.** $v = -600x + 4{,}000$ **b.** $(0, 4{,}000)$: it gives the value of the copier when new: \$4,000 **21.** no **22.** yes **23.** D: the set of real numbers, R: the set of nonnegative real numbers **24.** D: the set of real numbers, R: the set of real numbers **25.** 10 **26.** -1 **27.** 3 **28.** $r^2 - 2r - 1$ **29.** function **30.** not a function

31.



32.

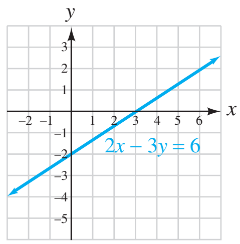


33. a. Find the x-coordinate of the x-intercept of the graph of $y = 3(x - 2) - 2(-2 + x)$; 2 **b.** Find the x-coordinate of the point of intersection of the graph of $y = 3(x - 2) - 2(-2 + x)$ and $y = 1$; 3 **34.** answers vary **35.** answers vary **36.** answers vary

Chapters 1–2 Cumulative Review (page 213)

1. 1, 2, 6, 7 **2.** 0, 1, 2, 6, 7 **3.** $-2, 0, 1, 2, \frac{13}{12}, 6, 7$ **4.** $\sqrt{5}, \pi$ **5.** -2 **6.** $-2, 0, 1, 2, \frac{13}{12}, 6, 7, \sqrt{5}, \pi$ **7.** 2, 7 **8.** 6 **9.** $-2, 0, 2, 6$ **10.** 1, 7 **11.** -2 **12.** -2 **13.** 22 **14.** -2 **15.** $\frac{24}{25}$ **16.** 2 **17.** 4 **18.** -5 **19.** associative property of addition **20.** distributive property **21.** commutative property of addition **22.** associative property of multiplication **23.** $-5y$ **24.** $-28st$ **25.** 0 **26.** $z - 4$ **27.** 8 **28.** -27 **29.** -1 **30.** 6 **31.** $\frac{8}{3}$ **32.** 24 **33.** $a = \frac{2S}{n} - l$ or $a = \frac{2S - ln}{n}$ **34.** $h = \frac{2A}{b_1 + b_2}$ **35.** \$14,000 **36.** 39 mph going, 65 mph returning

37.



38. $-\frac{5}{6}$ **39.** $y = -\frac{7}{5}x + \frac{11}{5}$ **40.** $y = -3x - 3$ **41.** 5 **42.** -1 **43.** 3 **44.** $3r^2 + 2$ **45.** a function; D: the set of real numbers, R: the set of all real numbers less than or equal to 1 **46.** a function; D: the set of real numbers, R: the set of nonnegative real numbers **47.** The points (2, 20), (3, 40), and (6, 60) do not lie on a straight line. **48. a.** Find the x-coordinate of the x-intercept of the graph of $y = -7 - 5(x - 1) + 4x$; -2 **b.** Find the x-coordinate of the point of intersection of the graph of $y = -7 - 5(x - 1) + 4x$ and $y = -4$; 2

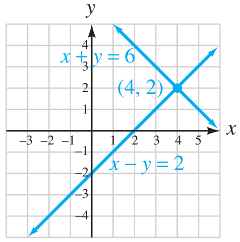
Think It Through (page 220)

(1981, 50); in 1981, 50% of the bachelor's degrees that were awarded went to men and 50% went to women

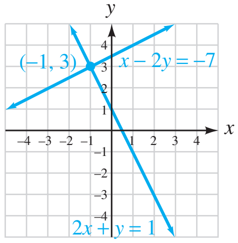
Study Set Section 3.1 (page 223)

1. system **3.** inconsistent **5.** dependent **7. a.** true **b.** false **c.** true **d.** true **9. a.** $-4, (-4, 0), 2, (0, 2), 3, (2, 3)$ **b.** $(-4, 0), (0, 2)$ **11. a.** $\begin{cases} x + y = 5 \\ x - y = -1 \end{cases}$ (answers may vary) **b.** $\begin{cases} x + y = 5 \\ 2x + 2y = 10 \end{cases}$ (answers may vary) **c.** $\begin{cases} x + y = 5 \\ x + y = 4 \end{cases}$ (answers may vary) **13.** brace **15.** yes **17.** no

19.

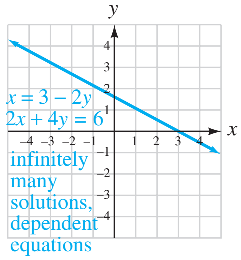


21.

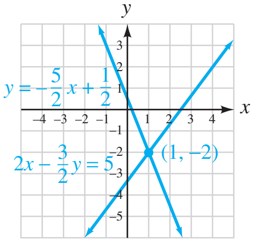


23. no solution, inconsistent system

25.



27.



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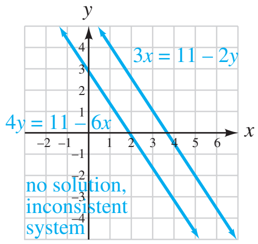

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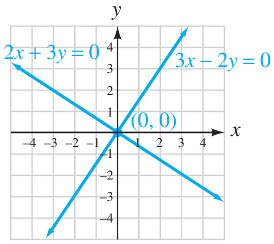

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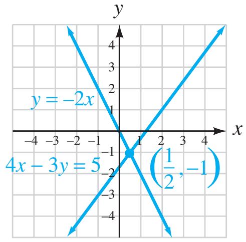

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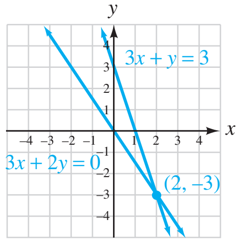
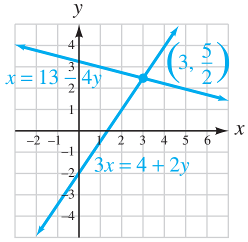


37.



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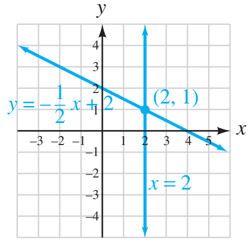
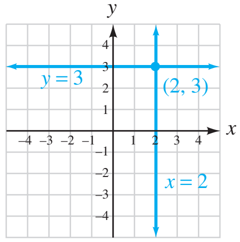
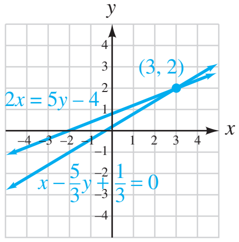


41.



43.



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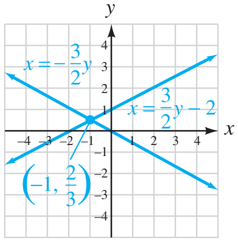
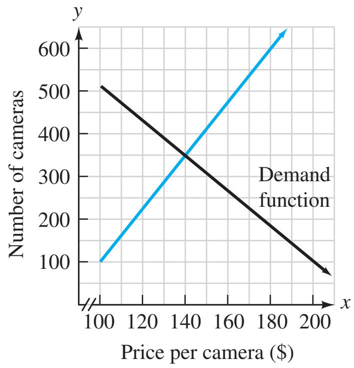


47. infinitely many solutions, dependent equations **49.** $(-0.37, -2.69)$ **51.** $(-7.64, -7.04)$ **53.** Gallup, Grants, Albuquerque, Tucumcari; Las Vegas, Santa Fe, Albuquerque, Socorro, Las Cruces; Albuquerque **55.** $(2{,}000, 50)$

57. a.



b. $140 **c.** Supply increases and demand decreases **59. a.** yes **b.** $(3.75, -0.5)$ **c.** no **63.** -3 **65.** 0

67. D: the set of real numbers, R: the set of all real numbers greater than or equal to -2 **69.** 40.5 cm^2

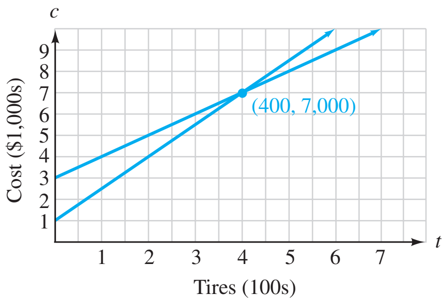
Study Set Section 3.2 (page 234)

1. standard (general) **3.** eliminated **5.** y, second equation **7.** addition method **9. a.** ii **b.** iii **c.** i **11.** $(2, 2)$ **13.** $(-1, -3)$ **15.** $(5, 3)$ **17.** $(-2, 4)$ **19.** $(3, 0)$ **21.** $(4, -7)$ **23.** $(5, 2)$ **25.** $(-4, -2)$ **27.** $\left(5, \frac{3}{2}\right)$ **29.** $(3, -2)$ **31.** $\left(\frac{1}{2}, \frac{2}{3}\right)$ **33.** $\left(-2, \frac{3}{2}\right)$ **35.** no solution, inconsistent system **37.** infinitely many solutions, dependent equations **39.** $(1, 2)$ **41.** $(0, 5)$ **43.** $\left(0, -\frac{7}{3}\right)$ **45.** $(-2, -3)$ **47.** $(-2, 9)$ **49.** $(-6, 16)$ **51.** infinitely many solutions, dependent equations **53.** $\left(\frac{5}{3}, \frac{3}{5}\right)$ **55.** $(4, -7)$ **57.** no solution, inconsistent system **59.** $\left(\frac{4}{5}, \frac{3}{4}\right)$ **61.** $(2, -3)$ **63.** $(9, -1)$ **65.** $(2, 3)$ **67.** $\left(-\frac{1}{3}, 1\right)$ **75.** $-\frac{5}{2}$ **77.** $-\frac{8}{5}$ **79.** $\frac{4}{3}$

Study Set Section 3.3 (page 247)

1. parallel **3. a.** 70°, 75°, **b.** alternate interior **5. a.** $x + c$ **b.** $x - c$ **7. a.** 0.06 **b.** 0.048 **c.** 0.135 **9.** $0.05x$, $0.04y$, 25,000, 1,050, $\begin{cases} x + y = 25{,}000 \\ 0.05x + 0.04y = 1{,}050 \end{cases}$ **11.** $13.90x$, $5.10y$, 61.50, $\begin{cases} x + y = 6 \\ 13.90x + 5.10y = 61.50 \end{cases}$ **13.** 750 ohms, 625 ohms. **15.** Montana: 406, Idaho: 208 **17.** Canada: 8, United States: 24 **19.** 75°, 25° **21.** 150°, 30° **23.** 16 m by 20 m **25.** 15 sec: $475, 30 sec: $800

27. 85 racing bikes, 120 mountain bikes
29. Rolling Stones: $264; Jimmy Buffet: $132
31. a. 400 tires
b.

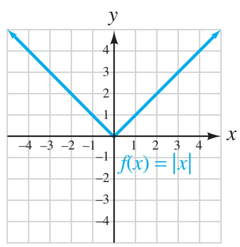


c. the second mold

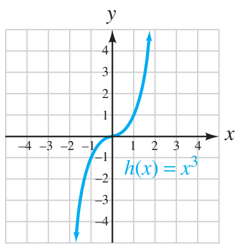
33. a. $4{,}666\frac{2}{3}$ books **b.** the newer press **35.** 4,031
37. a. 590 units per month **b.** 620 units per month
c. A (smaller loss) **39.** $3,000 at 10%, $5,000 at 12%
41. credit union: $8,000, money market: $24,000
43. VISA: $11,800, Robinsons-May: $4,700
45. 525 mph, 75 mph **47.** walking: 6 ft per sec, moving walkway: 2 ft per sec **49.** 25 mph, 5 mph **51.** gummy bears: 45 lb, jelly beans: 15 lb **53.** regular: $14\frac{2}{3}$ lb, Kona: $5\frac{1}{3}$ lb
55. small flake: 900 lb, mylar: 1,100 lb **57.** 10%: 8 pints, 40%: 16 pints **59.** 148 g of the 0.2%, 37 g of the 0.7%
67. rational **69.** identity **71.** isosceles

Study Set Section 3.4 (page 260)

1. system **3.** three **5.** dependent **7. a.** no solution **b.** no solution **9.** $x + 2y - 3z = -6$ **11.** yes **13.** no
15. (1, 1, 2) **17.** (−1, 3, 0) **19.** (0, 2, 2) **21.** (−3, 0, 5)
23. (7, −6, 3) **25.** $\left(\frac{1}{2}, 2, -1\right)$ **27.** no solution, inconsistent system **29.** infinitely many solutions, dependent equations
31. (8, 4, 5) **33.** (60, 30, 90) **35.** (2, 4, 8) **37.** (2, 6, 9)
39. no solution, inconsistent system **41.** infinitely many solutions, dependent equations **43.** (3, 2, 1)
47.



49.



Study Set Section 3.5 (page 266)

1. satisfy **3.** $\begin{cases} x + y + z = 50 \\ 5x + 6y + 7z = 295 \\ 2x + 3y + 4z = 145 \end{cases}$ **5.** $-3 = 4a + 2b + c$
7. 30 large, 50 medium, 100 small **9.** Food A: 2, Food B: 3, Food C: 1 **11.** 120 coats, 200 shirts, 150 slacks **13.** Young: 85, Montana: 55, Gannon: 16 **15.** nitrogen: 78%, oxygen: 21%, other gases: 1% **17.** $\angle A = 40°$, $\angle B = 60°$, $\angle C = 80°$
19. *X-Files:* 201, *Will & Grace:* 194, *Seinfeld:* 180
21. 6 lb rose petals, 3 lb lavender, 1 lb buckwheat hulls
23. nickels: 20, dimes: 40, quarters: 4 **25.** $y = \frac{1}{2}x^2 - 2x - 1$

27. $x^2 + y^2 - 2x - 2y - 2 = 0$ **31.** yes **33.** yes
35. no; (1, 2), (1, −2) **37.** no; (4, 2), (4, −2)

Study Set Section 3.6 (page 277)

1. matrix **3.** rows, columns **5.** augmented **7.** 2×3
9. $\begin{cases} x - y = -10 \\ y = 6 \end{cases}$, (−4, 6) **11.** It has no solution. The system is inconsistent.
13. a. multiply row 1 by $\frac{1}{3}$, $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 5 & -2 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

b. add −1 times row 1 to row 2, $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 3 & 1 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$

15. −1, 1, −5, 2, y, 4 **17.** $\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 3 & -1 & -10 \end{array}\right]$
19. $\begin{cases} x + 6y = 7 \\ y = 4 \end{cases}$ **21.** $\left[\begin{array}{cc|c} 1 & -4 & 4 \\ -3 & 1 & -6 \end{array}\right]$
23. $\left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 2 \\ 1 & -4 & 4 \end{array}\right]$ **25.** $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ -2 & 2 & -2 & 5 \\ 1 & 5 & -2 & 1 \end{array}\right]$
27. $\left[\begin{array}{ccc|c} 3 & 6 & -9 & 0 \\ -2 & -1 & 7 & 1 \\ -2 & 2 & -2 & 5 \end{array}\right]$ **29.** (1, 1) **31.** (2, −3)
33. (1, 2, 3) **35.** (4, 5, 4) **37.** no solution, inconsistent system **39.** infinitely many solutions, dependent equations
41. no solution, inconsistent system **43.** infinitely many solutions, dependent equations **45.** (−1, −1) **47.** (0, −3)
49. (2, 1, 0) **51.** (−1, −1, 2) **53.** (0, 1, 3) **55.** (−4, 8, 5)
57. no solution, inconsistent system **59.** 22, 68
61. 40, 65, 75 **63.** 76, 104 **67.** $m = \frac{y_2 - y_1}{x_2 - x_1}$ $(x_2 \neq x_1)$
69. $y - y_1 = m(x - x_1)$

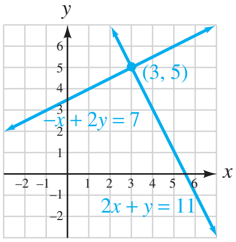
Study Set Section 3.7 (page 287)

1. determinant **3.** minor **5.** rows, columns
7. dependent, inconsistent **9.** $ad - bc$ **11.** $\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}$
13. $\left(\frac{7}{11}, -\frac{5}{11}\right)$ **15.** 6, 30 **17.** 8 **19.** −2 **21.** 200 **23.** 6
25. 1 **27.** 26 **29.** 0 **31.** −79 **33.** (4, 2) **35.** $\left(-\frac{1}{2}, \frac{1}{3}\right)$
37. no solution, inconsistent system **39.** infinitely many solutions, dependent equations **41.** (1, 1, 2) **43.** (3, 2, 1)
45. (2, −1) **47.** (3, −2, 1) **49.** $\left(-\frac{1}{2}, -1, -\frac{1}{2}\right)$
51. infinitely many solutions, dependent equations
53. (−2, 3, 1) **55.** no solutions, inconsistent system
57. 200 of the $67 phones, 160 of the $100 phones
59. $5,000 in HiTech, $8,000 in SaveTel, $7,000 in OilCo
61. −23 **63.** 26 **67.** no **69.** yes **71.** x **73.** y-intercept
75. x, y

Chapter 3 Review Exercises (page 291)

1. yes **2.** no **3.** (1, 3), (2, 1), (4, −3) (answers may vary) **4.** (0, −4), (2, −2), (4, 0) (answers may vary) **5.** (3, −1) **6.** President Clinton's job approval and disapproval ratings were the same: approximately 47% in 5/94 and approximately 48% in 5/95.

7.



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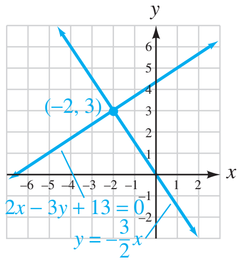

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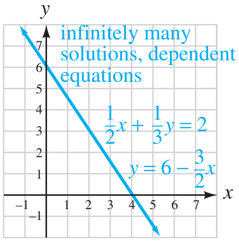

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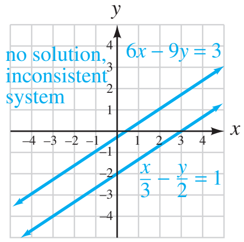
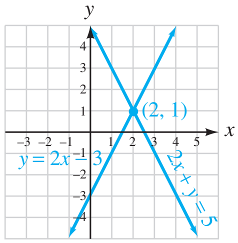


11. (−1, 3) **12.** (−3, −1) **13.** (3, 4) **14.** infinitely many solutions, dependent equations **15.** (−3, 1) **16.** no solution, inconsistent system **17.** (9, −4) **18.** $\left(4, \frac{1}{2}\right)$ **19.** Using the addition method, the computations are easier. **20.** (−1, 0.7), (answers may vary); $\left(-1, \frac{2}{3}\right)$ **21.** 162 mi, 83 mi **22.** 8 mph, 2 mph **23.** 500 oz of 6%, 250 oz of 18% **24.** \$4,000 at 6%, \$6,000 at 12% **25.** teaspoon: 5 ml, tablespoon: 15 ml **26.** 17,500 bottles **27.** no **28.** yes, infinitely many solutions **29.** (2, 0, −3) **30.** (2, −1, 3) **31.** $\left(\frac{1}{2}, 4, -6\right)$ **32.** no solution, inconsistent system **33.** (−1, 1, 3) **34.** infinitely many solutions, dependent equations **35.** 2 cups mix A, 1 cup mix B, 1 cup mix C **36.** 50 small bears, 60 medium bears, 40 large bears **37.** \$5,000 at 5%, \$7,000 at 6%, and \$10,000 at 7% **38.** $y = -\frac{1}{16}x^2 + 2x$ **39.** $\left[\begin{array}{cc|c} 5 & 4 & 3 \\ 1 & -1 & -3 \end{array}\right]$ **40.** $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -3 & -1 & 4 \\ 6 & 1 & -2 & -1 \end{array}\right]$ **41.** (1, −3) **42.** (5, −3, −2) **43.** infinitely many solutions, dependent equations **44.** no solution, inconsistent system **45.** 18 **46.** 38 **47.** −3 **48.** 28 **49.** (2, 1) **50.** no solution, inconsistent system **51.** (1, −2, 3) **52.** (−3, 2, 2)

Chapter 3 Test (page 301)

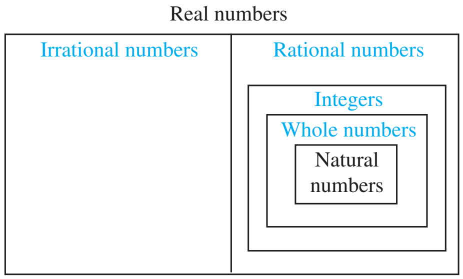
1.



2. (7, 0) **3.** (2, −3) **4.** dependent **5.** no **6.** (3, 2, −1) **7.** 55, 70 **8.** 15 gal 40%, 5 gal 80% **9.** (2, 2) **10.** (1, 0, −1) **11.** 22 **12.** 4 **13.** $\begin{vmatrix} -6 & -1 \\ -6 & 1 \end{vmatrix}$ **14.** $\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$ **15.** −3 **16.** 3 **17.** −1 **18.** C: 60, GA: 30, S: 10 **19.** 25 lb peanuts, 10 lb cashews, 15 lb Brazil nuts **20.** Substitution, the second equation is solved for y. **21.** The system has no solution. **23.** (2035, 25)

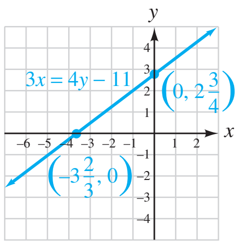
Chapters 1–3 Cumulative Review (page 303)

1.

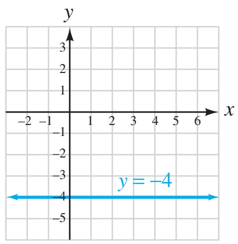


2. \$504,000,000,000 **3.** 70 **4.** 2 **5.** $-12.1x^2 + 12.7x$ **6.** −4 **7.** 20 mph **8.** 5 lb apple slices, 5 lb banana chips **9.** −28 **10.** $-\frac{1}{3}$ **11.** −2 **12.** all real numbers, identity **13.** $B = \frac{C - Ax}{A}$ **14.** $n = \frac{l - a + d}{d}$ or $n = \frac{l - a}{d} + 1$

15.



16.



17. $-\frac{4}{5}$ **18.** $y = -3x + 17$ **19.** −105 **20.** −95 **21.** $f(x) = x^3$ (answer may vary) **22.** no **23.** no **24.** $v = 17.5x + 300$ **25.** (2, −1) **26.** (−1, 0, 2) **27.** 26 **28.** 26

Think It Through (page 307)

from 1979 to 2003

Study Set Section 4.1 (page 314)

1. inequality **3.** parenthesis **5.** linear **7.** is less than, is greater than or equal to **9.** equation **11.** inequality **13.** inequality **15.** true **17.** false **19.** true **21.** yes **23.** yes **25.** yes **27.** yes **29.** false **31.** true

33. $<, 7x, 14, >, -2$ **35.** $x > \frac{7}{8}$

37. $(4, \infty)$; $\{x \mid x > 4\}$

39. $(-\infty, 4]$; $\{x \mid x \le 4\}$

41. $(-\infty, 1)$ **43.** $(-3, \infty)$

45. $[-11, \infty)$ **47.** $(-\infty, 2]$

49. $(-\infty, 1)$ **51.** $[2, \infty)$

53. $[20, \infty)$ **55.** $\left[-\frac{2}{5}, \infty\right)$

57. $[-2, \infty)$ **59.** $(-\infty, 7]$

61. $(-\infty, 10)$ **63.** $[-36, \infty)$

65. no solution, $\varnothing$ **67.** $(-\infty, \infty)$

69. the number of seriously injured ≤ 16
71. the age of the car ≥ 25

73. $[60, \infty)$ **75.** $\left(-\frac{10}{3}, \infty\right)$

77. $(6, \infty)$ **79.** $(-\infty, 1.5]$

81. $(-\infty, 20]$ **83.** no solution, $\varnothing$

85. $\left(-\infty, \frac{45}{7}\right]$ **87.** Midwest, South

89. $6 + 45 > 52$ **91.** 8 hr **93.** 88 or higher **95.** 13 hr
97. 7 hr **99.** $x < 1$ **101.** $x \ge -4$ **107.** 4, 5, 3
109. $6, -6$

Think It Through (page 326)

1. 2001, 2002 **2.** 1999, 2001, 2002 **3.** 1999, 2000

Study Set Section 4.2 (page 327)

1. intersection, union **3.** double **5.** closed **7.** both
9. reversed **11. a.** no **b.** yes **13. a.** no solution
b. $(-\infty, \infty)$ **15. a.** ii **b.** iii **c.** i

17. $(-\infty, -3) \cup (3, \infty)$

19. $[-3, 3]$ **21.** $A \cap B$

23. **25.**

27. half-open **29.** open **31.** $\{4, 6\}$ **33.** $\{-3, 1, 2\}$
35. $\{-3, -1, 0, 1, 2, 4, 6, 8, 10\}$ **37.** $\{-3, 0, 1, 2, 3, 4, 5, 6, 8\}$
39. $(-2, 5]$

41. $[-8, 4]$ **43.** $[2, \infty)$

45. $[5, \infty)$ **47.** $\varnothing$ **49.** $\varnothing$

51. $[1, 4]$

53. $[-21, -3)$

55. $(-\infty, -2] \cup (6, \infty)$

57. $(-\infty, -1) \cup (2, \infty)$

59. $(-\infty, 1)$ **61.** $(-\infty, \infty)$

63. $(-10, -9)$

65. $(-3, 1)$ **67.** $(-2, 5)$

69. $[-4, 6)$ **71.** $[2, 2]$

73. $(-0.7, 0.2]$

75. $\left[\frac{1}{3}, 3\right]$

77. $(-\infty, -2) \cup (2, \infty)$

79. $(-\infty, \infty)$ **81. a.** 128, 192

b. $32 \le s \le 48$ **83.** See doctor today. **85. a.** 1999
b. 1998–2001 **c.** 1999, 2000 **d.** 1998–2000 **91.** 85.7, 86, 86
93. 13.3 pts/game

Study Set Section 4.3 (page 339)

1. equation **3.** isolate **5.** 0 **7.** more than **9.** less than 5
11. yes **13.** yes **15.** $x = 8$ or $x = -8$ **17.** $-8 \le x \le 8$
19. a. ii **b.** iii **c.** i **21.** $|x| < 4$ **23.** $|x + 3| > 6$ **25.** 8

27. -0.02 **29.** $-\frac{31}{16}$ **31.** π **33.** 25 **35.** -2

37. $23, -23$ **39.** $\varnothing$ **41.** $\frac{14}{3}, -6$ **43.** $\varnothing$ **45.** $-\frac{49}{4}, \frac{55}{4}$

47. 4 **49.** $0, -6$ **51.** $-4, -28$ **53.** $-\frac{5}{3}, 1$ **55.** -8

57. $-2, -\frac{4}{5}$ **59.** $0, -2$ **61.** 0 **63.** $\frac{4}{3}$

65. $(-4, 4)$ **67.** $[-21, 3]$

69. $\left(-\frac{8}{3}, 4\right)$ **71.** $\varnothing$

73. $(-\infty, -3) \cup (3, \infty)$

75. $(-\infty, -12) \cup (36, \infty)$

77. $\left(-\infty, -\frac{16}{3}\right) \cup (4, \infty)$

79. $(-\infty, \infty)$ **81.** $9.1, -2.9$

83. $40, -20$ **85.** $(-\infty, -2] \cup \left[\frac{10}{3}, \infty\right)$

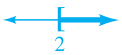
87. $(-\infty, -2) \cup (5, \infty)$ 

89. $40, -20$
91. $[-10, 14]$ 

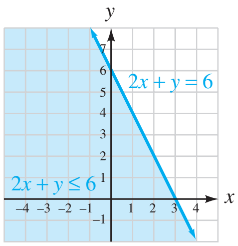
93. $-3, -\frac{3}{4}$
95. $(-\infty, -24) \cup (-18, \infty)$ 

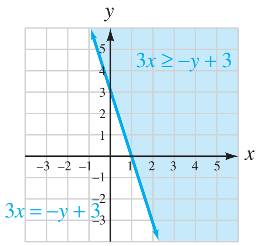
97. $\varnothing$
99. $\varnothing$ **101.** $70° \le t \le 86°$ **103. a.** $|c - 0.6°| \le 0.5°$ **b.** $[0.1°, 1.1°]$ **105. a.** 26.45%, 24.76% **b.** It is less than or equal to 1%. **113.** 50°, 130°

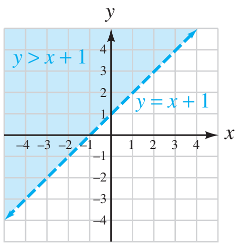
Study Set Section 4.4 (page 347)

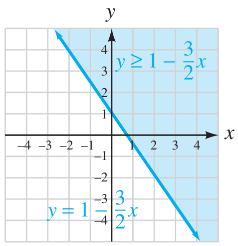
1. linear, two **3.** edge **5.** yes **7.** yes **9.** yes **11.** yes **13.** $m = 3, (0, -1)$ **15.** no
17. 

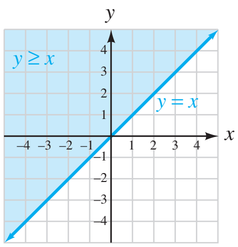
19. no, dashed **21.** yes, solid

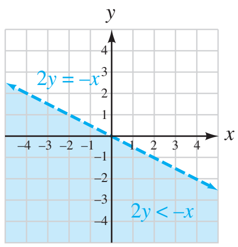
23. 

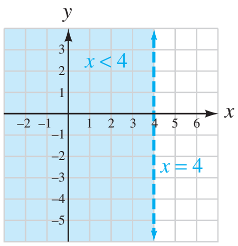
25. 

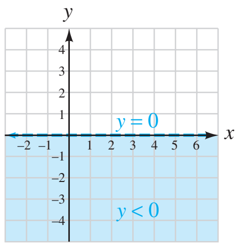
27. 

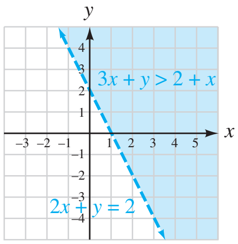
29. 

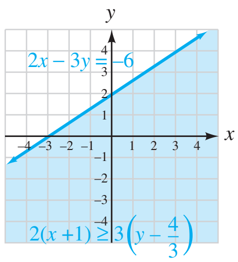
31. 

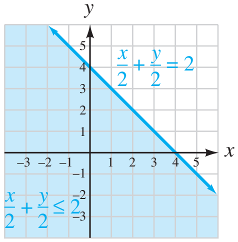
33. 

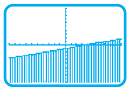
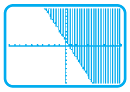
35. 

37. 

39. 

41. 

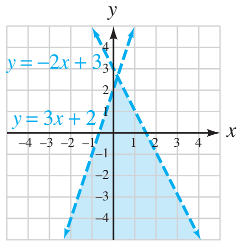
43. 

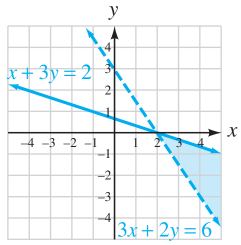
45. $3x + 2y > 6$ **47.** $x \le 3$
49. 
51. 

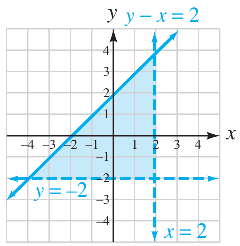
53. a. the Mississippi River **b.** the area of the U.S. west of the Mississippi River **55.** (5, 15), (15, 10), (20, 5); answers may vary **57.** (40, 80), (80, 80), (120, 40); answers may vary **61.** yes **63.** (3, 1)

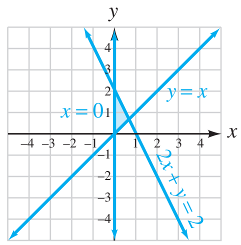
Study Set Section 4.5 (page 356)

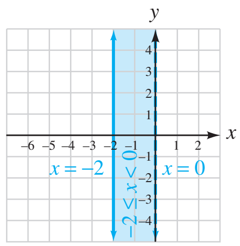
1. inequalities **3.** intersect **5.** yes **7.** no **9.** yes **11.** false **13.** true **15.** true **17.** iii **19.** i

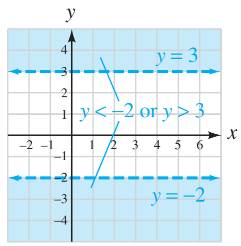
21. 

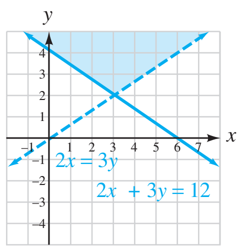
23. 

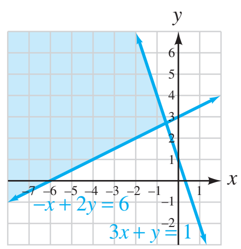
25. 

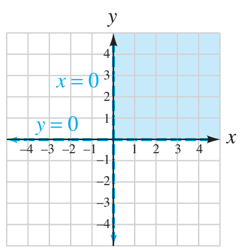
27. 

29. 

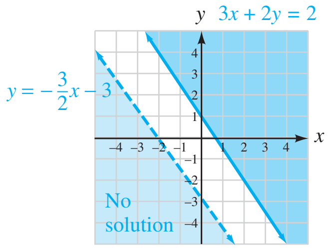
31. 

33. 

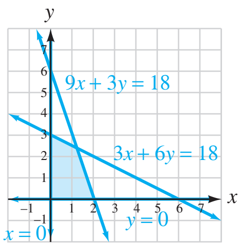
35. 

37. 

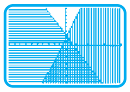
39.



41.

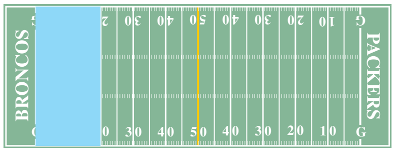


43.



45.

47.



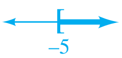
Packers moving this direction

49.

51. 1 \$10 CD and 2 \$15 CDs, 4 \$10 CDs and 1 \$15 CD **53.** 2 desk chairs and 4 side chairs, 1 desk chair and 5 side chairs **57.** IV **59.** II

Chapter 4 Review Exercises (page 361)

1. a. no **b.** yes

2. $[-5, \infty)$, $\{x \mid x \geq -5\}$ 

3. $(-\infty, 3]$, $\{x \mid x \leq 3\}$ 

4. $[4, \infty)$, $\{x \mid x \geq 4\}$ 

5. $(-\infty, 20)$, $\{t \mid t < 20\}$ 

6. $\left(-\infty, -\frac{51}{11}\right)$, $\left\{x \mid x < -\frac{51}{11}\right\}$ 

7. $(-\infty, \infty)$, $\mathbb{R}$ 

8. no solution, $\varnothing$ **9.** \$20,000 or more **10.** She needs to receive a score that is greater than 5.2. **11.** 5 hr **13.** $\{-3, 3\}$ **14.** $\{-6, -5, -3, 0, 3, 6, 8\}$ **15.** yes **16.** no

17. 

18. 

19. $[-10, -4)$ 

20. $(-\infty, -11)$ 

21. no solution, $\varnothing$

22. $[0, 0]$ 

23. $\left(-\frac{1}{3}, 2\right)$ 

24. $[1, 9]$ 

25. yes **26.** no

27. $(-\infty, -5) \cup (4, \infty)$ 

28. $(-\infty, \infty)$ 

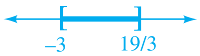
29. $17 \leq 4x \leq 25$, $4.25 \text{ ft} \leq x \leq 6.25 \text{ ft}$, $[4.25, 6.25]$

30. a. ii, iv **b.** i, iii **31.** $2, -2$ **32.** $3, -\frac{11}{3}$ **33.** $\frac{26}{3}, -\frac{10}{3}$

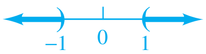
34. no solution, $\varnothing$ **35.** 3 **36.** $\frac{1}{8}, -\frac{19}{8}$ **37.** $\frac{1}{5}, -5$ **38.** $\frac{13}{12}$

39. $[-3, 3]$ 

40. $(-5, -2)$ 

41. $\left[-3, \frac{19}{3}\right]$ 

42. no solution, $\varnothing$

43. $(-\infty, -1) \cup (1, \infty)$ 

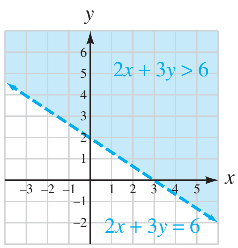
44. $(-\infty, -4] \cup \left[\frac{22}{5}, \infty\right)$ 

45. $\left(-\infty, \frac{4}{3}\right) \cup (4, \infty)$ 

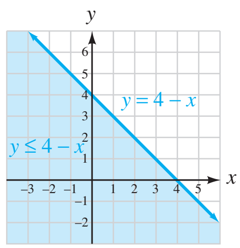
46. $(-\infty, \infty)$, $\mathbb{R}$ 

47. Since $|0.04x - 8.8|$ is always greater than or equal to 0 for any real number x, this absolute value inequality has no solution. **48.** Since $\left|\frac{3x}{50} + \frac{1}{45}\right|$ is always greater than or equal to 0 for any real number x, this absolute value inequality is true for all real numbers. **49. a.** 8, 2 **b.** $[6, 10]$ **50.** $3, -3$ **51.** no **52.** yes

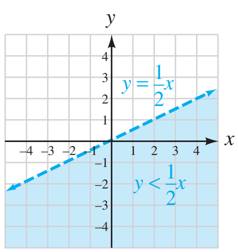
53.



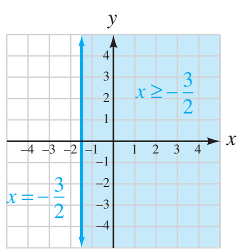
54.



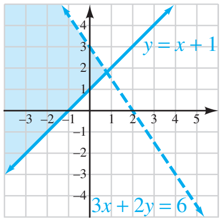
55.

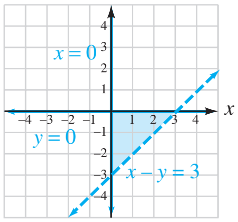


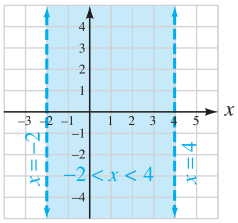
56.

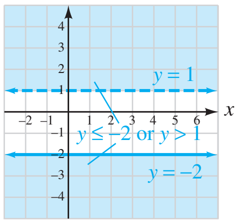


57. $6x + 4y \geq 10{,}200$; (1,800, 0), (1,000, 1,500), (2,000, 2,000)
58. $3x - 4y > 12$ **59.** yes **60. a.** true **b.** false **c.** true
d. false **e.** true **f.** true

61.


62.


63.


64.


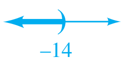
65. 5 \$20 shirts and 15 \$30 shirts, 15 \$20 shirts and 10 \$30 shirts **66.** ≥, ≤, ≥, ≤

Chapter 4 Test (page 369)

1. a. inequality **b.** infinity **c.** compound **d.** union, intersection **e.** system, two **2.** yes

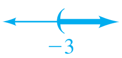
3. $(12, \infty)$,


4. $(-\infty, -5]$,


5. $(-\infty, -14)$,


6. $(-\infty, \infty)$,


7. more than 78 **8.** 11 hr **9.** no **10.** $\{-4, 0, 11\}$
11. $\{-5, -4, 0, 7, 8, 9, 10, 11\}$

12. a.


b.


13. $\left[1, \frac{9}{4}\right]$


14. $(-\infty, -3) \cup (8, \infty)$


15. $(-2, 16)$


16. no solution, $\varnothing$

17. $-5, \frac{23}{3}$ **18.** 4, −4 **19.** $\frac{8}{9}, -\frac{8}{9}$ **20.** no solution, $\varnothing$
21. 10 **22.** $|x - 0.0625| \leq 0.0015$; [0.0610, 0.0640]

23. $[-7, 1]$

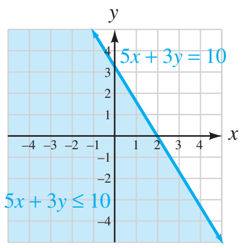

24. $(-\infty, -9) \cup (13, \infty)$

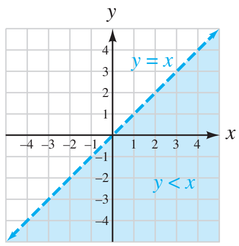

25. $(-\infty, 1) \cup (3, \infty)$

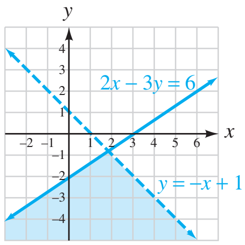

26. $\left[\frac{4}{3}, \frac{8}{3}\right]$

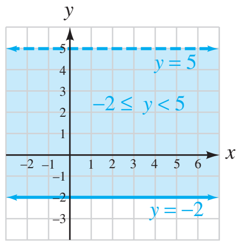

27. $(-\infty, \infty)$, $\mathbb{R}$

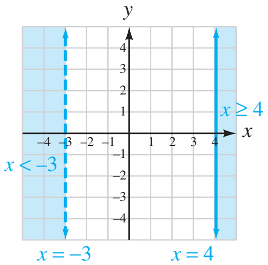

28. $(-6, -3)$

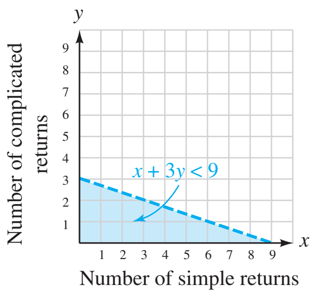
29.


30.


31.


32.


33.


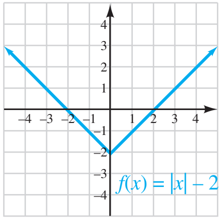
34. (1, 1), (2, 1), (2, 2); answers may vary


35. a. (3, −4) is a solution of inequality 2. **b.** No, it does not lie in the doubly shaded region. **36.** ≤, ≥, ≥, ≤

Chapters 1–4 Cumulative Review (page 371)

1. rational numbers: terminating and repeating decimals; irrational numbers: nonterminating, nonrepeating decimals
2. 0.125, 0.0625, 0.03125, 0.054125 **3.** 10 **4.** −6
5. $-2a + b - 2$ **6.** $8t - 9$ **7.** 3 **8.** 6 **9.** no solution, $\varnothing$
10. −2 **11.** $d = \frac{l - a}{n - 1}$ **12.** 201 ft^2 **13.** \$20,000 **14.** $\frac{1}{4}$ hr
15. $-\frac{8}{5}$ **16. a.** 26,000 prisoners/yr **b.** 1990–1995, 67,200 prisoners/yr **17.** parallel **18.** $y = \frac{1}{3}x + \frac{11}{3}$
19. a. $y = -0.06x + 8.2$ **b.** 2.8 L/min
20. D: $\{-12, -6, 5, 8\}$, R: $\{-6, 4, 6\}$ **21.** 14 **22.** $3t^2 - t$

23. D: the set of real numbers, R: the set of all real numbers greater than or equal to -2



24. yes **25.** $(2, 1)$ **26.** $(7, 23)$, in 1907 the percent of U.S. workers in white-collar and farming jobs was the same (23%): $(45, 42)$, in 1945 the percent of U.S. workers in white-collar and blue-collar jobs was the same (42%). **27.** $(3, 1)$ **28.** $(1, 1)$ **29.** 750 **30.** $(-1, -1, 3)$ **31.** 250 \$5 tickets, 375 \$3 tickets, 125 \$2 tickets **32.** $(-1, -1)$ **33.** -10 **34.** $(3, -2)$ **35.** $(-\infty, 11]$, $\{x \mid x \le 11\}$

36. $(-3, 3)$, $\{x \mid -3 < x < 3\}$

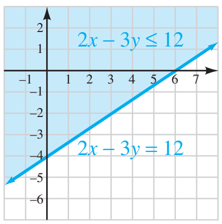
37. $(-\infty, 2) \cup (7, \infty)$

38. $\left[1, \frac{9}{4}\right]$

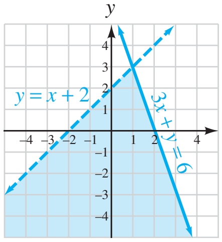
39. $3, -\frac{3}{2}$ **40.** $-5, -\frac{3}{5}$ **41.** $\left[-\frac{2}{3}, 2\right]$

42. $(-\infty, -4) \cup (1, \infty)$

43.

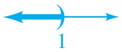


44.



Study Set Section 5.1 (page 385)

1. power **3.** product, quotient **5.** 9 **7.** $\frac{1}{16}$ **9.** -9 **11.** 9 **13.** x^{m+n} **15.** $x^n y^n$ **17.** 1 **19.** x^{m-n} **21.** $2x = 2 \cdot x$; $x^2 = x \cdot x$ **23. a.** add **b.** subtract **c.** multiply **25.** $9, -(-2), 11$ **27.** 5; 3 **29.** x; 5 **31.** b; 6 **33.** $\frac{n}{4}$; 3 **35.** x^5 **37.** x^{10} **39.** $2a^4b^5$ **41.** $-5y^7$ **43.** 64 **45.** z^{24} **47.** a^{20} **49.** b^{12} **51.** $27x^9y^{12}$ **53.** $\frac{1}{729}m^6n^{12}$ **55.** $\frac{a^{15}}{b^{10}}$ **57.** $\frac{16a^4}{9b^6}$ **59.** 1 **61.** 1 **63.** 3 **65.** $-6b$ **67.** $\frac{1}{25}$ **69.** $-\frac{3}{p^2}$ **71.** $\frac{1}{y^2}$ **73.** r^8 **75.** $2a^3$ **77.** $\frac{8}{9}$ **79.** $-\frac{b^5}{4a^3}$ **81.** $-\frac{p^3q^3}{5}$ **83.** p^4 **85.** $\frac{3}{y^{16}}$ **87.** a^7 **89.** 1 **91.** $\frac{64b^{12}}{27a^9}$ **93.** $-\frac{8a^{21}}{b^3}$ **95.** $\frac{9}{4}$ **97.** $\frac{b^8}{a^{12}}$ **99.** $\frac{4}{9a^4b^8}$ **101.** $\frac{4p^2r^{12}}{9q^8}$ **103.** $\frac{1}{625}$ **105.** $\frac{1}{25}$ **107.** $-32x^5$ **109.** k^7 **111.** p^{10} **113.** x^5y^4 **115.** $\frac{1}{b^{72}}$ **117.** $\frac{s^3}{r^9}$ **119.** $-\frac{1}{d^3}$ **121.** $\frac{a^6}{b^4}$ **123.** a^5 **125.** c^7 **127.** a^4 **129.** $\frac{1}{9x^3}$ **131.** 1 **133.** $-\frac{27}{8a^{18}}$ **135.** $\frac{8y^{13}}{9x^{16}}$ **137.** $\frac{8x^5y^{14}}{9}$ **139.** 3.462825992 **141.** -244.140625 **149.** $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}$ **151.** $10^3 \cdot 26^3$; 17,576,000 **153. a.** x^6 ft^2 **b.** x^9 ft^3 **159.** $(-\infty, 1)$



161. $(-\infty, 20]$

Think It Through (page 390)

5.36×10^7, 1.56×10^7, 7.45×10^{11}

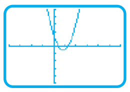
Study Set Section 5.2 (page 393)

1. scientific, standard **3.** 10^n, integer **5.** left **7.** 60.22 is not between 1 and 10. **9.** 3.9×10^3 **11.** 7.8×10^{-3} **13.** 1.73×10^{14} **15.** 9.6×10^{-6} **17.** 3.23×10^7 **19.** 6.0×10^{-4} **21.** 5.27×10^3 **23.** 3.17×10^{-4} **25.** 270 **27.** 0.00323 **29.** 796,000 **31.** 0.00037 **33.** 5.23 **35.** 23,650,000 **37.** 1.817×10^{12} **39.** 5.005×10^8 **41.** 5×10^{-8} **43.** 1.9×10^{-10} **45.** 4.005×10^{20}; 400,500,000,000,000,000,000 **47.** 4.3×10^{-3}; 0.0043 **49.** 6×10^3; 6,000 **51.** 3.6×10^{-5}; 0.000036 **53.** 2,600,000 to 1 **55.** \$$1.7 \times 10^{12}$, \$$3.9 \times 10^9$, \$$2.75 \times 10^8$, \$$3.12 \times 10^8$ **57.** 8.5×10^{-28} g **59.** about 9.5×10^{15} m **61. a.** 2.5×10^9 sec = 2,500,000,000 sec **b.** about 79 years **63.** 1.209×10^8 mi **65.** 1.0×10^{21} **71.** $\left[1, \frac{9}{4}\right]$

73. $(-\infty, 2) \cup (7, \infty)$

Study Set Section 5.3 (page 406)

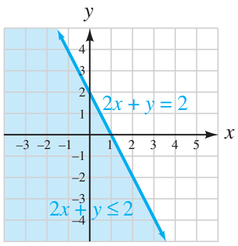
1. polynomial **3.** binomial **5.** degree **7.** cubic **9.** like **11.** $-2x^4 - 5x^2 + 3x + 7$ **13.** $7y + 4y^2 - 5y^3 - 2y^5$ **15.** $9x^2 + 3x - 2$ **17.** $-1, -1, 1, 3$ **19.** monomial, 2 **21.** trinomial, 3 **23.** binomial, 2 **25.** monomial, 0 **27.** none of these, 10 **29.** binomial, 9 **31.** -4 **33.** -2 **35.** D: $(-\infty, \infty)$ R: $[0, \infty)$; 8, 2, 0, 2, 8 **37.** D: $(-\infty, \infty)$ R: $(-\infty, \infty)$; $-42, 0, 12, 6, -6, -12, 0, 42$ **39.**

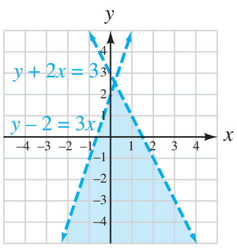


41. $10x$ **43.** $2x^2 + 4x$ **45.** $-3r^2t^3$ **47.** $\frac{1}{3}x^2y^3 + \frac{1}{4}x^2y^2$ **49.** $x^2 - 5x + 6$ **51.** $5a^2 + 3a - 9$ **53.** $5x^2 + x - 1$ **55.** $7x^2 + 6x + 10$ **57.** $2a^2$ **59.** $13a^2b$ **61.** $-5a^2 + 4a + 4$ **63.** $4x^3 - 3x^2 + 3x - 7$ **65.** $x^2 - 8x + 22$ **67.** $-3y^3 + 18y^2 - 28y + 35$ **69.** $2pq - 2q$ **71.** $2x^2y^3 + 13xy + 3y^2$ **73.** $4x^2 - 11$ **75.** $6x^3 - 6x^2 + 14x - 17$ **77.** $2x$ **79.** $12x^2 + 9x - 14$ **81.** 20 ft **83.** 872 ft^3 **85.** 2,160 in.^{3} **87. a.** $V(x) = 2{,}500x + 275{,}000$ **b.** \$325,000

89. a. 4 **b.** ascending **c.** $-\frac{1}{720}$ **d.** fourth **e.** sixth
97. $[-5, 5]$ **99.** $(-1, 9)$

Study Set Section 5.4 (page 418)

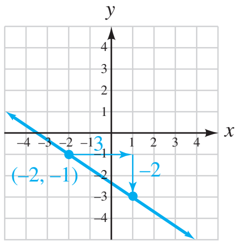
1. product **3.** terms **5.** factors **7.** term
9. $x^2 + 2xy + y^2$ **11.** $x^2 - y^2$ **13.** square, square
15. $\frac{1}{2}b^2 + \frac{3}{2}b - 5$ **17.** $4a^2 - 9$ **19.** $2x, -3$ **21.** $4, -3$
23. $8b^2 - 6b + 1$ **25.** $-6a^3b$ **27.** $-15a^2b^2c^3$
29. $-120a^9b^3$ **31.** $-405x^7y^4$ **33.** $3x + 6$ **35.** $3x^3 + 9x^2$
37. $-6x^3 + 6x^2 - 4x$ **39.** $7r^3st + 7rs^3t - 7rst^3$
41. $x^2 + 5x + 6$ **43.** $6t^2 + 5t - 6$ **45.** $6y^2 - 5yz + z^2$
47. $b^4 + b^3 - b - 1$ **49.** $x^3 - y^3$
51. $6y^3 + 11y^2 + 9y + 2$ **53.** $4a^2 + 4a - 3$
55. $12x^3 - 2x^2 - 7x + 2$ **57.** $x^2 + 2xz - y^2 + z^2$
59. $2a^2 - ab - az - b^2 - 2bz - z^2$
61. $18p^4 + 30p^3 - 72p^2$ **63.** $-10m^3n + 15m^2n^2 + 45mn^3$
65. $x^2 + 4x + 4$ **67.** $a^2 - 8a + 16$ **69.** $4a^2 + 4ab + b^2$
71. $25r^4 + 60r^2 + 36$ **73.** $x^2 - 4$ **75.** $y^6 - 4$
77. $3x^2 + 12x$ **79.** $3x^2 + 3x - 11$ **81.** $2b^2 + 35b + 48$
83. $0.2t^2 - 2.7t + 9$ **85.** $-6t^2u^2 + 11tu - 3$
87. $27b^5 - 9b^3c - 3b^2c + c^2$
89. $55m^3 + 22m^2n^2 + 15mn^3 + 6n^5$
91. $24m^3y - 20m^2y^2 + 4my^3$ **93.** $-18a^3b^4 - 6a^2b^5$
95. $81a^2b^2 - 72ab + 16$ **97.** $\frac{1}{16}b^2 + b + 4$
99. $16k^2 - 10.4k + 1.69$ **101.** $x^2y^2 - 36$ **103.** $\frac{1}{4}x^2 - 256$
105. $5.76 - y^2$ **107.** $8a^3 - b^3$
109. $a^3 - 3a^2b - ab^2 + 3b^3$
111. $2a^2 + ab - b^2 - 3bc - 2c^2$ **113.** $r^4 - 2r^2s^2 + s^4$
115. $-p^2 + 4pq$ **117.** $-b^2 - 4b - 1$
119. $9.2127x^2 - 7.7956x - 36.0315$
121. $299.29y^2 - 150.51y + 18.9225$
123. a. $(x + y)(x - y)$ **b.** $x(x - y); x^2 - xy$
c. $y(x - y); xy - y^2$ **d.** They represent the same area.
$(x + y)(x - y) = x^2 - y^2$
125. $x(12 - 2x)(12 - 2x)\text{ in.}^3 = (144x - 48x^2 + 4x^3)\text{ in.}^3$
131.


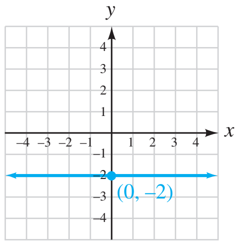
133.


Study Set Section 5.5 (page 426)

1. factored **3.** factor **5.** greatest common factor **7.** $6xy^2$
9. a. The terms within the parentheses have a common factor 2. **b.** The terms within the parentheses have a common factor t. **11.** 4 **13.** $x^2, 2$ **15.** $2 \cdot 3$ **17.** $3^3 \cdot 5$
19. 2^7 **21.** $5^2 \cdot 13$ **23.** 12 **25.** 2 **27.** $4a^2$ **29.** $6xy^2z^2$
31. $2(x + 4)$ **33.** $2x(x - 3)$ **35.** $5x^2y(3 - 2y)$
37. $7ab(4a^2 - 3b^2)$ **39.** $9x^7y^3(5x^3 - 7y^4 + 1)$
41. $2a^3b^2(6a^3b^3 + 4a - 3b^2)$ **43.** prime **45.** prime
47. $-3(a + 2)$ **49.** $-x(3x + 1)$ **51.** $-3x(2x + y)$
53. $-6ab(3a + 2b)$ **55.** $(x + y)(4 + t)$
57. $(m + n + p)(3 + x)$ **59.** $(x + y)(a + b)$
61. $(x + 2)(x + y)$ **63.** $(v - u)(v - 7)$
65. $(x + y)(x + y + z)$ **67.** $x(m + n)(p + q)$
69. $y(x + y)(x + y + 2z)$ **71.** $r_1 = \frac{rr_2}{r_2 - r}$
73. $f = \frac{d_1d_2}{d_2 + d_1}$ **75.** $a^2 = \frac{b^2x^2}{b^2 - y^2}$ **77.** $r = \frac{S - a}{S - l}$
79. $7x(x + 2)$ **81.** $5t^2(5t^4 - 2t + 1)$
83. $-7u^2v^3z^2(9uv^3z^7 - 4v^4 + 3uz^2)$ **85.** $-(x + y)(a - b)$
87. $(u + v)(u + v - 1)$ **89.** $-6xy(x^2 + 2xy + 3y^2)$
91. $(3 - c)(c + d)$ **93.** $n(2n - p + 2m)(n^2p - 1)$
95. $(a - b)(r - s)$ **97.** $(x - y)(x - 4)$
99. a. $\frac{1}{2}b_1h$ **b.** $\frac{1}{2}b_2h$ **c.** $\frac{1}{2}h(b_1 + b_2)$; the formula for the area of a trapezoid **101.** $r^2(4 - \pi)$ **107.** a line
109. $(-\infty, -3)$ **111.** They are the same.

Study Set Section 5.6 (page 435)

1. squares **3.** 1, 4, 9, 16, 25, 36, 49, 64, 81, 100
5. $(x + 5)(x + 5) = x^2 + 10x + 25$ **7.** A common factor of 2 can be factored out of each binomial. **9.** $p^2 - pq + q^2$
11. $p - q$ **13.** $6y, 6y, 6y$ **15.** $(x + 2)(x - 2)$
17. $(3y + 8)(3y - 8)$ **19.** $(4x^2 + 9y)(4x^2 - 9y)$
21. $(25a + 13b^2)(25a - 13b^2)$ **23.** $(s^2 + 4)(s + 2)(s - 2)$
25. $(m^2 + 9)(m + 3)(m - 3)$ **27.** $(x + y + z)(x + y - z)$
29. $[(m + n)^2 + p^2](m + n + p)(m + n - p)$
31. $2x(x + 4)(x - 4)$ **33.** $2(4a^2 + 9b^2)(2a + 3b)(2a - 3b)$
35. $(a + b)(a - b + 1)$ **37.** $(a - b)(a + b + 2)$
39. $(t + 3)(t^2 - 3t + 9)$ **41.** $(r + s)(r^2 - rs + s^2)$
43. $(r - 5)(r^2 + 5r + 25)$ **45.** $(2a - 3b)(4a^2 + 6ab + 9b^2)$
47. $(3 - x - y)(9 + 3x + 3y + x^2 + 2xy + y^2)$
49. $(t + 1)(t^2 - t + 1)(t - 1)(t^2 + t + 1)$
51. $5(x + 5)(x^2 - 5x + 25)$ **53.** $2(x - 4)(x^2 + 4x + 16)$
55. $5(p^2 + 4)$ **57.** $5(p^3 + 4)$ **59.** prime
61. $(9a + 7b)(9a - 7b)$
63. $(2ab^2c^3 + 3d^4)(2ab^2c^3 - 3d^4)$
65. $(x^2 + y^2)(x + y)(x - y)$
67. $(15a^2 + 4b^4c^6)(15a^2 - 4b^4c^6)$ **69.** $8(x + 3)(x - 3)$
71. $6x^2(x + 6)(x - 6)$ **73.** $16a^2bc^4(ab + 2c)(ab - 2c)$
75. $(m - 2n)(1 + m + 2n)$ **77.** $(3a + b)(9a^2 - 3ab + b^2)$
79. $(5xy^2 + 6z^3)(25x^2y^4 - 30xy^2z^3 + 36z^6)$
81. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$
83. $2u^2(4v - t)(16v^2 + 4tv + t^2)$
85. $(a + b)(x + 3)(x^2 - 3x + 9)$
87. $\frac{4}{3}\pi(r_1 - r_2)(r_1{}^2 + r_1r_2 + r_2{}^2)$
91.


93.


95. $y = -4$

Study Set Section 5.7 (page 446)

1. trinomial **3.** leading **5.** positive **7.** positive
9. $2xy + y^2$ **11.** $x^2 - y^2$ **13.** $4, -4, 1$ **15.** $(x + 2)$
17. $(x - 3)$ **19.** $(2a + 1)$ **21.** $(x + 1)^2$ **23.** $(a - 9)^2$
25. $(2y + z)^2$ **27.** $(3a - 2b)^2$ **29.** $(x + 5)(x - 4)$
31. $(p - 9)(p - 8)$ **33.** $(a + 3)(a - 4)$ **35.** $(p + 7)(p - 6)$
37. $x(3x - 1)(x - 3)$ **39.** $-3a(b + 5)(b - 1)$
41. $(x - 3)(x - 2)$ **43.** $(3y + 2)(2y + 1)$ **45.** prime
47. prime **49.** $x^2(7x - 8)(3x + 2)$
51. $-2(4x + 3y)(x + 3y)$ **53.** $-2(p + 2q)(p - q)$
55. $x^2(b - 7)(b - 5)$ **57.** $(x + 2 + y)(x + 2 - y)$
59. $(x + 1 + 3z)((x + 1 - 3z)$ **61.** $(x + a + 1)^2$
63. $(3a + 3b + 4)(a + b - 6)$ **65.** $(a + 9)(a - 5)$
67. $(2z + 3)(3z + 4)$ **69.** $3(x + 7)(x - 3)$
71. $(x - 2)(x - 5)$ **73.** $-(x + 5)(x - 3)$
75. $-2(y + 10)(y - 2)$ **77.** $(x + 3y)(x - 7y)$
79. $(4a - 3)(2a + 3)$ **81.** $(5b - 2)(3b + 2)$
83. $(6y - 5z)(3y + 2z)$ **85.** $(a + b)(a - 4b)$
87. $-(3a + 2b)(a - b)$ **89.** $5(a - 3b)^2$
91. $(x^2 + 5)(x^2 + 3)$ **93.** $(y^2 - 10)(y^2 - 3)$
95. $(a + 5)(a - 5)(a + 2)(a - 2)$ **97.** $(a - 16)(2a - 1)$
99. $(2u + 3)(u + 1)$ **101.** $(5r + 2s)(4r - 3s)$
103. $(c + 2a - b)(c - 2a + b)$ **105.** $x + 3$ **109.** 5
111. $\frac{8}{3}$ **113.** $-26p^2 - 6p$

Study Set Section 5.8 (page 450)

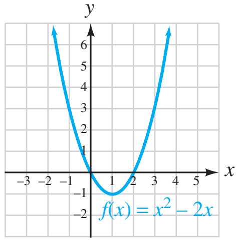
1. factoring **3.** cubes **5.** common **7.** trinomial
9. Multiply the factors of $y^2z^3(x + 6)(x + 1)$ to see if the product is $x^2y^2z^3 + 7xy^2z^3 + 6x^2z^3$ **11.** $3ab, 2b, 2a$
13. $(x + 4)^2$ **15.** $(2xy - 3)(4x^2y^2 + 6xy + 9)$
17. $(x - t)(y + s)$ **19.** $(5x + 4y)(5x - 4y)$
21. $(6x + 5)(2x + 7)$ **23.** $2(3x - 4)(x - 1)$
25. $y^2(2x + 1)^2$ **27.** $(x + a^2y)(x^2 - a^2xy + a^4y^2)$
29. $2(x - 3)(x^2 + 3x + 9)$ **31.** $(a + b)(f + e)$
33. $(2x + 2y + 3)(x + y - 1)$
35. $(25x^2 + 16y^2)(5x + 4y)(5x - 4y)$
37. $36(x^2 + 1)(x + 1)(x - 1)$
39. $(a + 3)(a - 3)(a + 2)(a - 2)$
41. $(x + 3 + y)(x + 3 - y)$
43. $(2x + 1 + 2y)(2x + 1 - 2y)$
45. $(x + y + 1)(x - y - 1)$ **47.** $(p + 2)(p + 4)(x + y)$
51. perpendicular **53.** -225

Study Set Section 5.9 (page 457)

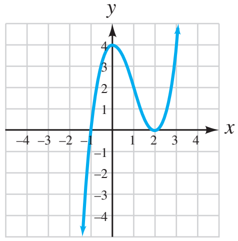
1. quadratic **3.** At least one is 0. **5.** yes **7.** yes
9. If the product of two numbers is 8, neither number need be 8. **11.** $y + 6, y - 9, -6$ **13.** $0, -2$ **15.** $4, -4$
17. $0, -1$ **19.** $0, 5$ **21.** $-3, -5$ **23.** $\frac{1}{2}, 2$ **25.** $\frac{1}{3}, -1$
27. $2, \frac{1}{2}$ **29.** $0, 0, -1$ **31.** $0, 7, -3$ **33.** $3, -3, 2, -2$
35. $2, -2, 4, -4$ **37.** $-2, -4$ **39.** $-\frac{1}{3}, -3$ **41.** $1, -\frac{1}{2}$
43. $0, 7, -7$ **45.** $3, 3$ **47.** $\frac{1}{4}, -\frac{3}{2}$ **49.** $2, -\frac{5}{6}$ **51.** $\frac{1}{5}, -\frac{5}{3}$
53. $0, -2, -3$ **55.** $0, \frac{5}{6}, -7$ **57.** $16, 18$ or $-18, -16$
59. $2.78, 0.72$ **61.** 1 **63.** 10 in., 16 in. **65.** 3 ft
67. 20 ft by 40 ft **69.** 11 sec and 19 sec **71.** 2 sec
73. 6 m/sec **75.** 4 **83.** 25 ft^2

Chapter 5 Review Exercises (page 462)

1. 243 **2.** -32 **3.** -64 **4.** $\frac{4}{9}$ **5.** x^6 **6.** $\frac{m^3}{n^5}$ **7.** $64m^{16}$
8. t^{13} **9.** $9x^4y^6$ **10.** $\frac{x^{16}}{b^4}$ **11.** -10 **12.** $\frac{h^{12}}{5}$ **13.** $\frac{b}{2a}$
14. $\frac{x^5}{5}$ **15.** $\frac{x^6}{9}$ **16.** $\frac{2}{9x}$ **17.** $-c^{10}$ **18.** $\frac{25}{16}$ **19.** $\frac{16}{27}$
20. 64 **21.** s^{73} **22.** $-\frac{b^3c^3}{8a^{21}}$ **23.** 1.93×10^{10}
24. 2.735×10^{-8} **25.** 72,770,000 **26.** 0.0000000083
27. 7.6×10^2 sec **28.** 1.67248×10^{-18} g **29.** 8.4×10^6
30. 1.875×10^{-21} **31.** no **32.** yes **33.** yes **34.** no
35. binomial, 2 **36.** monomial, 4 **37.** none of these, 4
38. trinomial, 8 **39.** 134 in.^3 **40.** -29
41.



42.



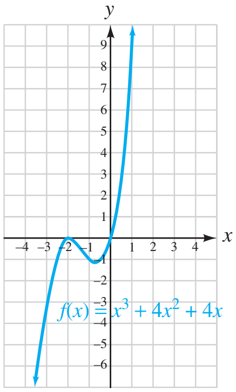
$f(x) = x^3 - 3x^2 + 4$

43. $2t^3 - 2t^2 - 5t$ **44.** $\frac{3}{4}ab^2c - \frac{5}{6}abc$
45. $3x^2y^3 - 8x^2y + 8y$ **46.** $19m^2 - 13m - 9$
47. $\frac{1}{2}s^6 - \frac{5}{3}s^4$ **48.** $6k^4 - 6k^3 + 9k^2 - 2$ **49.** $4c^2d^2 + 4cd^2$
50. a. -1 **b.** $-2, -1, 1$ **c.** D: $(-\infty, \infty)$, R: $[-1, \infty)$
51. $-4a^3$ **52.** $6x^3y^2z^5$ **53.** $2x^4y^3 - 8x^2y^7$
54. $a^4b + 2a^3b^2 - a^2b^3$ **55.** $6x^3 - 12x^2 + 4x - 8$
56. $25a^2t^2 - 60at + 36$ **57.** $49c^8d^6 - d^2$
58. $15x^4 - 22x^3 + 58x^2 - 40x$ **59.** $r^3 - 3r^2s - rs^2 + 3s^3$
60. $9c^2 - \frac{9}{2}c + \frac{9}{16}$ **61.** $25 - 10a + 10b + a^2 - 2ab + b^2$
62. $2x^2 + xy - y^2 - 3yz - 2z^2$ **63.** $41a^2 - 12a + 5$
64. $b^2 - b - 7$ **65. a.** $(12x - 2)$ in. **b.** $(8x^2 - 2x)$ in.^2
c. $(16x^3 + 12x^2 - 4x)$ in.^3 **66. a.** $f(x) = x^3 + 3x^2 + 2x$
b. 210 in.^3 **67.** 6 **68.** $3xy^3$ **69.** $4(x^4 + 2)$
70. $\frac{x}{5}(3x^2 - 6x + 1)$ **71.** prime **72.** $7a^3b(ab + 7)$
73. $5x^2(x + y)(1 - 3x)$ **74.** $9x^2y^3z^2(3xz + 9x^2y^2 - 10z^{5)}$
75. $-(x + 9)$ **76.** $-(-4r + 7)$ or $-(7 - 4r)$
77. $-7(b^3 - 2c)$ **78.** $-7a^2b^2(a - b)^3(7a^2 - 7ab - 9b^2)$
79. $(x + 2)(y + 4)$ **80.** $(ry - a + 1)(r - 1)$
81. $(t^2 + 1)(t - 9)$ **82.** $(1 - x)(1 - 3z)$
83. $m_1 = \frac{mm_2}{m_2 - m}$ **84.** $A = 2\pi r(r + h)$
85. $(z + 4)(z - 4)$ **86.** $(xy^2 + 8z^3)(xy^2 - 8z^3)$ **87.** prime
88. $(c + a + b)(c - a - b)$
89. $10m^2(m^2 + 4)(m + 2)(m - 2)$

90. $(m + n)(m - n + 1)$
91. $2c(4a^2 + 9b^2)(2a + 3b)(2a - 3b)$
92. $(k + 1 + 3m)(k + 1 - 3m)$ **93.** $(t + 4)(t^2 - 4t + 16)$
94. $(2a - 5b^3)(4a^2 + 10ab^3 + 25b^6)$
95. $4d^4(d + 1)(d^2 - d + 1)$
96. $(b + c + 3)(b^2 + 2bc + c^2 - 3b - 3c + 9)$
97. $(x + 5)^2$ **98.** $(7a^3 + 6b^2)^2$ **99.** $(y - 20)(y - 1)$
100. $(z - 5)(z - 6)$ **101.** $-(x + 7)(x - 4)$
102. $(a - 8b)(a + 3b)$ **103.** $(4a - 1)(a - 1)$ **104.** prime
105. $y^6(y + 2)(y - 1)$ **106.** $9st(3r - 2)(r + 4)$
107. $(r + s)(6t - 5)(t + 3)$ **108.** $(v^2 - 7)(v^2 - 6)$
109. $(w^4 - 10)(w^4 + 9)$ **110.** $(s + t - 1)^2$
111. $4rs(q - 5t)(q + 6t)$ **112.** $(2m + 2n + 3)(m + n - 1)$
113. $(z - 2)(z + x + 2)$ **114.** prime
115. $(x + 2 + 2p^2)(x + 2 - 2p^2)$ **116.** $(y + 2)(y + 1 + x)$
117. $4c^2(ab + 4)(a^2b^2 - 4ab + 16)$
118. $(a + 3)(a - 3)(a + 2)(a - 2)$
119. $(2x + 3)(2x^3 + 3x^2 + 1)$ **120.** $\frac{\pi}{2}h(r_1 + r_2)(r_1 - r_2)$
121. $0, \frac{3}{4}$ **122.** $6, -6$ **123.** $\frac{1}{2}, -\frac{5}{6}$ **124.** $3, -3, 1, -1$
125. $0, -\frac{2}{3}, \frac{4}{5}$ **126.** $-\frac{2}{3}, 7, 0$
127. a repeated solution of -8 **128.** $-7, -1, 1$
129. 1, 3 **130.** 7 m by 10 m **131.** 5 ft **132.** $-\frac{1}{2}, 1$

Chapter 5 Test (page 473)

1. a. perfect **b.** product **c.** binomials **d.** quadratic **e.** difference, cubes **f.** greatest common **2. a.** x^{10} ft^2 **b.** x^{15} ft^3 **3.** $\frac{a^5}{9m^5}$ **4.** $-\frac{8x^6y^9}{125}$ **5.** $-\frac{64}{b^8}$ **6.** $\frac{m^4}{9n^{10}}$
7. $2.9 \times 10^6 = 2{,}900{,}000$ **8.** 1.116×10^7 mi
9. a. $3, -4, -3, -\frac{5}{3}; 5$ **b.** $8, -1, 1, -6, 4; 13$ **10.** 110 ft
11. a.

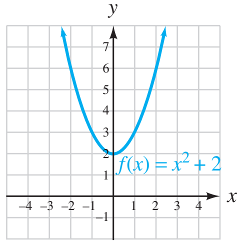


b. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$ **c.** $-2, 0$ **12. a.** 0 **b.** 2, 6 **c.** D: $(-\infty, \infty)$, R: $(-\infty, 2]$
13. $2x^2y^3 + 13xy + 3y^2$
14. $2x^5 + \frac{2}{15}x^2$
15. $-12a^4y^{12}z^{12}$
16. $-15a^3b^4 + 10a^3b^5$
17. $6y^3 + 11y^2 + 9y + 2$
18. $0.06d^2 + 1.6d - 6$
19. $36 + 12m - 12n + m^2 - 2mn + n^2$ **20.** $32s^3 - 50st^2$
21. $15t^2 - 21t + 13$ **22.** $c^2 - c + 4$
23. $3abc(4a^2b - abc + 2c^2)$ **24.** $(k + z)(h + b)$
25. $(x - 6)(x + 5)$ **26.** $(y^2 + 9)(y + 3)(y - 3)$
27. $(5m - 6n)^2$ **28.** $(s + 3)(s - 3)(s + 2)(s - 2)$
29. $-x^2(7x - 8)(3x + 2)$ **30.** prime
31. $5(x + 5)(x^2 - 5x + 25)$
32. $(4a - 5b^2)(16a^2 + 20ab^2 + 25b^4)$
33. $(x - y + 5)(x - y - 2)$ **34.** $(3b + 2c)(2b - c)$

35. $(a + b)(a - b + 1)$ **36.** $v = \frac{v_1v_3}{v_3 + v_1}$ **37.** 0, 5
38. $-5, \frac{7}{2}$ **39.** $\frac{3}{4}, -\frac{3}{2}$ **40.** $1, 0, -9$ **41.** $-1, -4, 4$
42. 1.5 ft

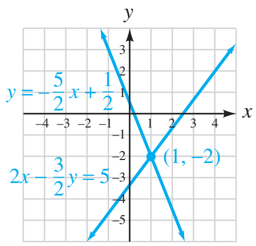
Chapters 1–5 Cumulative Review (page 475)

1. true **2.** false **3.** false **4.** true **5.** true **6.** -27 **7.** 1
8. $C = \frac{5}{9}(F - 32)$ **9.** 1,687.22 cm^3 **10.** 12 oz
11. $y = 8x + 21$ **12.** $y = -\frac{10}{13}x + \frac{2}{13}$
13. -1.2% per year **14.** -3 **15.** $f(x) = 0.025x + 95$
16.



D: the set of real numbers, R: the set of all real numbers greater than or equal to 2

17.



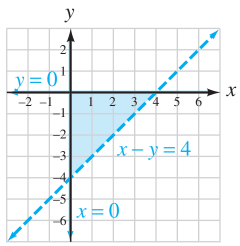
18. $\left(\frac{3}{4}, \frac{1}{3}\right)$ **19.** $(-2, 0, 2)$
20. -23
21. $(-\infty, 5]$

22. $[-21, -3)$

23. $(-\infty, -12] \cup [2, \infty)$

24. $(-\infty, -2)$

25. iii
26.



27. $\frac{9a^2}{b^8}$ **28.** 0.000000090895
29. $8a^3 - b^3$
30. $13k^2 - 10k + 17$
31. $(x - y)(x - 4)$
32. $6s^2(s + 6)(s - 6)$
33. $(2x^2 + 5y)(4x^4 - 10x^2y + 25y^2)$
34. $-(3a + 2b)(a - b)$ **35.** $1, -\frac{1}{2}$ **36.** $m_1 = \frac{m_2g}{m_2 - g}$

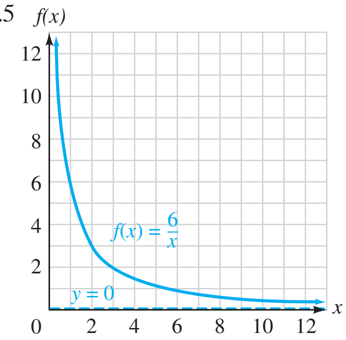
Think It Through (page 481)

1. about 25% **2.** about 88% **3.** about 10% **4.** yes

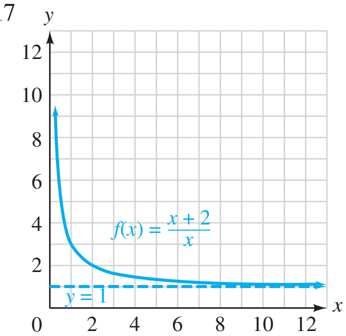
Study Set Section 6.1 (page 487)

1. rational **3.** simplify **5.** opposites **7. a.** 1 **b.** 0.5 **c.** 0.25 **d.** D: $(0, \infty)$, R: $(0, \infty)$ **9. a.** $\frac{3y}{7x^2}$ **b.** $\frac{x - 3}{x + 2}$

c. $-\dfrac{a^3}{a+9}$ **11. a.** iii **b.** i **c.** iv **d.** ii
13. yes, yes, yes, yes, yes
15. 6, 3, 1.5, 1, 0.75, 0.6, 0.5



17. 3, 2, 1.5, 1.33, 1.25, 1.2, 1.17



19. all real numbers except 0, $(-\infty, 0) \cup (0, \infty)$
21. all real numbers except -2, $(-\infty, -2) \cup (-2, \infty)$
23. all real numbers except 0 and 1, $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$
25. all real numbers except -7 and 8, $(-\infty, -7) \cup (-7, 8) \cup (8, \infty)$ **27.** $\dfrac{2a^2}{3}$ **29.** $\dfrac{3}{5a^6}$ **31.** $\dfrac{3}{4t}$
33. $\dfrac{4y^7}{3x}$ **35.** $\dfrac{2}{x-6}$ **37.** $\dfrac{3n}{2n+3}$ **39.** $\dfrac{2}{x-9}$ **41.** $\dfrac{2a+5}{10}$
43. $\dfrac{5x}{x-2}$ **45.** $\dfrac{x+1}{x+3}$ **47.** $\dfrac{d+4}{d+2}$ **49.** $\dfrac{h+5}{2h-1}$
51. $\dfrac{t^2-3t+9}{t^2+4}$ **53.** $\dfrac{s^2-6}{s^2-s+1}$
55. $-\dfrac{3m+n}{m}$ or $\dfrac{-(3m+n)}{m}$ **57.** $-b-a$
59. $-\dfrac{x+2}{x+1}$ or $\dfrac{-(2+x)}{x+1}$ **61.** $-\dfrac{20x^3}{x-1}$
63. D: $(-\infty, 2) \cup (2, \infty)$, R: $(-\infty, 1) \cup (1, \infty)$
65. D: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$, R: $(-\infty, \infty)$
67. $\dfrac{x+6}{3(x+4)}$ **69.** $\dfrac{a+2}{a^2+2a+4}$ **71.** $-\dfrac{m+n}{n+2m}$ or $\dfrac{-m-n}{n+2m}$
73. $\dfrac{s-3}{s+6}$ **75.** Does not simplify. **77.** 3
79. $\dfrac{x^2}{x-3}$ **81.** $\dfrac{2}{x+2}$ **83.** $-\dfrac{2x+1}{x-2}$ or $\dfrac{2x+1}{2-x}$
85. $-\dfrac{x-3}{(9+x^2)(3+x)}$ or $-\dfrac{x-3}{(x^2+9)(x+3)}$
87. $\dfrac{2p^2}{3q^6}$ **89.** $-\dfrac{t-2}{3+t}$ or $-\dfrac{t-2}{t+3}$ **91. a.** \$50,000
b. \$200,000 **93. a.** $c(n) = 0.09n + 7.50$
b. $c(n) = \dfrac{0.09n + 7.50}{n}$ **c.** about 10¢ **95. a.** about 2.5 hr
b. about 4.6 hr **99.** $a^3 - 6a^2 + 5a + 6$
101. $-3m^4n^2 + 21m^2n^3 + 6m^3n^2$

Study Set Section 6.2 (page 498)

1. rational **3.** invert **5.** numerators, denominators, $\dfrac{AC}{BD}$
7. $(x-5)$, $(x+3)$, x, 5, x **9.** yes, no, yes **11.** $\dfrac{11}{4}$ **13.** $\dfrac{2}{5}$
15. $\dfrac{1}{25a^3}$ **17.** $\dfrac{3x^4}{14y^3}$ **19.** $\dfrac{y(y+3)}{10}$ **21.** $\dfrac{(x-2)^2}{x}$
23. $\dfrac{x+1}{9x^2}$ **25.** $\dfrac{(t+3)^2}{(t-3)(t+2)}$ **27.** 1 **29.** $\dfrac{t-1}{t+1}$ **31.** $\dfrac{3}{2x}$
33. $\dfrac{a-b}{a}$ **35.** $x+1$ **37.** $2y + 16$ or $2(y+8)$
39. $-\dfrac{a^4}{2(a^2+3)}$ **41.** $-\dfrac{(x+1)^2(x+2)}{x+2c}$
43. $\dfrac{x^2-6x+9}{x^4+8x^2+16}$ **45.** $\dfrac{4m^4-4m^3-11m^2+6m+9}{x^4-2x^2+1}$
47. $\dfrac{5}{6}$ **49.** $\dfrac{9}{2}$ **51.** $\dfrac{2y^5}{3x^6}$ **53.** $\dfrac{3}{10p^9}$ **55.** $\dfrac{x^{11}}{x^2+2x+4}$
57. $\dfrac{2x(x-4)}{x+5}$ **59.** $\dfrac{n+2}{n+1}$ **61.** $\dfrac{c+d}{d(25c^2-5cd+d^2)}$
63. $\dfrac{y(y+3)}{y+2}$ **65.** $\dfrac{2}{x+1}$ **67.** $\dfrac{a+1}{a-1}$ **69.** $\dfrac{3x}{2}$
71. $-\dfrac{(x-3)(x-6)}{(x+2)(x+3)}$ **73.** $\dfrac{2(x+1)}{x(x+3)}$ **75.** $\dfrac{q}{p}$ **77.** $\dfrac{5r^3}{2s^2}$
79. $-10h + 30$ or $-10(h-3)$ **81.** $\dfrac{x+y}{x-y}$ **83.** $\dfrac{x+2}{x+1}$
85. $\dfrac{2x(x-5)}{x+5}$ **87.** $k_1(k_1+2)$, k_2+6 **93.** x^{m+n} **95.** x^ny^n
97. 1 **99.** x^{m-n} **101.** $\dfrac{y^n}{x^m}$

Study Set Section 6.3 (page 509)

1. denominator **3.** build **5.** numerators, denominator, $A+B$, $A-B$ **7.** factor, greatest **9. a.** ii **b.** adding or subtracting rational expressions **c.** simplifying a rational expression **11. a.** twice **b.** once **13. a.** $2 \cdot 2 \cdot 2 \cdot 5 \cdot x \cdot x$
b. $2x(x-3)$ **c.** $(n+8)(n-8)$ **15.** $3x-2$, 3, $3x-1$, 3
17. $\dfrac{13}{3x}$ **19.** $\dfrac{t}{2r}$ **21.** 4 **23.** $\dfrac{3}{x+3}$ **25.** $\dfrac{15q+2p}{pq}$
27. $\dfrac{21a-22b}{6ab}$ **29.** $\dfrac{8x-2}{(x+2)(x-4)}$ **31.** $\dfrac{2x^2-30x}{(x+3)(x-3)}$
33. $\dfrac{x}{x-3}$ **35.** $\dfrac{9m+2}{m-n}$ **37.** $\dfrac{4x-7}{x-2}$ **39.** $\dfrac{7x^2+x}{7x-3}$
41. $36x^2y$ **43.** $x(x+3)(x-3)$ **45.** $(x+3)^2(x^2-3x+9)$
47. $(2x+3)^2(x+1)^2$ **49.** $\dfrac{41}{30m}$ **51.** $\dfrac{3a-10b}{4a^2b^2}$ **53.** $\dfrac{1}{x+1}$
55. $\dfrac{m-5}{(m+3)(m+5)}$ **57.** $\dfrac{2x^2+x}{(x+3)(x+2)(x-2)}$
59. $\dfrac{-x+28}{30(x-7)}$ **61.** 3
63. $\dfrac{-4x^2+14x+54}{x(x+3)(x-3)}$ or $\dfrac{-2(2x^2-7x-27)}{x(x+3)(x-3)}$
65. $\dfrac{11x^2+7x-3}{(2x-1)(3x+2)}$ **67.** 0 **69.** $\dfrac{14s+58}{(s+3)(s+7)}$

71. $\frac{5x-y}{6}$ **73.** $\frac{6}{x-3}$ **75.** $\frac{1}{a-b}$ **77.** $\frac{2x^2+5x+4}{x+1}$
79. $\frac{-x^2+11x+8}{(3x+2)(x+1)(x-3)}$ **81.** $\frac{2}{x+1}$ **83.** $\frac{16y^2+3}{18y^4}$
85. $\frac{3}{(b+2)(b-3)}$ **87.** $\frac{a^2-a}{a-5}$
89. $\frac{3(a-1)}{3a-2}$ or $\frac{3a-3}{3a-2}$ **91.** $\frac{2x}{x+1}$ **93.** $\frac{m-3}{(m+3)(m+1)}$
95. $\frac{9p^3}{5}$ **97.** $\frac{6-3d}{5d(d-1)}$ **99.** $\frac{2s-3t}{t(2s+3t)}$
101. $\frac{10r+20}{r}$; $\frac{3t+9}{t}$ **107.** a repeated solution of 3
109. a repeated solution of 0, -1

Study Set Section 6.4 (page 519)

1. rational, complex **3.** $\frac{t^2}{t^2}$ **5.** $\div$, $\frac{3}{25m}$, m, 3, 3, m, 10
7. a. $\div$ **b.** $6-k-\frac{5}{k}$; k^2-9 **9.** $\frac{2a^5}{3}$ **11.** $\frac{5y}{9}$ **13.** $\frac{27y^3}{5x^3}$
15. $\frac{20}{49c^6d^4}$ **17.** $\frac{3-2a}{a^2-3a}$ **19.** $\frac{p^2-1}{3p-1}$ **21.** $y-x$
23. $-\frac{1}{a+b}$ **25.** $\frac{2+a}{2a+1}$ **27.** $\frac{b-1}{b+2}$ **29.** $\frac{y+x}{xy}$
31. $\frac{a^3b^2-a^2}{a^2b^3-b^2}$ or $\frac{a^2(ab^2-1)}{b^2(a^2b-1)}$ **33.** $\frac{5z-12}{5z}$
35. $\frac{x-9}{3}$ **37.** $\frac{y+x}{y-x}$ **39.** $\frac{2x+4}{3-x}$ **41.** $\frac{x+2}{x-3}$ **43.** $\frac{1}{c-d}$
45. $\frac{a-1}{a+1}$ **47.** $125b$ **49.** -1 **51.** $\frac{3a+7}{2a}$ **53.** $\frac{xy^2}{y-x}$
55. $\frac{y-2x}{2y+x}$ **57.** $\frac{5x^2y^2}{xy+1}$ **59.** $\frac{1}{2y}$ **61.** $\frac{1}{x-y}$
63. $\frac{4}{c+d}$ **65.** $\frac{k_1k_2}{k_2+k_1}$ **67.** $\frac{4d-1}{3d-1}$ **73.** 8
75. 2, -2, 3, -3

Study Set Section 6.5 (page 529)

1. monomial, polynomial, binomial **3.** Divisor, Quotient, Dividend, Remainder **5. a.** term **b.** 9, 9 **c.** 6, 6, 6
7. $(2x-1)(x^2+3x-4)=2x^3+5x^2-11x+4$
9. x^2, $7x$, 28 **11.** $3a^2+5+\frac{6}{3a-2}$
13. $\frac{x^2-x-12}{x-4}$, $x-4\overline{)x^2-x-12}$, $(x^2-x-12)\div(x-4)$ **15.** $\frac{y}{2x^3}$ **17.** $\frac{3}{4a^2}$ **19.** $2x^4+3x$
21. $\frac{2x}{3}-\frac{x^2}{6}$ **23.** $2a-\frac{2a^2y}{3}$ **25.** $-\frac{x^4y^4}{2}+\frac{x^3y^9}{4}-\frac{3}{4xy^2}$
27. $x+2$ **29.** $x-3$ **31.** $4x-5$ **33.** $3x^2+4x+3$
35. $t^2+2t+1+\frac{3}{t+6}$ **37.** $2x^2+5x-3+\frac{-8}{3x-2}$
39. $a+1$ **41.** $2y+2$ **43.** $3x^2-x+2$
45. $4x^3-3x^2+3x+1$ **47.** $4a^2-2a+1$
49. $5a^2-3a-4$ **51.** x^2+3x+4
53. $2x+3+\frac{20x-13}{3x^2-7x+4}$ **55.** $6y-12$
57. $4a^2-3a+\frac{7}{a+1}$ **59.** x^4+x^2+4 **61.** $\frac{a^2}{5}-\frac{2}{5a^2}$
63. $s+5+\frac{-10}{2s+3}$ **65.** $\frac{8m^2}{7n^{10}}$ **67.** $m-4+\frac{m+3}{m^2+1}$
69. $y^2+4y+16$ **71.** a^4-a^2+1
73. $10x^2z-2x-\frac{1}{x}$ **75.** $x^2-2+\frac{-x^2+7x+4}{x^3+2x+1}$
77. $2x^2-x+1$ **79.** $\frac{5x^2}{9}+\frac{x}{3}-\frac{1}{9}$ **81.** x^4-x^2-3
83. x^2-5x+6 **85.** $3x-2$, $x+5$ **89.** $8x^2+2x+4$
91. $-2y^3-y^2+10y-14$

Study Set Section 6.6 (page 538)

1. synthetic **3.** divisor **5.** theorem
7. a. $(5x^3+x-3)\div(x+2)$ **b.** $5x^2-10x+21+\frac{-45}{x+2}$
9. $6x^3-x^2-17x+9$, $x-8$ **11.** 2, 1, 2, 12, 26, 6, 8
13. $2x+3$ **15.** $5x-2$ **17.** $3x-4$ **19.** $5x+6$
21. a^2-a-2 **23.** $3a^2+12a+1$
25. $3b^2+9b-4+\frac{1}{b-3}$ **27.** $4t^2+8t+15+\frac{12}{t-2}$
29. x^2-5x+6 **31.** $3x^2-4x-4+\frac{-10}{x+8}$
33. $2x^2-3x+12+\frac{-52}{x+5}$ **35.** $x^2-2x+3+\frac{-3}{x+10}$
37. $7.2x-0.66+\frac{0.368}{x-0.2}$
39. $9x^2-513x+29{,}241-\frac{1{,}666{,}762}{x+57}$ **41.** -1 **43.** -37
45. 23 **47.** -1 **49.** 2 **51.** -1 **53.** 18 **55.** 174 **57.** -8
59. 59 **61.** 44 **63.** $\frac{29}{32}$ **65.** yes **67.** no
69. $6x^2-x+1+\frac{3}{x+1}$ **71.** $x-7+\frac{28}{x+2}$
73. $a^4+a^3+a^2+a+1$ **75.** $-6c^4-10c^3-2c^2-4c+9$
77. $9a^2-21$ **79.** $4x^3-x+2+\frac{6}{x+3}$ **81.** $3x^2-x+2$
83. $2x^2+4x+5$ **85.** $4x^2-3x+6+\frac{-13}{x+2}$
87. $8a^2-16a-20$ **93.** 0 **95.** 2

Study Set Section 6.7 (page 547)

1. rational **3. a.** yes **b.** no **5.** the LCD, $10(y-5)$
7. $30y$, $\frac{9}{2y}$, $30y$, $30y$, $30y$, $7y$, 35 **9.** 12 **11.** 5 **13.** 4 **15.** $\frac{5}{3}$
17. 5 **19.** $-\frac{2}{3}$ **21.** 1 **23.** 7 **25.** 4, -1 **27.** 2, -3
29. $-\frac{1}{3}$, 7 **31.** $-\frac{1}{2}$, 6 **33.** -8 **35.** 6
37. no solution; 9 is extraneous. **39.** no solution; -3 is extraneous. **41.** $A=LQ+I$ **43.** $r=\frac{E-IR_L}{I}$

45. $n_2 = \dfrac{2\mu_R - n_1^2 - n_1}{n_1}$ **47.** $Q_1 = \dfrac{PQ_2}{1 + P}$

49. $R = \dfrac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$ **51.** $r = \dfrac{eR}{E - e}$

53. 2 **55.** $\frac{1}{2}$ **57.** no solution; 0 is extraneous. **59.** 1, −11

61. 1, 6 **63.** a repeated solution of 2 **65.** $-\frac{1}{2}$ **67.** 0

69. 5 **71.** 5; 3 is extraneous. **73.** 1 **75.** 6 **77.** 1 **79.** 2

81. $\frac{1}{6}$ **83.** $-\frac{3}{2}, 2$ **85. a.** $\dfrac{11x + 30}{12x}$ **b.** 6

87. a. $\dfrac{m^2 - 4m + 2}{(m - 2)(m - 3)}$ **b.** 4 **89. a.** $f = \dfrac{s_1s_2}{s_1 + s_2}$ **b.** $4\frac{8}{13}$ in.

91. a. $L = \dfrac{SN - CN}{V - C}$ **b.** 8 yr **97.** 9.0×10^9

99. 4.4×10^{-22}

Study Set Section 6.8 (page 557)

1. work, motion **3.** x **5.** $\frac{x}{15}, \frac{x}{8}$ **7.** $\frac{12}{x}, \frac{12}{x + 15}$ **9.** $4\frac{5}{9}$ hr

11. $2\frac{6}{11}$ days **13. a.** $1\frac{7}{8}$ days **b.** Santos: \$412.50, Mays: \$375 **15.** $5\frac{5}{6}$ min **17.** $4\frac{8}{13}$ sec **19.** $2\frac{4}{13}$ weeks

21. waiter: 10 min, busboy: 15 min **23.** faster worker: 6 hr, slower worker: 12 hr **25.** experienced plumber: 6 days, apprentice: 12 days **27.** $1\frac{1}{2}$ hours **29.** about 110 sec

31. going: 40 mph, returning: 30 mph **33.** going: 60 mph, returning: 40 mph **35.** 35 mph and 45 mph **37.** 150 mph

39. 2 mph **41.** 3 mph **45.** $\dfrac{m^{80}}{n^8}$ **47.** $-\dfrac{1}{w^2}$ **49.** −4

51. $-x^8y^{10}$

Study Set Section 6.9 (page 570)

1. ratio **3.** extremes, means, extremes, means **5.** similar **7.** inverse, decreases **9.** Direct **11.** Inverse

13. 7, 6, 18, 66, 11 **15.** 3 **17.** 5 **19.** $\frac{21}{5}$ **21.** −4

23. 2, −2 **25.** 4, −1 **27.** $-\frac{5}{2}, -1$ **29.** 0, −3

31. $A = kp^2$ **33.** $z = \dfrac{k}{t^3}$ **35.** $C = kxyz$ **37.** $P = \dfrac{ka^2}{j^3}$

39. r varies directly as t. **41.** b varies inversely as h. **43.** U varies jointly as r, the square of s, and t. **45.** P varies directly as m and inversely as n.

47. $\frac{7}{2}$ **49.** −5, 2 **51.** no solution **53.** $-\frac{1}{2}, 0, 5$ **55.** 6

57. $-\frac{5}{2}, -1, 1$ **59.** −0.1 **61.** −10, 10 **63.** 202 mg, 139 mg, 125 mg **65.** about 2 gal **67.** eye: 49.9 in., seat: 17.6 in., elbow: 27.8 in. **69.** 12.5 in. **71.** 25 ft **73.** $46\frac{7}{8}$ ft

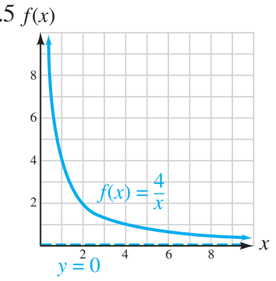
75. 4 cm **77. a.** false **b.** false **c.** true **79.** 1,600 ft **81.** 25 days **83.** 12 in.^{3} **85.** \$9,000 **87.** 3 ohms

89. 0.275 in. **91.** 12.8 lb **95.** $\frac{13}{6}w^3 - \frac{1}{4}w^2 + \frac{4}{5}$

97. $6y^3 + 11y^2 + 9y + 2$

Chapter 6 Review Exercises (page 576)

1. 8, 4, 2, 1.33, 1, 0.8, 0.67, 0.57, 0.5



2. a. 1 **b.** 2 **c.** D: $(0, \infty)$, R: $(0, \infty)$ **3.** The domain is the set of all real numbers except −6 and 4: $(-\infty, -6) \cup (-6, 4) \cup (4, \infty)$. **4.** $y = 3, x = 0$; D: $(-\infty, 0) \cup (0, \infty)$, R: $(-\infty, 3) \cup (3, \infty)$ **5.** $\dfrac{12x}{19y^7}$ **6.** $\dfrac{x - 7}{x + 7}$

7. $\dfrac{1}{2x^2(x + 2)}$ **8.** $\dfrac{1}{x^4(x - 6)}$ **9.** $\dfrac{-a - b}{c + d}$ **10.** $\dfrac{m + 2n}{2m + n}$

11. $\dfrac{2x + 1}{3x - 4}$ **12.** −2 **13.** does not simplify

14. $-\dfrac{3m - 4}{m + 3}$ or $\dfrac{4 - 3m}{m + 3}$ **15.** $\dfrac{2}{49x^2}$ **16.** $-x$ **17.** $\dfrac{2a - 1}{4a(a + 2)}$

18. $\dfrac{t - 2}{t}$ **19.** $\dfrac{h^2 - 4h + 4}{h^6 + 8h^3 + 16}$ **20.** $\dfrac{a + 6}{(m + 4)(m - 3)}$

21. $\dfrac{2m - n}{m + n}$ **22.** $\dfrac{3x(x - 1)}{(x - 3)(x + 1)}$ **23.** $\dfrac{5y - 3}{x - y}$ **24.** $\dfrac{1}{c - d}$

25. $-\dfrac{2}{t - 3}$ **26.** $-\dfrac{1}{p + 12}$ **27.** $60a^2h^3$ **28.** $ab^2(b - 1)$

29. $(x - 5)(x + 5)(x + 1)$ **30.** $(m^2 + 2m + 4)(m - 2)^2$

31. $\dfrac{9a + 8}{a + 1}$ **32.** $\dfrac{40x + 7y^2z}{112z^2}$ **33.** $\dfrac{4x^2 + 9x + 12}{(x - 4)(x + 3)}$

34. $\dfrac{2a^2 + 8a - 19}{3(a + 2)}$ **35.** $\dfrac{14y + 58}{(y + 3)(y + 7)}$ **36.** $\dfrac{12y + 20}{15y(x - 2)}$

37. $\dfrac{1}{(a + 3)(a + 2)}$ **38.** $\dfrac{2a - 3}{2a}$ **39.** $\dfrac{2bc^3}{7}$ **40.** $\dfrac{p - 3}{2(p + 2)}$

41. $\dfrac{b + 2a}{2b - a}$ **42.** $\dfrac{x - 2}{x + 3}$ **43.** $\dfrac{x^2y^2}{(x - y)^2(y^2 - x^2)}$

44. $\dfrac{1 + b + d}{1 - b - d}$ **45.** $\dfrac{2b^2 - 3b + 3}{3b - 2}$ **46.** $\dfrac{4r - 8}{2r + 5}$ **47.** $\dfrac{5h^3}{11k^2}$

48. $\dfrac{1}{2y^3z^{10}}$ **49.** $6a + \dfrac{16}{3}$ **50.** $-3x^2y + \dfrac{3x}{2} + y$ **51.** $b + 4$

52. $v^2 - 3v - 10$ **53.** $x^2 - 2x + 4$ **54.** $2m - 5 + \dfrac{-4}{4m + 1}$

55. $3a - 2 + \dfrac{-15a + 2}{a^2 + 5}$ **56.** $m^4 - m^2 + 1 + \dfrac{2}{m^4 + 2m^2 - 3}$

57. $x - 7$ **58.** $m^2 - 3m + 2$

59. $-3n^4 - 2n^3 - n^2 - 2n + 1$

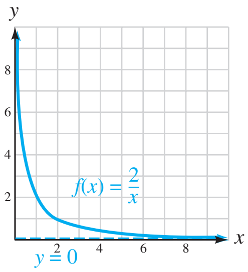
60. $4x^2 - 3x + 6 + \dfrac{-13}{x + 2}$ **61.** $3a^3 - a + 4 + \dfrac{6}{a + 1}$

62. $x^3 + 3x^2 + 9x + 27 + \dfrac{82}{x - 3}$ **63.** 54 **64.** −227

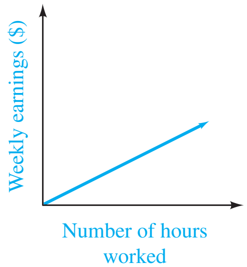
65. yes **66.** no **67.** 5 **68.** $\frac{77}{5}$ **69.** $-2, 3$ **70.** $-1, -2$ **71.** $\frac{3}{2}$ **72.** 0 **73.** No solution; 3 is extraneous. **74. a.** $\frac{x + 26}{x(x - 4)}$ **b.** $\frac{26}{9}$ **75.** $b = \frac{Ha}{2a - H}$ **76.** $y^2 = \frac{x^2b^2 - a^2b^2}{a^2}$ **77.** $R = \frac{R_1R_2}{R_2 + R_1}$ **78.** $F = \frac{ma}{k}$ **79. a.** $\frac{1}{10}$ of the job per hour **b.** $\frac{x}{10}$ of the job is completed **80.** $14\frac{2}{5}$ hr **81.** experienced electrician: 10 days, apprentice: 15 days **82.** 6 min **83.** 5 mph **84.** 50 mph **85.** 5 **86.** $-4, -12$ **87.** $0, -1$ **88.** $-2, 3$ **89.** 70.4 ft **90.** 20 **91.** 66 in. **92.** \$5,460 **93.** 1.25 amps **94.** 0.2 **95.** 126.72 lb **96.** inverse variation

Chapter 6 Test (page 589)

1. a. rational **b.** reciprocal **c.** complex **d.** extraneous **e.** proportion **3.** $\frac{2}{3xy}$ **4.** $\frac{2}{x - 2}$ **5.** -3 **6.** $\frac{2x + y}{4y}$ **7.**

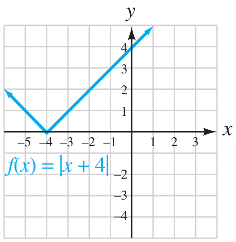


8. $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$ **9.** $\frac{xz}{y^4}$ **10.** 1 **11.** $\frac{13}{x + 1}$ **12.** $\frac{(x + y)^2}{2x}$ **13.** -1 **14.** $\frac{(2x - 3)^2}{x^6}$ **15.** $\frac{2}{t - 4}$ **16.** 2 **17.** $\frac{24b^2 - 2b - 1}{3b + 1}$ **18.** $\frac{6a - 17}{(a + 1)(a - 2)(a - 3)}$ **19.** $\frac{u^2}{2vw}$ **20.** $\frac{3k^2 + 4k + 4}{3k^2 - 9k - 9}$ **21.** $-\frac{6x}{y} + \frac{4x^2}{y^2} - \frac{3}{y^3}$ **22.** $y^2 - 2y + 4 - \frac{56}{y + 2}$ **23.** 41 **24.** $x + 3$ is a factor of $P(x)$. **25.** 40 **26.** 5; 3 is extraneous. **27.** $6, -1$ **28.** 26 **29.** $r_2 = \frac{rr_1}{r_1 - r}$ **30.** $a^2 = \frac{x^2b^2}{b^2 - y^2}$ **31.** no, $\frac{5}{11}$ of an hour **32.** supervisor: 45 min, technician: 90 min **33.** 3 mph **34.** 2 mph **35.** 80 ft **36.** \$77.32 **37.** 25 decibels **38.**

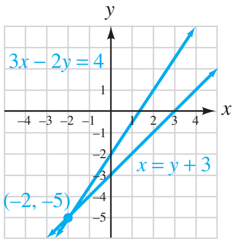


Chapters 1–6 Cumulative Review (page 591)

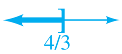
1. -372 **2.** $22a^3 + 20a$ **3.** -2 **4.** $n = \frac{l - a + d}{d}$ **5.** 490.9 in.^{2} **6.** 24 **7.** Life expectancy will increase 0.1 year each year during this period. **8.** $y = -7x + 54$ **9.** $y = -\frac{11}{6}x - \frac{7}{3}$ **10. a.** $(-3, 0), (0, -7)$ **b.** $-\frac{7}{3}$ **c.** I **d.** $y = -\frac{7}{3}x - 7$ **e.** yes **11.** 18 **12.** no; $(1, 1), (1, -1)$ **13.** D: $(-\infty, \infty)$, R: $[0, \infty)$



14. $(-2, -5)$



15. $T_1 = 80, T_2 = 60$ **16.** 12 oz 50% alloy, 8 oz 25% alloy **17.** $(2, -3, 5)$ **18.** 8 **19.** $\left(-\infty, \frac{4}{3}\right]$

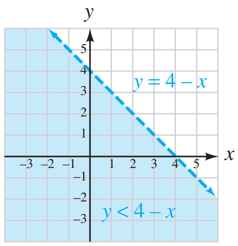


20. $(-\infty, -11)$

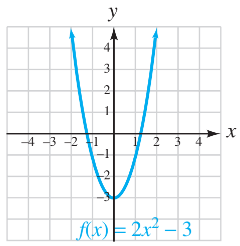
21. $(-\infty, 1)$

22. $(-\infty, 3] \cup \left[\frac{11}{3}, \infty\right)$

23.



24. yes **25.** $81b^{24}$ **26.** $81d^{10}$ **27.** $-\frac{k^7}{20x^9}$ **28.** $\frac{x^{21}y^3}{8}$ **29.** penny: $\$10^{-2}$, dime: $\$10^{-1}$, one-dollar bill: $\$10^0$, one-hundred-thousand-dollar bill: $\$10^5$ **30.** 2.05×10^{12} **31. a.** yes **b.** about 17 **c.** about 14 **d.** 4 **32.** D: $(-\infty, \infty)$, R: $[-3, \infty)$



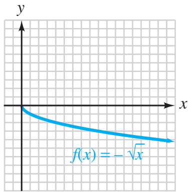
33. 7 **34.** $\frac{7}{4}c^2 - \frac{5}{6}c$ **35.** $2x^3 + x^2 + 12$ **36.** $4m^2n^2 + 4m - 2n$ **37.** $2r^{11}s^{20}$ **38.** $-12a^{12} - 9a^{11} + 12a^{10}$ **39.** $4x^6 - 4x^3 + 1$ **40.** $2a^2 + ab - b^2 - 3bc - 2c^2$ **41.** $3rs^3(r - 2s)$ **42.** $(x - y)(5 - a)$ **43.** $(x + y)(u + v)$ **44.** $(9x^2 + 4y^2)(3x + 2y)(3x - 2y)$ **45.** $(2x - 3y^2)(4x^2 + 6xy^2 + 9y^4)$ **46.** $(4x^2 - 3)(2x^2 - 1)$ **47.** $(x - y + 5)(x - y - 2)$ **48.** $(x + 5 + 4z^4)(x + 5 - 4z^4)$ **49.** $b^2 = \frac{a^2y^2}{a^2 - x^2}$ **50.** $-\frac{1}{3}, -\frac{7}{2}$ **51.** $0, 2, -2$

52. 9 in. by 12 in. **53.** $\frac{2x-3}{3x-1}$ **54.** all real numbers except 0 and 2: $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$ **55.** $-\frac{n^6}{n^2-2}$ **56.** $-\frac{q}{p}$ **57.** $\frac{4}{x-y}$ **58.** $\frac{c+9}{c-5}$ **59.** $\frac{y^2}{3} + 3y$ **60.** $4x^2 - x - 1$ **61.** -17 **62.** 0 **63.** It will rise sharply. **64.** It has dropped. **65.** 13 cups **66.** 1 day

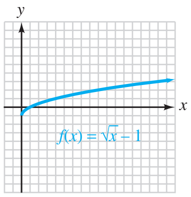
Study Set Section 7.1 (page 607)

1. square root **3.** radical **5.** odd **7.** simplified **9.** a **11.** two, positive **13.** x **15.** 3, up **17.** $\sqrt{x^2} = |x|$ **19.** $f(x) = \sqrt{x-5}$ **21.** 9, -9 **23.** 16, -16 **25.** 11 **27.** -8 **29.** $\frac{1}{3}$ **31.** -0.5 **33.** $2|x|$ **35.** $|t+5|$ **37.** $5|b|$ **39.** $|a+3|$

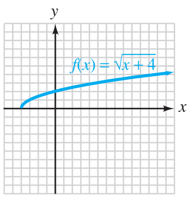
41. 0, -1, -2, -3, -4; D: $[0, \infty)$, R: $(-\infty, 0)$



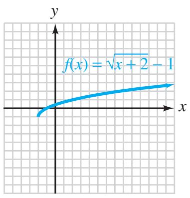
43. -1, 0, 1, 2, 3; D: $[0, \infty)$, R: $[-1, \infty)$



45. D: $[-4, \infty)$, R: $[0, \infty)$

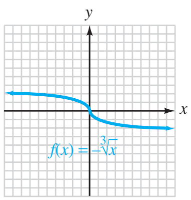


47. D: $[-2, \infty)$, R: $[-1, \infty)$

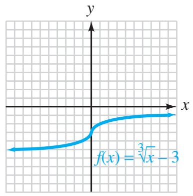


49. 1 **51.** $-5a$ **53.** $-\frac{2a^2}{3b}$ **55.** $0.4ab^2$

57. 2, 1, 0, -1, -2; D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



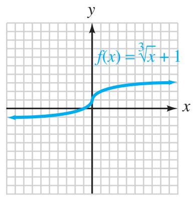
59. -5, -4, -3, -2, -1; D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



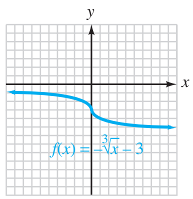
61. 3 **63.** -3 **65.** not real **67.** $\frac{2}{5}$ **69.** $\frac{1}{2}$ **71.** 10 **73.** $2a$ **75.** $2|a|$ **77.** $|k^3|$

79. $\frac{1}{2}|m|$ **81.** $|x+2|$ **83.** $x+2$ **85.** not real **87.** 4 **89.** $2a$ **91.** $-10pq$ **93.** $-\frac{1}{2}m^2n$ **95.** $-0.4s^2t^2$ **97.** 3.4641 **99.** 26.0624

101. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



103. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



105. 7.0 in. **107.** about 61.3 beats/min **109.** 13.4 ft **111.** 3.5% **117.** 1 **119.** $\frac{3(m^2+2m-1)}{(m+1)(m-1)}$

Study Set Section 7.2 (page 618)

1. like **3.** factor **5.** $\sqrt[n]{a}\sqrt[n]{b}$ **7. a.** $\sqrt{4 \cdot 5}$ **b.** $\sqrt{4}\sqrt{5}$ **c.** $\sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5}$ **9. a.** $\sqrt{5}, \sqrt[3]{5}$ (answers may vary); no **b.** $\sqrt{5}$, $\sqrt{6}$ (answers vary); no **11.** $8k^3, 8k^3, 4k$ **13.** $2\sqrt{5}$ **15.** $-10\sqrt{2}$ **17.** $2\sqrt[3]{10}$ **19.** $-2\sqrt[4]{2}$ **21.** $a^3\sqrt{a}$ **23.** $5x\sqrt{2}$ **25.** $4\sqrt{2b}$ **27.** $-3x^2\sqrt[3]{2}$ **29.** $\frac{x^2\sqrt[5]{3}}{2}$ **31.** $5ab\sqrt{7b}$ **33.** $6\sqrt{5}$ **35.** $2y\sqrt[3]{2y}$ **37.** $-10\sqrt{3xy}$ **39.** $2x^3y\sqrt[4]{2y}$ **41.** $\frac{\sqrt{7}}{3}$ **43.** $\frac{\sqrt[3]{7}}{4}$ **45.** $\frac{z}{4x}$ **47.** $\frac{\sqrt[4]{5x}}{2z}$ **49.** $7x$ **51.** $6b$ **53.** $2a\sqrt[3]{2a}$ **55.** $3a\sqrt[3]{3}$ **57.** $2\sqrt{2}$ **59.** $9\sqrt{6}$ **61.** $3\sqrt{5}$ **63.** $\sqrt{7} + 6\sqrt{2}$ **65.** $-\sqrt[3]{4}$ **67.** -10 **69.** $\sqrt[3]{2}$ **71.** $-5\sqrt[4]{3}$ **73.** $10\sqrt{2x}$ **75.** $3\sqrt{2t} + \sqrt{3t}$ **77.** $3\sqrt[3]{3x}$ **79.** $29a\sqrt{2}$ **81.** $m^2\sqrt[7]{m^5}$ **83.** $ab^5\sqrt[3]{a^2b}$ **85.** $5\sqrt[3]{2}$ **87.** 10 **89.** $-3\sqrt[3]{3}$ **91.** $2\sqrt[5]{3}$ **93.** $\frac{\sqrt[4]{3}}{10}$ **95.** $-4a\sqrt{7a}$ **97.** $10\sqrt{2x}$ **99.** $\sqrt[5]{7a^2}$ **101.** $-\sqrt{2}$ **103.** $-17\sqrt[4]{2}$ **105.** $16\sqrt[4]{2}$ **107.** $y\sqrt{z}$ **109.** $13y\sqrt{x}$ **111.** $12\sqrt[3]{a}$ **113.** $-7y^2\sqrt{y}$ **115.** $-3x\sqrt[5]{xy^2}$ **117.** $3\sqrt{2t} + \sqrt{3t}$ **119.** $8\pi\sqrt{5}$ ft^2; 56.2 ft^2 **121.** $5\sqrt{3}$ amps; 8.7 amps **123.** $\left(26\sqrt{5} + 10\sqrt{3}\right)$ in.; 75.5 in. **129.** $-\frac{15x^5}{y}$ **131.** $3p + 4 - \frac{5}{2p-5}$

Study Set Section 7.3 (page 629)

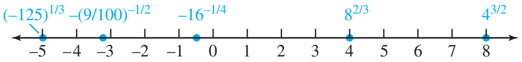
1. multiply **3.** irrational **5.** rationalize **7.** $6\sqrt{6}$ **9.** can't be simplified **11.** can't be simplified **13.** $6\sqrt[3]{15}$ **15.** When the numerator and the denominator of a fraction are multiplied by the same nonzero number, the value of the fraction is not changed. **17.** A radical appears in the denominator. **19.** $\sqrt{6}, 48, 16, 4$ **21.** $5\sqrt{2}$ **23.** $24a\sqrt{10}$ **25.** $6\sqrt[3]{3}$ **27.** $18a$ **29.** $12\sqrt{5} - 15$ **31.** $24\sqrt{3} + 6\sqrt{14}$ **33.** $-1 - 2\sqrt{2}$ **35.** $-1 - \sqrt{15}$ **37.** $5z + 2\sqrt{15z} + 3$ **39.** $b\sqrt{10} + \sqrt{30b} - \sqrt{6b} - 3\sqrt{2}$ **41.** 7 **43.** $18x$ **45.** $135x^2$ **47.** $x + 6\sqrt{x-2} + 7$ **49.** $\frac{\sqrt{7}}{7}$ **51.** $\frac{\sqrt{30}}{5}$ **53.** $\frac{\sqrt[3]{4}}{2}$ **55.** $\sqrt[3]{3}$ **57.** $\frac{\sqrt[3]{6}}{3}$ **59.** $\frac{\sqrt[4]{2}}{2}$ **61.** $\frac{2\sqrt{2xy}}{xy}$ **63.** $\frac{\sqrt{5y}}{y}$ **65.** $\frac{\sqrt{14a}}{4a^2}$ **67.** $-\frac{\sqrt{65p}}{5p^4}$ **69.** $\frac{\sqrt[3]{2ab^2}}{b}$ **71.** $\frac{\sqrt[3]{14p^2}}{2p}$ **73.** $\sqrt{2} + 1$ **75.** $\frac{3\sqrt{2} - \sqrt{10}}{4}$ **77.** $2 + \sqrt{3}$ **79.** $\frac{x - 2\sqrt{xy} + y}{x - y}$ **81.** $\frac{a - 4}{a + 2\sqrt{a}}$ **83.** $\frac{9 - b}{3\sqrt{b} - b}$ **85.** 11 **87.** 4 **89.** $6\sqrt{2}$ **91.** 5 **93.** 8 **95.** ab^2 **97.** $5a\sqrt{b}$ **99.** $-20r\sqrt[3]{10s}$ **101.** $x^2(x + 3)$ **103.** $-8x\sqrt{10} + 6\sqrt{15x}$ **105.** $3x - 2y$ **107.** $6a + 5\sqrt{3ab} - 3b$ **109.** $18r - 12\sqrt{2r} + 4$ **111.** $-6x - 12\sqrt{x} - 6$ **113.** $\frac{\sqrt{10}}{4}$ **115.** 2 **117.** $\frac{\sqrt[3]{2}}{2}$ **119.** $\frac{9 - 2\sqrt{14}}{5}$ **121.** $\frac{2(\sqrt{x} - 1)}{x - 1}$ **123.** $\sqrt{2z} + 1$ **125.** $\frac{\sqrt{2\pi}}{2\pi\sigma}$ **127.** $\frac{\sqrt{2}}{2}$ **129.** $f(L) = \frac{\pi\sqrt{2L}}{4}$ **133.** 1 **135.** $\frac{1}{3}$

Study Set Section 7.4 (page 639)

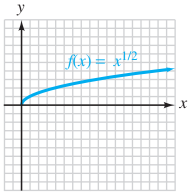
1. radical **3.** extraneous **5.** Square both sides. **7.** x **9.** $32x$ **11.** x **13.** $-2x$ **15.** x **17.** 6 **19.** $2\sqrt{x-2}, 4, 16, 8, 24$ **21.** 2 **23.** 4 **25.** 1 **27.** 64 **29.** $14, \cancel{6}$ **31.** $2, \cancel{7}$ **33.** $2, -1$ **35.** $4, -1$ **37.** -7 **39.** 85 **41.** 16 **43.** 0 **45.** 9 **47.** 1 **49.** $h = \frac{v^2}{2g}$ **51.** $l = \frac{8T^2}{\pi^2}$ **53.** $A = P(r + 1)^3$ **55.** $v^2 = c^2\left(1 - \frac{L^2{}_A}{L^2{}_B}\right)$ **57.** 4 **59.** 8 **61.** $\frac{5}{2}, \frac{1}{2}$ **63.** 1 **65.** 4, 3 **67.** $-1, \cancel{1}$ **69.** $\cancel{1}$ **71.** 4 **73.** $0, \cancel{4}$ **75.** $\cancel{-3}$ **77.** 1, 9 **79.** $4, \cancel{0}$ **81.** $2, \cancel{142}$ **83.** $\cancel{6}$ **85.** 178 ft **87.** about 488 watts **89.** 10 lb **91.** \$5 **97.** 2.5 foot-candles **99.** 0.41511 in.

Study Set Section 7.5 (page 651)

1. rational (or fractional) **3.** index, radicand **5.** $25^{1/5}, (\sqrt[3]{-27})^2, 16^{-3/4}, (\sqrt{81})^3, -\left(\frac{9}{64}\right)^{1/2}$

7.


9. x^{m+n} **11.** x^{m-n} **13.** $\sqrt[n]{x}$ **15.** 0, 1, 2, 3, 4



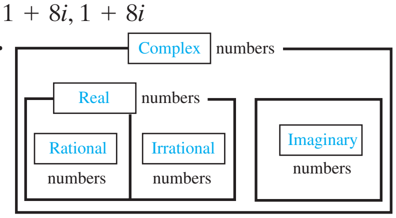
17. $\sqrt{100a^4}, 10a^2$ **19.** 2 **21.** $\frac{1}{4}$ **23.** -3 **25.** 3 **27.** $\frac{1}{2}$ **29.** 2 **31.** 0 **33.** $-\sqrt[3]{3p^2}$ **35.** $m^{1/2}$ **37.** $(3a)^{1/4}$ **39.** $\left(\frac{1}{7}abc\right)^{1/6}$ **41.** $(a^2 - b^2)^{1/3}$ **43.** $c = \sqrt{a^2 + b^2}$ **45.** $d = \sqrt[3]{\frac{12V}{\pi}}$ **47.** $5|y|$ **49.** $2|x|$ **51.** $3x$ **53.** not real **55.** 216 **57.** 27 **59.** 1,728 **61.** $\frac{1}{4}$ **63.** $125x^6$ **65.** $\frac{4x^2}{9}$ **67.** $\frac{1}{2}$ **69.** not a real number **71.** $\frac{1}{8}$ **73.** $\frac{1}{64x^3}$ **75.** 4 **77.** $64x^3$ **79.** $5^{5/7}$ **81.** $4^{3/5}$ **83.** $a^{3/4}b^{1/2}$ **85.** b **87.** $y + y^2$ **89.** $x^2 - x + x^{3/5}$ **91.** $\sqrt{p}$ **93.** $\sqrt{5b}$ **95.** 5 **97.** 2 **99.** -2 **101.** not real **103.** $\frac{1}{9y^2}$ **105.** $\frac{16}{81}$ **107.** $-\frac{3}{2x}$ **109.** $\sqrt[4]{3x}$ **111.** $\sqrt[4]{17x^3y}$ **113.** $\sqrt{x^2 + y^2}$ **115.** 2.47 **117.** 1.01 **119.** $9^{1/5}$ **121.** $\frac{1}{36}$ **123.** a **125.** $a^{2/9}$ **127.** $\frac{2x}{3}$ **129.** $\frac{1}{3}x$ **131.** $\frac{32x^6}{y^{25/12}}$ **133.** 736 ft/sec **135.** 1.96 units **137.** 4,608 in.^{2}, 32 ft^2 **141.** $(-\infty, 3)$ **143.** $(28, \infty)$

Study Set Section 7.6 (page 662)

1. hypotenuse **3.** Pythagorean **5.** $a^2 + b^2 = c^2$ **7.** $\sqrt{2}$ **9.** $\sqrt{3}$ **11.** 30°, 60° **13.** Take the positive square root of both sides. **15.** 6, 52, 4, 2 **17.** 10 ft **19.** 80 m **21.** $h = 2\sqrt{2} \approx 2.83, x = 2$ **23.** $x = 12.11, y = 12.11$ **25.** $x = 4.69, y = 8.11$ **27.** $x = 5\sqrt{3} \approx 8.66, h = 10$ **29.** 5 **31.** 13 **33.** 10 **35.** $2\sqrt{26}$ **37.** $7\sqrt{2}$ cm **41.** $(5\sqrt{2}, 0), (0, 5\sqrt{2}), (-5\sqrt{2}, 0), (0, -5\sqrt{2})$; (7.07, 0), (0, 7.07)(−7.07,0), (0, −7.07) **43.** $10\sqrt{3}$ mm, 17.32 mm **45.** $10\sqrt{181}$ ft, 134.54 ft **47.** about 0.13 ft **49. a.** 21.21 units **b.** 8.25 units **c.** 13.00 units **51.** yes **55.** 7 **57.** 9

Study Set Section 7.7 (page 674)

1. imaginary, power **3.** real, imaginary **5. a.** $\sqrt{-1}$ **b.** -1 **c.** $-i$ **d.** 1 **e.** four **7.** real, imaginary **9.** $1 + 8i, 1 + 8i$

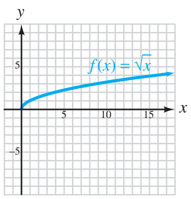
11.


13. $9, 6i, 2, 11$ **15. a.** true **b.** false **c.** false **d.** false **17.** $3i$ **19.** $\sqrt{7}i$ or $i\sqrt{7}$

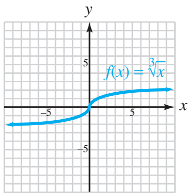
21. $2\sqrt{6}i$ or $2i\sqrt{6}$ **23.** $-6\sqrt{2}i$ or $-6i\sqrt{2}$ **25.** $45i$
27. $\frac{5}{3}i$ **29. a.** $5 + 0i$ **b.** $0 + 7i$ **31. a.** $1 + 5i$
b. $-3 + 2i\sqrt{2}$ **33. a.** $76 - 3i\sqrt{6}$ **b.** $-7 + i\sqrt{19}$
35. a. $-6 - 3i$ **b.** $3 + i\sqrt{6}$ **37.** $8 - 2i$ **39.** $15 + 2i$
41. $3 - 5i$ **43.** $1 + 3i$ **45.** -6 **47.** $-2\sqrt{6}$ **49.** $6 - 27i$
51. $35 - 28i$ **53.** $6 + 14i$ **55.** $-25 - 25i$ **57.** $7 + i$
59. $12 + 5i$ **61.** $13 - i$ **63.** $3 + 4i$ **65.** 40 **67.** 65
69. $\frac{45}{26} - \frac{9}{26}i$ **71.** $-\frac{77}{65} + \frac{44}{65}i$ **73.** $\frac{14}{17} - \frac{5}{17}i$
75. $-\frac{6}{29} + \frac{43}{29}i$ **77.** $\frac{11}{10} + \frac{13}{10}i$ **79.** $0 - i$ **81.** $4 + 0i$
83. $-2 + 0i$ **85.** $0 - \frac{5}{3}i$ **87.** $0 + \frac{2}{7}i$ **89.** i **91.** $-i$
93. 1 **95.** -1 **97.** $4 - 11i$ **99.** $14 - 8i$ **101.** $-2 - 6i$
103. $-\frac{4}{13} - \frac{6}{13}i$ **105.** $18 + 12i$ **107.** $0 + \frac{4}{5}i$
109. $8 + \sqrt{2}i$ or $8 + i\sqrt{2}$ **111.** $-2 + 7i$
113. $-48 - 64i$ **115.** $\frac{1}{4} - \frac{\sqrt{15}}{4}i$ **117.** $-1 + i$
123. 20 mph

Chapter 7 Review Exercises (page 679)

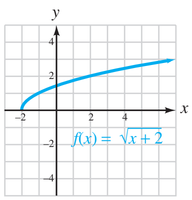
1. 7 **2.** -11 **3.** $\frac{15}{7}$ **4.** not a real number **5.** $10a^6$
6. $5|x|$ **7.** x^4 **8.** $|x + 2|$ **9.** -3 **10.** -6 **11.** $4x^2y$
12. $\frac{x^3}{5}$ **13.** 2 **14.** -2 **15.** $4x^2|y|$ **16.** $x + 1$ **17.** $-\frac{1}{2}$
18. not a real number **19.** not a real number **20.** 0
21. 13 ft **22.** 24 cm^2
23. D: $[0, \infty)$, R: $[0, \infty)$



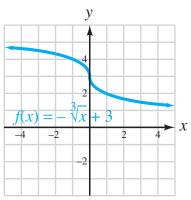
24. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



25. D: $[-2, \infty)$, R: $[0, \infty)$



26. D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



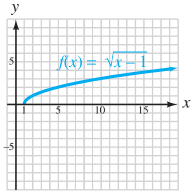
27. $4\sqrt{5}$
28. $3\sqrt[3]{2}$
29. $2\sqrt[4]{10}$
30. $-2\sqrt[5]{3}$
31. $2x^2\sqrt{2x}$
32. $r^4\sqrt[4]{r}$

33. $-3j^2\sqrt[3]{jk}$ **34.** $-2xy\sqrt[3]{2x^2y}$ **35.** $\frac{\sqrt{m}}{12n^6}$
36. $\frac{\sqrt{17xy}}{8a^2}$ **37.** $2x$ **38.** $3x^3$ **39.** $3\sqrt{2}$ **40.** $11\sqrt{5}$
41. 0 **42.** $-8a\sqrt[4]{2a}$ **43.** $29x\sqrt{2}$ **44.** $13x\sqrt[3]{2}$
45. $9\sqrt[4]{2t^3} - 8\sqrt[4]{6t^3}$ **46.** $12x^2\sqrt[4]{x} + 5x\sqrt[4]{x}$
48. $\left(6\sqrt{2} + 2\sqrt{10}\right)$ in., 14.8 in. **49.** 7 **50.** $6\sqrt{10}$
51. 32 **52.** $6\sqrt{10}$ **53.** $3x$ **54.** $x + 1$ **55.** $-2x^3\sqrt[3]{x}$
56. 3 **57.** $42t + 9t\sqrt{21t}$ **58.** $-2x^3y^3\sqrt[4]{2x^2y^2}$
59. $3b + 6\sqrt{b} + 3$ **60.** $\sqrt[3]{9p^2} - \sqrt[3]{6p} - 2\sqrt[3]{4}$
61. $\frac{10\sqrt{3}}{3}$ **62.** $\frac{\sqrt{15xy}}{5xy}$ **63.** $\frac{\sqrt[3]{6u^2}}{u^2}$ **64.** $\frac{\sqrt[4]{27ab^2}}{3b}$
65. $2\left(\sqrt{2} + 1\right)$ or $2\sqrt{2} + 2$ **66.** $\frac{12\sqrt{xz} - 16x - 2z}{z - 16x}$
67. $\frac{a - b}{a + \sqrt{ab}}$ **68.** $r = \frac{\sqrt[3]{6\pi^2V}}{2\pi}$ **69.** 22 **70.** $16, 9$
71. $\frac{13}{2}$ **72.** $\frac{9}{16}$ **73.** $2, -4$ **74.** 7 **75.** 1 **76.** $-\frac{3}{2}, 1$
77. $\cancel{3}$, no solution **78.** $6, \cancel{-2}$ **79.** $-\frac{1}{2}, 4$ **80.** 2
81. $P = \frac{A}{(r + 1)^2}$ **82.** $I = \frac{h^3b}{12}$ **83.** $\sqrt{t}$ **84.** $\sqrt[4]{5xy^3}$
85. 5 **86.** -6 **87.** not a real number **88.** 1 **89.** $\frac{3}{x}$
90. -2 **91.** 5 **92.** $3cd$ **93.** 27 **94.** $\frac{1}{4}$ **95.** $-16{,}807$
96. 10 **97.** $\frac{27}{8}$ **98.** $\frac{1}{3{,}125}$ **99.** $125x^3y^6$ **100.** $\frac{1}{4u^4v^2}$
101. $5^{3/4}$ **102.** $a^{1/7}$ **103.** k^8 **104.** $3^{2/3}$ **105.** $u - 1$
106. $v + v^2$ **107.** $\sqrt{a}$ **108.** $\sqrt[6]{c}$ **109.** 183 mi
110. Two true statements result: $32 = 32$. **111.** 17 ft
112. 88 yd **113.** $7\sqrt{2}$ m ≈ 9.90 m
114. $\frac{15\sqrt{2}}{2}$ yd ≈ 10.61 yd **115.** shorter leg: 6 cm, longer leg: $6\sqrt{3}$ cm ≈ 10.39 cm **116.** $40\sqrt{3}$ ft ≈ 69.28 ft, $20\sqrt{3}$ ft ≈ 34.64 ft **117.** $x = 5\sqrt{2} \approx 7.07, y = 5$
118. $x = 25\sqrt{3} \approx 43.30, y = 25$ **119.** 13 **120.** $2\sqrt{2}$
121. $5i$ **122.** $3i\sqrt{2}$ **123.** $-i\sqrt{6}$ **124.** $\frac{3}{8}i$
125. Real, Imaginary **126. a.** true **b.** true **c.** false **d.** false **127. a.** $3 - 6i$ **b.** $0 - 19i$ **128. a.** $-1 + 7i$ **b.** $0 + i$ **129.** $8 - 2i$ **130.** $3 - 5i$ **131.** $3 + 6i$
132. $22 + 29i$ **133.** $-3\sqrt{3} + 0i$ **134.** $-81 + 0i$
135. $4 + i$ **136.** $0 - \frac{3}{11}i$ **137.** -1 **138.** i

Chapter 7 Test (page 689)

1. a. radical **b.** imaginary **c.** extraneous **d.** isosceles **e.** rationalize **f.** complex **2. a.** If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$. **b.** If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, $(b \neq 0)$.

3. D: $[1, \infty)$, R: $[0, \infty)$



4. 46 ft/sec **5. a.** -1 **b.** 2 **c.** 1 **d.** D: $(-\infty, \infty)$, R: $(-\infty, \infty)$

6. No real number raised to the fourth power is -16.

7. $|x|$ **8.** $|y - 5|$ **9.** $-4xy^2$ **10.** $\frac{2}{3}a$ **11.** $t + 8$

12. $6xy^2\sqrt{15xy}$ **13.** $2x^5y\sqrt[3]{3}$ **14.** $2\sqrt[4]{2}$ **15.** $2y^2\sqrt{3y}$

16. $14\sqrt[3]{5}$ **17.** $5z^3\sqrt[4]{3z}$ **18.** $-6x\sqrt{y} - 2xy^2$

19. $3 - 7\sqrt{6}$ **20.** $\sqrt[3]{4a^2} + 18\sqrt[3]{2a} + 81$

21. $\frac{4\sqrt{10}}{5}$ **22.** $\sqrt{3t} + 1$ **23.** $\frac{\sqrt[3]{18a^2}}{2a}$

24. $\frac{1}{\sqrt{2}(\sqrt{5} - 3)} = \frac{1}{\sqrt{10} - 3\sqrt{2}}$ **25.** $\frac{1}{15}$ **26.** 10

27. $\cancel{4}$, no solution **28.** 3, $\cancel{-3}$ **29.** 2, 3 **30.** $\cancel{108}$, no solution

31. -2 **32.** $G = \frac{4\pi^2r^3}{Mt^2}$ **33.** $7x^2$ **34.** -9 **35.** $\frac{1}{216}$

36. $\frac{25n^4}{4}$ **37.** $2^{4/3}$ **38.** $a^{1/9}$

39. $x = \frac{8\sqrt{3}}{3}$ cm ≈ 4.62 cm, $h = \frac{16\sqrt{3}}{3}$ cm ≈ 9.24 cm

40. $x = \frac{(12.26)\sqrt{2}}{2}$ in. ≈ 8.67 in., $y = \frac{(12.26)\sqrt{2}}{2}$ in. ≈ 8.67 in.

41. 25 **42.** 28 in. **43.** $3i\sqrt{5}$ **44.** -1 **45.** $-4 + 11i$

46. $4 - 7i$ **47.** $75 + 45i$ **48.** $-46 - 78i$ **49.** $0 - \frac{\sqrt{2}}{2}i$

50. $\frac{1}{2} + \frac{1}{2}i$

Chapters 1–7 Cumulative Review (page 691)

1. a. A rational number is any number that can be written as a fraction with an integer numerator and a nonzero integer denominator. **b.** An irrational number is a nonterminating, nonrepeating decimal. **c.** A real number is any number that is either a rational number or an irrational number.

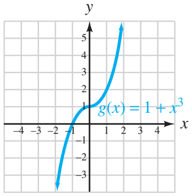
2. $\frac{1}{10}$ **3.** $\ell = \frac{a - S + Sr}{r}$ **4.** 22

5. 10%: $3\frac{3}{4}$ cups, 18%: $6\frac{1}{4}$ cups **6. a.** $-\frac{3}{5}$

b. $(0, -3)$ **c.** $y = -\frac{3}{5}x - 3$ **7. a.** 3

b. $y = -\frac{1}{3}x - \frac{2}{3}$ **8.** -30.75 **9.** D: $(-\infty, \infty)$, R: $(-\infty, -2)$

10.



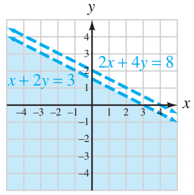
11. no solution

12. 10, 15, 5

13. $(-\infty, -15)$

14. $(-\infty, -3) \cup (2, \infty)$

15.



16. $\frac{64b^{12}}{27a^9}$ **17.** 2.379×10^{14}

18. $2x^2y^3 + 13xy + 3y^2$

19. $6y^3 + 11y^2 + 9y + 2$

20. $3x^2 + 3x - 11$

21. $(3 - c)(c + d)$

22. $(x - 2y)(x^2 + 2xy + 4y^2)$

23. $(a + b - 1)^2$ **24.** $(x + 1)(x - 1)(x + 4)(x - 4)$

25. 0, 10, -10 **26.** $-\frac{1}{3}$, -3 **27.** $\frac{-3x - 2y}{y}$ **28.** -7, 8

29. $x - 5$ **30.** $\frac{3x + 2}{(x + 2)(x - 1)}$ **31.** $2\frac{2}{5}$ hr

32. Direct variation: As one quantity increases, the other increases, in a predictable way. Inverse variation: As one quantity increases, the other decreases, in a predictable way.

33. $10x^2y\sqrt{2yz}$ **34.** $6\sqrt[3]{2}$ **35.** $5z + 2\sqrt{15z} + 3$

36. $\frac{\sqrt{6}}{10}$ **37.** 4, 3 **38.** $-\frac{3}{2x}$

Study Set Section 8.1 (page 703)

1. quadratic **3.** perfect **5.** $\sqrt{c}, -\sqrt{c}$ **7.** yes **9. a.** 36 **b.** $\frac{25}{4}$ **c.** $\frac{1}{16}$ **11. a.** Subtract 35 from both sides. **b.** Add 36 to both sides. **13.** $\pm\frac{\sqrt{10}}{2}$

15. 4 is not a factor of the numerator. It cannot be divided out.

17. $\frac{3 \pm \sqrt{10}}{5} = \frac{3}{5} \pm \frac{\sqrt{10}}{5}$ **19. a.** 2; $2\sqrt{5}, -2\sqrt{5}$ **b.** ± 4.47

21. 0, -2 **23.** 5, -5 **25.** -2, -4 **27.** 2, $\frac{1}{2}$ **29.** ± 9

31. ± 6 **33.** $\pm 5\sqrt{2}$ **35.** $\pm\frac{4\sqrt{3}}{3}$ **37.** $d = \frac{3\sqrt{2}}{2}$

39. $t = \sqrt{3}$ **41.** 0, -2 **43.** 4, 10 **45.** $\pm 4i$ **47.** $\pm\frac{9}{2}i$

49. $x^2 + 24x + 144 = (x + 12)^2$

51. $a^2 - 7a + \frac{49}{4} = \left(a - \frac{7}{2}\right)^2$ **53.** 2, -4 **55.** 2, 6

57. 1, -6 **59.** -1, 4 **61.** 1, $-\frac{1}{2}$ **63.** $\frac{1}{2}, -\frac{2}{3}$ **65.** $\frac{3 \pm \sqrt{29}}{2}$

67. $\frac{3 \pm 2\sqrt{3}}{3}$ **69.** $-1 \pm i$ **71.** $-4 \pm i\sqrt{2}$ **73.** $-5 \pm \sqrt{3}$

75. $-4 \pm \sqrt{10}$ **77.** $1 \pm 3\sqrt{2}$ **79.** $\frac{1 \pm 2\sqrt{2}}{2}$

81. $\frac{-5 \pm \sqrt{41}}{4}$ **83.** $\frac{-7 \pm \sqrt{29}}{10}$ **85.** $\frac{3 \pm 2\sqrt{3}}{3}$

87. $-\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$ **89.** $-\frac{1}{4} \pm \frac{\sqrt{11}}{4}i$

91. width: $7\frac{1}{4}$ ft, length: $13\frac{3}{4}$ ft **93.** 1.6 sec **95.** 1.70 in.

97. 0.92 in. **99.** 2.9 ft, 6.9 ft **103.** $2ab^2\sqrt[3]{5}$ **105.** x^3

107. $5ab\sqrt{7b}$

Study Set Section 8.2 (page 714)

1. quadratic **3. a.** $x^2 + 2x + 5 = 0$ **b.** $3x^2 + 2x - 1 = 0$ **5. a.** true **b.** true **7.** $c = \dfrac{-s \pm \sqrt{s^2 - 4rt}}{2r}$ **9. a.** $x = 1 \pm 2\sqrt{2}$ **b.** $x = \dfrac{-3 \pm \sqrt{7}}{2}$ **11. a.** The fraction bar wasn't drawn under both parts of the numerator. **b.** A $\pm$ symbol wasn't written between b and the radical. **13.** $-1, -2$ **15.** $-6, -6$ **17.** $\frac{1}{2}, -3$ **19.** $\frac{2}{3}, -\frac{1}{4}$ **21.** $-\frac{3}{2}, -\frac{1}{2}$ **23.** $\dfrac{3 \pm \sqrt{17}}{4}$ **25.** $-\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$ **27.** $\frac{1}{3} \pm \frac{\sqrt{2}}{3}i$ **29.** $3x^2 + 6x + 1 = 0$ **31.** $x^2 - 2x + 6 = 0$ **33.** $\sqrt{5}$ **35.** 6 **37.** $\dfrac{-5 \pm \sqrt{5}}{10}$ **39.** $\frac{1}{4}, -\frac{3}{4}$ **41.** $\frac{4}{3}, -\frac{2}{5}$ **43.** $\dfrac{-5 \pm \sqrt{17}}{2}$ **45.** $5 \pm \sqrt{7}$ **47.** $23, -17$ **49.** $1 \pm i$ **51.** $-\frac{2}{3}, \frac{5}{2}$ **53.** $-\frac{1}{2} \pm i$ **55.** $\dfrac{-3 \pm \sqrt{29}}{10}$ **57.** $\dfrac{9 \pm \sqrt{89}}{2}$ **59.** $\dfrac{10 \pm \sqrt{55}}{30}$ **61.** $-0.68, -7.32$ **63.** $1.22, -0.55$ **65.** $8.98, -3.98$ **67.** 97 ft by 117 ft **69.** 0.5 mi by 2.5 mi **71.** 34 in. **73.** \$4.80 or \$5.20 **75.** 4,000 **77.** 9% **79.** early 1976 **83.** $n^{1/2}$ **85.** $(3b)^{1/4}$ **87.** $\sqrt[3]{t}$ **89.** $\sqrt[4]{3t}$

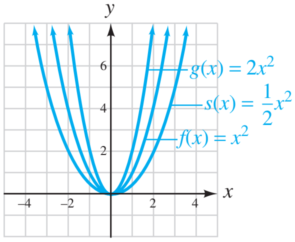
Think It Through Section 8.3 (page 726)

We estimate that 50-year old drivers are involved in the least number of accidents, 12 per million miles driven.

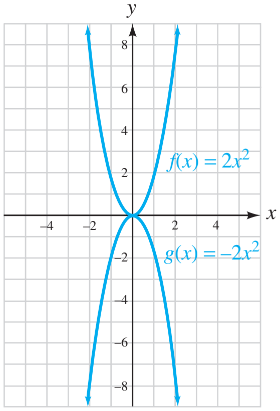
Study Set Section 8.3 (page 728)

1. quadratic **3.** vertex **5.** a parabola **7.** $(0, -3)$ **9.** $x = 2$ **11.** $-3, 5$ **13.** $h = -1$; $f(x) = 2[x - (-1)]^2 + 6$ **15.** 256 **17.** 172

19.



21.



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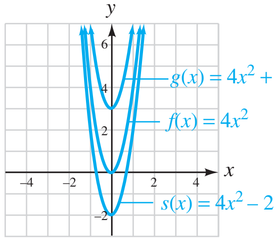
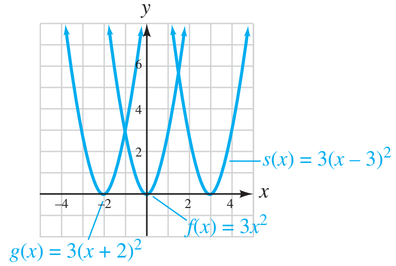
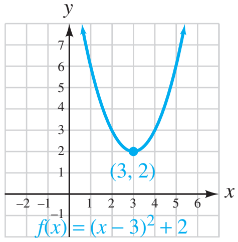


25.

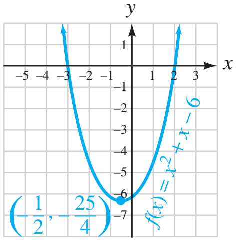


27.

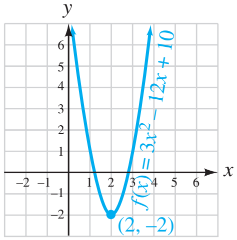


29. $(1, 2)$, $x = 1$, upward **31.** $(-3, -4)$, $x = -3$, upward

33. axis: $x = -\frac{1}{2}$

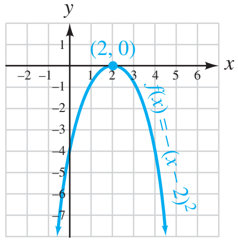


35. axis: $x = 2$

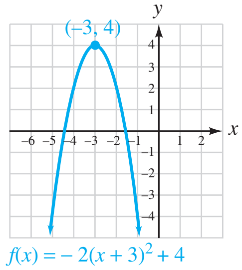


37. $(1, -2)$, $x = 1$, upward **39.** $(2, 21)$, $x = 2$, downward

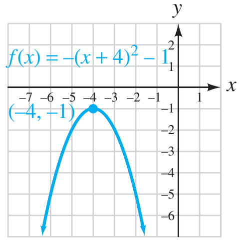
41. $\left(-\frac{2}{3}, \frac{2}{3}\right)$, $x = -\frac{2}{3}$, upward **43.** $(2, 0)$, axis: $x = 2$



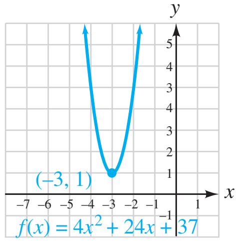
45. $(-3, 4)$, axis: $x = -3$



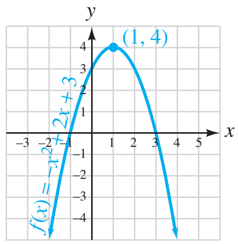
47. $(-4, -1)$, axis: $x = -4$



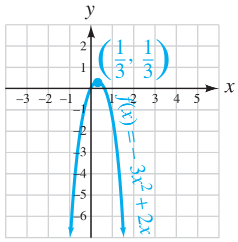
49. $(-3, 1)$, axis: $x = -3$



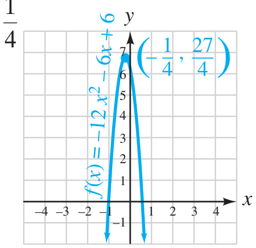
51. $(1, 4)$, axis: $x = 1$



53. $\left(\frac{1}{3}, \frac{1}{3}\right)$, axis: $x = \frac{1}{3}$



55. $\left(-\frac{1}{4}, \frac{27}{4}\right)$, axis: $x = -\frac{1}{4}$



57. $(0.25, 0.88)$ **59.** $(0.50, 7.25)$ **61.** $2, -3$ **63.** $-1.85, 3.25$ **65.** 3.75 sec, 225 ft **67.** 250 ft by 500 ft **69.** 15 min, \$160 **71.** 1968, 1.5 million; the U.S. involvement in the war in Vietnam was at its peak **73.** 200, \$7,000 **81.** $4a^2\sqrt{b}$ **83.** $\frac{\sqrt{6}}{10}$ **85.** $15b - 6\sqrt{15b} + 9$

Study Set Section 8.4 (page 740)

1. $b^2 - 4ac$ **3.** conjugates **5.** rational, unequal **7.** $x^4 = (x^2)^2$ **9.** $x^{2/3} = (x^{1/3})^2$ **11.** 5, 6, 24, rational **13.** rational, equal **15.** complex conjugates **17.** irrational, unequal **19.** rational, unequal **21.** $1, -1, 4, -4$ **23.** $2, -2, 3i, -3i$ **25.** 25, 64 **27.** 1 **29.** $-8, -27$ **31.** $-1, 27$ **33.** 0, 2 **35.** $\frac{1}{4}, \frac{1}{2}$ **37.** $-1, -4$ **39.** $-1, -\frac{27}{13}$ **41.** yes **43.** $2, -2, i\sqrt{7}, -i\sqrt{7}$ **45.** $1, -1, \sqrt{5}, -\sqrt{5}$ **47.** no solution **49.** 16, 4 **51.** $-64, 8$ **53.** $1 - 2\sqrt{3}, 1 + 2\sqrt{3}$ **55.** $\frac{5 \pm \sqrt{65}}{2}$ **57.** $\frac{3 + \sqrt{57}}{6}, \frac{3 - \sqrt{57}}{6}$ **59.** $1, 1, -1, -1$ **61.** $1 \pm i$ **63.** 12.1 in. **65.** 30 mph **67.** 14.3 min **71.** -2 **73.** $\frac{9}{5}$

Study Set Section 8.5 (page 751)

1. quadratic **3.** two **5.** $(-\infty, -1), (-1, 4), (4, \infty)$ **7. a.** yes **b.** no **c.** yes **d.** no **9. a.** $(-3, 2)$ **b.** $(-\infty, -1] \cup [1, \infty)$ **11. a.** solid **b.** yes **13.** $x^2 - 6x - 7 \geq 0$

15. $(1, 4)$

17. $(-\infty, 3) \cup (5, \infty)$

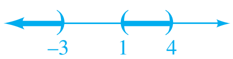
19. $(-\infty, -6] \cup [7, \infty)$

21. $[-4, 3]$

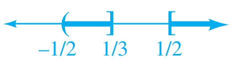
23. $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$

25. $\left(-\infty, -\frac{5}{3}\right] \cup (0, \infty)$

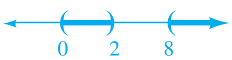
27. $(-\infty, -3) \cup (1, 4)$



29. $\left(-\frac{1}{2}, \frac{1}{3}\right] \cup \left[\frac{1}{2}, \infty\right)$

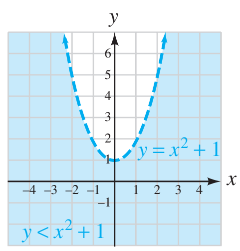


31. $(0, 2) \cup (8, \infty)$

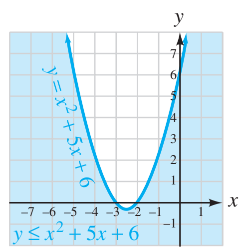


33. $\left[\frac{-34}{5}, -4\right) \cup (3, \infty)$

35.



37.



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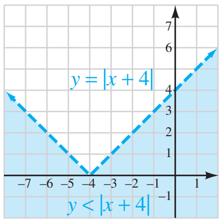
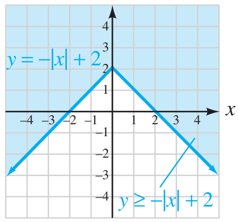


41.



43. $(-1, 3)$ **45.** $(-\infty, -3) \cup (2, \infty)$

47. $(-4, -2] \cup (-1, 2]$

49. $(-\infty, -3] \cup [3, \infty)$

51. $(-\infty, \infty)$

53. $(-2, 1) \cup (3, \infty)$

55. $(-5, 5)$,

57. $\left(-\frac{1}{3}, \frac{3}{2}\right)$

59. no solutions

61. $(-1, 4) \cup \left(\frac{23}{2}, \infty\right)$

63. $(-2{,}100, -900) \cup (900, 2{,}100)$ **69.** $x = ky$ **71.** $t = kxy$

Chapter 8 Review Exercises (page 756)

1. $-5, -4$ **2.** $-\frac{1}{3}, -\frac{5}{2}$ **3.** $\pm 2\sqrt{7}$ **4.** $4, -8$ **5.** $\pm 5i$

6. $\pm\frac{7\sqrt{5}}{5}$ **7.** $r = \frac{\sqrt{\pi A}}{\pi}$ **8.** $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

9. $-4, -2$ **10.** $\frac{3 \pm \sqrt{3}}{2}$ **11.** $\frac{6 \pm \sqrt{30}}{6}$ **12.** $1 \pm 2i\sqrt{3}$

13. Because 7 is an odd number and not divisible by 2, the computations involved in completing the square on $x^2 + 7x$ create fractions. The computations involved in completing the square on $x^2 + 6x$ do not. **14.** 2 is not a factor of the numerator—it is a term. Only common factors of the numerator and denominator can be removed.

15. a. $(2 + 2x)$ ft **b.** $(6 + 2x)$ ft **16.** 6 seconds before midnight **17.** $\frac{1}{2}, -7$ **18.** $5 \pm \sqrt{7}$ **19.** $0, 10$

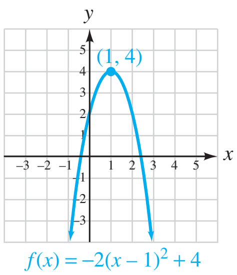
20. $\frac{13 \pm \sqrt{163}}{3}$ **21.** $-\frac{3}{4} \pm \frac{\sqrt{15}}{4}i$ **22.** $\frac{2}{3} \pm \frac{\sqrt{2}}{3}i$

23. $\frac{-3 \pm \sqrt{29}}{10}$ **24.** $\frac{3 \pm 3\sqrt{13}}{2}$ **25.** \$24 or \$26

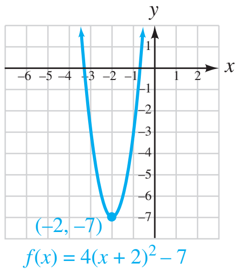
26. sides: 1.25 in. wide; top/bottom: 2.5 in. wide

27. 0.7 sec, 1.8 sec **28.** 33 in., 56 in. **29.** 30.8 million

30. h, k, x **31.** $(1, 4), x = 1$,

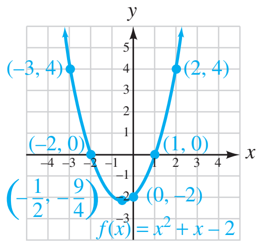


32. $f(x) = 4(x + 2)^2 - 7; (-2, -7), x = -2$



33. $(1, -6)$

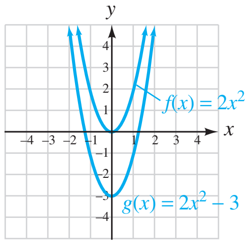
34. $\left(-\frac{1}{2}, -\frac{9}{4}\right), x = -\frac{1}{2}; (-2, 0), (1, 0); (0, -2)$



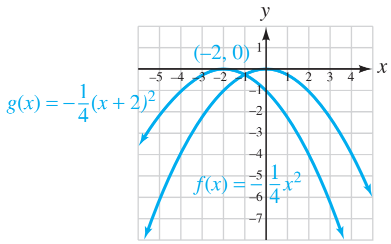
35. 1921; 6,469,326

36. $-2, \frac{1}{3}$

37.



38.



39. two different irrational-number solutions

40. two imaginary-number solutions that are complex conjugates **41.** one repeated solution, a rational number

42. two different rational-number solutions **43.** 1, 144

44. $8, -27$ **45.** $i, -i, \frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3}$ **46.** $1, -\frac{8}{5}$ **47.** $4 \pm i$

48. repeated solutions of -1 and 1

49. a repeated solution of $-\frac{2}{5}$ **50.** $\frac{1}{32}, 32$

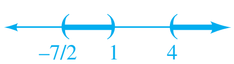
51. about 81 min **52.** 30 mph

53. $(-\infty, -7) \cup (5, \infty)$

54. $[-9, 9]$

55. $(-\infty, 0) \cup \left[\frac{3}{5}, \infty\right)$

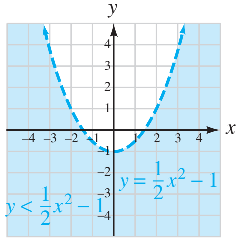
56. $\left(-\frac{7}{2}, 1\right) \cup (4, \infty)$



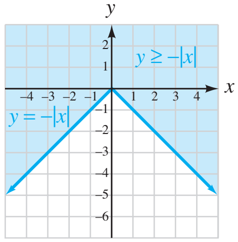
57. $\left[-4, \frac{2}{3}\right]$

58. $(-\infty, 0) \cup (1, \infty)$

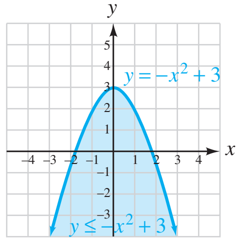
59.



60.



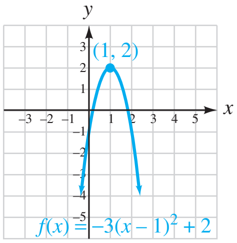
30.



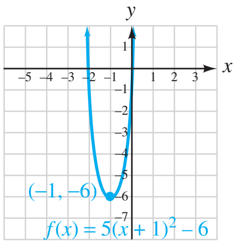
31. $-2, 3$ **32.** $[-2, 3]$

Chapter 8 Test (page 764)

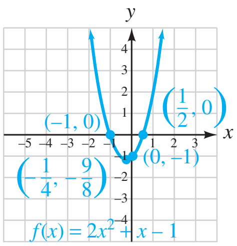
1. a. quadratic **b.** completed, square **c.** vertex **d.** rational **e.** nonlinear **2.** $\pm 3\sqrt{7} \approx \pm 7.94$ **3.** $-7 \pm 5\sqrt{2}$ **4.** $\pm 2i$ **5.** $x^2 + 11x + \frac{121}{4} = \left(x + \frac{11}{2}\right)^2$ **6.** $\frac{3}{2}, \frac{5}{2}$ **7.** $\frac{-1 \pm \sqrt{2}}{2}$ **8.** $1 \pm \sqrt{5}$ **9.** $2 \pm 3i$ **10.** $-5, -3$ **11.** $1, \frac{1}{4}$ **12.** $-1, \frac{1}{3}$ **13.** $2, -2, i\sqrt{3}, -i\sqrt{3}$ **14.** $-\frac{4}{5}, \frac{4}{7}$ **15.** $2 \pm \sqrt{10}$ **16.** $-8, -\frac{1}{125}$ **17.** $c = \frac{\sqrt{Em}}{m}$ **18. a.** two different imaginary-number solutions that are complex conjugates **b.** one repeated solution, rational number **19.** 4.5 ft by 1,502 ft **20.** about 46 min **21.** 20 in. **22.** iii **23.** $(1, -2), x = 1$



24. $f(x) = 5(x + 1)^2 - 6, (-1, -6), x = -1$



25. $\left(-\frac{1}{4}, -\frac{9}{8}\right), x = -\frac{1}{4}, (-1, 0), \left(\frac{1}{2}, 0\right); (0, -1)$



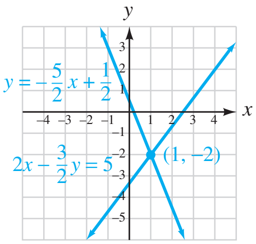
26. 211 ft **27.** $(-\infty, -2) \cup (4, \infty)$

28. $(-3, 2]$

29. 11,160 gal

Chapters 1–8 Cumulative Review (page 766)

1. -3 **2.** 6 L **3.** $y = 3x + 2$ **4.** $y = -\frac{2}{3}x - 2$ **5.** supply: a decrease of 15,000 nurses per year; demand: an increase of 55,000 nurses per year **6. a.** domain: $[0, 24]$ **b.** 1.5 **c.** At noon, the low tide mark was -2.5 m. **d.** 0, 2, 9, 17 **7.** $(1, -2)$

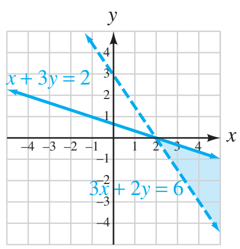


8. $(3, 8)$ **9.** $(2, -1, 1)$ **10.** 6 **11.** $\left(-\infty, -\frac{10}{9}\right)$

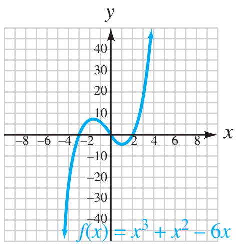
12. $\left[1, \frac{9}{4}\right]$

13. $(-\infty, -10] \cup [15, \infty)$

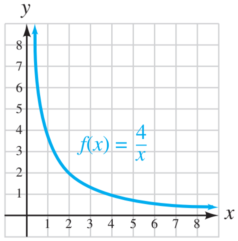
14.



15. $\frac{125c^{21}}{8a^{12}b^{12}}$ **16.** 1.44×10^{10}; 14,400,000,000 **17.** $-11t^3 - 0.8t^2 - 1.4t$ **18.** $8a^3 - b^3$ **19.** $(x - y)(x - 4)$ **20.** $(x^2 + 4y^2)(x + 2y)(x - 2y)$ **21.** $(2x^2 + 5y)(4x^4 - 10x^2 + 25y^2)$ **22.** $2a^2(3a + 2)(5a - 4)$ **23.** $(7s^3 - 6n^2)^2$ **24.** $(x + 5 + y^4)(x + 5 - y^4)$ **25.** $2, -\frac{5}{2}$ **26.** $0, \frac{2}{3}, -\frac{1}{2}$ **27.** D: $(-\infty, \infty)$, R: $(-\infty, \infty)$



28. D: $(0, \infty)$, R: $(0, \infty)$



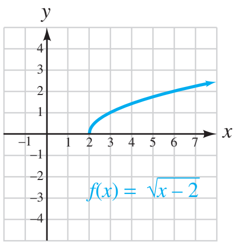
29. $\frac{3x - 5}{x + 2}$ **30.** $\frac{x + y}{x - y}$ **31.** 0 **32.** $\frac{1}{2r(r + 2)}$

33. $\frac{x^4y^4}{2} - \frac{x^3y^9}{4} + \frac{3}{4xy^2}$ **34.** $5a^2 - 3a - 4$

35. 5; 3 is extraneous **36.** $R = \frac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

37. 18 sec **38.** 21 tons **39.** \$9,000

40. D: $[2, \infty)$, R: $[0, \infty)$



41. $-3x$ **42.** $4t\sqrt{3t}$

43. $-12\sqrt[4]{2} + 10\sqrt[4]{3}$ **44.** $-18\sqrt{6}$ **45.** $\frac{x + 3\sqrt{x} + 2}{x - 1}$

46. $\frac{5\sqrt[3]{x^2}}{x}$ **47.** $\frac{1}{16}$ **48.** $x^{17/12}$ **49.** 2, 7 **50.** $\frac{1}{4}$

51. $3\sqrt{2}$ in. **52.** $2\sqrt{3}$ in. **53.** 10 **54.** $-i$ **55.** $-5 + 17i$

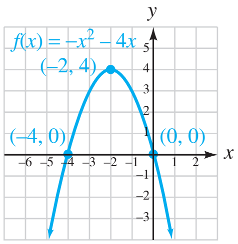
56. $\frac{3}{2} + \frac{1}{2}i$ **57.** $3 + 4i$ **58.** $0 - \frac{2}{3}i$ **59.** $\pm 2\sqrt{7}$

60. $19 \pm i\sqrt{5}$ **61.** $\frac{3 \pm \sqrt{3}}{2}$ **62.** $\frac{1}{5} \pm \frac{2}{5}i$ **63.** 10 ft by 18 ft

64. 50 m and 120 m **65.** $-8, 27$

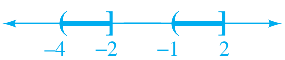
66. repeated solutions of -1 and 1

67. $(-2, 4)$, $x = -2$; $(-4, 0)$, $(0, 0)$; $(0, 0)$



68. $(-9, 9)$

69. $(-4, -2] \cup (-1, 2]$



70. a. $-\frac{3}{4}$ **b.** no solution

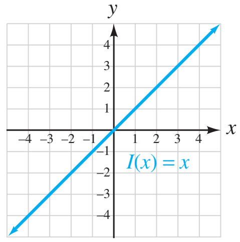
Think It Through (page 771)

The graph of $f + g$ gives the total number of students enrolled in historically Black colleges and universities.

Study Set Section 9.1 (page 776)

1. sum, $f(x) + g(x)$ **3.** product, $f(x)g(x)$ **5.** domain
7. identity **9.** $g(x)$ **11.** $-14, -19$

13. $-3, -2, -1, 0, 1, 2, 3$



15. $(3x - 1), 9x, 2x$ **17.** $7x, (-\infty, \infty)$ **19.** $x, (-\infty, \infty)$

21. $12x^2, (-\infty, \infty)$ **23.** $\frac{4}{3}, (-\infty, 0) \cup (0, \infty)$

25. $3x - 2, (-\infty, \infty)$ **27.** $2x^2 - 5x - 3, (-\infty, \infty)$

29. $-x - 4, (-\infty, \infty)$

31. $\frac{x - 3}{2x + 1}, \left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$ **33.** 7

35. $2x^2 - 1$ **37.** 0 **39.** $4x^2 + 4x$ **41.** $7x + 5$

43. $-3x + 6$ **45.** $-2x^2 + 3x - 3, (-\infty, \infty)$

47. $\frac{3x - 2}{2x^2 + 1}, (-\infty, \infty)$ **49.** $3, (-\infty, \infty)$

51. $\frac{x^2 - 4}{x^2 - 1}, (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

53. 58 **55.** 110 **57.** 2 **59.** $9x^2 - 9x + 2$

63. $C(t) = \frac{5}{9}(2{,}668 - 200t)$ **65. a.** about \$37.50

b. $C(m) = \frac{1.50m}{20}$ **71.** $-\frac{3x + 7}{x + 2}$ **73.** $\frac{x - 4}{3x^2 - x - 12}$

Study Set Section 9.2 (page 786)

1. one-to-one **3.** inverses **5.** once **7.** 2 **9.** x

11. $-6, -4, 0, 2, 8$ **13.** no **15.** 2

17. The graphs are not symmetric about the line $y = x$.

19. $2x, 2y, 3, y, f^{-1}(x)$ **21.** the inverse of, inverse

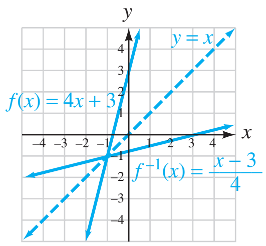
23. yes, $\{(2, 3), (1, 2), (0, 1)\}$, yes

25. yes $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$, no **27.** yes **29.** no

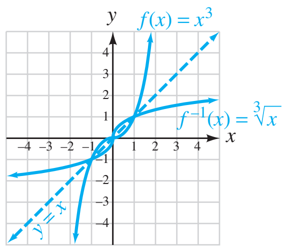
31. one-to-one **33.** not one-to-one **35.** not one-to-one

37. one-to-one **39.** $f^{-1}(x) = \frac{5}{4}x + 5$ **41.** $f^{-1}(x) = 5x - 4$

43.



45.



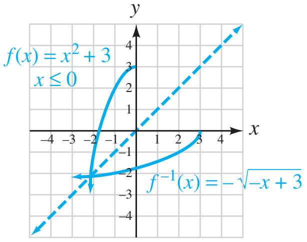
47. $y = \pm\sqrt{x - 4}$, no

49. $f^{-1}(x) = \sqrt[3]{x}$, yes

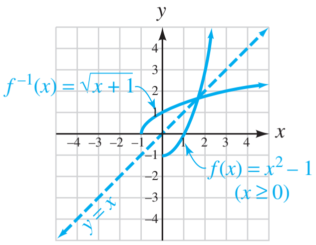
51. $x = |y|$, no

53. $f^{-1}(x) = \sqrt[3]{\frac{x + 3}{2}}$, yes

55.



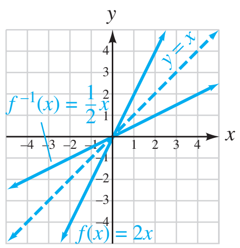
57.



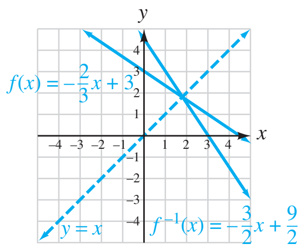
59. $f^{-1}(x) = \dfrac{x+1}{5}$

61. $f^{-1}(x) = 5x + 4$

63. $f^{-1}(x) = \dfrac{1}{2}x$



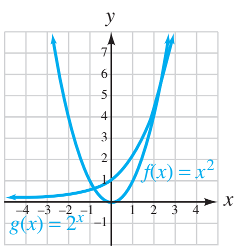
65. $f^{-1}(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$



67. a. yes, no **b.** No. Twice during this period, the person's anxiety level was at the maximum threshold value.
73. $3 - 8i$ **75.** $18 - i$ **77.** $-28 - 96i$

Study Set Section 9.3 (page 799)

1. exponential **3.** $(0, \infty)$ **5.** none **7.** increasing
9.



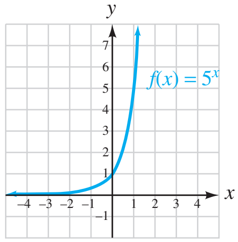
11. $A = P\left(1 + \dfrac{r}{k}\right)^{kt}$, $FV = PV(1 + i)^n$

13. $g(x) = 2^x + 3, h(x) = 2^x - 2$ **15.** $\left(1 + \dfrac{r}{k}\right), kt$

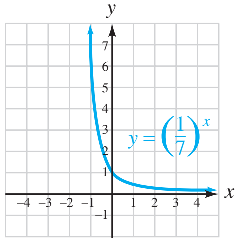
17. 2.6651 **19.** 36.5548 **21.** 8 **23.** b^5 **25.** $7^{3\sqrt{3}}$

27. $b^{5\sqrt{10}}$ **29.** $3^{\sqrt{7}}$ **31.** $\dfrac{1}{5^{\sqrt{5}}}$

33.



35.



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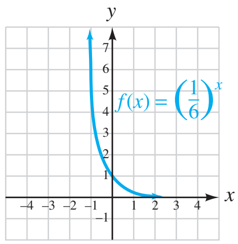
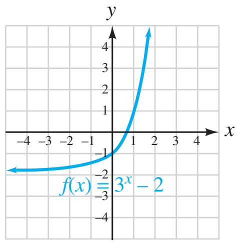
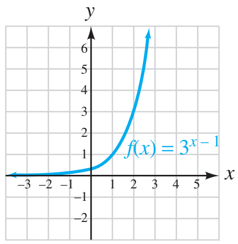


39. increasing
41. decreasing

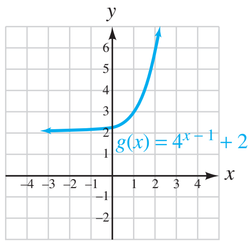
43.



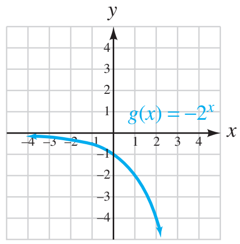
45.



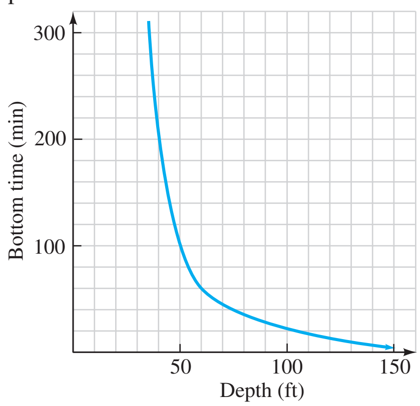
47.



49.



51. \$22,080.40 **53.** \$32.03 **55.** \$2,273,996.13
57. a. about 1500, about 1825 **b.** 6.5 billion
c. exponential
59.

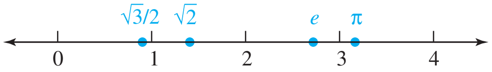


61. $\dfrac{32}{243}A_0$ **63.** 2,295 **65.** about \$2,346,230 **69.** 40
71. 120°

Study Set Section 9.4 (page 808)

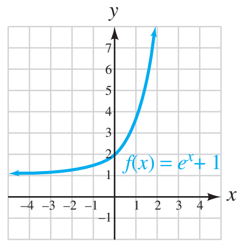
1. the natural exponential function **3.** $(0, \infty)$ **5.** none
7. increasing **9.** continuous **11.** 2.72

13.



15. an exponential function **17.** 2.7182818 . . . ; e
19. 1,000, 10, 0.9, 1,000 **21.** 24,765.16 **23.** 3,811,325.37

25.



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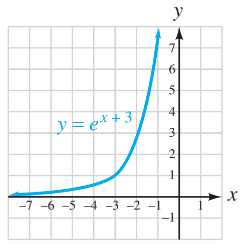

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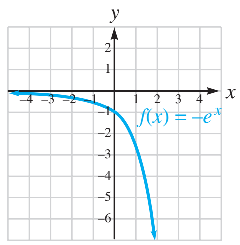

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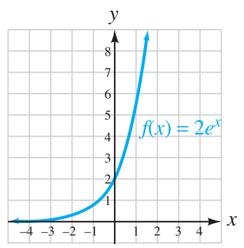
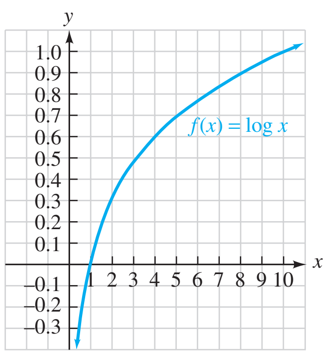


33. 11,542.14 **35.** 542.85 **37.** about 168 years
39. about 57 years **41.** \$13,375.68
43. \$7,518.28 from annual compounding, \$7,647.95 from continuous compounding **45.** \$6,849.16
47. about 9.3 billion **49.** 315 **51.** 31.5 mm **53.** 12 hr
55. 49 mps **57.** 14 **59.** about 72 yr **63.** $4x^2\sqrt{15x}$
65. $10y\sqrt{3y}$

Study Set Section 9.5 (page 822)

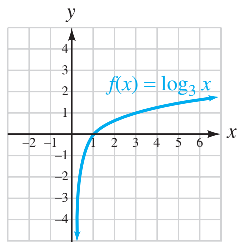
1. logarithmic **3.** $(-\infty, \infty)$ **5.** yes **7.** increasing **9.** x, b^y
11. inverse **13.** 2, −2 **15.** 1, undefined, undefined
17. a. −0.30, 0, 0.30, 0.60, 0.78, 0.90, 1
b.



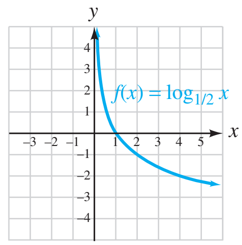
19. a. 10 **b.** x
21. $3^4 = 81$
23. $10^1 = 10$
25. $4^{-3} = \frac{1}{64}$
27. $5^{1/2} = \sqrt{5}$
29. $\log_8 64 = 2$
31. $\log_4 \frac{1}{16} = -2$
33. $\log_{1/2} 32 = -5$
35. $\log_x z = y$
37. 9

39. 64 **41.** 3 **43.** $\frac{1}{25}$ **45.** $\frac{1}{6}$ **47.** 10 **49.** $\frac{2}{3}$ **51.** 5
53. 1,000 **55.** 4 **57.** 1 **59.** $\frac{1}{3}$ **61.** 3 **63.** 2 **65.** 6
67. −1 **69.** 5 **71.** $\frac{1}{2}$ **73.** 0.5119 **75.** −2.3307
77. 6,043.6597 **79.** 0.1726 **81.** 0.3162 **83.** 0.0195

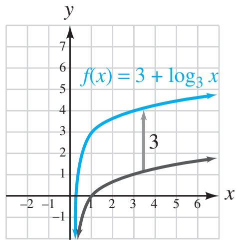
85. increasing



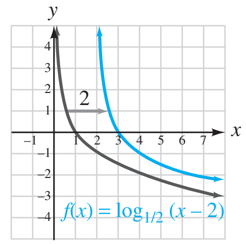
87. decreasing



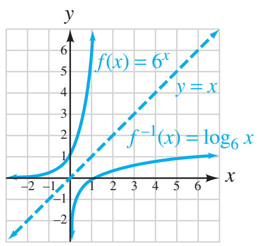
89.



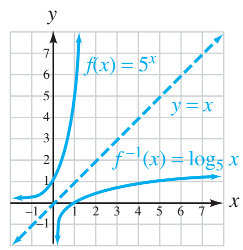
91.



93.



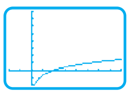
95.



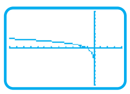
97. 29 db **99.** 49.5 db **101.** 4.4 **103.** 2,500 micrometers
105. 77.8% **107.** 10.8 yr **113.** 10
115. 0; $-\frac{5}{9}$ does not check

Study Set Section 9.6 (page 830)

1. natural
3. −0.69, 0, 0.69, 1.10, 1.39, 1.61, 1.79, 1.95, 2.08, 2.20, 2.30
5. y-axis **7.** $(-\infty, \infty)$ **9.** x **11.** The logarithm of a negative number or 0 is not defined. **13.** 2.7182818 . . . , e
15. e **17.** $\frac{\ln 2}{r}$ **19.** 5 **21.** 6 **23.** −6 **25.** $\frac{1}{4}$ **27.** $\frac{2}{3}$
29. −7 **31.** 3.5596 **33.** −5.3709 **35.** 0.5423
37. undefined **39.** 4.0645 **41.** 69.4079 **43.** 0.0245
45. 2.7210
47.



49.



51. 5.8 yr **53.** 9.2 yr
55. about 3.5 hr **59.** $y = 5x$ **61.** $y = -\frac{3}{2}x + \frac{13}{2}$
63. $x = 2$

Study Set Section 9.7 (page 840)

1. product **3.** power **5.** 0 **7.** M, N **9.** x, y **11.** x
13. $\neq$ **15.** $\log_a b$ **17.** $10^0 = 1, 10^1 = 10, 10^2 = 10^2$
19. rs, t, r, s **21.** 0 **23.** 7 **25.** 10 **27.** 2 **29.** 1 **31.** 7
33. $2 + \log_2 5$ **35.** $1 + \log x$ **37.** $\ln 12 - \ln 5$
39. $\log_6 x - 2$ **41.** $\log x + \log y + \log z$
43. $1 + \log_2 x - \log_2 y$
45. $2 \log_4 5$ **47.** $\frac{1}{2} \log 5$ **49.** $3 \log x + 2 \log y$
51. $\frac{1}{2}(\log_b x + \log_b y)$ **53.** $\frac{1}{2}(\log_b x + \log_b y)$
55. $2 \ln x + \frac{1}{2} \ln y - \ln z$ **57.** $\log_a(x^3 y^{1/3})$ **59.** $\log_b \frac{z^{1/2}}{x^3 y^2}$
61. 1.4472 **63.** -1.1972 **65.** 1.1972 **67.** 1.8063
69. 1.7712 **71.** -1.000 **73.** 1.8928 **75.** 2.3219 **83.** false
85. false **87.** true **89.** $\frac{1}{3} \log_a x - \frac{1}{4} \log_a y - \frac{1}{4} \log_a z$
91. $\ln \dfrac{\frac{x}{z} + x}{\frac{y}{z} + y} = \ln \dfrac{x}{y}$ **93.** $\ln x + \frac{1}{2} \ln z$ **95.** 4.8
97. from 2.5×10^{-8} to 1.6×10^{-7} **101.** $-\frac{7}{6}$ **103.** $\left(1, -\frac{1}{2}\right)$

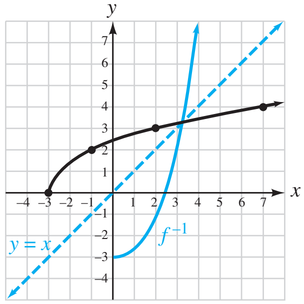
Study Set Section 9.8 (page 850)

1. exponential **3.** $A_o 2^{-th}$ **5.** about 1.8
7. logarithm, exponent **9.** 1.2920 **11.** 1 **13.** $10^2 = x + 1$
15. ± 3.46 **17.** 1,001 **19.** no **21.** log, log 2, log 2 **23.** 8
25. 3, -1 **27.** 1.1610 **29.** 1.2091 **31.** 1.7095 **33.** 0
35. 0.7324 **37.** -13.2662 **39.** 20 **41.** 100 **43.** -7 **45.** 3
47. 50 **49.** 20 **51.** 2, 4 **53.** 7 **55.** 1.2702 **57.** $-\frac{3}{4}$
59. ± 1.0878 **61.** 0, 1.0566 **63.** $-2, -2$ **65.** 9,998
67. -93 **69.** $e \approx 2.7183$ **71.** 19.0855 **73.** 2 **75.** 4
77. 10, -10 **79.** 10 **81.** 10 **83.** no solution **85.** 9
87. 4 **89.** 1, 7 **91.** 1.8 **93.** 8.8, 0.2 **95.** 20 **97.** 8
99. 5.1 yr **101.** 42.7 days **103.** about 4,200 yr
105. 5.6 yr **107.** 5.4 yr **109.** because $\ln 2 \approx 0.7$
111. 25.3 yr **113.** 2.828 times larger **115.** 13.3
117. $\frac{1}{3} \ln 0.75$ **121.** 0, 5 **123.** $\frac{2}{3}, -4$ **125.** $\sqrt{137}$ in.

Chapter 9 Review Exercises (page 854)

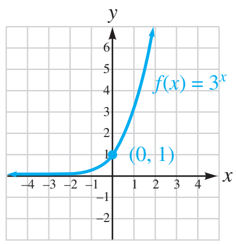
1. $(f + g)(x) = 3x + 1, (-\infty, \infty)$
2. $(f - g)(x) = x - 1, (-\infty, \infty)$
3. $(f \cdot g)(x) = 2x^2 + 2x, (-\infty, \infty)$
4. $(f/g)(x) = \frac{2x}{x + 1}, (-\infty, -1) \cup (-1, \infty)$ **5.** 3 **6.** 5
7. $(f \circ g)(x) = 4x^2 + 4x + 3$ **8.** $(g \circ f)(x) = 2x^2 + 5$
9. no **10.** yes **11.** yes **12.** no **13.** yes **14.** no
15. $-6, -1, 7, 20$

16.

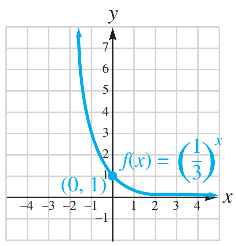


17. $f^{-1}(x) = \frac{x + 3}{6}$
18. $f^{-1}(x) = \frac{4}{x} + 1$
19. $f^{-1}(x) = \sqrt[3]{x} - 2$
20. $f^{-1}(x) = 6x + 1$
21. $5^{4\sqrt{6}}$
22. $2^{2\sqrt{7}}$

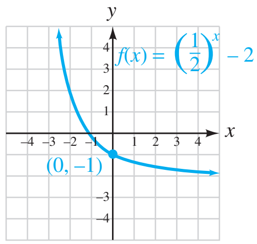
23. D: $(-\infty, \infty)$; R: $(0, \infty)$



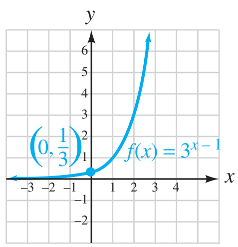
24. D: $(-\infty, \infty)$; R: $(0, \infty)$



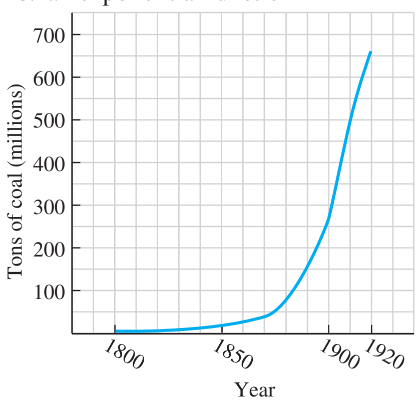
25. D: $(-\infty, \infty)$; R: $(-2, \infty)$



26. D: $(-\infty, \infty)$; R: $(0, \infty)$

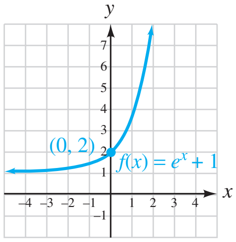


27. the x-axis $(y = 0)$
28. an exponential function

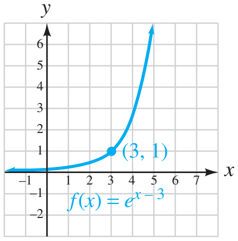


29. $2,189,703.45
30. about $2,015

31. D: $(-\infty, \infty)$; R: $(1, \infty)$

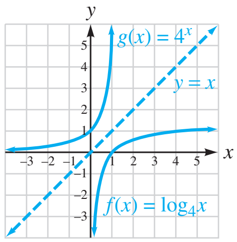


32. D: $(-\infty, \infty)$; R: $(0, \infty)$

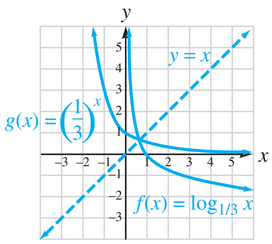


33. 2,324,767.37 **34.** 8,838,365 **35.** 13.9%, 11.67%, 9.80%
36. The exponent on the base e is negative.
37. D: $(0, \infty)$; R: $(-\infty, \infty)$
38. Since there is no real number such that $10^? = 0$, log 0 is undefined. **39.** $4^3 = 64$ **40.** $\log_7 \frac{1}{7} = -1$ **41.** 2 **42.** -2
43. 0 **44.** undefined **45.** $\frac{1}{2}$ **46.** 3 **47.** 32 **48.** $\frac{1}{81}$
49. 4 **50.** 10 **51.** $\frac{1}{2}$ **52.** $\frac{1}{3}$ **53.** 0.6542 **54.** 26.9153

55.



56.



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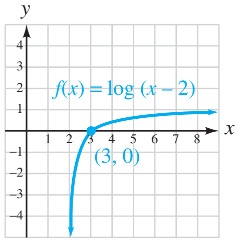

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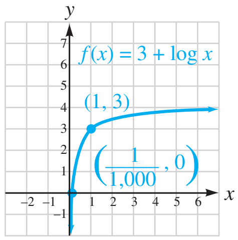
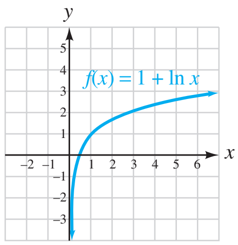
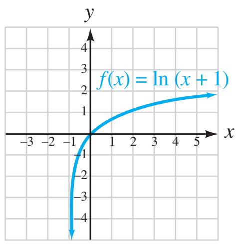


59. about 53 **60.** about 4.4 **61.** 1 **62.** 2 **63.** -5
64. $\frac{1}{2}$ **65.** undefined **66.** undefined **67.** 0 **68.** -7
69. 6.1137 **70.** -0.1625 **71.** 10.3398 **72.** 0.0002
73. They have different bases: $\log x = \log_{10} x$ and $\ln x = \log_e x$. **74.** $f^{-1}(x) = e^x$

75.



76.

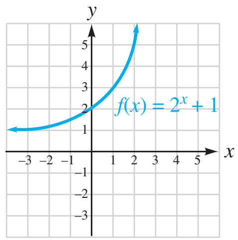


77. about 60 yr **78.** About 72 in. (6 ft) **79.** 0 **80.** 1
81. 3 **82.** 4 **83.** $3 + \log_3 x$ **84.** $2 - \log x$ **85.** $\frac{1}{2}\log_5 27$
86. $\log_b 10 + \log_b a + 1$ **87.** $2\log_b x + 3\log_b y - \log_b z$
88. $\frac{1}{2}(\ln x - \ln y - 2\ln z)$ **89.** $\log_2 \frac{x^3 z^7}{y^5}$ **90.** $\log_b \frac{\sqrt{x+2}}{y^3 z^7}$
91. 2.6609 **92.** 3.0000 **93.** 1.7604 **94.** about 7.9×10^{-4} gram-ions/liter **95.** -4 **96.** $-3, -1$ **97.** 1.7712
98. 2.7095 **99.** 1.9459 **100.** -8.0472 **101.** 104 **102.** 9
103. 25, 4 **104.** 4, -2 extraneous **105.** 4, 3
106. 2 **107.** 6 **108.** 31 **109.** $0.76787 \neq -0.27300$
110. about 3,300 yr **111.** about 91 days **112.** 2, 5

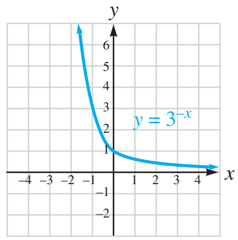
Chapter 9 Test (page 866)

1. $(g + f)(x) = 5x - 1$ **2.** $(g \cdot f)(x) = 4x^2 - 4x$ **3.** 3
4. $4(x - 1)$ **5.** $f^{-1}(x) = \frac{2x - 12}{3}$ **6.** $f^{-1}(x) = \sqrt{\frac{x-4}{3}}$

7.

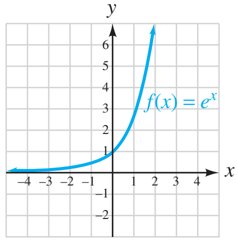


8.



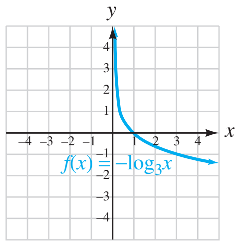
9. $\frac{3}{64} g = 0.046875\, g$ **10.** $1,060.90

11.

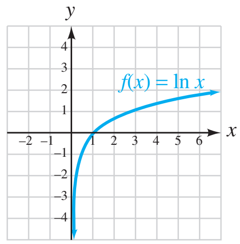


12. $4,451.08 **13.** 2 **14.** 3
15. $\frac{1}{27}$ **16.** e **17.** $6^{-2} = \frac{1}{36}$
18. D: $(0, \infty)$, R: $(-\infty, \infty)$

19.



20.



21. $2\log a + \log b + 3\log c$ **22.** $\ln \frac{b\sqrt{a+2}}{c^3}$ **23.** 0.5646
24. $y = \log x$ **25.** 6.4 **26.** 46 **27.** 0.6826 **28.** 4 **29.** 1
30. 10 **31.** 5 **32.** $\frac{1}{2}\ln(5 - 1) = \frac{1}{2}\ln 4 = 0.69314718\ \ldots,$ which is ln 2. **33.** answers may vary **34.** yes **35.** yes
36. 80; when the temperature of the tire tread is 260°, the vehicle is traveling 80 mph **37.** about 1,631,973,737
38. about 20 min.

Chapters 1–9 Cumulative Review (page 868)

1. $P = 2l + 2w$ **2.** $A = \pi r^2$ **3.** $A = \frac{1}{2}bh$ **4.** $V = s^3$
5. $I = Prt$ **6.** $d = rt$ **7.** $\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$ **8.** $y = mx + b$
9. $y - y_1 = m(x - x_1)$ **10.** $m = \frac{y_2 - y_1}{x_2 - x_1}$ **11.** $a^2 + b^2 = c^2$
12. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **13.** $y = kx$ **14.** $y = \frac{k}{x}$
15. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **16.** $A = Pe^{rt}$
17. $\log_b x = \frac{\log_a x}{\log_a b}$ **18.** x **19.** x^{m+n} **20.** x^{mn} **21.** $x^n y^n$
22. 1 **23.** $\frac{x^n}{y^n}$ **24.** x^{m-n} **25.** $\frac{1}{x^n}$ **26.** x^n **27.** $\left(\frac{y}{x}\right)^n$
28. $\sqrt[n]{x}$ **29.** $\left(\sqrt[n]{x}\right)^m$ **30.** $(x + y)(x - y)$
31. $(x - y)(x^2 + xy + y^2)$ **32.** $(x + y)(x^2 - xy + y^2)$
33. $x^2 + 2xy + y^2$ **34.** $x^2 - 2xy + y^2$ **35.** $x^2 - y^2$
36. $\sqrt[n]{a}\sqrt[n]{b}$ **37.** $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ **38.** 3.14 **39.** 1.41 **40.** 2.72
41. Pick a test point on one side of the boundary line. In $y \geq x$, replace x and y with the coordinates of that point. If the inequality is satisfied, shade the side that contains that point. If the inequality is not satisfied, shade the other side.
42. 0 **43.** 1 **44.** x **45.** x **46.** $\log_b M + \log_b N$
47. $\log_b M - \log_b N$ **48.** $p \log_b M$ **49.** k, k
50. to build up the fractions so they have the same denominator **51.** to rationalize the denominator
52. to simplify a complex fraction **53.** It was used to simplify a rational expression. The slashes and 1's show a common factor of $3a - 2$ being divided out. **54.** bc
55. an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$
56. $x + 1 = 0$ or $x - 7 = 0$ **57.** Multiply both sides of the equation by the LCD, which is $x(x - 2)$. **58.** the addition method **59.** $x, -x$ **60.** b^2 **61.** $|x|$ **62.** x **63.** $+$
64. $\sqrt{-1}$ **65.** domain, range, one **66.** both
67. one, both

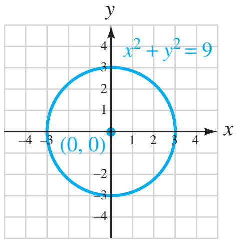
Think It Through (page 876)

$\frac{21}{2}$ in. $= 10.5$ in.

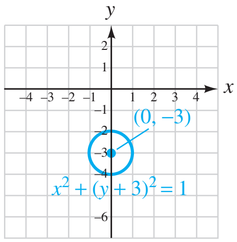
Study Set Section 10.1 (page 883)

1. conic sections **3.** circle, radius
5. a. $(x - h)^2 + (y - k)^2 = r^2$ **b.** $x^2 + y^2 = r^2$
7. a. $(2, -1), r = 4$ **b.** $(x - 2)^2 + (y + 1)^2 = 16$
9. a. $y = a(x - h)^2 + k$ **b.** $x = a(y - k)^2 + h$
11. a. circle **b.** parabola **c.** parabola **d.** circle
13. $6, -2, 3$

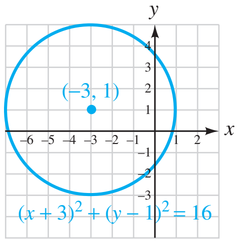
15. $(0, 0), r = 3$



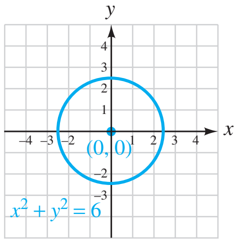
17. $(0, -3), r = 1$



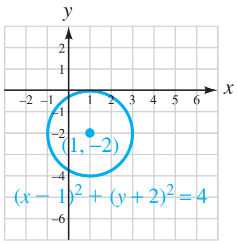
19. $(-3, 1), r = 4$



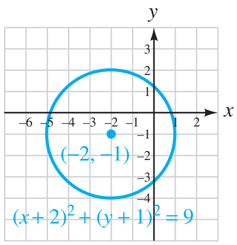
21. $(0, 0), r = \sqrt{6} \approx 2.4$



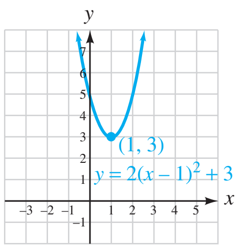
23. $x^2 + y^2 = 1$ **25.** $(x - 6)^2 + (y - 8)^2 = 25$
27. $(x + 2)^2 + (y - 6)^2 = 144$ **29.** $x^2 + y^2 = \frac{1}{16}$
31. $\left(x - \frac{2}{3}\right)^2 + \left(y + \frac{7}{8}\right)^2 = 2$ **33.** $x^2 + y^2 = 8$
35. $(x - 1)^2 + (y + 2)^2 = 4; (1, -2), r = 2$



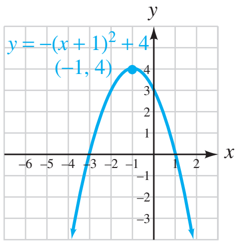
37. $(x + 2)^2 + (y + 1)^2 = 9; (-2, -1), r = 3$



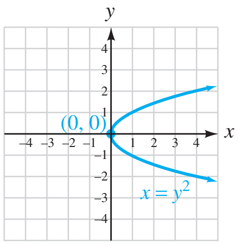
39. $y = 2(x - 1)^2 + 3$; vertex: $(1, 3)$



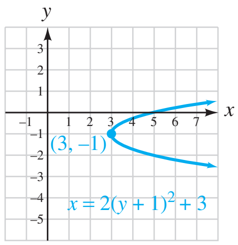
41. $y = -(x + 1)^2 + 4$; vertex: $(-1, 4)$



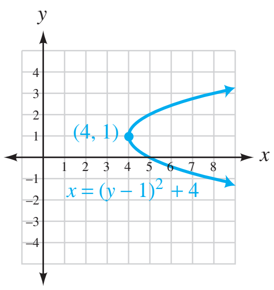
43. vertex: $(0, 0)$



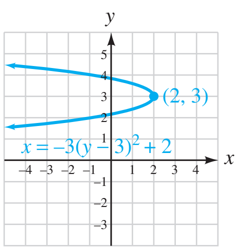
45. vertex: $(3, -1)$



47. $x = (y - 1)^2 + 4$, vertex: $(4, 1)$



49. $x = -3(y - 3)^2 + 2$, vertex: $(2, 3)$



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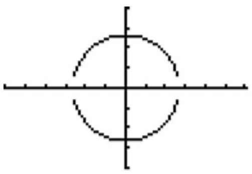

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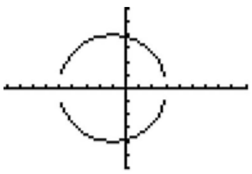

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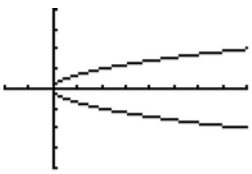

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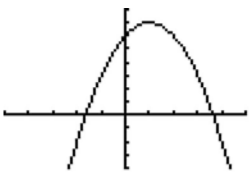
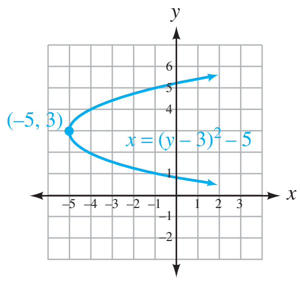
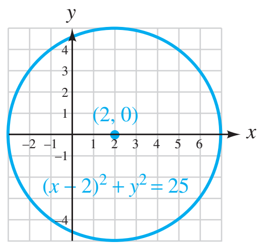


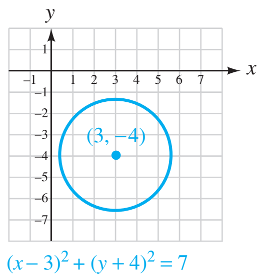
59. $x = (y - 3)^2 - 5$, vertex: $(-5, 3)$



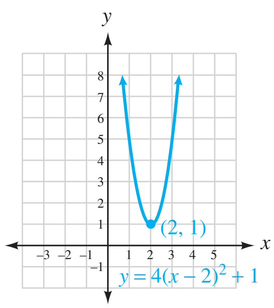
61. $(2, 0)$, $r = 5$



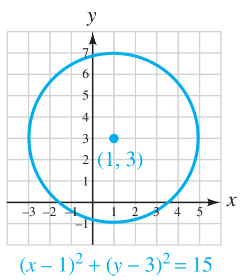
63. $(x - 3)^2 + (y + 4)^2 = 7$, $(3, -4)$, $r = \sqrt{7} \approx 2.6$



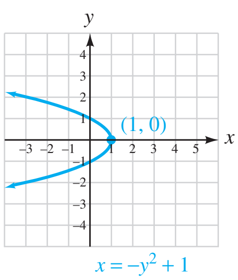
65. $y = 4(x - 2)^2 + 1$, vertex: $(2, 1)$



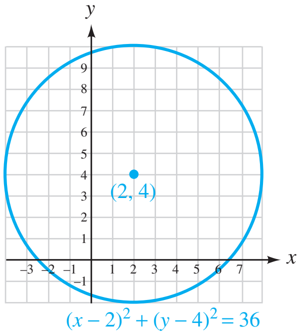
67. $(1, 3)$, $r = \sqrt{15} \approx 3.9$



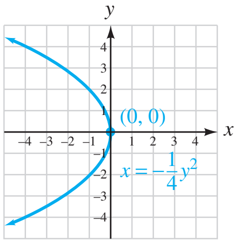
69. vertex: $(1, 0)$



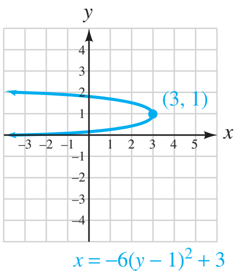
71. $(2, 4)$, $r = 6$



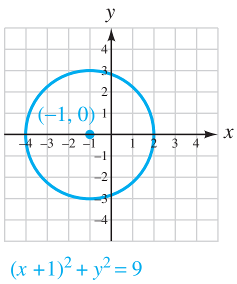
73. vertex: $(0, 0)$



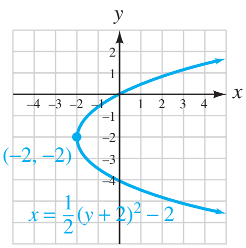
75. vertex: $(3, 1)$



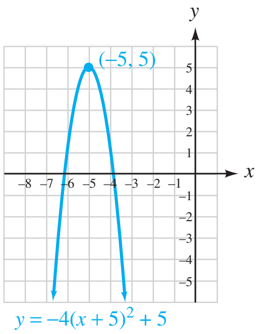
77. $(x + 1)^2 + y^2 = 9$, $(-1, 0)$, $r = 3$



79. $x = \frac{1}{2}(y + 2)^2 - 2$, vertex: $(-2, -2)$



81. vertex: $(-5, 5)$



83. no **85.** **a.** 8 mi **b.** 9 mi **87.** 5 ft **89.** 2 AU

95. $5, -\frac{7}{3}$ **97.** $3, -\frac{1}{4}$

Study Set Section 10.2 (page 896)

1. ellipse **3.** foci, focus **5.** major **7.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

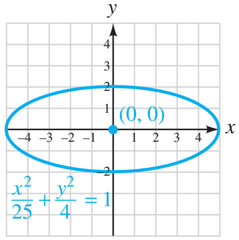
9. x-intercepts: $(a, 0)$, $(-a, 0)$; y-intercepts: $(0, b)$, $(0, -b)$

11. **a.** $(-2, 1)$; $a = 2$, $b = 5$ **b.** vertical

c. $\frac{(x + 2)^2}{4} + \frac{(y - 1)^2}{25} = 1$ **13.** $\frac{(x - 1)^2}{16} + \frac{(y + 5)^2}{1} = 1$

15. $h = -8$, $k = 6$, $a = 10$, $b = 12$

17.



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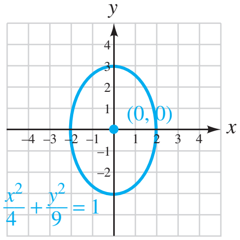

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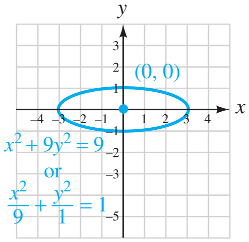

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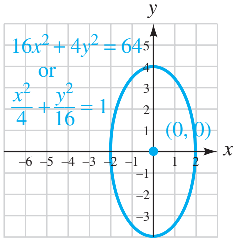

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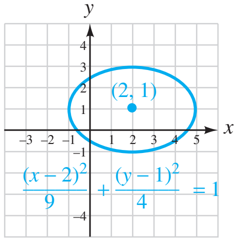

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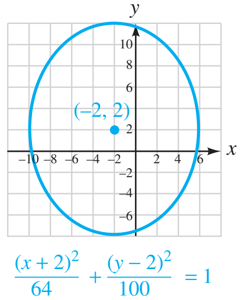

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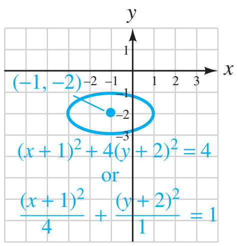

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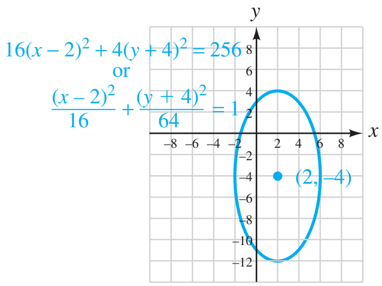

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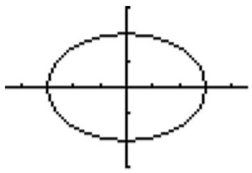

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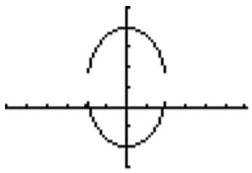

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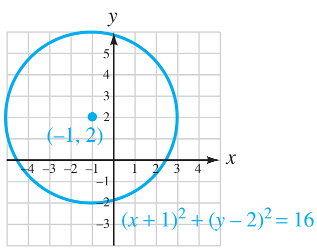

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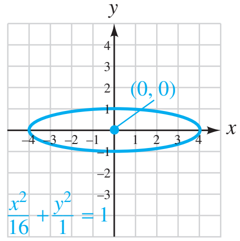

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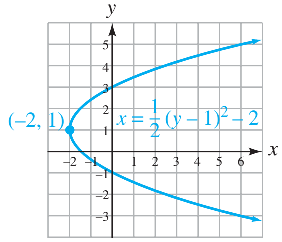

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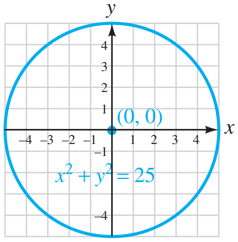

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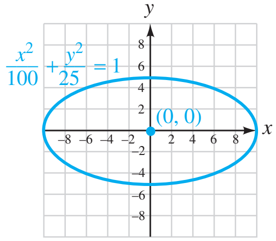

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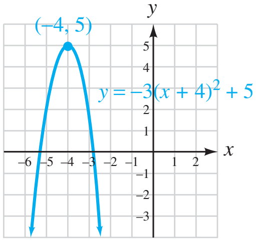

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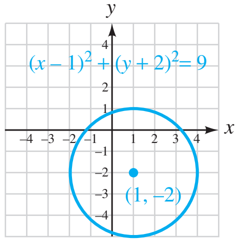

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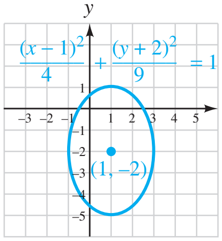


53. $\dfrac{x^2}{144} + \dfrac{y^2}{25} = 1$ **55.** $5\sqrt{3}$ ft ≈ 8.7 ft

57. 12π sq. units ≈ 37.7 sq. units **63.** $12y^2 + \dfrac{9}{x^2}$

65. $\dfrac{y^2 + x^2}{y^2 - x^2}$

Study Set Section 10.3 (page 909)

1. hyperbola **3.** vertices **5.** diagonals

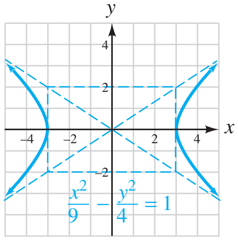
7. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ **9.** $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

11. a. $(-1, -2)$; $a = 3, b = 1$ **b.** $\dfrac{(y+2)^2}{9} - \dfrac{(x+1)^2}{1} = 1$

13. $\dfrac{(x+1)^2}{1} - \dfrac{(y-5)^2}{4} = 1$

15. $h = 5, k = -11, a = 5, b = 6$

17.



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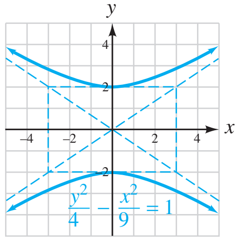

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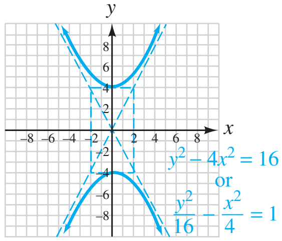

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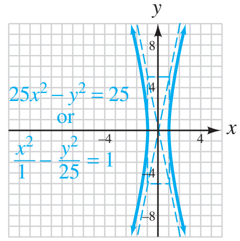

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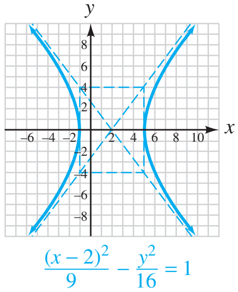

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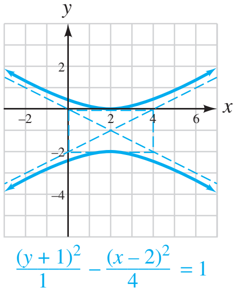

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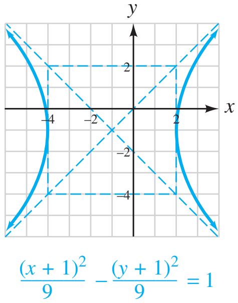

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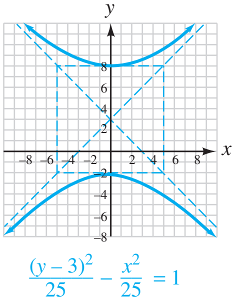

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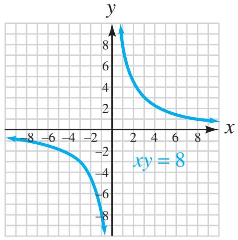

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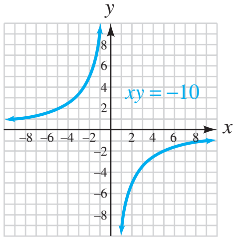

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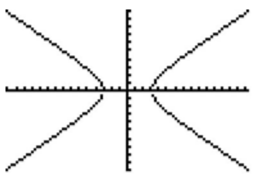

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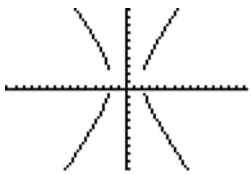

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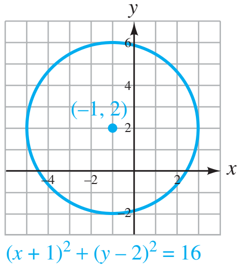

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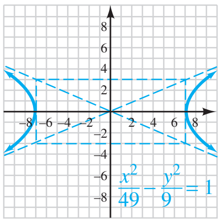

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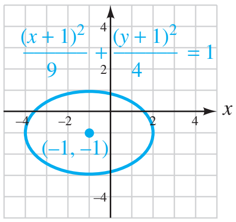

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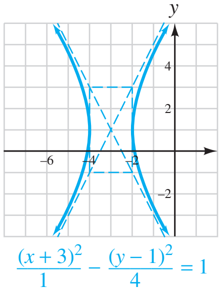

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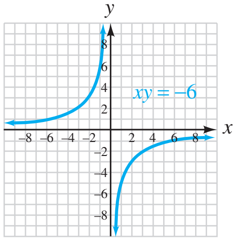

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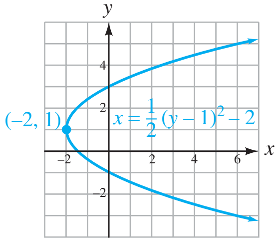
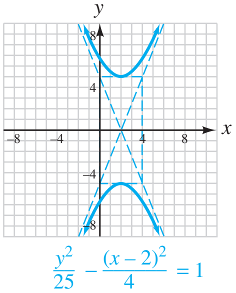


53.



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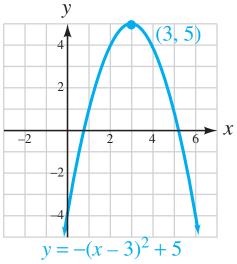

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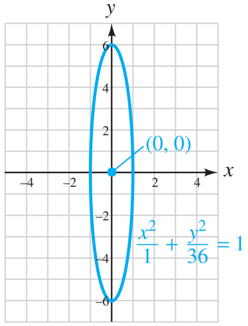
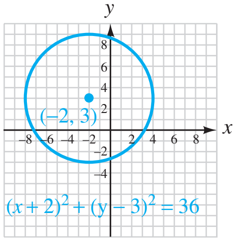


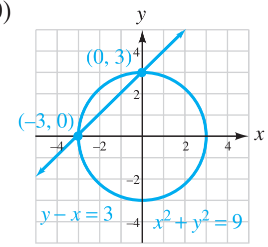
59.



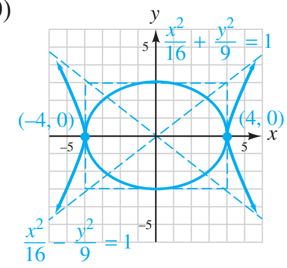
61. 3 units **63.** $10\sqrt{3}$ miles **69.** 64 **71.** 3 **73.** $\frac{3}{2}$
75. 10

Study Set Section 10.4 (page 917)

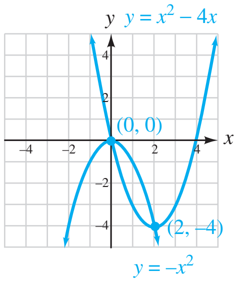
1. system **3.** intersection **5.** secant **7. a.** two **b.** four **c.** four **d.** four **9.** $(-3, 2), (3, 2), (-3, -2), (3, -2)$
11. a. -4 **b.** -2 **13.** $2x, 5, 5, 1, 1, -1, -2$
15. $(0, 3), (-3, 0)$



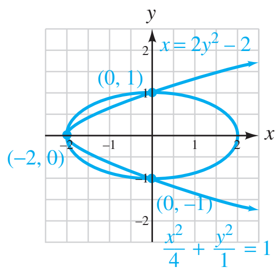
17. $(-4, 0), (4, 0)$



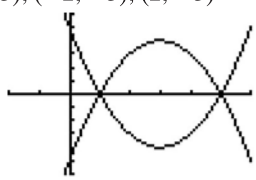
19. $(0, 0), (2, -4)$



21. $(-2, 0), (0, -1), (0, 1)$



23. $(1, 2), (2, 1)$ **25.** $(-6, 7), (-2, -1)$ **27.** $(-2, 3), (2, 3)$
29. $(\sqrt{5}, 5), (-\sqrt{5}, 5)$
31. $(2, 4), (2, -4), (-2, 4), (-2, -4)$ **33.** $(3, 0), (-3, 0)$
35. $(\sqrt{3}, 0), (-\sqrt{3}, 0)$
37. $(-2, 3), (2, 3), (-2, -3), (2, -3)$
39. $(1, 0), (5, 0)$

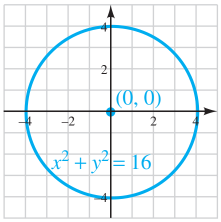


41. $(2, 1), (-2, 1), (2, -1), (-2, -1)$
43. $(0, -4), (-3, 5), (3, 5)$
45. $(6, 2), (-6, -2), (-\sqrt{42}, 0), (\sqrt{42}, 0)$
47. $(-\sqrt{15}, 5), (\sqrt{15}, 5), (-2, -6), (2, -6)$
49. $(0, 3), \left(-\frac{25}{12}, -\frac{13}{4}\right)$ **51.** $(0, 0), (2, 4)$
53. no solution, $\varnothing$ **55.** $(3, 5)$

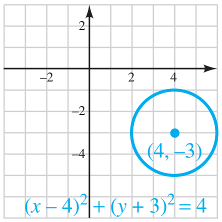
57. $\left(\frac{\sqrt{10}}{4}, \frac{3\sqrt{6}}{4}\right), \left(\frac{\sqrt{10}}{4}, -\frac{3\sqrt{6}}{4}\right), \left(-\frac{\sqrt{10}}{4}, \frac{3\sqrt{6}}{4}\right), \left(-\frac{\sqrt{10}}{4}, -\frac{3\sqrt{6}}{4}\right)$ **59.** $\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right)$ **61.** $(-1, 3), (1, 3)$
63. $\left(\frac{5}{2}, \frac{3}{2}\right)$ **65.** 4, 8 **67.** $\left(10, \frac{10}{3}\right), \frac{10}{3}\sqrt{10}$ m
69. 80 ft by 100 ft or 50 ft by 160 ft
71. \$2,500 at 9% **75.** 2,000 **77.** 7

Chapter 10 Review Exercises (page 922)

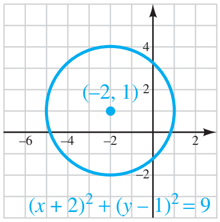
1.



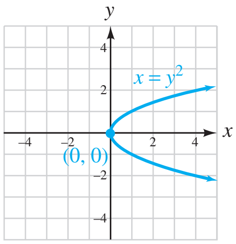
2.

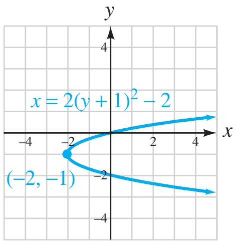


3. $(x + 2)^2 + (y - 1)^2 = 9$

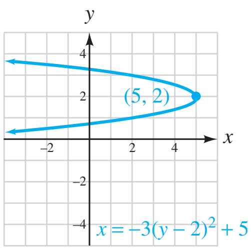


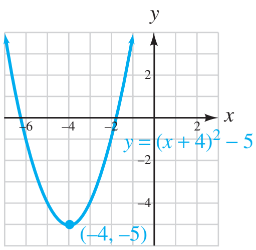
4. $(x - 9)^2 + (y - 9)^2 = 9^2$ or $(x - 9)^2 + (y - 9)^2 = 81$
5. $(-6, 0), r = 2\sqrt{6}$ **6.** center, radius
7. $(0, 0)$ **8.** $(-2, -1)$





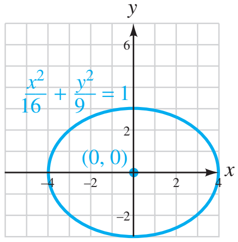
9. $x = -3(y - 2)^2 + 5$ **10.** $y = (x + 4)^2 - 5$



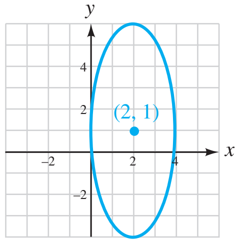


11. $(2, -2), (8, -3)$
12. When $x = 22$, $y = 0$: $-\frac{5}{121}(22 - 11)^2 + 5 = 0$

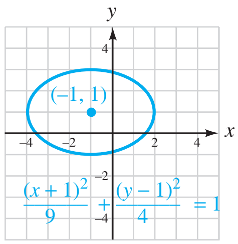
13.



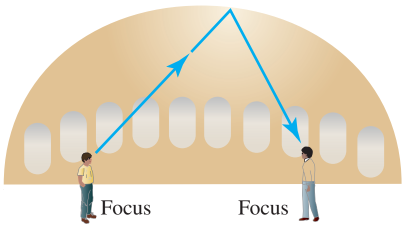
14.



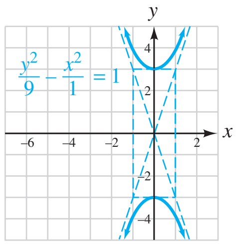
15.



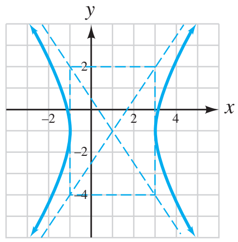
16. $\frac{x^2}{12^2} + \frac{y^2}{1^2} = 1$
17. a. circle **b.** ellipse **c.** parabola **d.** ellipse
18. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
19. ellipse, focus
20. answers may vary



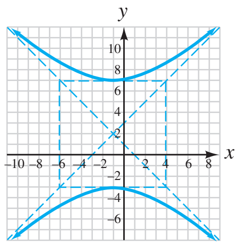
21.



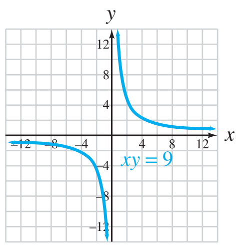
22.



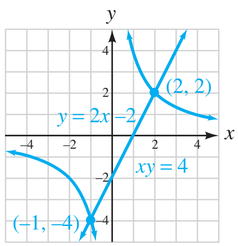
23.



24.



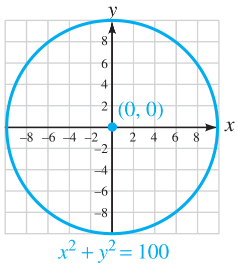
25. 4 units **26. a.** ellipse **b.** hyperbola **c.** parabola
d. circle **27.** yes **28.** $(0, 3), (0, -3)$
29. $(2, 2), (-1, -4)$



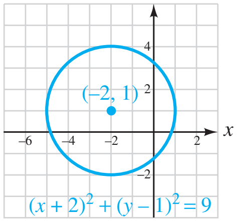
30. a. 2 **b.** 4 **c.** 4 **d.** 4 **31.** $(0, 1), (0, -1)$
32. $y = 3x - 1$ **33.** $(0, -4), (-3, 5), (3, 5)$
34. $(\sqrt{2}, 0), (-\sqrt{2}, 0)$ **35.** $(2, 2), \left(-\frac{2}{9}, -\frac{22}{9}\right)$
36. $(4, 2), (4, -2), (-4, 2), (-4, -2)$
37. $(2, 3), (2, -3), (-2, 3), (-2, -3)$
38. $(2\sqrt{2}, \sqrt{2}), (-2\sqrt{2}, -\sqrt{2}), (1, 4), (-1, -4)$
39. no solution, $\varnothing$ **40.** $(-2, 1)$

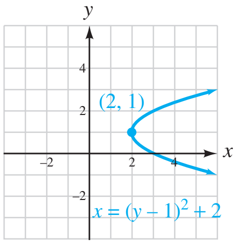
Chapter 10 Test (page 929)

1. a. conic **b.** center, radius **c.** hyperbola **d.** nonlinear **e.** ellipse **2.** $(0, 0), r = 10$



3. $(-2, 3), r = 3\sqrt{2}$ **4.** $(x - 4)^2 + (y - 3)^2 = 9$

5.


6.


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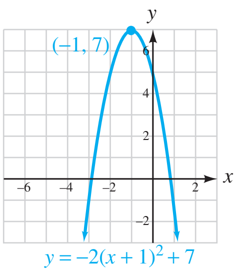
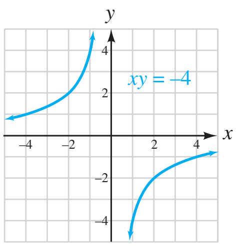
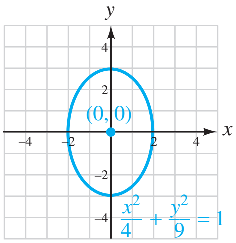
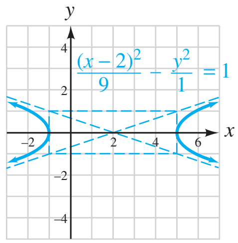


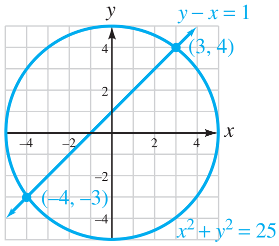
8.


9.


10.


11. $\frac{(x - 1)^2}{16} + \frac{(y + 2)^2}{9} = 1$ **12.** 2.5 in.
13. answers may vary **14.** $(-1, 1)$; length: 4 units, width: 4 units **15.** $\frac{y^2}{16} - \frac{x^2}{36} = 1$ **16. a.** ellipse **b.** hyperbola **c.** circle **d.** parabola

17. $(-4, -3), (3, 4)$

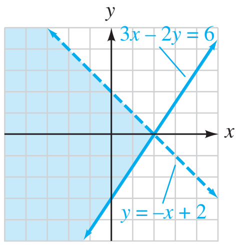


18. $(2, 6), (-2, -2)$
19. $(1, \sqrt{2}), (1, -\sqrt{2}), (-1, \sqrt{2}), (-1, -\sqrt{2})$
20. $\left(-1, \frac{9}{2}\right), \left(3, -\frac{3}{2}\right)$ **21.** $(-1, 0)$ **22.** no solution; $\varnothing$

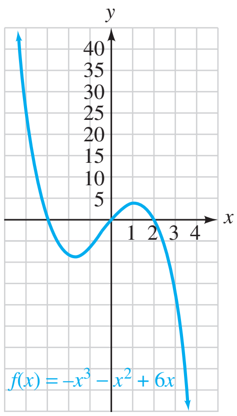
Chapters 1–10 Cumulative Review (page 931)

1. 0 **2.** $-\frac{4}{3}, 5.6, 0, -23$ **3.** $\pi, \sqrt{2}, e$
4. $-\frac{4}{3}, \pi, 5.6, \sqrt{2}, 0, -23, e$ **5.** \$8,250 **6.** $\frac{1}{120}$ db/rpm
7. parallel **8.** perpendicular **9.** $y = -2x + 5$
10. $y = -\frac{9}{13}x + \frac{7}{13}$ **11.** 3 **12.** $(1, 1)$ **13.** $(2, -2)$
14. $(3, 2, 1)$ **15.** 85°, 80°, 15° **16.** $(-1, -1)$ **17.** -1
18. $(-\infty, -2)$


19. $\left[-3, \frac{19}{3}\right]$


20.


21. 24, 0, −8, −6, 0, 4, 0, −18; $(-3, 0), (0, 0), (2, 0)$; $(0, 0)$



22. a. a quadratic function **b.** at about 85% and 120% of the suggested inflation
23. $12x^2 - 5xy - 3y^2$
24. $2x^2y^3 + 13xy + 3y^2$
25. $a^2 - 4ab + 4b^2$
26. $3a^3 + 10a^2 + 6a - 4$
27. $xy(3x - 4y)(x - 2)$
28. $(16x^2y^2 + z^4)(4xy + z^2)(4xy - z^2)$ **29.** $\lambda = \frac{4d - 2}{A - 6}$
30. $\frac{64b^{12}}{27a^9}$ **31.** $-\frac{3x + 2}{3x - 2}$ **32.** $-\frac{q}{p}$ **33.** $\frac{4a - 1}{(a + 2)(a - 2)}$

34. 5; 3 is extraneous **35.** $R = \dfrac{R_1R_2R_3}{R_2R_3 + R_1R_3 + R_1R_2}$

36. $-x^2 + x + 5 + \dfrac{8}{x-1}$ **37.** 2 **38.** $5\sqrt{2}$ **39.** $81x\sqrt[3]{3x}$

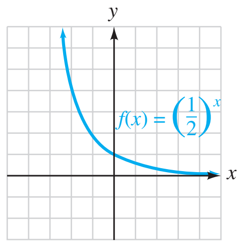
40. $\sqrt{3t} - 1$ **41.** 5, 0 is extraneous **42.** 0 **43.** $\dfrac{343}{125}$

44. about $21\frac{1}{2}$ in. **45.** $-5 + 17i$ **46.** $-\dfrac{21}{29} - \dfrac{20}{29}i$

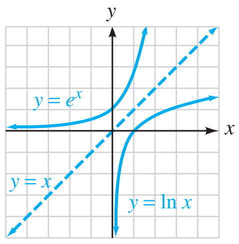
47. $\dfrac{2}{3}, -\dfrac{3}{2}$ **48.** $\dfrac{-3 \pm \sqrt{5}}{4}$ **49.** $\dfrac{2 \pm i\sqrt{2}}{3}$ **50.** $\dfrac{1}{4}, \dfrac{1}{2}$

51. $(f \circ g)(x) = 4x^2 + 4x - 1$ **52.** $f^{-1}(x) = \sqrt[3]{\dfrac{x+1}{2}}$

53.



54.



55. $2^y = x$ **56.** $f^{-1}(x) = 2^x$ **57.** $(\log_6 x) - 2$ **58.** x

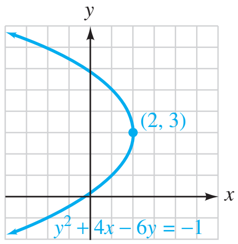
59. 1.9912 **60.** 0.301 **61.** 1.16056 **62.** 5 **63.** 3

64. $\dfrac{1}{27}$ **65.** 1 **66.** 3.4190 **67.** 1, -10 is extraneous

68. $-\dfrac{3}{4}$ **69.** 9 **70.** \$2,848.31

71. $x^2 + y^2 - 2x - 6y - 15 = 0$

72.



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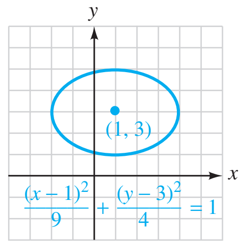
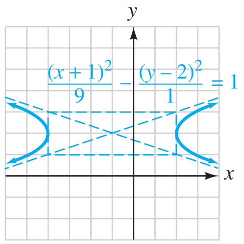


74.



INDEX

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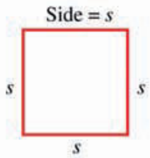
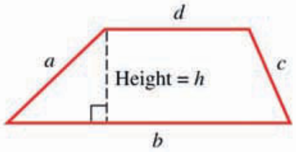
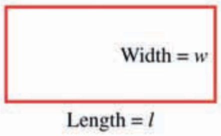
Perimeter and Area Formulas



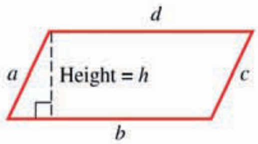
Square
$P = 4s$
$A = s^2$



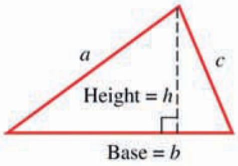
Trapezoid
$P = a + b + c + d$
$A = \frac{1}{2}h(b + d)$



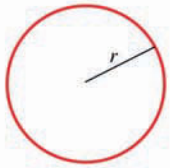
Rectangle
$P = 2l + 2w$
$A = lw$



Parallelogram
$P = a + b + c + d$
$A = bh$

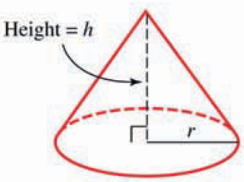


Triangle
$P = a + b + c$
$A = \frac{1}{2}bh$

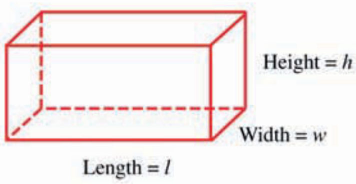


Circle
$C = 2\pi r$ or $C = \pi D$
where $\pi \approx 3.14$
$A = \pi r^2$

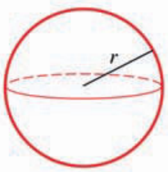
Volume Formulas



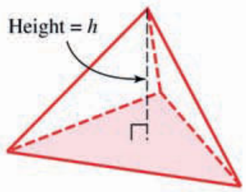
Cone
$V = \frac{1}{3}\pi r^2 h$



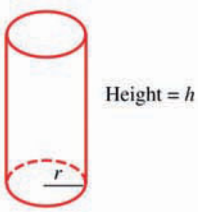
Rectangular solid
$V = lwh$



Sphere
$V = \frac{4}{3}\pi r^3$



Pyramid
$V = \frac{1}{3}Bh$*
*B represents the area of the base.



Cylinder
$V = \pi r^2 h$